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Green's Function for Lamb's Problem

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Summary

The complete solution to the three-dimensional Lamb's problem, the problem of determining the elastic disturbance resulting from a point force in a half space, is derived using the Cagniard–de Hoop method. In addition, spatial derivatives of this solution with respect to both the source co-ordinates and the receiver co-ordinates are derived. The solutions are quite amenable to numerical calculations and a few results of such calculations are given.

1. Introduction

Since the classic paper of Lamb (1904) the problem of the elastic displacements resulting from a point force in a half space has been the subject of numerous studies. A partial list of authors who have treated the three-dimensional problem includes Cagniard (1939), Dix (1954), Pinney (1954), Pekeris (1955a, b), Pekeris & Lifson (1957), de Hoop (1961) ,and Aggarwal & Ablow (1967). More recently, Kawasaki, Suzuki & Sato (1972a, b) and Sato (1972) have solved for the surface displacements resulting from a double-couple source in a half space. Solutions, or at least clear outlines of how the solutions are to be obtained, can be found in the literature for most of the various cases which comprise the general problem. However, it seems that nowhere are all of these solutions collected together with a uniform notation and in a form suitable for numerical calculations. This is one of the purposes of this paper.

The solutions of Lamb's problem can be thought of as the Green's function for the elastic wave equation in a uniform half space, and as such it is the starting point for the consideration of sources more complicated than the simple point force. For instance, such a Green's function is a basic ingredient in the Knopoff–de Hoop representation theorem (Burridge & Knopoff 1964), which is one of the more elegant approaches to the problem of modelling an earthquake. However, in most cases the boundary conditions at an earthquake source are specified in terms of displacements, and we find that it is not the Green's function itself but rather the spatial derivatives of the Green's function which are required. Thus a second purpose of this paper is to present formulas for these spatial derivatives.

In this paper we consider only the three-dimensional problem. The solution to the two-dimensional problem can be obtained by integrating the three-dimensional solution over one spatial dimension, although it is just as easy to derive it from first principles using the methods outlined in this paper.

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2. Statement of the problem

For a uniform elastic material and a Cartesian co-ordinate system the equation for the conservation of linear momentum can be written

$$\rho \frac{\partial^2}{\partial t^2} \mathbf{u}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) + (\lambda+\mu)\nabla(\nabla \mathbf{u}(\mathbf{x}, t)) + \mu \nabla^2 \mathbf{u}(\mathbf{x}, t) \tag{1}$$

where t is time, $\mathbf{x}$ is the location vector, $\mathbf{u}$ is the displacement, $\mathbf{f}$ is the force which is the source of the elastic disturbance, ρ is the density, and λ and μ are the Lamé constants. The vector equation (1) can be separated into three scalar equations, and thus we actually have three independent problems to solve. In what follows it will be convenient to derive the solutions to these three problems simultaneously, but it should be kept in mind that we are actually doing three independent problems at the same time.

We consider a half space with $x_3 = 0$ defining the free surface and positive x_3 pointing into the half space (Fig. 1). Initially we will formulate the problem in Cartesian co-ordinates but later on the cylindrical and spherical co-ordinates which are also shown in Fig. 1 will be useful.

The stresses on any plane of constant x_3 are

$$\left.\begin{aligned} T_{13}(\mathbf{x}, t) &= \mu\left(\frac{\partial}{\partial x_3} u_1(\mathbf{x}, t) + \frac{\partial}{\partial x_1} u_3(\mathbf{x}, t)\right) \\ T_{23}(\mathbf{x}, t) &= \mu\left(\frac{\partial}{\partial x_3} u_2(\mathbf{x}, t) + \frac{\partial}{\partial x_2} u_3(\mathbf{x}, t)\right) \\ T_{33}(\mathbf{x}, t) &= \lambda\left(\frac{\partial}{\partial x_1} u_1(\mathbf{x}, t) + \frac{\partial}{\partial x_2} u_2(\mathbf{x}, t) + \frac{\partial}{\partial x_3} u_3(\mathbf{x}, t)\right) + 2\mu \frac{\partial}{\partial x_3} u_3(\mathbf{x}, t) \end{aligned}\right\} \tag{2}$$

The basic problem is to solve equation (1) subject to the condition that the stresses of equation (2) vanish when $x_3 = 0$.

We will consider the case where the source function is localized in both time and space.

$$\mathbf{f}(\mathbf{x}, t) = (f_1 \hat{\mathbf{x}}_1 + f_2 \hat{\mathbf{x}}_2 + f_3 \hat{\mathbf{x}}_3)\,\delta(x_1 - x'_1)\,\delta(x_2 - x'_2)\,\delta(x_3 - x'_3)\,\delta(t - t'). \tag{3}$$

For such a source we will refer to the displacement solution as a Green's function and use the standard notation

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \mathbf{g}(\mathbf{x}, t; \mathbf{x}', t') \\ &= g_1(\mathbf{x}, t; \mathbf{x}', t')\,\hat{\mathbf{x}}_1 + g_2(\mathbf{x}, t; \mathbf{x}', t')\,\hat{\mathbf{x}}_2 + g_3(\mathbf{x}, t; \mathbf{x}', t')\,\hat{\mathbf{x}}_3. \end{aligned} \tag{4}$$

For the problem being considered there is no loss in generality if we take $x'_1 = x'_2 = t' = 0$ as implied by Fig. 1.

3. The solution in the transform domain

We now proceed to solve equation (1) with the particular source given by equation (3). This differential equation can be reduced to the following set of algebraic equations (written in matrix form) by taking Laplace transforms with respect to t, x_1,

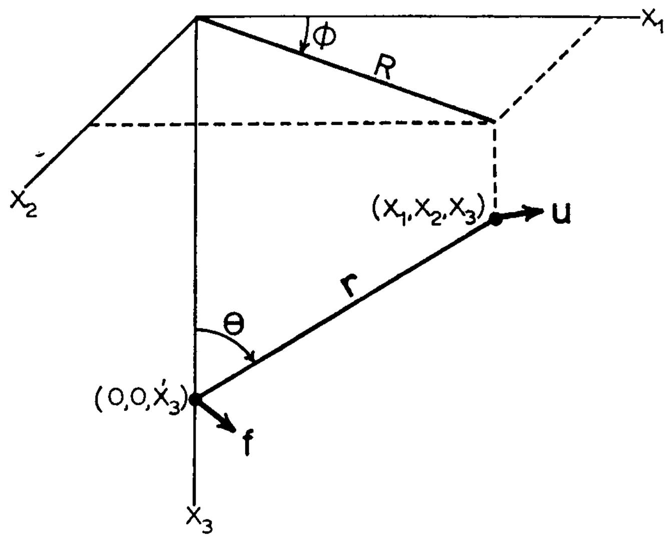

FIG. 1. The geometry of the problem. The displacement **u** at the position (x_1, x_2, x_3) resulting from the force **f** at the position $(0, 0, x'_3)$ is to be determined. The plane $x_3 = 0$ is a free surface.

x_2, and x_3. The corresponding transform variables are s, ξ_1, ξ_2, and ξ_3.

$$\begin{bmatrix} \xi_1^2+\frac{\mu}{\lambda+\mu}(\xi_3^2-\nu_\beta^2) & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2+\frac{\mu}{\lambda+\mu}(\xi_3^2-\nu_\beta^2) & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2+\frac{\mu}{\lambda+\mu}(\xi_3^2-\nu_\beta^2) \end{bmatrix}$$

$$\times \mathbf{G}(\boldsymbol{\xi}, s; 0, 0, x'_3, 0) = \frac{-\exp(-\xi_3 x'_3)}{\lambda+\mu}\mathbf{F} \qquad (5)$$

where **G** and **F** are the column matrices

$$\mathbf{G}(\boldsymbol{\xi}, s; 0, 0, x'_3, 0) = \begin{Bmatrix} g_1(\boldsymbol{\xi}, s; 0, 0, x'_3, 0) \\ g_2(\boldsymbol{\xi}, s; 0, 0, x'_3, 0) \\ g_3(\boldsymbol{\xi}, s; 0, 0, x'_3, 0) \end{Bmatrix} \qquad \mathbf{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}. \qquad (6)$$

Note that the Laplace transform with respect to t is the ordinary one-sided Laplace transform while those with respect to x_1, x_2, and x_3 are the generalized two-sided Laplace transforms. In the above equations we have used the following definitions.

$$\left.\begin{array}{ll} \alpha = \left(\frac{\lambda+2\mu}{\rho}\right)^{\frac{1}{2}} & \beta = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}} \\ \kappa_\alpha = s/\alpha & \kappa_\beta = s/\beta \\ \mathrm{Re}\{(\kappa_\alpha^2-\xi_1^2)^{\frac{1}{2}}\} \geqslant 0 & \mathrm{Re}\{(\kappa_\beta^2-\xi_1^2)^{\frac{1}{2}}\} \geqslant 0 \\ \nu_\alpha = (\kappa_\alpha^2-\xi_1^2-\xi_2^2)^{\frac{1}{2}} & \mathrm{Re}\{\nu_\alpha\} \geqslant 0 \\ \nu_\beta = (\kappa_\beta^2-\xi_1^2-\xi_2^2)^{\frac{1}{2}} & \mathrm{Re}\{\nu_\beta\} \geqslant 0 \end{array}\right\} \qquad (7)$$

The following conditions insure convergence of the transforms and thus define the region of the transform space in which our solutions are valid.

$$\left.\begin{array}{ll}\text{(a)} & s \text{ is positive real}\\ \text{(b)} & |\mathrm{Re}\{\xi_1\}| < \kappa_\alpha\\ \text{(c)} & |\mathrm{Re}\{\xi_2\}| < \mathrm{Re}\{(\kappa_\alpha^2-\xi_1^2)^{\frac{1}{2}}\}\\ \text{(d)} & |\mathrm{Re}\{\xi_3\}| < \mathrm{Re}\{\nu_\alpha\}\end{array}\right\} \tag{8}$$

Obtaining the solution of equation (5) is simply a matter of algebra. Doing this and also performing the inverse transform of the (x_3, ξ_3) pair leads to the general solution.

$$\begin{aligned}
&\mathbf{G}(\xi_1, \xi_2, x_3, s; 0, 0, x'_3, 0)\\
&= \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\nu_\alpha \operatorname{sgn}(x'_3-x_3)\\ \xi_1\xi_2 & \xi_2^2 & \xi_2\nu_\alpha \operatorname{sgn}(x'_3-x_3)\\ \xi_1\nu_\alpha \operatorname{sgn}(x'_3-x_3) & \xi_2\nu_\alpha \operatorname{sgn}(x'_3-x_3) & \nu_\alpha^2\end{bmatrix} \times \frac{\exp(-\nu_\alpha|x'_3-x_3|)}{\nu_\alpha}\mathbf{F}\\
&+ \begin{bmatrix} \xi_2^2+\nu_\beta^2 & -\xi_1\xi_2 & -\xi_1\nu_\beta \operatorname{sgn}(x'_3-x_3)\\ -\xi_1\xi_2 & \xi_1^2+\nu_\beta^2 & -\xi_2\nu_\beta \operatorname{sgn}(x'_3-x_3)\\ -\xi_1\nu_\beta \operatorname{sgn}(x'_3-x_3) & -\xi_2\nu_\beta \operatorname{sgn}(x'_3-x_3) & \xi_1^2+\xi_2^2\end{bmatrix} \times \frac{\exp(-\nu_\beta|x'_3-x_3|)}{\nu_\beta}\mathbf{F}\\
&+ \begin{bmatrix} a_1\xi_1 & a_2\xi_1 & a_3\xi_1\\ a_1\xi_2 & a_2\xi_2 & a_3\xi_2\\ -a_1\nu_\alpha & -a_2\nu_\alpha & -a_3\nu_\alpha\end{bmatrix} \times \frac{\exp(-\nu_\alpha(x'_3+x_3))}{\nu_\alpha}\mathbf{F}\\
&+ \begin{bmatrix} b_1\nu_\beta & b_2\nu_\beta & b_3\nu_\beta\\ c_1\nu_\beta & c_2\nu_\beta & c_3\nu_\beta\\ b_1\xi_1+c_1\xi_2 & b_2\xi_1+c_2\xi_2 & b_3\xi_1+c_3\xi_2\end{bmatrix} \times \frac{\exp-(\nu_\beta(x'_3+x_3))}{\nu_\beta}\mathbf{F}.
\end{aligned} \tag{9}$$

The first two terms represent the solution to the inhomogeneous problem while the last two terms with the nine arbitrary constants $a_1, a_2, \ldots, c_3$ are solutions to the homogeneous problem that remain bounded as x_3 goes to positive infinity.

The next step is to apply the stress boundary conditions at the free surface. Transforming equations (2) into the (ξ_1, ξ_2, x_3, s) domain, substituting in the solution of equation (9) for $\mathbf{u}$, and requiring that the stresses vanish when $x_3 = 0$ leads to a set of algebraic equations which can be solved for the nine constants $a_1, a_2, \ldots, c_3$.

$$\left.\begin{aligned}
a_1 &= (-\xi_1/d)\left[h^2-4\nu_\alpha\nu_\beta(\xi_1{}^2+\xi_2{}^2)-4\nu_\alpha\nu_\beta h\exp\left((\nu_\alpha-\nu_\beta)x'_3\right)\right]\\
b_1 &= (1/\nu_\beta d)\left[h^2(\kappa_\beta{}^2-\xi_1{}^2)+4\nu_\alpha\nu_\beta(\xi_2{}^2\kappa_\beta{}^2-\xi_1{}^2\nu_\beta{}^2)\right.\\
&\qquad\left.+4\xi_1{}^2\nu_\beta{}^2 h\exp\left((\nu_\beta-\nu_\alpha)x'_3\right)\right]\\
c_1 &= (-\xi_1\xi_2/\nu_\beta d)\left[h^2+4\nu_\alpha\nu_\beta(\kappa_\beta{}^2+\nu_\beta{}^2)-4\nu_\beta{}^2 h\exp\left((\nu_\beta-\nu_\alpha)x'_3\right)\right]\\
a_2 &= (-\xi_2/d)\left[h^2-4\nu_\alpha\nu_\beta(\xi_1{}^2+\xi_2{}^2)-4\nu_\alpha\nu_\beta h\exp\left((\nu_\alpha-\nu_\beta)x'_3\right)\right]\\
b_2 &= c_1\\
c_2 &= (1/\nu_\beta d)\left[h^2(\kappa_\beta{}^2-\xi_2{}^2)+4\nu_\alpha\nu_\beta(\xi_1{}^2\kappa_\beta{}^2-\xi_2{}^2\nu_\beta{}^2)\right.\\
&\qquad\left.+4\xi_2{}^2\nu_\beta{}^2 h\exp\left((\nu_\beta-\nu_\alpha)x'_3\right)\right]\\
a_3 &= (-\nu_\alpha/d)\left[h^2-4\nu_\alpha\nu_\beta(\xi_1{}^2+\xi_2{}^2)+4(\xi_1{}^2+\xi_2{}^2)h\exp\left((\nu_\alpha-\nu_\beta)x'_3\right)\right]\\
b_3 &= (-\xi_1/d)\left[h^2-4\nu_\alpha\nu_\beta(\xi_1{}^2+\xi_2{}^2)-4\nu_\alpha\nu_\beta h\exp\left((\nu_\beta-\nu_\alpha)x'_3\right)\right]\\
c_3 &= (-\xi_2/d)\left[h^2-4\nu_\alpha\nu_\beta(\xi_1{}^2+\xi_2{}^2)-4\nu_\alpha\nu_\beta h\exp\left((\nu_\beta-\nu_\alpha)x'_3\right)\right]
\end{aligned}\right\} \quad (10)$$

where

$$h = \nu_\beta{}^2-\xi_1{}^2-\xi_2{}^2 \quad (11)$$

and

$$d = h^2+4\nu_\alpha\nu_\beta(\xi_1{}^2+\xi_2{}^2). \quad (12)$$

Now if the expressions of equation (10) are substituted into equation (9) we have the complete solution to the problem in the transform domain. It is convenient to write the results in the form

$$\begin{aligned}
\mathbf{G}(\xi_1,\xi_2,x_3,s;0,0,x'_3,0) &= \mathbf{P}(\xi_1,\xi_2,x_3,s,x'_3)\,\mathbf{F}+\mathbf{S}(\xi_1,\xi_2,x_3,s,x'_3)\,\mathbf{F}\\
&\quad+\mathbf{PP}(\xi_1,\xi_2,x_3,s,x'_3)\,\mathbf{F}+\mathbf{SS}(\xi_1,\xi_2,x_3,s,x'_3)\,\mathbf{F}\\
&\quad+\mathbf{PS}(\xi_1,\xi_2,x_3,s,x'_3)\,\mathbf{F}+\mathbf{SP}(\xi_1,\xi_2,x_3,s,x'_3)\,\mathbf{F} \quad (13)
\end{aligned}$$

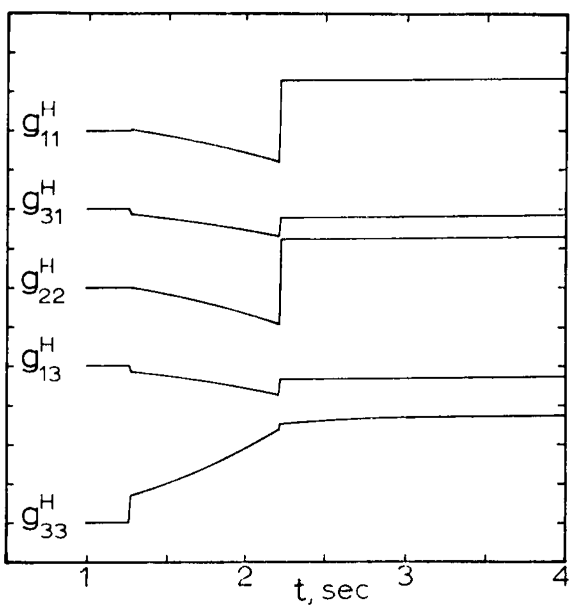

FIG. 2. The components of $\mathbf{G}^H(2, 0, 0, t; 0, 0, 10, 0)$. All components not shown are identically zero. For a force of 1 dyne a division on the vertical scale is equal to 10^{-19} cm.

where

$$\left.\begin{aligned}
&\mathbf{P}(\xi_1,\xi_2,x_3,s,x'_3)\\
&\quad=\begin{bmatrix}\xi_1^2 & \xi_1\xi_2 & \xi_1\nu_\alpha\,\mathrm{sgn}\,(x'_3-x_3)\\ \xi_1\xi_2 & \xi_2^2 & \xi_2\nu_\alpha\,\mathrm{sgn}\,(x'_3-x_3)\\ \xi_1\nu_\alpha\,\mathrm{sgn}\,(x'_3-x_3) & \xi_2\nu_\alpha\,\mathrm{sgn}\,(x'_3-x_3) & \nu_\alpha^2\end{bmatrix}\\
&\qquad\qquad\times\frac{\exp\,(-\nu_\alpha|x'_3-x_3|)}{2\rho s^2\nu_\alpha}\\
&\mathbf{S}(\xi_1,\xi_2,x_3,s,x'_3)\\
&\quad=\begin{bmatrix}\xi_2^2+\nu_\beta^2 & -\xi_1\xi_2 & -\xi_1\nu_\beta\,\mathrm{sgn}\,(x'_3-x_3)\\ -\xi_1\xi_2 & \xi_1^2+\nu_\beta^2 & -\xi_2\nu_\beta\,\mathrm{sgn}\,(x'_3-x_3)\\ -\xi_1\nu_\beta\,\mathrm{sgn}\,(x'_3-x_3) & -\xi_2\nu_\beta\,\mathrm{sgn}\,(x'_3-x_3) & \xi_1^2+\xi_2^2\end{bmatrix}\\
&\qquad\qquad\times\frac{\exp\,(-\nu_\beta|x'_3-x_3|)}{2\rho s^2\nu_\beta}\\
&\mathbf{PP}(\xi_1,\xi_2,x_3,s,x'_3)\\
&\quad=\begin{bmatrix}-\xi_1^2 & -\xi_1\xi_2 & -\xi_1\nu_\alpha\\ -\xi_1\xi_2 & -\xi_2^2 & -\xi_2\nu_\alpha\\ \xi_1\nu_\alpha & \xi_2\nu_\alpha & \nu_\alpha^2\end{bmatrix}\big(h^2-4\nu_\alpha\nu_\beta(\xi_1^2+\xi_2^2)\big)\frac{\exp\big(-\nu_\alpha(x'_3+x_3)\big)}{2\rho s^2\nu_\alpha d}\\
&\mathbf{SS}(\xi_1,\xi_2,x_3,s,x'_3)\\
&\quad=\begin{bmatrix}(\nu_\beta^2+\xi_2^2)d-8\nu_\alpha\nu_\beta^3\xi_1^2 & -\xi_1\xi_2(d+8\nu_\alpha\nu_\beta^3) & -\xi_1\nu_\beta\big(h^2-4\nu_\alpha\nu_\beta(\xi_1^2+\xi_2^2)\big)\\ -\xi_1\xi_2(d+8\nu_\alpha\nu_\beta^3) & (\nu_\beta^2+\xi_1^2)d-8\nu_\alpha\nu_\beta^3\xi_2^2 & -\xi_2\nu_\beta\big(h^2-4\nu_\alpha\nu_\beta(\xi_1^2+\xi_2^2)\big)\\ \xi_1\nu_\beta\big(h^2-4\nu_\alpha\nu_\beta(\xi_1^2+\xi_2^2)\big) & \xi_2\nu_\beta\big(h^2-4\nu_\alpha\nu_\beta(\xi_1^2+\xi_2^2)\big) & -(\xi_1^2+\xi_2^2)\big(h^2-4\nu_\alpha\nu_\beta(\xi_1^2+\xi_2^2)\big)\end{bmatrix}\\
&\qquad\qquad\times\frac{\exp\big(-\nu_\beta(x'_3+x_3)\big)}{2\rho s^2\nu_\beta d}\\
&\mathbf{PS}(\xi_1,\xi_2,x_3,s,x'_3)\\
&\quad=\begin{bmatrix}\xi_1^2\nu_\beta & \xi_1\xi_2\nu_\beta & \xi_1\nu_\alpha\nu_\beta\\ \xi_1\xi_2\nu_\beta & \xi_2^2\nu_\beta & \xi_2\nu_\alpha\nu_\beta\\ \xi_1(\xi_1^2+\xi_2^2) & \xi_2(\xi_1^2+\xi_2^2) & \nu_\alpha(\xi_1^2+\xi_2^2)\end{bmatrix}2h\frac{\exp\,(-\nu_\alpha x'_3-\nu_\beta x_3)}{\rho s^2 d}\\
&\mathbf{SP}(\xi_1,\xi_2,x_3,s,x'_3)\\
&\quad=\begin{bmatrix}\xi_1^2\nu_\beta & \xi_1\xi_2\nu_\beta & -\xi_1(\xi_1^2+\xi_2^2)\\ \xi_1\xi_2\nu_\beta & \xi_2^2\nu_\beta & -\xi_2(\xi_1^2+\xi_2^2)\\ -\xi_1\nu_\alpha\nu_\beta & -\xi_2\nu_\alpha\nu_\beta & \nu_\alpha(\xi_2^2+\xi_2^2)\end{bmatrix}2h\frac{\exp\,(-\nu_\beta x'_3-\nu_\alpha x_3)}{\rho s^2 d}.
\end{aligned}\right\}\quad(14)$$

As the notation indicates, the six different terms of the solution in equation (13) consist of the direct P wave, the direct S wave, the reflected PP wave, the reflected SS wave, the reflected PS wave, and the reflected SP wave.

4. The solution at the free surface

For most seismological problems we require only the solution at the free surface, and in that case the solution is a little more compact than in the general case. We will carry through the details for this particular case in the present section, and in the following section the results for the more general case will be presented.

Upon setting $x_3 = 0$ in equation (13) the solution simplifies to the extent that it can be written in the form

$$\mathbf{G}(\xi_1, \xi_2, 0, s; 0, 0, x'_3, 0) = \frac{\exp(-v_\alpha x'_3)}{\mu d} \mathbf{M}(\xi_1, \xi_2, 0, s, x'_3)\,\mathbf{F} + \frac{\exp(-v_\beta x'_3)}{\mu d} \mathbf{N}(\xi_1, \xi_2, 0, s, x'_3)\,\mathbf{F} \tag{15}$$

where

$$\left.\begin{aligned} \mathbf{M}(\xi_1, \xi_2, 0, s, x'_3) &= \begin{bmatrix} 2\xi_1^2 v_\beta & 2\xi_1 \xi_2 v_\beta & 2\xi_1 v_\alpha v_\beta \\ 2\xi_1 \xi_2 v_\beta & 2\xi_2^2 v_\beta & 2\xi_2 v_\alpha v_\beta \\ \xi_1 h & \xi_2 h & v_\alpha h \end{bmatrix} \\ \mathbf{N}(\xi_1, \xi_2, 0, s, x'_3) &= \begin{bmatrix} (1/v_\beta)(h(v_\beta^2-\xi_2^2)+4v_\alpha v_\beta \xi_2^2) & (\xi_1 \xi_2/v_\beta)(h-4v_\alpha v_\beta) & -\xi_1 h \\ (\xi_1 \xi_2/v_\beta)(h-4v_\alpha v_\beta) & (1/v_\beta)(h(v_\beta^2-\xi_1^2)+4v_\alpha v_\beta \xi_1^2) & -\xi_2 h \\ -2\xi_1 v_\alpha v_\beta & -2\xi_2 v_\alpha v_\beta & 2v_\alpha(\xi_1^2+\xi_2^2) \end{bmatrix}. \end{aligned}\right\} \tag{16}$$

The next step is to perform the inverse x_1 and x_2 transforms.

$$\mathbf{G}(x_1, x_2, 0, s; 0, 0, x'_3, 0) = \frac{-1}{4\pi^2 \mu} \int\!\!\int_{-i\infty}^{i\infty} \left[\frac{\exp(-v_\alpha x'_3)}{d} \mathbf{M}(\xi_1, \xi_2, 0, s, x'_3)\,\mathbf{F} + \frac{\exp(-v_\beta x'_3)}{d} \mathbf{N}(\xi_1, \xi_2, 0, s, x'_3)\,\mathbf{F}\right] \exp(\xi_1 x_1 + \xi_2 x_2)\, d\xi_1\, d\xi_2. \tag{17}$$

By a series of substitutions it is possible to transform the right-hand side of this equation into a form such that the inverse s transform can be solved by inspection. This general approach is commonly referred to as the Cagniard method, but the particular method to be followed here is a modification of the Cagniard method due to de Hoop (1960, 1961). The method is thoroughly expounded in the papers by de Hoop so we will only list the essential steps of the method as applied to the present problem.

(1) Make the following substitutions (see Fig. 1):

$$\left.\begin{aligned} x_1 &= R\cos(\phi) & x_2 &= R\sin(\phi) & R &= (x_1^2+x_2^2)^{\frac{1}{2}} \\ R &= r\sin(\theta) & x'_3 &= r\cos(\theta) & r &= (R^2+x'^2_3)^{\frac{1}{2}}. \end{aligned}\right\} \tag{18}$$

(2) Change the variables of integration from ξ_1 and ξ_2 to q and p through the substitutions:

$$\left.\begin{aligned} \xi_1 &= sq\cos(\phi) - isp\sin(\phi) \\ \xi_2 &= sq\sin(\phi) + isp\cos(\phi) \end{aligned}\right\} \tag{19}$$

(3) Note that only the terms of the integrand which are even in p will contribute so discard the odd parts.

(4) Note that the integrand is symmetric about the real q axis so only the imaginary part of the integration with respect to q has to be retained.

(5) In the first term of the integrand of equation (17) let

$$\tau_\alpha = -qr\sin(\theta)+\eta_\alpha r\cos(\theta) \tag{20}$$

where

$$\eta_\alpha = (\alpha^{-2}+p^2-q^2)^{\frac{1}{2}} \qquad \mathrm{Re}\{\eta_\alpha\} \geqslant 0 \tag{21}$$

and in the second term let

$$\tau_\beta = -qr\sin(\theta)+\eta_\beta r\cos(\theta) \tag{22}$$

where

$$\eta_\beta = (\beta^{-2}+p^2-q^2)^{\frac{1}{2}} \qquad \mathrm{Re}\{\eta_\beta\} \geqslant 0. \tag{23}$$

(6) Handling the two terms of the integrand of equation (17) separately, deform the path of integration in the q plane onto paths such that τ_α and τ_β are positive real.

(7) Change the variable of integration from q to τ_α in the first term and from q to τ_β in the second term.

(8) Interchange the order of the τ_α or τ_β integration with the p integration.

At this point the integral of equation (17) has been manoeuvred into the general form

$$\mathbf{G}(x_1, x_2, 0, s; 0, 0, x'_3, 0) = \int_0^\infty \mathbf{W}_\alpha(q, p, 0, \tau_\alpha, x'_3)\, s\exp(-s\tau_\alpha)\, d\tau_\alpha + \int_0^\infty \mathbf{W}_\beta(q, p, 0, \tau_\beta, x'_3)\, s\exp(-s\tau_\beta)\, d\tau_\beta \tag{24}$$

and from this it is clear that

$$\mathbf{G}(x_1, x_2, 0, t; 0, 0, x'_3, 0) = \frac{\partial}{\partial t}\mathbf{W}_\alpha(q, p, 0, t, x'_3)+\frac{\partial}{\partial t}\mathbf{W}_\beta(q, p, 0, t, x'_3). \tag{25}$$

In the present case the actual results are

$$\mathbf{G}(x_1, x_2, 0, t; 0, 0, x'_3, 0) = \frac{1}{\pi^2\mu r}\frac{\partial}{\partial t}\int_0^{((t/r)^2-\alpha^{-2})^{1/2}} H(t-r/\alpha) \times \mathrm{Re}\left[\eta_\alpha\sigma^{-1}((t/r)^2-\alpha^{-2}-p^2)^{-\frac{1}{2}}\mathbf{M}(q, p, 0, t, x'_3)\right]\mathbf{F}\,dp$$
$$+\frac{1}{\pi^2\mu r}\frac{\partial}{\partial t}\int_0^{p_2} H(t-t_2) \times \mathrm{Re}\left[\eta_\beta\sigma^{-1}((t/r)^2-\beta^{-2}-p^2)^{-\frac{1}{2}}\mathbf{N}(q, p, 0, t, x'_3)\right]\mathbf{F}\,dp \tag{26}$$

where $H(t)$ is the unit step function and

$$p_2 = \begin{cases} ((t/r)^2-\beta^{-2})^{\frac{1}{2}} & \sin(\theta) \leqslant \beta/\alpha \\ \left[\left(\dfrac{t/r-(\beta^{-2}-\alpha^{-2})^{\frac{1}{2}}\cos(\theta)}{\sin(\theta)}\right)^2-\alpha^{-2}\right]^{\frac{1}{2}} & \sin(\theta) > \beta/\alpha \end{cases} \tag{27}$$

$$t_2 = \begin{cases} r/\beta & \sin(\theta) \leqslant \beta/\alpha \\ r/\alpha\sin(\theta)+r(\beta^{-2}-\alpha^{-2})^{\frac{1}{2}}\cos(\theta) & \sin(\theta) > \beta/\alpha \end{cases} \tag{28}$$

$$\gamma = \eta_\beta{}^2 + p^2 - q^2 \tag{29}$$

$$\sigma = \gamma^2 + 4\eta_\alpha \eta_\beta (q^2 - p^2). \tag{30}$$

In the first integral of equation (26) q is given by the expression

$$q = -t/r \sin(\theta) + i\big((t/r)^2 - \alpha^{-2} - p^2\big)^{\frac{1}{2}} \cos(\theta) \tag{31}$$

while in the second integral it is given by

$$q = -t/r \sin(\theta) + i\big((t/r)^2 - \beta^{-2} - p^2\big)^{\frac{1}{2}} \cos(\theta). \tag{32}$$

Finally, the expressions for the individual elements of the three-by-three matrices **M** and **N** are as follows:

$$\left.\begin{aligned}
M_{11} &= 2\eta_\beta\big((q^2+p^2)\cos^2(\phi) - p^2\big)\\
M_{12} &= 2\eta_\beta(q^2+p^2)\sin(\phi)\cos(\phi)\\
M_{13} &= 2q\eta_\alpha\eta_\beta\cos(\phi)\\
M_{21} &= M_{12}\\
M_{22} &= 2\eta_\beta\big((q^2+p^2)\sin^2(\phi) - p^2\big)\\
M_{23} &= 2q\eta_\alpha\eta_\beta\sin(\phi)\\
M_{31} &= q\gamma\cos(\phi)\\
M_{32} &= q\gamma\sin(\phi)\\
M_{33} &= \eta_\alpha\gamma
\end{aligned}\right\} \tag{33}$$

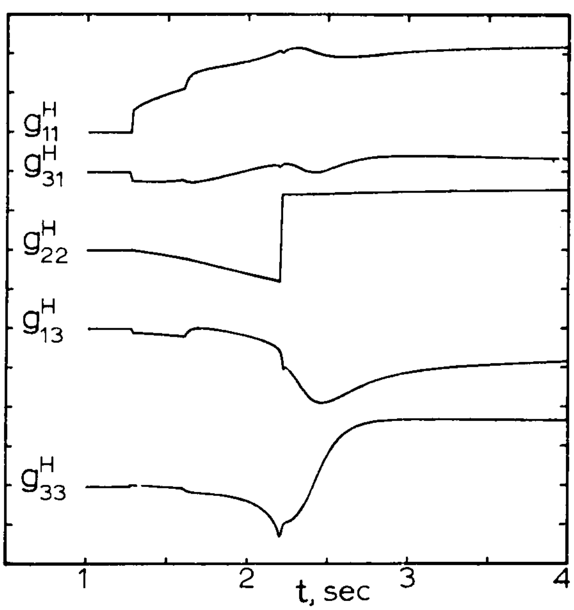

FIG. 3. The components of $\mathbf{G}^H(10, 0, 0, t; 0, 0, 2, 0)$. All components not shown are identically zero. For a force of 1 dyne a division on the vertical scale is equal to 10^{-19} cm.

$$\left.\begin{aligned}
N_{11} &= \eta_\beta^{-1}[\eta_\beta^2\gamma-(\gamma-4\eta_\alpha\eta_\beta)((q^2+p^2)\sin^2(\phi)-p^2)]\\
N_{12} &= \eta_\beta^{-1}(q^2+p^2)(\gamma-4\eta_\alpha\eta_\beta)\sin(\phi)\cos(\phi)\\
N_{13} &= -q\gamma\cos(\phi)\\
N_{21} &= N_{12}\\
N_{22} &= \eta_\beta^{-1}[\eta_\beta^2\gamma-(\gamma-4\eta_\alpha\eta_\beta)((q^2+p^2)\cos^2(\phi)-p^2)]\\
N_{23} &= -q\gamma\sin(\phi)\\
N_{31} &= -2q\eta_\alpha\eta_\beta\cos(\phi)\\
N_{32} &= -2q\eta_\alpha\eta_\beta\sin(\phi)\\
N_{33} &= 2\eta_\alpha(q^2-p^2).
\end{aligned}\right\} \tag{34}$$

The physical interpretation of the various parts of equation (26) is straightforward. The first integral represents that part of the solution which results from the compressional waves generated at the source, while the second integral is that part which results from the shear waves generated at the source. The time t_2 (equation (28)) is the arrival time of the direct shear wave when $\sin(\theta) \leqslant \beta/\alpha$ and when $\sin(\theta) > \beta/\alpha$ it is the arrival time of the diffracted *SP* wave which was first pointed out for the two-dimensional case by Nakano (1925). The expression denoted by σ (equation (30)) is the equivalent of Rayleigh's equation in this problem.

As written in equation (26) the Green's function **G** consists of the three components of displacement (equations (6) and (4)) which result from the application of the three components of force represented by **F** (equations (6) and (3)). Viewed in a less compact form, each component of **F** by itself gives rise to three components of displacement and thus there are a total of nine such displacements which add up to give **G**. These nine displacements have a one-to-one correspondence with the elements of the matrices **M** and **N** (equations (33) and (34)). If we follow the usual convention and let $g_{ij}(x_1, x_2, 0, t; 0, 0, x'_3, 0)$ denote the displacement in the i direction at the receiver $(x_1, x_2, 0, t)$ due to a unit force in the j direction at the source $(0, 0, x'_3, 0)$, then g_{ij} is the result of equation (26) when only the M_{ij} and N_{ij} terms are included in the integrals. While representing the Green's function as a matrix **G** is convenient for illustrating the development of the solution, a representation in terms of the individual components g_{ij} is usually more convenient when it comes to computing and using the Green's function.

As mentioned earlier, the fact that we have obtained a solution for $x_1 = x_2 = t = 0$ is no restriction upon the generality of the results. Because the problem is invariant with respect to a translation in either the x or y direction it is obvious that

$$g_{ij}(x_1, x_2, 0, t; x'_1, x'_2, x'_3, t') = g_{ij}(x_1-x'_1, x_2-x'_2, 0, t-t'; 0, 0, x'_3, 0). \tag{35}$$

In this section we have solved the problem of a source at depth and a receiver at the free surface, but the same solution can be used for the problem of a source at the free surface and a receiver at depth. Starting with the general reciprocal relation for Green's functions (Burridge & Knopoff 1964)

$$g_{ij}(\mathbf{x}, t; \mathbf{x}', t') = g_{ji}(\mathbf{x}', -t'; \mathbf{x}, -t) \tag{36}$$

we can use equation (35) to show that

$$g_{ij}(x_1, x_2, x_3, t; 0, 0, 0, 0) = g_{ji}(-x_1, -x_2, 0, t; 0, 0, x_3, 0). \tag{37}$$

Finally, note that equation (26) is the Green's function for a source which is a delta function in time (equation (3)). Should we prefer the solution corresponding to a source which is a step function in time, then the result is identical to equation (26) except that the differentiation with respect to time is omitted.

5. The solution within the half space

In the previous section the solution at the free surface due to a source at depth was derived. By reciprocity arguments it was shown that this is equivalent to the problem where the source is at the surface and the receiver is at depth. While this solution is sufficient for many seismological problems, on occasions it is necessary to consider the problem where both the source and receiver are at depth. The solution to this problem is presented in the present section.

We begin with the general solution in the transform domain given by equations (13) and (14). The present task is to transform this solution back to the physical domain where it can be written in the form

$$\begin{aligned}\mathbf{G}(x_1, x_2, x_3, t; 0, 0, x'_3, 0) = {} & \mathbf{P}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} + \mathbf{S}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} \\ & + \mathbf{PP}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} + \mathbf{SS}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} \\ & + \mathbf{PS}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} + \mathbf{SP}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F}. \end{aligned} \tag{38}$$

The six individual terms of this solution will be derived in turn in the remainder of this section. All of the terms can be obtained using the Cagniard–de Hoop method which was outlined in the previous section, so, aside from pointing out a couple of slight modifications in the method, we will only list the final results.

For the direct P and S waves the solutions are easily obtained using the method outlined in section (4). Moreover, the final integrals with respect to the variable p (see for example equation (26)) can be evaluated analytically to yield the following solutions.

$$\mathbf{P}(x_1, x_2, x_3, t, x'_3) = \frac{1}{8\pi\rho r}\frac{\partial}{\partial t} H(t - r/\alpha)\,\mathbf{D}(x_1, x_2, x_3, t, x'_3) \tag{39}$$

$$\mathbf{S}(x_1, x_2, x_3, t, x'_3) = \frac{1}{8\pi\rho r}\frac{\partial}{\partial t} H(t - r/\beta)\,\mathbf{E}(x_1, x_2, x_3, t, x'_3). \tag{40}$$

The quantities r and θ now have the more general definitions

$$\left.\begin{aligned} r &= \left(R^2 + (x'_3 - x_3)^2\right)^{\frac{1}{2}} \\ \theta &= \tan^{-1}\left(\frac{R}{x'_3 - x_3}\right) \end{aligned}\right\} \tag{41}$$

whereas R and ϕ remain the same as defined earlier in equation (18). **D** and **E** are three-by-three matrices with the following individual elements.

$$\left.\begin{aligned} D_{11} &= \left(3(t/r)^2 - \alpha^{-2}\right)\sin^2(\theta)\cos^2(\phi) - \left((t/r)^2 - \alpha^{-2}\right) \\ D_{12} &= \left(3(t/r)^2 - \alpha^{-2}\right)\sin^2(\theta)\sin(\phi)\cos(\phi) \\ D_{13} &= -\left(3(t/r)^2 - \alpha^{-2}\right)\sin(\theta)\cos(\theta)\cos(\phi) \\ D_{21} &= D_{12} \\ D_{22} &= \left(3(t/r)^2 - \alpha^{-2}\right)\sin^2(\theta)\sin^2(\phi) - \left((t/r)^2 - \alpha^{-2}\right) \\ D_{23} &= -\left(3(t/r)^2 - \alpha^{-2}\right)\sin(\theta)\cos(\theta)\sin(\phi) \\ D_{31} &= D_{13} \\ D_{32} &= D_{23} \\ D_{33} &= \left(3(t/r)^2 - \alpha^{-2}\right)\cos^2(\theta) - \left((t/r)^2 - \alpha^{-2}\right) \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned}
E_{11} &= -\left(3(t/r)^2-\beta^{-2}\right)\sin^2(\theta)\cos^2(\phi)+(t/r)^2+\beta^{-2}\\
E_{12} &= -\left(3(t/r)^2-\beta^{-2}\right)\sin^2(\theta)\sin(\phi)\cos(\phi)\\
E_{13} &= \left(3(t/r)^2-\beta^{-2}\right)\sin(\theta)\cos(\theta)\cos(\phi)\\
E_{21} &= E_{12}\\
E_{22} &= -\left(3(t/r)^2-\beta^{-2}\right)\sin^2(\theta)\sin^2(\phi)+(t/r)^2+\beta^{-2}\\
E_{23} &= \left(3(t/r)^2-\beta^{-2}\right)\sin(\theta)\cos(\theta)\sin(\phi)\\
E_{31} &= E_{13}\\
E_{32} &= E_{23}\\
E_{33} &= \left(3(t/r)^2-\beta^{-2}\right)\sin^2(\theta)-2\left((t/r)^2-\beta^{-2}\right).
\end{aligned}\right\} \tag{43}$$

It is worth noticing that these solutions for the direct P and S waves taken together represent the Green's function for a homogeneous elastic whole space.

The reflected PP and SS waves can also be obtained by a straightforward application of the method of Section 4. The results for the PP wave follow.

$$\mathbf{PP}(x_1, x_2, x_3, t, x'_3) = \frac{1}{2\pi^2\rho r'}\frac{\partial}{\partial t}\int_0^{((t/r')^2-\alpha^{-2})^{1/2}} H(t-r'/\alpha)$$

$$\times \operatorname{Re}\left\{\sigma^{-1}\left((t/r')^2-\alpha^{-2}-p^2\right)^{-\frac{1}{2}}\mathbf{I}(q, p, x_3, t, x'_3)\right\}dp. \tag{44}$$

We have introduced the new variables

$$\left.\begin{aligned}
r' &= \left(R^2+(x'_3+x_3)^2\right)^{\frac{1}{2}}\\
\theta' &= \tan^{-1}\left(\frac{R}{x'_3+x_3}\right)
\end{aligned}\right\} \tag{45}$$

but R and ϕ are the same as defined in equation (18). For q we have the expression

$$q = -t/r'\sin(\theta')+i\left((t/r')^2-\alpha^{-2}-p^2\right)^{\frac{1}{2}}\cos(\theta') \tag{46}$$

while the expressions for η_α, η_β, γ ,and σ are the same as defined in equations (21), (23), (27) and (30). The individual elements of the three-by-three matrix $\mathbf{I}$ are listed below.

$$\left.\begin{aligned}
I_{11} &= -\left(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)\right)\left((q^2+p^2)\cos^2(\phi)-p^2\right)\\
I_{12} &= -\left(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)\right)(q^2+p^2)\sin(\phi)\cos(\phi)\\
I_{13} &= -\left(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)\right)q\eta_\alpha\cos(\phi)\\
I_{21} &= I_{12}\\
I_{22} &= -\left(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)\right)\left((q^2+p^2)\sin^2(\phi)-p^2\right)\\
I_{23} &= -\left(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)\right)q\eta_\alpha\sin(\phi)\\
I_{31} &= I_{13}\\
I_{32} &= I_{23}\\
I_{33} &= \left(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)\right)\eta_\alpha^2.
\end{aligned}\right\} \tag{47}$$

The results for the reflected SS wave are obtained in a similar manner.

$$\mathbf{SS}(x_1, x_2, x_3, t, x'_3) = \frac{1}{2\pi^2\rho r'}\frac{\partial}{\partial t}\int_0^{p'_2} H(t-t'_2)$$

$$\times \operatorname{Re}\left\{\sigma^{-1}\left((t/r')^2-\beta^{-2}-p^2\right)^{-\frac{1}{2}}\mathbf{J}(q, p, x_3, t, x'_3)\right\}dp. \tag{48}$$

The definitions and remarks following equation (44) apply here also except that now

$$q = -t/r' \sin(\theta') + i((t/r')^2 - \beta^{-2} - p^2)^{\frac{1}{2}} \cos(\theta') \tag{49}$$

and we have the two additional definitions

$$p'_2 = \begin{cases} ((t/r')^2 - \beta^{-2})^{\frac{1}{2}} & \sin(\theta') \leqslant \beta/\alpha \\ \left[\left(\dfrac{t/r' - (\beta^{-2} - \alpha^{-2})^{\frac{1}{2}} \cos(\theta')}{\sin(\theta')}\right)^2 - \alpha^{-2}\right]^{\frac{1}{2}} & \sin(\theta') > \beta/\alpha \end{cases} \tag{50}$$

$$t'_2 = \begin{cases} r'/\beta & \sin(\theta') \leqslant \beta/\alpha \\ r'/\alpha \sin(\theta') + r'(\beta^{-2} - \alpha^{-2})^{\frac{1}{2}} \cos(\theta') & \sin(\theta') > \beta/\alpha. \end{cases} \tag{51}$$

The individual elements of the three-by-three matrix **J** follow.

$$\left.\begin{aligned} J_{11} &= -(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 - p^2))((q^2 + p^2)\cos^2(\phi) - p^2) \\ &\qquad + \beta^{-2}(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 + p^2)(2\cos^2(\phi) - 1)) \\ J_{12} &= -(q^2 + p^2)(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 - p^2) + 8\beta^{-2}\eta_\alpha \eta_\beta) \sin(\phi)\cos(\phi) \\ J_{13} &= -q\eta_\beta(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 - p^2))\cos(\phi) \\ J_{21} &= J_{12} \\ J_{22} &= -(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 - p^2))((q^2 + p^2)\sin^2(\phi) - p^2) \\ &\qquad + \beta^{-2}(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 + p^2)(2\sin^2(\phi) - 1)) \\ J_{23} &= -q\eta_\beta(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 - p^2))\sin(\phi) \\ J_{31} &= -J_{13} \\ J_{32} &= -J_{23} \\ J_{33} &= -(q^2 - p^2)(\gamma^2 - 4\eta_\alpha \eta_\beta(q^2 - p^2)). \end{aligned}\right\} \tag{52}$$

For the cases of the reflected *PS* and *SP* waves we can once again follow the general method of solution outlined in Section 4. An additional complication is encountered with regard to determining the point where the reflection occurs, but Cagniard (1962, Chapter 5 and Appendix I) gives a technique for handling this difficulty. The results for the *PS* wave are

$$\mathbf{PS}(x_1, x_2, x_3, t, x'_3) = \frac{1}{2\pi^2 \rho} \frac{\partial}{\partial t} \int_0^{p_3} H(t - t_3)$$
$$\times \mathrm{Re}\{i\sigma^{-1}(R + q(x'_3/\eta_\alpha + x_3/\eta_\beta))^{-1} \mathbf{K}(q, p, x_3, t, x'_3)\}\, dp. \tag{53}$$

The quantities R, ϕ, η_α, η_β, γ, and σ are the same as defined earlier, and we now introduce R_α and R_β which are defined as the pair of quantities that satisfy the two relations

$$\left.\begin{aligned} R_\alpha + R_\beta &= R \\ (R + {x'_3}^2/R_\alpha + x_3^2/R_\beta)\, m &= t \end{aligned}\right\} \tag{54}$$

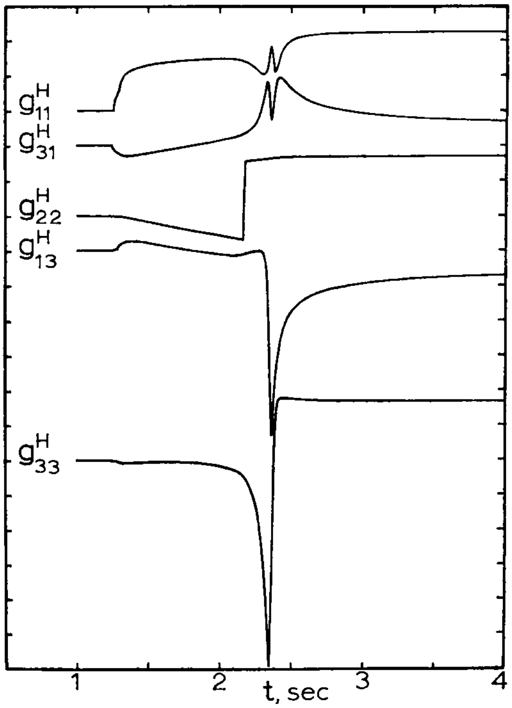

FIG. 4. The components of $\mathbf{G}^H(10, 0, 0, t; 0, 0, \cdot 2, 0)$. All components not shown are identically zero. For a force of 1 dyne a division on the vertical scale is equal to 10^{-19} cm.

where

$$m^2 = \frac{1}{\left(1+\left(\frac{x'_3}{R_\alpha}\right)^2\right)\left(1+\left(\frac{x_3}{R_\beta}\right)^2\right)}$$

$$\left[\frac{\beta^{-2}\left(\frac{x_3}{R_\beta}\right)^2\left(1+\left(\frac{x'_3}{R_\alpha}\right)^2\right)-\alpha^{-2}\left(\frac{x'_3}{R_\alpha}\right)^2\left(1+\left(\frac{x_3}{R_\beta}\right)^2\right)}{\left(\frac{x_3}{R_\beta}\right)^2-\left(\frac{x'_3}{R_\alpha}\right)^2}+p^2\right]. \tag{55}$$

The expression for q is now

$$q = -m+i\left[m^2-\frac{\alpha^{-2}+p^2}{1+(x'_3/R_\alpha)^2}\right]^{\frac{1}{2}} x'_3/R_\alpha. \tag{56}$$

The upper limit of integration p_3 is defined as the value of p for which

$$m^2 = \frac{\alpha^{-2}+p^2}{1+(x'_3/R_\alpha)^2} \tag{57}$$

and t_3 is the value of t for which $p_3 = 0$. The individual terms of the three-by-three matrix **K** are listed below.

$$\left.\begin{aligned}
K_{11} &= 4\gamma\eta_\beta\big((q^2+p^2)\cos^2(\phi)-p^2\big)\\
K_{12} &= 4\gamma\eta_\beta(q^2+p^2)\sin(\phi)\cos(\phi)\\
K_{13} &= 4q\gamma\eta_\alpha\eta_\beta\cos(\phi)\\
K_{21} &= K_{12}\\
K_{22} &= 4\gamma\eta_\beta\big((q^2+p^2)\sin^2(\phi)-p^2\big)\\
K_{23} &= 4q\gamma\eta_\alpha\eta_\beta\sin(\phi)\\
K_{31} &= 4q\gamma(q^2-p^2)\cos(\phi)\\
K_{32} &= 4q\gamma(q^2-p^2)\sin(\phi)\\
K_{33} &= 4\gamma\eta_\alpha(q^2-p^2).
\end{aligned}\right\} \tag{58}$$

For the reflected SP wave we proceed similarly and obtain the result

$$\mathbf{SP}(x_1, x_2, x_3, t, x'_3) = \frac{1}{2\pi^2\rho}\frac{\partial}{\partial t}\int_0^{p_3} H(t-t_3)$$
$$\operatorname{Re}\left\{i\sigma^{-1}\big(R+q(x_3/\eta_\alpha+x'_3/\eta_\beta)\big)^{-1}\mathbf{L}(q, p, x_3, t, x'_3)\right\}dp. \tag{59}$$

In this integral we use relations identical to equations (54) through (57) except that the positions of x_3 and x'_3 are interchanged. The individual terms of **L** follow.

$$\left.\begin{aligned}
L_{11} &= 4\gamma\eta_\beta\big((q^2+p^2)\cos^2(\phi)-p^2\big)\\
L_{12} &= 4\gamma\eta_\beta(q^2+p^2)\sin(\phi)\cos(\phi)\\
L_{13} &= -4q\gamma(q^2-p^2)\cos(\phi)\\
L_{21} &= L_{12}\\
L_{22} &= 4\gamma\eta_\beta\big((q^2+p^2)\sin^2(\phi)-p^2\big)\\
L_{23} &= -4q\gamma(q^2-p^2)\sin(\phi)\\
L_{31} &= -4q\gamma\eta_\alpha\eta_\beta\cos(\phi)\\
L_{32} &= -4q\gamma\eta_\alpha\eta_\beta\sin(\phi)\\
L_{33} &= 4\gamma\eta_\alpha(q^2-p^2).
\end{aligned}\right\} \tag{60}$$

6. Spatial derivatives of the solutions

As mentioned in the introduction, there are many situations in seismology where we need the spatial derivatives of the Green's function. For instance, if the Knopoff–de Hoop representation theorem (Burridge & Knopoff 1964) is applied to the case of a simple shear dislocation in a uniform elastic medium it can be shown that the resulting displacement at any point can be written in the form

$$u_i(\mathbf{x}, t) = \int_{-\infty}^{\infty}\int_{\Sigma} \mu n_k(\mathbf{x}')[u_j(\mathbf{x}', t')]\big(g_{ij,k'}(\mathbf{x}, t; \mathbf{x}', t')+g_{ik,j'}(\mathbf{x}, t; \mathbf{x}', t')\big)\,d\mathbf{x}'\,dt' \tag{61}$$

where $[u_j]$ is the dislocation in displacement which is specified over the surface Σ, n_k is the unit normal to this surface, and repeated indices imply summation. Note that $g_{ij,k'}$ indicates partial differentiation with respect to the source co-ordinates

$$g_{ij,k'}(\mathbf{x}, t; \mathbf{x}', t') = \frac{\partial}{\partial x'_k}g_{ij}(x_1, x_2, x_3, t; x'_1, x'_2, x'_3, t'). \tag{62}$$

In this section we show that these spatial derivatives of the Green's function can be derived in the same manner that the Green's function itself was derived.

To illustrate the method let us consider the spatial derivatives with respect to the source co-ordinates of the solution at the free surface. From equation (35) it is easy to see that

$$g_{ij,k'}(\mathbf{x}, t; \mathbf{x}', t') = -g_{ij,k}(\mathbf{x}, t; \mathbf{x}', t') \tag{63}$$

so long as k is restricted to the 1 and 2 directions. With this relation we can return to equation (17) and differentiate beneath the integral sign to get, for example

$$\mathbf{G}_{,1'}(x_1, x_2, 0, s; 0, 0, x'_3, 0) = \frac{1}{4\pi^2 \mu} \int\int_{-i\infty}^{i\infty} \left[\frac{\exp(-\nu_\alpha x'_3)}{d} \mathbf{M}(\xi_1, \xi_2, 0, s, x'_3)\,\mathbf{F} + \frac{\exp(-\nu_\beta x'_3)}{d} \mathbf{N}(\xi_1, \xi_2, 0, s, x'_3)\,\mathbf{F} \right] \xi_1 \exp(\xi_1 x_1 + \xi_2 x_2)\, d\xi_1\, d\xi_2 \tag{64}$$

with expressions of a similar form for $\mathbf{G}_{,2'}$ and $\mathbf{G}_{,3'}$. From this point on the procedure is identical to that followed for the Green's function itself except that an extra differentiation with respect to time appears in the answer. The equivalent of equation (26) becomes

$$\begin{aligned}
\mathbf{G}_{,k'}&(x_1, x_2, 0, t; 0, 0, x'_3, 0) \\
&= \frac{1}{\pi^2 \mu r} \frac{\partial^2}{\partial t^2} \int_0^{((t/r)^2-\alpha^{-2})^{1/2}} H(t-r/\alpha) \\
&\quad \times \operatorname{Re}\left\{\eta_\alpha \sigma^{-1}\left((t/r)^2-\alpha^{-2}-p^2\right)^{-\frac{1}{2}} \mathbf{M}_{,k'}(q, p, 0, t, x'_3)\right\} \mathbf{F}\, dp \\
&\quad + \frac{1}{\pi^2 \mu r} \frac{\partial^2}{\partial t^2} \int_0^{p_2} H(t-t_2) \\
&\quad \times \operatorname{Re}\left\{\eta_\beta \sigma^{-1}\left((t/r)^2-\beta^{-2}-p^2\right)^{-\frac{1}{2}} \mathbf{N}_{,k'}(q, p, 0, t, x'_3)\right\} \mathbf{F}\, dp
\end{aligned} \tag{65}$$

and equations (21), (23), and (27) to (30) all apply here. The individual terms of $\mathbf{M}_{,k'}$ and $\mathbf{N}_{,k'}$ are listed below.

$$\left.\begin{aligned}
M_{11,1'} &= -2q\eta_\beta\left((q^2+3p^2)\cos^2(\phi)-3p^2\right)\cos(\phi) \\
M_{12,1'} &= -2q\eta_\beta\left((q^2+3p^2)\cos^2(\phi)-p^2\right)\sin(\phi) \\
M_{13,1'} &= -2\eta_\alpha\eta_\beta\left((q^2+p^2)\cos^2(\phi)-p^2\right) \\
M_{21,1'} &= M_{12,1'} \\
M_{22,1'} &= -2q\eta_\beta\left((q^2+3p^2)\sin^2(\phi)-p^2\right)\cos(\phi) \\
M_{23,1'} &= -2\eta_\alpha\eta_\beta(q^2+p^2)\sin(\phi)\cos(\phi) \\
M_{31,1'} &= -\gamma\left((q^2+p^2)\cos^2(\phi)-p^2\right) \\
M_{32,1'} &= -\gamma(q^2+p^2)\sin(\phi)\cos(\phi) \\
M_{33,1'} &= -q\eta_\alpha\gamma\cos(\phi)
\end{aligned}\right\} \tag{66}$$

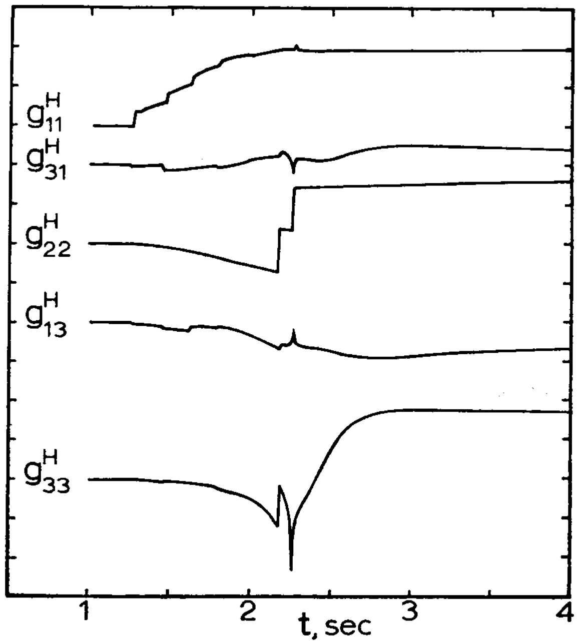

FIG. 5. The components of $\mathbf{G}^H(10, 0, 1, t; 0, 0, 2, 0)$. All components not shown are identically zero. For a force of 1 dyne a division on the vertical scale is equal to 10^{-19} cm.

$$\left.\begin{aligned}
N_{11,1'} &= -q\eta_\beta^{-1}[\eta_\beta^2\gamma-(\gamma-4\eta_\alpha\eta_\beta)((q^2+3p^2)\sin^2(\phi)-p^2)]\cos(\phi)\\
N_{12,1'} &= -q\eta_\beta^{-1}(\gamma-4\eta_\alpha\eta_\beta)((q^2+3p^2)\cos^2(\phi)-p^2)\sin(\phi)\\
N_{13,1'} &= \gamma((q^2+p^2)\cos^2(\phi)-p^2)\\
N_{21,1'} &= N_{12,1'}\\
N_{22,1'} &= -q\eta_\beta^{-1}[\eta_\beta^2\gamma-(\gamma-4\eta_\alpha\eta_\beta)((q^2+3p^2)\cos^2(\phi)-3p^2)]\cos(\phi)\\
N_{23,1'} &= \gamma(q^2+p^2)\sin(\phi)\cos(\phi)\\
N_{31,1'} &= 2\eta_\alpha\eta_\beta((q^2+p^2)\cos^2(\phi)-p^2)\\
N_{32,1'} &= 2\eta_\alpha\eta_\beta(q^2+p^2)\sin(\phi)\cos(\phi)\\
N_{33,1'} &= -2q\eta_\alpha(q^2-p^2)\cos(\phi)
\end{aligned}\right\} \quad (67)$$

$$\left.\begin{aligned}
M_{11,2'} &= M_{12,1'}\\
M_{12,2'} &= M_{22,1'}\\
M_{13,2'} &= M_{23,1'}\\
M_{21,2'} &= M_{12,2'}\\
M_{22,2'} &= -2q\eta_\beta((q^2+3p^2)\sin^2(\phi)-3p^2)\sin(\phi)\\
M_{23,2'} &= -2\eta_\alpha\eta_\beta((q^2+p^2)\sin^2(\phi)-p^2)\\
M_{31,2'} &= M_{32,1'}\\
M_{32,2'} &= -\gamma((q^2+p^2)\sin^2(\phi)-p^2)\\
M_{33,2'} &= -q\eta_\alpha\gamma\sin(\phi)
\end{aligned}\right\} \quad (68)$$

$$\left.\begin{aligned}
N_{11,2'} &= -q\eta_\beta^{-1}[\eta_\beta^2\gamma-(\gamma-4\eta_\alpha\eta_\beta)((q^2+3p^2)\sin^2(\phi)-3p^2)]\sin(\phi)\\
N_{12,2'} &= -q\eta_\beta^{-1}(\gamma-4\eta_\alpha\eta_\beta)((q^2+3p^2)\sin^2(\phi)-p^2)\cos(\phi)\\
N_{13,2'} &= N_{23,1'}\\
N_{21,2'} &= N_{12,2'}\\
N_{22,2'} &= -q\eta_\beta^{-1}[\eta_\beta^2\gamma-(\gamma-4\eta_\alpha\eta_\beta)((q^2+3p^2)\cos^2(\phi)-p^2)]\sin(\phi)\\
N_{23,2'} &= \gamma((q^2+p^2)\sin^2(\phi)-p^2)\\
N_{31,2'} &= N_{32,1'}\\
N_{32,2'} &= 2\eta_\alpha\eta_\beta((q^2+p^2)\sin^2(\phi)-p^2)\\
N_{33,2'} &= -2q\eta_\alpha(q^2-p^2)\sin(\phi)
\end{aligned}\right\} \quad (69)$$

$$M_{ij,3'} = -\eta_\alpha M_{ij} \quad (70)$$

$$N_{ij,3'} = -\eta_\beta N_{ij}. \quad (71)$$

With the foregoing formulas the spatial derivatives of the Green's function with respect to the source co-ordinates can be obtained with a degree of difficulty that is no greater than that required to obtain the Green's function itself. One can think of these spatial derivatives as the solution to the problem where the source is a force couple with unit moment rather than a simple force. Furthermore, the solution to the problem where the source is a double couple without moment can be obtained by combining two of the spatial derivatives as indicated in the integral of equation (61).

The solution for a point source of dilatation, such as a symmetric explosion, can also be obtained quite easily from the spatial derivatives of the Green's function. The solution for a unit source of dilatation is

$$\mathbf{G}_\Delta(x_1, x_2, 0, t; 0, 0, x'_3, 0) = \frac{3\lambda+2\mu}{3\pi^2\mu r}\frac{\partial^2}{\partial t^2}\int_0^{((t/r)^2-\alpha^{-2})^{1/2}} H(t-r/\alpha)$$

$$\times \operatorname{Re}\{\eta_\alpha\sigma^{-1}((t/r)^2-\alpha^{-2}-p^2)^{-\frac{1}{2}}\mathbf{M}_\Delta(q, p, 0, t, x'_3)\}\,dp \quad (72)$$

where $\mathbf{M}_\Delta$ is the column matrix

$$\mathbf{M}_\Delta(q, p, 0, t, x'_3) = \begin{bmatrix} M_{1\Delta} \\ M_{2\Delta} \\ M_{3\Delta} \end{bmatrix} \quad (73)$$

and

$$M_{i\Delta} = M_{i1,1'} + M_{i2,2'} + M_{i3,3'}. \quad (74)$$

In the process of combining the spatial derivatives to obtain $\mathbf{M}_\Delta$ there is a significant simplification and the individual terms turn out to be

$$\left.\begin{aligned}
M_{1\Delta} &= -2\alpha^{-2}q\eta_\beta\cos(\phi)\\
M_{2\Delta} &= -2\alpha^{-2}q\eta_\beta\sin(\phi)\\
M_{3\Delta} &= -\alpha^{-2}\gamma.
\end{aligned}\right\} \quad (75)$$

There are times when we also need the spatial derivatives of the Green's function with respect to the receiver co-ordinates. For the derivatives with respect to x_1 and x_2 the problem is already done because, as indicated by equation (63), in these cases the spatial derivative with respect to the receiver co-ordinate is just the negative of the spatial derivative with respect to the source co-ordinate. For the spatial derivative with respect to x_3 the problem is not quite so simple. We must return to equation (13) where the value of x_3 has not yet been set equal to zero, take the derivative with respect to x_3 at this point, and then follow through the analysis of Section 4. The net result is that the solution for $\mathbf{G}_{,k}$ is identical in form with the solution for $\mathbf{G}_{,k'}$ (equation (65)) with $\mathbf{M}_{,k}$ and $\mathbf{N}_{,k}$ replacing $\mathbf{M}_{,k'}$ and $\mathbf{N}_{,k'}$. The individual terms of these new matrices are given below.

$$M_{ij,1} = -M_{ij,1'} \tag{76}$$

$$N_{ij,1} = -N_{ij,1'} \tag{77}$$

$$M_{ij,2} = -M_{ij,2'} \tag{78}$$

$$N_{ij,2} = -N_{ij,2'} \tag{79}$$

$$\left.\begin{aligned}
M_{11,3} &= M_{31,1'}\\
M_{12,3} &= M_{32,1'}\\
M_{13,3} &= M_{33,1'}\\
M_{21,3} &= M_{12,3}\\
M_{22,3} &= M_{32,2'}\\
M_{23,3} &= M_{33,2'}\\
M_{31,3} &= -\frac{2\lambda}{\lambda+2\mu}\, q\eta_\beta\,(q^2-p^2)\cos(\phi)\\
M_{32,3} &= -\frac{2\lambda}{\lambda+2\mu}\, q\eta_\beta\,(q^2-p^2)\sin(\phi)\\
M_{33,3} &= -\frac{2\lambda}{\lambda+2\mu}\, \eta_\alpha\eta_\beta(q^2-p^2)
\end{aligned}\right\} \tag{80}$$

$$\left.\begin{aligned}
N_{11,3} &= N_{31,1'}\\
N_{12,3} &= N_{32,1'}\\
N_{13,3} &= N_{33,1'}\\
N_{21,3} &= N_{12,3}\\
N_{22,3} &= N_{32,2'}\\
N_{23,3} &= N_{33,2'}\\
N_{31,3} &= -\frac{\lambda}{\lambda+2\mu}\, q\eta_\beta\,\gamma\cos(\phi)\\
N_{32,3} &= -\frac{\lambda}{\lambda+2\mu}\, q\eta_\beta\,\gamma\sin(\phi)\\
N_{33,3} &= \frac{\lambda}{\lambda+2\mu}\, \gamma(q^2-p^2).
\end{aligned}\right\} \tag{81}$$

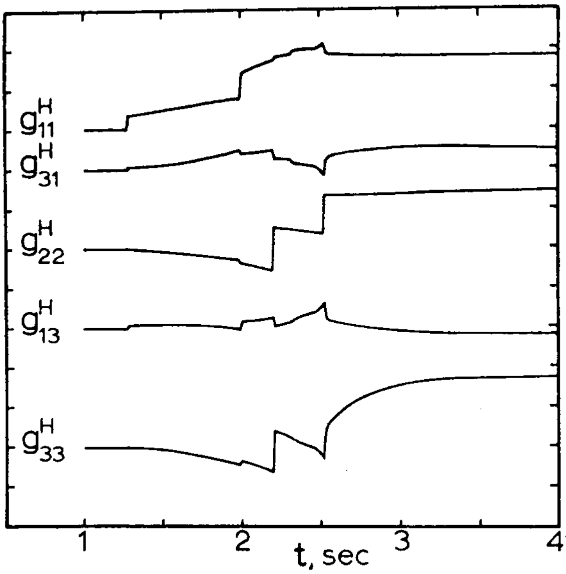

FIG. 6. The components of $\mathbf{G}^H(10, 0, 4, t; 0, 0, 2, 0)$. All components not shown are identically zero. For a force of 1 dyne a division on the vertical scale is equal to 10^{-19} cm.

With the foregoing results for the spatial derivatives of the Green's function with respect to the receiver co-ordinates one can compute the stresses and strains at the receiver directly. Also note that because the normal stress on the free surface vanishes, the dilatation at the receiver is simply $(-2\mu/\lambda)g_{3j,3}(x_1, x_2, 0, t; 0, 0, x'_3, 0)$.

The spatial derivatives of the solution within the half space can also be obtained following the procedure outlined in this section. The results can be written as

$$\begin{aligned}\mathbf{G}_{,k}(x_1, x_2, x_3, t; 0, 0, x'_3, 0) &= \mathbf{P}_{,k}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} + \mathbf{S}_{,k}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} \\ &+ \mathbf{PP}_{,k}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} + \mathbf{SS}_{,k}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} \\ &+ \mathbf{PS}_{,k}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} + \mathbf{SP}_{,k}(x_1, x_2, x_3, t, x'_3)\,\mathbf{F} \qquad (82)\end{aligned}$$

where

$$\mathbf{P}_{,k}(x_1, x_2, x_3, t, x'_3) = \frac{1}{8\pi\rho r}\frac{\partial^2}{\partial t^2} H(t - r/\alpha)\,\mathbf{D}_{,k}(x_1, x_2, x_3, t, x'_3) \qquad (83)$$

$$\mathbf{S}_{,k}(x_1, x_2, x_3, t, x'_3) = \frac{1}{8\pi\rho r}\frac{\partial^2}{\partial t^2} H(t - r/\beta)\,\mathbf{E}_{,k}(x_1, x_2, x_3, t, x'_3) \qquad (84)$$

$$\begin{aligned}\mathbf{PP}_{,k}(x_1, x_2, x_3, t, x'_3) &= \frac{1}{2\pi^2 \rho r'}\frac{\partial^2}{\partial t^2}\int_0^{((t/r')^2 - \alpha^{-2})^{1/2}} H(t - r'/\alpha) \\ &\times \mathrm{Re}\left\{\sigma^{-1}\left((t/r')^2 - \alpha^{-2} - p^2\right)^{-\frac{1}{2}}\,\mathbf{I}_{,k}(q, p, x_3, t, x'_3)\right\} dp \qquad (85)\end{aligned}$$

$$\begin{aligned}\mathbf{SS}_{,k}(x_1, x_2, x_3, t, x'_3) &= \frac{1}{2\pi^2 \rho r'}\frac{\partial^2}{\partial t^2}\int_0^{p'_2} H(t - t'_2) \\ &\times \mathrm{Re}\left\{\sigma^{-1}\left((t/r')^2 - \beta^{-2} - p^2\right)^{-\frac{1}{2}}\,\mathbf{J}_{,k}(q, p, x_3, t, x'_3)\right\} dp \qquad (86)\end{aligned}$$

$$\mathbf{PS}_{,k}(x_1, x_2, x_3, t, x'_3) = \frac{1}{2\pi^2 \rho} \frac{\partial^2}{\partial t^2} \int_0^{p_3} H(t-t_3)$$

$$\times \operatorname{Re}\{i\sigma^{-1}(R+q(x'_3/\eta_\alpha + x_3/\eta_\beta))^{-1}\, \mathbf{K}_{,k}(q, p, x_3, t, x'_3)\}\, dp \qquad (87)$$

$$\mathbf{SP}_{,k}(x_1, x_2, x_3, t, x'_3) = \frac{1}{2\pi^2 \rho} \frac{\partial^2}{\partial t^2} \int_0^{p_3} H(t-t_3)$$

$$\times \operatorname{Re}\{i\sigma^{-1}(R+q(x_3/\eta_\alpha + x'_3/\eta_\beta))^{-1}\, \mathbf{L}_{,k}(q, p, x_3, t, x'_3)\}\, dp. \qquad (88)$$

The expressions above yield the spatial derivatives with respect to the receiver co-ordinates, and k can be replaced everywhere by k' to obtain the spatial derivatives with respect to the source co-ordinates. All of the definitions and conventions of Section 5 apply here also, and, aside from the fact that the matrices are different, the only difference between the form of the solution in that section and in this one is the extra time differentiation that appears in equations (83)–(88). Expressions for the new matrix elements are given in the following paragraphs.

The spatial derivatives of the direct P wave (equation (83)) involve the following matrix elements.

$$\left.\begin{aligned}
D_{11,1} &= -(t/r)[(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\cos^2(\phi) - 3((t/r)^2 - \alpha^{-2})]\sin(\theta)\cos(\phi)\\
D_{12,1} &= -(t/r)[(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\cos^2(\phi) - ((t/r)^2 - \alpha^{-2})]\sin(\theta)\sin(\phi)\\
D_{13,1} &= (t/r)[(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\cos^2(\phi) - ((t/r)^2 - \alpha^{-2})]\cos(\theta)\\
D_{21,1} &= D_{12,1}\\
D_{22,1} &= -(t/r)[(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\sin^2(\phi) - ((t/r)^2 - \alpha^{-2})]\sin(\theta)\cos(\phi)\\
D_{23,1} &= (t/r)(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\cos(\theta)\sin(\phi)\cos(\phi)\\
D_{31,1} &= D_{13,1}\\
D_{32,1} &= D_{23,1}\\
D_{33,1} &= -(t/r)[(5(t/r)^2 - 3\alpha^{-2})\cos^2(\theta) - ((t/r)^2 - \alpha^{-2})]\sin(\theta)\cos(\phi)
\end{aligned}\right\} \qquad (89)$$

$$\left.\begin{aligned}
D_{11,2} &= D_{12,1}\\
D_{12,2} &= D_{22,1}\\
D_{13,2} &= D_{23,1}\\
D_{21,2} &= D_{12,2}\\
D_{22,2} &= -(t/r)[(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\sin^2(\phi) - 3((t/r)^2 - \alpha^{-2})]\sin(\theta)\sin(\phi)\\
D_{23,2} &= (t/r)[(5(t/r)^2 - 3\alpha^{-2})\sin^2(\theta)\sin^2(\phi) - ((t/r)^2 - \alpha^{-2})]\cos(\theta)\\
D_{31,2} &= D_{13,2}\\
D_{32,2} &= D_{23,2}\\
D_{33,2} &= -(t/r)[(5(t/r)^2 - 3\alpha^{-2})\cos^2(\theta) - ((t/r)^2 - \alpha^{-2})]\sin(\theta)\sin(\phi)
\end{aligned}\right\} \qquad (90)$$

$$\left.\begin{aligned}
D_{11,3} &= D_{13,1}\\
D_{12,3} &= D_{13,2}\\
D_{13,3} &= D_{33,1}\\
D_{21,3} &= D_{12,3}\\
D_{22,3} &= D_{23,2}\\
D_{23,3} &= D_{33,2}\\
D_{31,3} &= D_{13,3}\\
D_{32,3} &= D_{23,3}\\
D_{33,3} &= (t/r)[(5(t/r)^2 - 3\alpha^{-2})\cos^2(\theta) - 3((t/r)^2 - \alpha^{-2})]\cos(\theta)
\end{aligned}\right\} \qquad (91)$$

$$D_{ij,1'} = -D_{ij,1} \tag{92}$$

$$D_{ij,2'} = -D_{ij,2} \tag{93}$$

$$D_{ij,3'} = -D_{ij,3}. \tag{94}$$

Equation (84) for the spatial derivatives of the direct S wave involves the following matrix elements.

$$\left.\begin{aligned}
E_{11,1} &= (t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\cos^2(\phi)-(3(t/r)^2-\beta^{-2})]\sin(\theta)\cos(\phi)\\
E_{12,1} &= (t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\cos^2(\phi)-((t/r)^2-\beta^{-2})]\sin(\theta)\sin(\phi)\\
E_{13,1} &= -(t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\cos^2(\phi)-((t/r)^2-\beta^{-2})]\cos(\theta)\\
E_{21,1} &= E_{12,1}\\
E_{22,1} &= (t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\sin^2(\phi)-((t/r)^2+\beta^{-2})]\sin(\theta)\cos(\phi)\\
E_{23,1} &= -(t/r)(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\cos(\theta)\sin(\phi)\cos(\phi)\\
E_{31,1} &= E_{13,1}\\
E_{32,1} &= E_{23,1}\\
E_{33,1} &= (t/r)[(5(t/r)^2-3\beta^{-2})\cos^2(\theta)-((t/r)^2+\beta^{-2})]\sin(\theta)\cos(\phi)
\end{aligned}\right\} \tag{95}$$

$$\left.\begin{aligned}
E_{11,2} &= (t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\cos^2(\phi)-((t/r)^2+\beta^{-2})]\sin(\theta)\sin(\phi)\\
E_{12,2} &= (t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\sin^2(\phi)-((t/r)^2-\beta^{-2})]\sin(\theta)\cos(\phi)\\
E_{13,2} &= E_{23,1}\\
E_{21,2} &= E_{12,2}\\
E_{22,2} &= (t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\sin^2(\phi)-(3(t/r)^2-\beta^{-2})]\sin(\theta)\sin(\phi)\\
E_{23,2} &= -(t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\sin^2(\phi)-((t/r)^2-\beta^{-2})]\cos(\theta)\\
E_{31,2} &= E_{13,2}\\
E_{32,2} &= E_{23,2}\\
E_{33,2} &= (t/r)[(5(t/r)^2-3\beta^{-2})\cos^2(\theta)-((t/r)^2+\beta^{-2})]\sin(\theta)\sin(\phi)
\end{aligned}\right\} \tag{96}$$

$$\left.\begin{aligned}
E_{11,3} &= -(t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\cos^2(\phi)-((t/r)^2+\beta^{-2})]\cos(\theta)\\
E_{12,3} &= E_{23,1}\\
E_{13,3} &= (t/r)[(5(t/r)^2-3\beta^{-2})\cos^2(\theta)-((t/r)^2-\beta^{-2})]\sin(\theta)\cos(\phi)\\
E_{21,3} &= E_{12,3}\\
E_{22,3} &= -(t/r)[(5(t/r)^2-3\beta^{-2})\sin^2(\theta)\sin^2(\phi)-((t/r)^2+\beta^{-2})]\cos(\theta)\\
E_{23,3} &= (t/r)[(5(t/r)^2-3\beta^{-2})\cos^2(\theta)-((t/r)^2-\beta^{-2})]\sin(\theta)\sin(\phi)\\
E_{31,3} &= E_{13,3}\\
E_{32,3} &= E_{23,3}\\
E_{33,3} &= -(t/r)[(5(t/r)^2-3\beta^{-2})\cos^2(\theta)-(3(t/r)^2-\beta^{-2})]\cos(\theta)
\end{aligned}\right\} \tag{97}$$

$$E_{ij,1'} = -E_{ij,1} \tag{98}$$

$$E_{ij,2'} = -E_{ij,2} \tag{99}$$

$$E_{ij,3'} = -E_{ij,3}. \tag{100}$$

The following matrix elements are required for the spatial derivatives of the reflected PP wave (equation (85)).

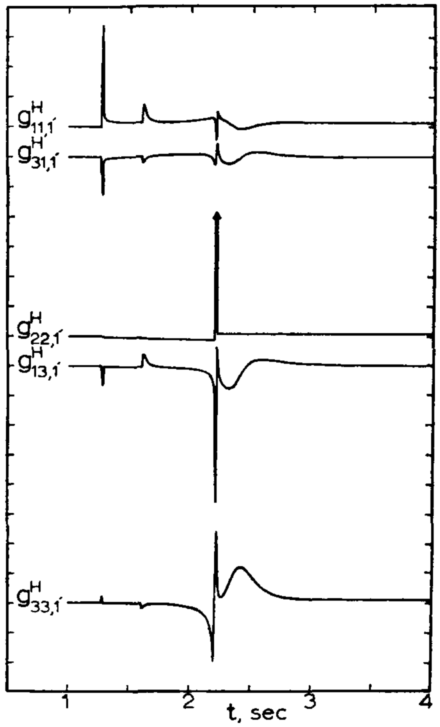

FIG. 7. The components of $\mathbf{G}^H{}_{,1'}(10, 0, 0, t; 0, 0, 2, 0)$. All components not shown are identically zero. For a couple of 1 dyne-cm a division on the vertical scale is equal to 2×10^{-24} cm and the spike on $g^H{}_{22,1'}$ attains a value of $46{\cdot}9\times10^{-24}$ cm.

$$\left.\begin{aligned}
I_{11,1} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+3p^2)\cos^2(\phi)-3p^2)\cos(\phi)\\
I_{12,1} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+3p^2)\cos^2(\phi)-p^2)\sin(\phi)\\
I_{13,1} &= -\eta_\alpha(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+p^2)\cos^2(\phi)-p^2)\\
I_{21,1} &= I_{12,1}\\
I_{22,1} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+3p^2)\sin^2(\phi)-p^2)\cos(\phi)\\
I_{23,1} &= -\eta_\alpha(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))(q^2+p^2)\sin(\phi)\cos(\phi)\\
I_{31,1} &= -I_{13,1}\\
I_{32,1} &= -I_{23,1}\\
I_{33,1} &= q\eta_\alpha{}^2(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))\cos(\phi)
\end{aligned}\right\} \quad (101)$$

$$\left.\begin{aligned}
I_{11,2} &= I_{12,1}\\
I_{12,2} &= I_{22,1}\\
I_{13,2} &= I_{23,1}\\
I_{21,2} &= I_{12,2}\\
I_{22,2} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+3p^2)\sin^2(\phi)-3p^2)\sin(\phi)\\
I_{23,2} &= -\eta_\alpha(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+p^2)\sin^2(\phi)-p^2)\\
I_{31,2} &= -I_{13,2}\\
I_{32,2} &= -I_{23,2}\\
I_{33,2} &= q\eta_\alpha{}^2(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))\sin(\phi)
\end{aligned}\right\} \quad (102)$$

$$I_{ij,3} = -\eta_\alpha I_{ij} \tag{103}$$

$$I_{ij,1'} = -I_{ij,1} \tag{104}$$

$$I_{ij,2'} = -I_{ij,2} \tag{105}$$

$$I_{ij,3'} = I_{ij,3}. \tag{106}$$

For the spatial derivatives of the reflected SS wave (equation (86)) we have the following matrix elements.

$$\left.\begin{aligned}
J_{11,1} &= -q[(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)+8\beta^{-2}\eta_\alpha\eta_\beta)((q^2+3p^2)\cos^2(\phi)-3p^2) \\
&\qquad -\beta^{-2}(\gamma^2+4\eta_\alpha\eta_\beta(q^2-p^2))]\cos(\phi) \\
J_{12,1} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)+8\beta^{-2}\eta_\alpha\eta_\beta)((q^2+3p^2)\cos^2(\phi)-p^2)\sin(\phi) \\
J_{13,1} &= -\eta_\beta(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+p^2)\cos^2(\phi)-p^2) \\
J_{21,1} &= J_{12,1} \\
J_{22,1} &= -q[(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)+8\beta^{-2}\eta_\alpha\eta_\beta)((q^2+3p^2)\sin^2(\phi)-p^2) \\
&\qquad -\beta^{-2}(\gamma^2+4\eta_\alpha\eta_\beta(q^2-p^2))]\cos(\phi) \\
J_{23,1} &= -\eta_\beta(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))(q^2+p^2)\sin(\phi)\cos(\phi) \\
J_{31,1} &= -J_{13,1} \\
J_{32,1} &= -J_{23,1} \\
J_{33,1} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))(q^2-p^2)\cos(\phi)
\end{aligned}\right\} \tag{107}$$

$$\left.\begin{aligned}
J_{11,2} &= -q[(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)+8\beta^{-2}\eta_\alpha\eta_\beta)((q^2+3p^2)\cos^2(\phi)-p^2) \\
&\qquad -\beta^{-2}(\gamma^2+4\eta_\alpha\eta_\beta(q^2-p^2))]\sin(\phi) \\
J_{12,2} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)+8\beta^{-2}\eta_\alpha\eta_\beta)((q^2+3p^2)\sin^2(\phi)-p^2)\cos(\phi) \\
J_{13,2} &= J_{23,1} \\
J_{21,2} &= J_{12,2} \\
J_{22,2} &= -q[(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)+8\beta^{-2}\eta_\alpha\eta_\beta)((q^2+3p^2)\sin^2(\phi)-3p^2) \\
&\qquad -\beta^{-2}(\gamma^2+4\eta_\alpha\eta_\beta(q^2-p^2))]\sin(\phi) \\
J_{23,2} &= -\eta_\beta(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))((q^2+p^2)\sin^2(\phi)-p^2) \\
J_{31,2} &= -J_{13,2} \\
J_{32,2} &= -J_{23,2} \\
J_{33,2} &= -q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))(q^2-p^2)\sin(\phi)
\end{aligned}\right\} \tag{108}$$

$$J_{ij,3} = -\eta_\beta J_{ij} \tag{109}$$

$$J_{ij,1'} = -J_{ij,1} \tag{110}$$

$$J_{ij,2'} = -J_{ij,2} \tag{111}$$

$$J_{ij,3'} = J_{ij,3} \tag{112}$$

The spatial derivatives of the reflected PS wave (equation (87)) involve the following matrix elements.

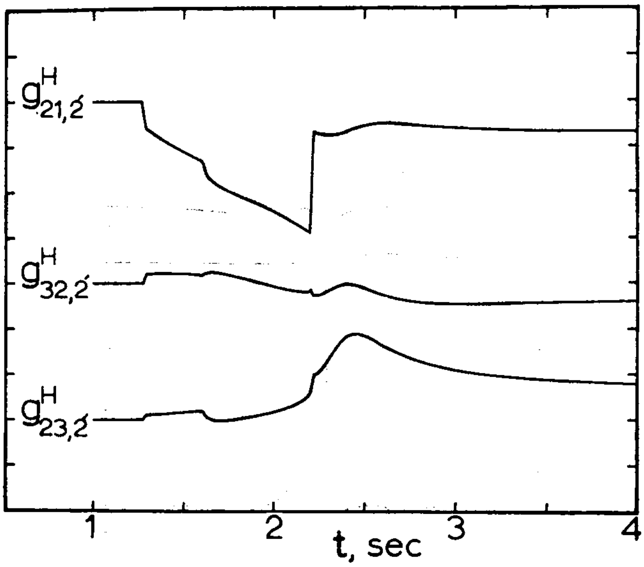

FIG. 8. The components of $\mathbf{G}^{H}{}_{,2'}(10, 0, 0, t; 0, 0, 2, 0)$. The component $g^{H}{}_{12,2'}$ is identical to $g^{H}{}_{21,2'}$ and all other components not shown are identically zero. For a couple of 1 dyne-cm a division on the vertical scale is equal to 10^{-25} cm.

$$\left.\begin{aligned}
K_{11,1} &= 4q\gamma\eta_\beta\big((q^2+3p^2)\cos^2(\phi)-3p^2\big)\cos(\phi)\\
K_{12,1} &= 4q\gamma\eta_\beta\big((q^2+3p^2)\cos^2(\phi)-p^2\big)\sin(\phi)\\
K_{13,1} &= 4\gamma\eta_\alpha\eta_\beta\big((q^2+p^2)\cos^2(\phi)-p^2\big)\\
K_{21,1} &= K_{12,1}\\
K_{22,1} &= 4q\gamma\eta_\beta\big((q^2+3p^2)\sin^2(\phi)-p^2\big)\cos(\phi)\\
K_{23,1} &= 4\gamma\eta_\alpha\eta_\beta(q^2+p^2)\sin(\phi)\cos(\phi)\\
K_{31,1} &= 4\gamma(q^2-p^2)\big((q^2+p^2)\cos^2(\phi)-p^2\big)\\
K_{32,1} &= 4\gamma(q^2-p^2)(q^2+p^2)\sin(\phi)\cos(\phi)\\
K_{33,1} &= 4q\gamma\eta_\alpha(q^2-p^2)\cos(\phi)
\end{aligned}\right\} \qquad (113)$$

$$\left.\begin{aligned}
K_{11,2} &= K_{12,1}\\
K_{12,2} &= K_{22,1}\\
K_{13,2} &= K_{23,1}\\
K_{21,2} &= K_{12,2}\\
K_{22,2} &= 4q\gamma\eta_\beta\big((q^2+3p^2)\sin^2(\phi)-3p^2\big)\sin(\phi)\\
K_{23,2} &= 4\gamma\eta_\alpha\eta_\beta\big((q^2+p^2)\sin^2(\phi)-p^2\big)\\
K_{31,2} &= K_{32,1}\\
K_{32,2} &= 4\gamma(q^2-p^2)\big((q^2+p^2)\sin^2(\phi)-p^2\big)\\
K_{33,2} &= 4q\gamma\eta_\alpha(q^2-p^2)\sin(\phi)
\end{aligned}\right\} \qquad (114)$$

$$K_{ij,3} = -\eta_\beta K_{ij} \qquad (115)$$

$$K_{ij,1'} = -K_{ij,1} \qquad (116)$$

$$K_{ij,2'} = -K_{ij,2} \qquad (117)$$

$$K_{ij,3'} = -\eta_\alpha K_{ij}. \qquad (118)$$

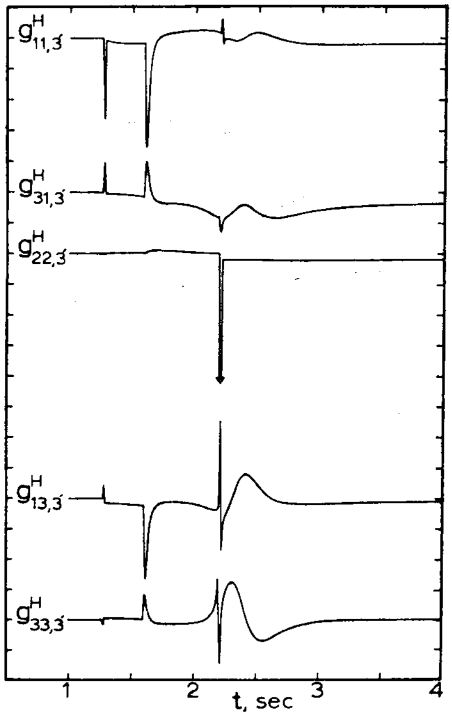

FIG. 9. The components of $\mathbf{G}^H_{,3'}(10, 0, 0, t; 0, 0, 2, 0)$. All components not shown are identically zero. For a couple of 1 dyne-cm a division on the vertical scale is equal to 5×10^{-25} cm and the spike on $g^H_{22,3'}$ attains a value of $-94{\cdot}2 \times 10^{-25}$ cm.

As one might expect the matrix elements for the spatial derivatives of the reflected *SP* wave (equation (88)) are closely related to those of the *PS* wave.

$$\left.\begin{aligned} L_{11,1} &= K_{11,1} \\ L_{12,1} &= K_{12,1} \\ L_{13,1} &= -K_{31,1} \\ L_{21,1} &= L_{12,1} \\ L_{22,1} &= K_{22,1} \\ L_{23,1} &= -K_{32,1} \\ L_{31,1} &= -K_{13,1} \\ L_{32,1} &= -K_{23,1} \\ L_{33,1} &= K_{33,1} \end{aligned}\right\} \qquad (119)$$

$$\left.\begin{aligned} L_{11,2} &= K_{11,2} \\ L_{12,2} &= K_{12,2} \\ L_{13,2} &= -K_{31,2} \\ L_{21,2} &= L_{12,2} \\ L_{22,2} &= K_{22,2} \\ L_{23,2} &= -K_{32,2} \\ L_{31,2} &= -K_{13,2} \\ L_{32,2} &= -K_{23,2} \\ L_{33,2} &= K_{33,2} \end{aligned}\right\} \qquad (120)$$

$$L_{ij,3} = -\eta_\alpha L_{ij} \tag{121}$$

$$L_{ij,1'} = -L_{ij,1} \tag{122}$$

$$L_{ij,2'} = -L_{ij,2} \tag{123}$$

$$L_{ij,3'} = -\eta_\beta L_{ij}. \tag{124}$$

The solution within the half space for a point source of dilatation can be obtained by combining the spatial derivatives of the Green's function in the same manner as in equations (72)–(75). The result is

$$\mathbf{G}_\Delta(x_1, x_2, x_3, t; 0, 0, x'_3, 0) = \mathbf{P}_\Delta(x_1, x_2, x_3, t, x'_3) + \mathbf{PP}_\Delta(x_1, x_2, x_3, t, x'_3) + \mathbf{PS}_\Delta(x_1, x_2, x_3, t, x'_3) \tag{125}$$

where

$$\mathbf{P}_\Delta(x_1, x_2, x_3, t, x'_3) = \frac{3\lambda+2\mu}{24\pi\rho r}\frac{\partial^2}{\partial t^2} H(t-r/\alpha)\, \mathbf{D}_\Delta(x_1, x_2, x_3, t, x'_3) \tag{126}$$

$$\mathbf{PP}_\Delta(x_1, x_2, x_3, t, x'_3) = \frac{3\lambda+2\mu}{6\pi^2 \rho r'}\frac{\partial^2}{\partial t^2} \int_0^{((t/r')^2-\alpha^{-2})^{1/2}} H(t-r'/\alpha) \times \mathrm{Re}\left\{\sigma^{-1}((t/r')^2-\alpha^{-2}-p^2)^{-\frac{1}{2}}\, \mathbf{I}_\Delta(q, p, x_3, t, x'_3)\right\} dp \tag{127}$$

$$\mathbf{PS}_\Delta(x_1, x_2, x_3, t, x'_3) = \frac{3\lambda+2\mu}{6\pi^2 \rho}\frac{\partial^2}{\partial t^2} \int_0^{p_3} H(t-t_3) \times \mathrm{Re}\left\{i\sigma^{-1}(R+q(x_3/\eta_\alpha+x_3/\eta_\beta))^{-1}\, \mathbf{K}_\Delta(q, p, x_3, t, x'_3)\right\} dp \tag{128}$$

and the column matrices $\mathbf{D}_\Delta$, $\mathbf{I}_\Delta$, and $\mathbf{K}_\Delta$ have the following elements.

$$\left.\begin{aligned} D_{1\Delta} &= 2\alpha^{-2}(t/r)\sin(\theta)\cos(\phi) \\ D_{2\Delta} &= 2\alpha^{-2}(t/r)\sin(\theta)\sin(\phi) \\ D_{3\Delta} &= -2\alpha^{-2}(t/r)\cos(\theta) \end{aligned}\right\} \tag{129}$$

$$\left.\begin{aligned} I_{1\Delta} &= \alpha^{-2} q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))\cos(\phi) \\ I_{2\Delta} &= \alpha^{-2} q(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2))\sin(\phi) \\ I_{3\Delta} &= -\alpha^{-2}\eta_\alpha(\gamma^2-4\eta_\alpha\eta_\beta(q^2-p^2)) \end{aligned}\right\} \tag{130}$$

$$\left.\begin{aligned} K_{1\Delta} &= -4\alpha^{-2} q\gamma\eta_\beta\cos(\phi) \\ K_{2\Delta} &= -4\alpha^{-2} q\gamma\eta_\beta\sin(\phi) \\ K_{3\Delta} &= -4\alpha^{-2}\gamma(q^2-p^2). \end{aligned}\right\} \tag{131}$$

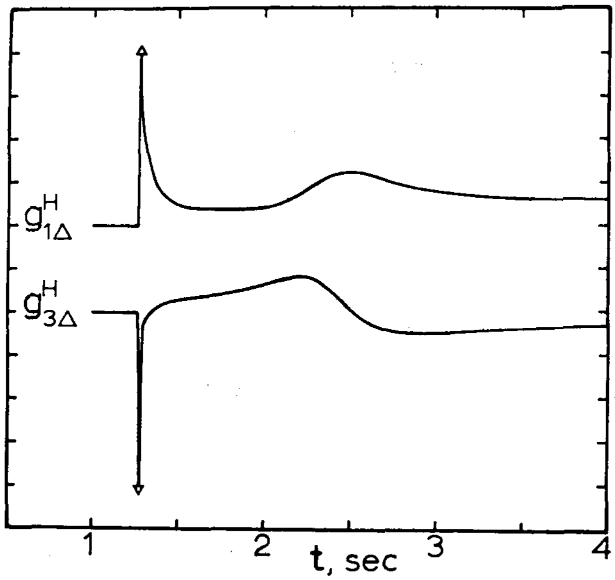

FIG. 10. The components of $\mathbf{G}_\Delta{}^H(10, 0, 0, t; 0, 0, 2, 0)$. The component $g_{2\Delta}{}^H$ is identically zero. For a unit dilatation a division on the vertical scale is equal to 2×10^{-13} cm and the spikes on $g_{1\Delta}{}^H$ and $g_{3\Delta}{}^H$ attain values of $82{\cdot}5\times10^{-13}$ cm and $-31{\cdot}2\times10^{-13}$ cm, respectively.

Finally, we note that in principle any of the higher-order spatial derivatives of the Green's function can be obtained by following the method outlined in this section. In practice the algebra increases at a rather moderate rate, so it would be relatively easy to derive a solution such as $\mathbf{G}_{,k'j}$, the strain resulting from a force couple.

7. Numerical results

All of the Green's functions and their spatial derivatives which have been presented in the preceding sections have been programmed for evaluation on a digital computer. In this section we point out some of the more important considerations involved in the numerical calculations and then give a few examples of the results.

With the exception of the portion of the solution arising from the direct P and S waves (equations (39) and (40)), all of the Green's functions of this paper involve an integral that must be evaluated by numerical methods. The integrands contain a singularity at either $((t/r)^2-\alpha^{-2})^{\frac{1}{2}}$ or $((t/r)^2-\beta^{-2})^{\frac{1}{2}}$, and, although it is an integrable singularity, it can lead to numerical problems. A simple transformation of the variable of integration helps avoid this problem. As an example, consider the first integral of equation (26)

$$\mathbf{W}_\alpha(x_1, x_2, 0, t; 0, 0, x'_3, 0) = \frac{1}{\pi^2\,\mu r}\int\limits_0^{((t/r)^2-\alpha^{-2})^{1/2}} H(t-r/\alpha)$$
$$\times \operatorname{Re}\{\eta_\alpha\,\sigma^{-1}((t/r)^2-\alpha^{-2}-p^2)^{-\frac{1}{2}}\,\mathbf{M}(q, p, 0, t, x'_3)\}\,\mathbf{F}\,dp. \qquad (132)$$

The substitution

$$p = ((t/r)^2-\alpha^{-2})^{\frac{1}{2}}-v^2 \qquad (133)$$

puts the above integral in the form

$$\mathbf{W}_\alpha(x_1, x_2, 0, t; 0, 0, x'_3, 0) = \frac{2}{\pi^2 \mu r} \int\limits_0^{((t/r)^2-\alpha^{-2})^{1/4}} H(t-r/\alpha)$$

$$\times \operatorname{Re}\{\eta_\alpha \sigma^{-1}[2((t/r)^2-\alpha^{-2})^{\frac{1}{2}}-v^2]^{-\frac{1}{2}} \mathbf{M}(q, v, 0, t, x'_3)\} \mathbf{F}\, dv \quad (134)$$

which no longer contains a singularity. Note that other transformations such as

$$p = ((t/r)^2-\alpha^{-2})^{\frac{1}{2}} \sin(v) \quad (135)$$

achieve the same effect.

After transformation into a form such as that of equation (134), the integrals can be readily evaluated with standard quadrature methods. We have achieved good success with the Romberg scheme but other methods would likely do just as well.

Another possible source of trouble in the numerical integration is associated with the Rayleigh pole (the root of equation (30)) which occupies a position on the real axis of the p plane. For values of θ near to $\pi/2$, which correspond to lateral distances large compared to the depths of the source and receiver, the path of integration passes very near to this pole and thus encounters fairly large values of the integrand. In order to maintain uniform numerical accuracy it is advisable to decrease the step size of the numerical integration procedure in the immediate vicinity of the Rayleigh pole.

It is also apparent from the form of the Green's functions in the previous sections that after the integral has been evaluated numerically it must be differentiated with respect to time at least once and sometimes twice. These time derivatives can be obtained by simple differences, but the numerical accuracy and stability of this approach are often unsatisfactory. Fortunately, this numerical differentiation of the Green's function can usually be avoided.

In most practical problems the desired solution is a combination of several factors of which the Green's function is only one. For instance, if we let $S(t)$ be the source time function, $I(t)$ be the instrument response, and $(\partial/\partial t)\, W(\mathbf{x}, t; \mathbf{x}', t')$ be the Green's function, then in a typical seismic problem the displacement on the seismogram can be represented in the form

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t} W(\mathbf{x}, t; \mathbf{x}', t') \otimes S(t) \otimes I(t) \quad (136)$$

where the symbol $\otimes$ denotes convolution in time. In most cases either $S(t)$ or $I(t)$ or both will be more band-limited in the frequency domain than $W(\mathbf{x}, t; \mathbf{x}', t')$, so from the viewpoint of numerical accuracy it is better to write equation (136) in the form

$$u(\mathbf{x}, t) = W(\mathbf{x}, t; \mathbf{x}', t') \otimes \frac{\partial}{\partial t} S(t) \otimes I(t) \quad (137)$$

or

$$u(\mathbf{x}, t) = W(\mathbf{x}, t; \mathbf{x}', t') \otimes S(t) \otimes \frac{\partial}{\partial t} I(t). \quad (138)$$

Often $S(t)$ or $I(t)$ is a simple expression which yields to analytic differentiation. The most obvious example of this is when $S(t)$ is a unit step function and then equation (137) is simply

$$u(\mathbf{x}, t) = W(\mathbf{x}, t; \mathbf{x}', t') \otimes I(t). \quad (139)$$

In other cases $S(t)$ or $I(t)$ is specified in the frequency domain and the time differentiation can be applied as a simple multiplication by frequency before transforming back to the time domain.

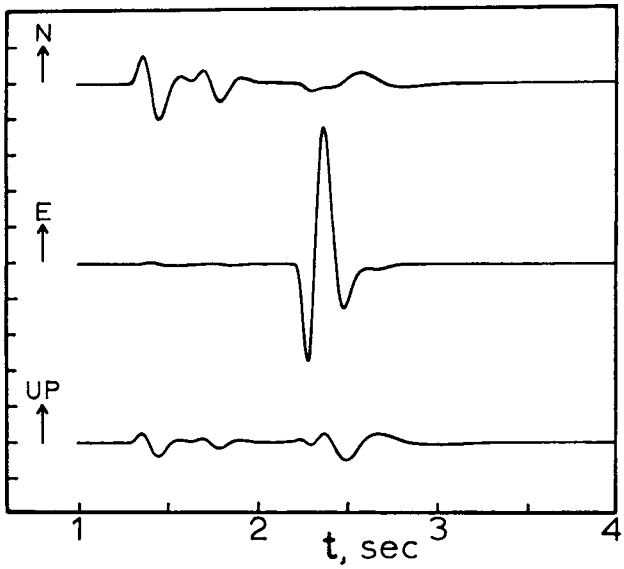

FIG. 11. Synthetic seismograms computed for a site 10 km north of a right-lateral point dislocation 2 km deep on a fault striking N30E. The dislocation is a step function in time. The seismographs are short-period Benioff systems with 1 s seismometers and 0·2 s galvonometers. For a seismic moment of $14{\cdot}08 \times 10^{18}$ erg (1 mm of dislocation over an area of $0{\cdot}02\ \mathrm{km}^2$) and instruments having magnifications of 25 000 at 1 Hz a division on the vertical scale is equal to 1 cm.

The case where equation (136) contains two differentiations with respect to time can be handled in a similar manner. The two differentiations can be applied to either $S(t)$ or $I(t)$ or distributed between them.

It is obvious that the solution at the free surface (Section 4) is just a special case of the solution within the half space (Section 5), and thus could be regarded as superfluous. However, for numerical calculations of the solution at the free surface, the special solution of Section 4 is much more satisfactory than the more general solution of Section 5. This is because several parts of the solution tend to cancel each other at the free surface, and in the special solution this cancellation has been achieved analytically while in the general solution it must be achieved numerically.

A total of 144 basic solutions have been presented in this paper, and each of these solutions exhibits markedly different behaviour in different azimuth and distance ranges. There is no practical way in which all of these results can be depicted in graphical form, so only a few representative results will be presented in the remainder of this section.

In Section 2 we adopted the convention that a Green's function is the solution resulting from a source which is a delta function in both time and space. However, for pedagogical purposes it is better to consider the solution which results from a source which is a step function in time, because such a solution contains the permanent displacements which are not present in the ordinary Green's function. Thus all of the calculations that follow have been performed assuming a source which is a step function in time, and to denote this fact the solutions have been given the symbol $\mathbf{G}^H$ in order to distinguish them from the ordinary Green's function $\mathbf{G}$.

All of the calculations illustrated in Figs 2–11 are for a half space having a compressional velocity α of $8{\cdot}00\ \mathrm{km\,s^{-1}}$, a shear velocity β of $4{\cdot}62\ \mathrm{km\,s^{-1}}$, and a density ρ of $3{\cdot}30\ \mathrm{g\,cm^{-3}}$. The solutions were calculated at points separated by 0·01 s in time and then linear interpolation was used to obtain continuous time traces.

In all of the calculations the geometry has been chosen so that the x_2 component of the solution is transverse to the line joining the source and receiver. This means that the *SH* wave (the horizontally polarized *S* wave) should appear exclusively on this component, while the *P*, *SV* (the component of the *S* wave orthogonal to *SH*), and Rayleigh waves should appear on the x_1 and x_3 components of the solution.

It is worth emphasizing that the solutions presented in this paper are complete solutions that contain all of the conventional 'far-field phases' such as the *P*, *S*, and Rayleigh waves, plus the 'near-field phases' which attenuate with distance at a rate greater than the inverse distance to the first power. The fact that the solutions are sums of all these phases can be somewhat of a nuisance when it comes to investigating one particular phase, and in such instances the approximate first-motion method of Gilbert & Knopoff (1961) can be very useful.

The Green's function at the free surface (which was derived in Section 4) exhibits a strong dependence on the ratio between the horizontal distance and the depth of the source. Results are shown in Figs 2, 3 and 4 for three different values of this ratio. In Fig. 2 this ratio is small (0·2) and the Green's function consists primarily of direct *P* and *S* waves plus the near-field parts of the solution. The Rayleigh wave is not apparent. In Fig. 3 the ratio of horizontal distance to source depth has a moderate value (5·0). The *P* and *SH* waves are much the same as in Fig. 2, but the *SV* wave has become much smaller and now has a delta-like waveform rather than the step-like waveform that it had in Fig. 2. A new phase, the diffracted *SP* wave, now arrives between the *P* and *S* waves at a time of 1·6 s. The Rayleigh wave is now apparent following the *S* wave and is best developed on the solutions resulting from a vertical force, $g_{13}{}^H$ and $g_{33}{}^H$. In Fig. 4 we have an example where the ratio of horizontal distance to source depth is fairly large (50·0). The direct *P* wave is now quite small and is followed immediately by the more prominent diffracted *SP* wave. The *SH* wave is still strong on the transverse component $g_{22}{}^H$, but the *SV* wave is little more than a subtle change of slope on the other components. The Rayleigh wave is well developed and shows the classical waveform that is associated with the far-field Rayleigh wave on a uniform half space.

Figs 5 and 6 show the Green's function at two points within the half space (which was derived in Section 5) which are directly below the point on the free surface for which the solution of Fig. 3 was calculated. The presence of the various reflected phases accounts for the more complicated appearance of the solution within the half space. The fact that the amplitude of the Rayleigh wave decreases as the depth of the receiver increases is also apparent in these figures.

Figs 7, 8 and 9 show the spatial derivatives with respect to the source co-ordinates (which were derived in Section 6) for the solution shown in Fig. 3. These results can be thought of as the displacements resulting from a source which is a force couple. In calculating these results it was necessary to perform one numerical differentiation with respect to time. As one might expect, the solution due to a force couple behaves much like the derivative with respect to time of the solution due to a simple force. Thus the phases such as *P*, diffracted *SP*, and *SH* which have a step-like appearance in Fig. 3 take on a delta-like appearance in Figs 7 and 9. And phases such as *SV* which have a delta-like appearance in Fig. 3 look something like a differentiated delta function in Figs 7 and 9. Consistent with this generalization is the fact that phases which are identically zero in Fig. 3 have a step-like appearance in Fig. 8. Also note that while Figs 2–6 all have the same scale factors, Figs 7, 8 and 9 have different and varying scale factors.

Fig. 10 represents the solution at a point on the free surface resulting from a point source of dilatation (equation (72)). The solution is dominated by the direct *P* wave, and, in the absence of *S* waves, the retrograde particle motion of the Rayleigh wave becomes fairly obvious.

Finally, in Fig. 11 we show a typical seismological application of the Green's functions. Assuming that an earthquake can be modelled as a simple point dislocation, the representation theorem of equation (61) was used to calculate ground displacements, and these were convolved with the response function of a typical short-period seismograph to yield the synthetic seismograms that are shown. The most pronounced phase is *SH* on the *EW* component, but the *P*, diffracted *SP*, *SV*, and Rayleigh phases are also apparent. Using the numbers given in the figure caption and an approximate empirical relation between seismic moment and magnitude, the synthetic seismograms of Fig. 11 are appropriate for an earthquake with a magnitude between 1·5 and 2·0 observed at an epicentral distance of 10 km. While the source model assumed for the purpose of calculating the results of Fig. 11 was extremely simple, it is clear that much more interesting sources that incorporate both finite dimensions and a propagation velocity can be handled merely by summing the contributions from point dislocations that are distributed in both time and space.

In many cases the limitation upon the practical application of the results presented in this paper will be the amount of computational time that is required. As a guideline to this aspect of the problem, the results shown in Fig. 11, which consist of three components of displacement calculated at 300 points in time with the local error of the integration algorithm set at 10^{-6}, required about 70 s of computational time on a CDC 6400 computer.

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