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GRIFFITHS-HARRIS RIGIDITY OF COMPACT HERMITIAN SYMMETRIC SPACES

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Abstract

I prove that any complex manifold that has a projective second fundmental form isomorphic to one of a rank two compact Hermitian symmetric space (other than a quadric hypersurface) at a general point must be an open subset of such a space. This contrasts the non-rigidity of all other compact Hermitian symmetric spaces observed in [**12**, **13**]. A key step is the use of higher order Bertini type theorems that may be of interest in their own right.

1. Introduction

Let $X \subset \mathbb{CP}^{n+a}$ be a variety and let $x \in X$ be a smooth point. The projective second fundamental form of X at x (see [**3**, **11**, **8**, **7**]) is a basic differential invariant that measures how X is moving away from its embedded tangent projective space at x to first order. It determines a system of quadrics $|II_{X,x}| \subset S^2T^*_xX$. I prove

Theorem 1.1. *Let $X^n \subset \mathbb{CP}^{n+a}$ be a complex submanifold. Let $x \in X$ be a general point. If $|II_{X,x}| \simeq |II_{Z,z}|$ where Z is a compact rank two Hermitian symmetric space in its natural embedding, other than a quadric hypersurface, then $\overline{X} = Z$.*

Let X be such that $|II_{X,x}|$ is an isolated point in the moduli space of a-dimensional linear subspaces of the space of quadratic forms on $\mathbb{C}^n$ up to linear equivalance. We say X is *infinitesimally rigid at order two* or is *Griffiths-Harris rigid* if whenever $Y \subset \mathbb{P}^N$ is a complex manifold, $y \in Y$ is a general point and $|II_{Y,y}| = |II_{X,x}|$, then $\overline{Y} = X$.

In [**3**], Griffiths and Harris posed the question as to whether the Segre variety $\mathrm{Seg}\,(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$ was infinitesimally rigid to order two, and in [**10**] I answered the question affirmatively and showed that all rank two compact Hermitian symmetric spaces (in their minimal homogeneous embeddings) except for the quadric hypersurface, and possibly the Grassmanian $G(\mathbb{C}^2, \mathbb{C}^5) \subset \mathbb{P}^9$, the Segre variety $Seg(\mathbb{P}^1 \times \mathbb{P}^n) \subset \mathbb{P}^{2n+1}$ and the *spinor variety* $D_5/P_5 = \mathbb{S}_{10} \subset \mathbb{P}^{15}$ were infinitesimally rigid at order two. The quadric is not rigid to order two and Fubini showed [**2**]

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it is rigid to order three when $n > 1$. (It is rigid to order five when $n = 1$.) In this paper I resolve the remaining cases, and explain shorter and less computational proofs for the other cases presented in [**10**]. I also reprove the rigidity of the three Severi varieties that are rigid to order two to illustrate the method. The new proofs use two tools, a higher order Bertini theorem, and elementary representation theory.

In [**12**, **13**] we showed that all rational homogeneous varieties other than the rank two compact Hermitian symmetric spaces fail to be rigid to order two, so the result of this paper is the best possible in this sense. One can compare this type of rigidity to that studied by Hwang and Mok, see [**4**, **5**]. Some differences are: in their study they require global hypotheses where here the hypotheses are at the level of germs (this is because the systems of quadrics under study admit no local deformations); in their study the objects of interest are not *a priori* given an embedding (although since they assume the Picard number is one, one gets something close to an embedding); and in their study the object of interest is the cone of minimal degree rational curves through a general point, which, *a priori*, has nothing to do with the cone of asymptotic directions I use here (in the systems under consideration, the base locus of II determines II).

Some open questions and relations with the Fulton-Hansen connectedness theorem are discussed in [**10**]. Another application of the techniques used here is given in [**15**].

After a preprint of this article was posted on the arXiv, Hwang and Yamaguchi [**6**] generalized the results of this paper to prive that all irreducible compact Hermitian symmetric spaces, except for the quadric hypersurface, are uniquely determined by their projective fundamental forms at a general point. Their method is basied on work of Se-ashi [**16**] and is quite elegant, but of more limited applicability. For example, by their methods one cannot determine the rigidity of rational homogeneous varieties that are not Hermitian symmetric, or of the Segre varieties $Seg(\mathbb{P}^1 \times \mathbb{P}^n) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^{n+1})$ with $n > 1$ (personal communication with Hwang).

While in [**10**] I did not calculate the rigidity of $Seg(\mathbb{P}^1 \times \mathbb{P}^n) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^{n+1})$, the rigidity follows from the same calculations, however one must take an additional derivative to get the appropriate vanishing of the Fubini forms. However there is also an elementary proof of rigidity in this case using the higher order Bertini theorem. Given a variety $X^{n+1} \subset \mathbb{P}^N$ such that at a general point $x \in X$, $\mathrm{Base}|II_{X,x}|$ contains a $\mathbb{C}^n$, by 3.1 the n-plane field is integrable and thus X is ruled by $\mathbb{P}^n$'s. Such a variety arises necessarily from a curve in the Grassmannian $G(n+1, N+1)$ (as the union of the points on the $\mathbb{P}^n$'s in the curve). But in order to also have the $\mathbb{C}$ factor in $\mathrm{Base}|II_{X,x}|$, such a curve must be a line and thus X must be the Segre.

2. Moving frames

For more details throughout this section, see any of [**3**, **9**, **11**, **7**].

Once and for all fix index ranges $1 \leq \alpha, \beta, \gamma \leq n$, $n+1 \leq \mu, \nu \leq n+a$.

Let $X^n \subset \mathbb{CP}^{n+a} = \mathbb{P}V$ be a complex submanifold and let $x \in X$ be a general point. Let $\pi : \mathcal{F}^1 \to X$ denote the bundle of bases of V (frames) preserving the flag

$$\hat{x} \subset \hat{T}_x X \subset V.$$

Here $\hat{x} \subset V$ denotes the line corresponding to x and $\hat{T}_x X$ denotes the affine tangent space (the cone over the embedded tangent projective space). Let $(e_0, \ldots, e_{n+a})$ be a basis of V with dual basis $(e^0, \ldots, e^{n+a})$ adapted such that $e_0 \in \hat{x}$ and $\{e_0, e_\alpha\}$ span $\hat{T}_x X$. I ignore twists and obvious quotients, writing e_α for $(e_\alpha \mathrm{mod}\ e_0) \otimes e^0 \in T_x X$ and e_μ for $(e_\mu \mathrm{mod}\ \hat{T}_x X) \otimes e^0 \in N_x X = T_x \mathbb{P}V / T_x X$. Moreover, if x and X are understood, I write $T = T_x X$ and $N = N_x X$.

The fiber of $\pi : \mathcal{F}^1 \to X$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix} g^0_0 & g^0_\beta & g^0_\nu \\ 0 & g^\alpha_\beta & g^\alpha_\nu \\ 0 & 0 & g^\mu_\nu \end{pmatrix} \,\middle|\, g \in GL(V) \right\}.$$

While $\mathcal{F}^1$ is not in general a Lie group, since $\mathcal{F}^1 \subset GL(V)$, we may pull back the Maurer-Cartan from on $GL(V)$ to $\mathcal{F}^1$. Write the pullback of the Maurer-Cartan form to $\mathcal{F}^1$ as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_\beta & \omega^0_\nu \\ \omega^\alpha_0 & \omega^\alpha_\beta & \omega^\alpha_\nu \\ \omega^\mu_0 & \omega^\mu_\beta & \omega^\mu_\nu \end{pmatrix}.$$

The adaptation implies that $\omega^\mu_0 = 0$ and the Maurer-Cartan equation $d\omega = -\omega \wedge \omega$ together with the Cartan Lemma imply that for all μ, α, $\omega^\mu_\alpha = q^\mu_{\alpha\beta}\omega^\beta_0$ for some functions $q^\mu_{\alpha\beta} = q^\mu_{\beta\alpha} : \mathcal{F}^1 \to \mathbb{C}$. These functions determine the projective second fundamental form $II = F_2 = \omega^\alpha_0 \omega^\mu_\alpha \otimes e_\mu = q^\mu_{\alpha\beta}\omega^\alpha_0\omega^\beta_0 \otimes e_\mu \in \Gamma(X, S^2T^*X \otimes NX)$.

While II descends to be a section of $S^2T^*X \otimes NX$, higher order derivatives provide relative differential invariants $F_k \in \Gamma(\mathcal{F}^1, \pi^*(S^kT^* \otimes N))$. For example,

$$F_3 = r^\mu_{\alpha\beta\gamma}\omega^\alpha_0\omega^\beta_0\omega^\gamma_0 \otimes e_\mu$$
$$F_4 = r^\mu_{\alpha\beta\gamma\delta}\omega^\delta_0\omega^\alpha_0\omega^\beta_0\omega^\gamma_0 \otimes e_\mu$$

where the functions $r^\mu_{\alpha\beta\gamma}, r^\mu_{\alpha\beta\gamma\delta}$ are given by

$$r^\mu_{\alpha\beta\gamma}\omega^\gamma_0 = -dq^\mu_{\alpha\beta} - q^\mu_{\alpha\beta}\omega^0_0 - q^\nu_{\alpha\beta}\omega^\mu_\nu + q^\mu_{\alpha\delta}\omega^\delta_\beta + q^\mu_{\beta\delta}\omega^\delta_\alpha \tag{1}$$

$$\begin{aligned} r^\mu_{\alpha\beta\gamma\delta}\omega^\delta_0 = &-dr^\mu_{\alpha\beta\gamma} - 2r^\mu_{\alpha\beta\gamma}\omega^0_0 - r^\nu_{\alpha\beta\gamma}\omega^\mu_\nu \\ &+ \mathfrak{S}_{\alpha\beta\gamma}(r^\mu_{\alpha\beta\epsilon}\omega^\epsilon_\gamma + 3q^\mu_{\alpha\beta}\omega^0_\gamma - q^\mu_{\alpha\epsilon}q^\nu_{\beta\gamma}\omega^\epsilon_\nu). \end{aligned} \tag{2}$$

If one chooses local affine coordinates $(x^1,\dots,x^{n+a})$ such that $x = (0,\dots,0)$ and $T_xX = \langle\frac{\partial}{\partial x^\alpha}\rangle$, and writes X as a graph

$$x^\mu = q^\mu_{\alpha\beta}x^\alpha x^\beta + r^\mu_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma + r^\mu_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta + \cdots$$

then there exists a local section of $\mathcal{F}^1$ such that

$$\begin{aligned} F_2|_x &= q^\mu_{\alpha\beta}dx^\alpha dx^\beta \otimes \frac{\partial}{\partial x^\mu} \\ F_3|_x &= r^\mu_{\alpha\beta\gamma}dx^\alpha dx^\beta dx^\gamma \otimes \frac{\partial}{\partial x^\mu} \\ F_4|_x &= r^\mu_{\alpha\beta\gamma\delta}dx^\alpha dx^\beta dx^\gamma dx^\delta \otimes \frac{\partial}{\partial x^\mu} \end{aligned}$$

etc...

Since an analytic variety is uniquely determined by its Taylor series at a point, to show Z is rigid to order two, it is sufficient to show that over varieties X with $|II_{X,x}| = |II_{Z,z}|$ there exists a subbundle of $\mathcal{F}^1$ such that the F_k's of X coincide with those of Z. Moreover, the cominuscule varieties, that is, the compact Hermitian symmetric spaces in their natural projective embeddings, have the property that on a reduced frame bundle all the differential invariants except for their fundamental forms are zero, and in our case the only nonzero fundamental form is II.

The method in [**10**] was first to use the equations above to calculate relations among the coefficients of F_3. These relations, combined with normalizations, eliminated most of the coefficients of F_3 (all in the Segre case). I then examined a particular subset of F_4 which equated an expression with coefficients of F_4 appearing with semi-basic forms with an expression with coefficients of F_3 appearing with vertical forms, which proved the remainder of F_3 to be zero (and part of F_4 to be zero as well). The same technique was used with higher order invariants.

The principle of calculation in this paper is the same, except that the calculations are minimized by using higher order Bertini theorems and the decomposition of the spaces $S^dT^*\otimes N$ into irreducible R-modules, where $R \subset GL(T)\times GL(N)$ is the subgroup preserving $II \in S^2T^*\otimes N$. The methods here should make the determination of the order of rigidity of other rational homogeneous varieties and of Schubert varieties a tractible problem.

3. Vanishing tools

3.1. Higher order Bertini. Let T be a vector space. The classical Bertini theorem implies that for a linear subspace $A \subset S^2T^*$, if $q \in A$ is generic, then $u \in q_{sing} := \{v \in T \mid q(v,w) = 0 \ \forall w \in T\}$ implies $u \in \mathrm{Base}\,(A) := \{v \in T \mid Q(v,v) = 0 \ \forall Q \in A\}$.

Theorem 3.1 (Higher order Bertini). *Let $X^n \subset \mathbb{P}V$ be a complex manifold and let $x \in X$ be a general point.*

1. *Let $q \in |II_{X,x}|$ be a generic quadric. Then $q_{sing} \subset \mathrm{Base}\{F_2, \dots, F_k\}$ for all k. I.e., q_{sing} is tangent to a linear space on the completion of X.*
2. *Let $q \in |II_{X,x}|$ be a any quadric, let $L \subset q_{sing} \cap \mathrm{Base}|II_{X,x}|$ be a linear subspace. Then for all $v, w \in L$, $F_3^q(v, w, \cdot) = 0$. Here and in what follows, F_k^q denotes the polynomial in F_k corresponding to the conormal direction of q. This is well defined by the lower order vanishing.*
3. *With L as in 2., if $L' \subset (\mathrm{Base}\{|II_{X,x}|, F_3\}) \cap L$ is a linear subspace then $F_4^q(u, v, w, \cdot) = 0$ for all $u, v, w \in L'$ and so on for higher orders.*
4. *With L' as in 3., if $L'' \subset L' \cap (F_3^q)_{sing}$ is a linear space, then for all $u, v \in L''$, $F_4^q(u, v, \cdot, \cdot) = 0$. The analogous result holds for higher orders.*

Proof. Note that 1. is classical, but we provide a proof for completeness. Assume $v = e_1$ and $q = q^\mu$. Our hypotheses imply $q^\mu_{1\beta} = 0$ for all β. Formula (1) reduces to

$$r^\mu_{11\beta}\omega^\beta_0 = -q^\nu_{11}\omega^\mu_\nu.$$

If q is generic we are working on a reduction of $\mathcal{F}^1$ where the ω^μ_ν are independent of the semi-basic forms; thus the coefficients on both sides of the equality are zero, proving both the classical Bertini theorem and 1 in the case $k = 3$.

If q is not generic, in order to have $q = q^\mu, v = e_1$ we have reduced to a subbundle of $\mathcal{F}^1$ where the forms ω^μ_ν are no longer necessarily independent of the semi-basic forms. However, hypothesis 2 states that $q^\nu_{11} = 0$ for all ν and the required vanishing still holds.

For 3., note that $r^\mu_{111\delta}\omega^\delta_0 = r^\nu_{111}\omega^\mu_\nu + r^\mu_{11\epsilon}\omega^\epsilon_1 + q^\mu_{1\epsilon}q^\nu_{11}\omega^\epsilon_\nu$ and the right hand side is zero under our hypotheses. Part 4 is proven similarly.

The extension to linear spaces holds by polarizing the forms. The analogous equation at each order proves the next higher order. q.e.d.

Example 3.2. Let $X = G(2, m)$ and let $V = \Lambda^2\mathbb{C}^m$ have basis e_{st} with $1 \le s < t \le m$. At $x = [e_{12}]$ we have the adapted flag

$$\{e_{12}\} \subset \{e_{1j}, e_{2j}\} \subset V$$

where $3 \leq i < j \leq m$, and $SL_2 \times SL_{m-2}$ acts transitively on $N_x \simeq \{e_{ij}\}$. So here $\alpha = \{(1j), (2,j)\}$, $\mu = \{(ij)\}$. In these frames $II = (\omega_0^{(1i)}\omega_0^{(2j)} - \omega_0^{(1j)}\omega_0^{(2i)}) \otimes e_{ij}$.

If $m = 5$, then q^{45} is a generic quadric with $e_{13} \subset q^{45}_{sing}$. Thus

$$r^{\mu}_{(13)(13)(13)} = 0 \ \forall \mu$$
$$r^{45}_{(13)(13)\beta} = 0 \ \forall \beta.$$

If $m > 5$ then q^{45} is no longer generic, but since $e_{(13)} \in \text{Base}|II_{X,x}|$ we still may conclude

$$r^{45}_{(13)(13)\beta} = 0 \ \forall \beta.$$

3.2. Normalizations. F_3 is translated in the fiber of $\mathcal{F}^1$ by the action of $T \otimes N^*$ and T^* (the g^{α}_{μ} and the g^0_{α}). We may decompose $T \otimes N^*$ and T^* into irreducible R modules and determine which of these act nontrivially. In the case where the variety is modeled on a rank two cominuscule variety, we will have that all of $T \otimes N^*$ acts effectively, but the T^* action duplicates a factor in $T \otimes N^*$. This is because in the homogeneous model, the forms ω^0_{β} are independent and the forms ω^{α}_{μ} are linear combinations of the ω^0_{β}. We will let $\mathcal{F}^n$ denote the bundle where the action of $T \otimes N^*$ has been used to kill the corresponding components of F_3. Similarly, on $\mathcal{F}^n$, F_4 is translated by the action of N^* and we will let $\mathcal{F}^N$ denote the subbundle of $\mathcal{F}^n$ where the component of N^* in F_4 has been normalized to zero.

3.3. Decomposition of the F_k and vanishing.

Proposition 3.3. *Let $II \in S^2T^* \otimes N$ arise from a trivial representation of a reductive group $R \subset GL(T) \times GL(N)$. Let $X^n \subset \mathbb{P}^{n+a}$ be a complex submanifold, let $x \in X$ be a general point and suppose $II_{X,x} = II$.*

1. *The component of F_k in an irreducible module $V \subset S^kT^* \otimes N$ is zero if the component in its highest weight vector is zero.*
2. *An irreducible module in $S^3T^* \otimes N$ can occur in F_3 only if it also occurs in $(T \otimes T^* \oplus N \otimes N^*)^{\mathfrak{r}^c} \otimes T^*$. Here $\mathfrak{r}$, the Lie algebra of R, occurs as a submodule of $T \otimes T^* \oplus N \otimes N^*$ and $(T \otimes T^* \oplus N \otimes N^*)^{\mathfrak{r}^c}$ denotes the complement of $\mathfrak{r}$ in $T \otimes T^* \oplus N \otimes N^*$.*
3. *If $F_3 = 0$, and the normalizations of F_3 are exactly by $T \otimes N^*$, then an irreducible module in $S^4T^* \otimes N$ can occur in F_4 only if it also occurs in $(T \otimes N^*)^{T^{*c}} \otimes T^*$, where $(T \otimes N^*)^{T^{*c}}$ is the complement of the image of T^* in $T \otimes N^*$.*
4. *If $F_3, F_4 = 0$ (after normalizations), then an irreducible module in $S^5T^* \otimes N$ can occur in F_4 only if it also occurs in N.*
5. *If $F_3, F_4, F_5 = 0$ (after normalizations), then all higher F_k are zero as well.*

Note that if $X = G/P$ is a rational homogeneous variety in its minimal homogeneous embedding, then R is just the Levi factor of P.

Proof. The first assertion follows because the orbit of a highest weight vector in any module spans the module.

Let $\mathcal{F}^2 \to X$ denote the bundle of frames adapted such that the coefficients of II are constant. $\mathcal{F}^2$ is a principal bundle with fiber group say H which contains R as a Levi factor. Write $\mathfrak{h} = \mathfrak{r} \oplus \mathfrak{n}$ where $\mathfrak{n}$ is nilpotent. Since we are working locally, if we take a basis ξ^s, η^p of the Maurer-Cartan form of H, with the forms ξ^s being $\mathfrak{r}$-valued and the forms η^p $\mathfrak{n}$-valued, a local coframing of $\mathcal{F}^2$ is given by $\xi^s, \eta^p, \omega_0^\alpha$. Thus we may write

$$\omega_\beta^\alpha = C_{\beta s}^\alpha \xi^s + E_{\beta p}^\alpha \eta^p + J_{\beta\gamma}^\alpha \omega_0^\gamma \tag{3}$$

$$\omega_\nu^\mu = G_{\nu s}^\mu \xi^s + H_{\nu p}^\mu \eta^p + I_{\nu\gamma}^\mu \omega_0^\gamma \tag{4}$$

for some functions $C_{\beta s}^\alpha, \dots, I_{\nu\gamma}^\mu$. Note that this decomposition is not unique and we can choose a different splitting of the cotangent bundle of $\mathcal{F}^2$ by translating the $\mathfrak{h}$-valued forms by semi-basic forms. Translating the $\mathfrak{n}$-valued forms is what gives rise to the normalizations of F_3, F_4 discussed above. By the same arguent, decomposing $T \otimes T^* \oplus N \otimes N^*$ into isotypic $\mathfrak{r}$-modules, the component that corresponds to the representation of $\mathfrak{r}$ in $T \otimes T^* \oplus N \otimes N^*$ is independent of the semi-basic forms.

We have the equality

$$F_3 = r_{\alpha\beta\gamma}^\mu \omega_0^\alpha \omega_0^\beta \omega_0^\gamma \otimes e_\mu = (-q_{\alpha\beta}^\mu \omega_0^0 - q_{\alpha\beta}^\nu \omega_\nu^\mu + q_{\alpha\delta}^\mu \omega_\beta^\delta + q_{\beta\delta}^\mu \omega_\alpha^\delta) \omega_0^\alpha \omega_0^\beta \otimes e_\mu.$$

Decomposing the right hand side and equating semi-basic forms we have

$$r_{\alpha\beta\gamma}^\mu \omega_0^\alpha \omega_0^\beta \omega_0^\gamma = (-q_{\alpha\beta}^\mu \omega_0^0 - q_{\alpha\beta}^\nu I_{\nu\gamma}^\mu \omega_0^\gamma + q_{\alpha\delta}^\mu J_{\beta\gamma}^\delta \omega_0^\gamma + q_{\beta\delta}^\mu J_{\alpha\gamma}^\delta \omega_0^\gamma) \omega_0^\alpha \omega_0^\beta.$$

While F_3 is not invariant under the action of H, it is invariant under the action of R and in particular the isotypic components are each invariant. Now consider the isotypic decomposition of $S^3T^* \otimes N$ as an R-module, say into the modules $V_1, \dots, V_q$. Write $F_3 = F_{3,1} + \dots + F_{3,q}$. Since each of these components is preserved by the action of R, it can be nonzero only if the corresponding module occurs on the right hand side (and moreover the multiplicity can be at most the multiplicity that occurs on the right hand side). These remarks prove the second assertion.

The third assertion follows because in the above normalizations, the forms ω_β^0 will remain independent and independent of the semi-basic forms while the forms ω_ν^ϵ will become dependent on thet semi-basic forms and the ω_β^0. Again, the components that depend on the semi-basic forms will have coefficients consisting of linear combinations of monomials in F_4 of the same weight. The fourth assertion is similar. The last assertion is proven in [**9**]. q.e.d.

4. Case of $G(2,5)$ and $\mathbb{S}_{10}$

4.1. Model for $G(2,5)$. Write $T = A^* \otimes B$. We index bases of T and N as above. $R = \mathfrak{sl}(A) + \mathfrak{sl}(B) + \mathbb{C} = \mathfrak{sl}_2 + \mathfrak{sl}_3 + \mathbb{C}$. We write A_j for the represention of $\mathfrak{sl}(A)$ with highest weight j and B_{ij} for the representation of $\mathfrak{sl}(B)$ of highest weight $i\omega_1 + j\omega_2$. Here and throughout we use the notations and ordering of the weights of [**1**]. The relevant modules are summarized in the table below.

4.2. Model for $\mathbb{S}_{10}$. Write $\mathbb{C}^{16} = \text{Clifford}\,(\mathbb{C}^5) \simeq \Lambda^{\text{even}}\mathbb{C}^5$ with $\hat{x} \simeq \Lambda^0 W$, $T \simeq \Lambda^2 W, N \simeq \Lambda^4 W \simeq W^*$. We let $e_{st} = e_s \wedge e_t$, $1 \leq s < t \leq 5$ index a basis of T and e^s index a basis of N. Note that, as with $G(2,5)$, R acts transitively on N so all quadrics in $|II|$ are generic.

Let V_{ijkl} denote the $\mathfrak{sl}_5$ module with highest weight $i\omega_1 + j\omega_2 + k\omega_3 + l\omega_4$. $|II|$ is given by the Pfaffians of the 4×4 minors centered about the diagonal with e^j corresponding to the Pfaffian obtained by removing the j-th row and column. The relevant modules are summarized in the following table. When there are two lines on the right hand side, the first is for $\mathbb{S}_{10}$ and the second for $G(2,5)$.

$$\begin{aligned}
T &= V_{0100}\\
&= A_1 \otimes B_{10}\\
T^* &= V_{0010}\\
&= A_1 \otimes B_{01}\\
N &= V_{0001}\\
&= A_0 \otimes B_{01}\\
N^* &= V_{1000}\\
&= A_0 \otimes B_{10}\\
U &= V_{0101}\\
&= A_2 \otimes B_{10}\\
S^2T^* &= T^{*2} \oplus N^*\\
S^3T^* &= T^{*3} \oplus N^*T^*\\
S^3T^* \otimes N &= (T^{*3}N \oplus TT^{*2}) \oplus ((N^*T^*)N \oplus N^*T \oplus T^*)\\
T \otimes N^* &= TN^* \oplus T^*\\
(T\otimes N^*)^{T^{*c}} \otimes T^* &= N^*TT^* \oplus N^{*2}N \oplus TN \oplus N^*\\
S^4T^* &= T^{*4} \oplus N^*T^{*2} \oplus N^{*2}\\
S^4T^* \otimes N &= T^{*4}N \oplus TT^{*3} \oplus NN^*T^{*2} \oplus TT^*N \oplus T^{*2} \oplus N^{*2}N \oplus N^*
\end{aligned}$$

The notation is such that if V_λ, W_μ are the irreducible representations with highest weights λ, μ, then V^k, VW are respectively the representations with highest weights $k\lambda$ and $\lambda+\mu$. $VW \subset V\otimes W$ is called the *Cartan component*.

To obtain the vanishing of F_3 we need to eliminate five modules. We first eliminate two by reducing to $\mathcal{F}^n$ as described above, so the last two factors are zero. Let $\mathcal{F}^n \subset \mathcal{F}^1$ denote our new frame bundle.

On our new bundle there remain three modules to eliminate.

The first module in S^3T^* is decomposably generated by $r_{(13)(13)(13)}$ in the $G(2,5)$ case and $r_{(12)(12)(12)}$ in the $\mathbb{S}_{10}$ case. We already saw the $G(2,5)$ case is covered by Bertini, and the $\mathbb{S}_{10}$ case is as well because $e_{(12)} \in q^1_{sing}$, and all quadrics in the system are generic. Thus the first two modules in $S^3T^*\otimes N$ don't appear in F_3.

At this point just $(N^*T^*)N$ remains. In the $G(2,5)$ case (N^*T^*) has highest weight a linear combination of $r_{(13)(13)(24)}$ and $r_{(13)(14)(23)}$. In the $\mathbb{S}_{10}$ case it has highest weight a linear combination of $r_{(12)(12)(34)}$, $r_{(12)(13)(24)}$ and $r_{(12)(14)(23)}$, and thus the Cartan components respectively have highest weights linear combinations of $r^{45}_{(13)(13)(24)}$, $r^{45}_{(13)(14)(23)}$ and $r^5_{(12)(12)(34)}$, $r^5_{(12)(13)(24)}$ and $r^5_{(12)(14)(23)}$, all of which are zero by Bertini. Thus F_3 is zero.

We normalize away the N^* factor in $S^4T^*\otimes N$ and study the remaining modules. Comparing $S^4T^*\otimes N$ and $(T\otimes N^*)^{T^{*c}}\otimes T^*$ modulo N^*, their intersection is N^{*2} which is eliminated by the higher order Bertini theorem and thus $F_4 = 0$ on $\mathcal{F}^N$.

$S^5T^*\otimes N$ does not contain a copy of N, so we are done. q.e.d.

5. Case of $\mathbb{AP}^2$

Let $\mathbb{A}_\mathbb{R}$ respectively denote $\mathbb{C}, \mathbb{H}, \mathbb{O}$, let $\mathbb{A} = \mathbb{A}_\mathbb{R}\otimes_\mathbb{R}\mathbb{C}$, and write $T = \mathbb{A}\oplus\mathbb{A}$. $\mathbb{AP}^2$ respectively denotes the Segre $\mathbb{P}^2\times\mathbb{P}^2$, the Grassmannian $G(2,6)$ and the complexified Cayley plane.

I use (a,b) as $\mathbb{A}$-valued coordinates. Then $|II| = \{a\overline{a}, b\overline{b}, a\overline{b}\}$ where $a\overline{b}$ represents dim$\mathbb{A}$ quadrics. Let $p = 1,3,7$. Write $a = a_0+a_1\epsilon_1+\ldots+a_p\epsilon_p$. We will need to work with null vectors so let $e_1 = 1+i\epsilon_1$, $\overline{e}_1 = 1-i\epsilon_1$, $e_2 = 1+i\epsilon_2$ denote elements of the first copy of $\mathbb{A}$ (with coordinate a). We let e_a denote the normal vector such that $q^a = a\overline{a}$ and similarly for e_b. Let e_0 denote the normal vector such that $q^0 = Re(a\overline{b})$ and e_{ϵ_j} the normal vector such that q^{ϵ_j} is the ϵ_j coefficient of $a\overline{b}$.

Let V_{ijklm} denote the $\mathfrak{d}_5$-module with highest weight $i\omega_1+j\omega_j+k\omega_3+l\omega_4+m\omega_5$, and the $\mathfrak{sl}(A)+\mathfrak{sl}(B)$ modules are indexed in the obvious way. For the Segre case write $T = U_{10}\oplus W_{10}$ and $N = U_{01}\otimes W_{01}$. The remaining relevant modules are as follows, where when there are two lines on the right hand side, the first is for the complexified Cayley plane and the second for the Grassmannian:

$$\begin{aligned}
T &= V_{00001}\\
&= A_1 \otimes B_{100}\\
T^* &= V_{00010}\\
&= A_1 \otimes B_{001}\\
N &= N^* = V_{10000}\\
&= A_0 \otimes B_{010}\\
S^2T^* &= T^{*2} \oplus N\\
S^3T^* &= T^{*3} \oplus NT^*\\
S^3T^* \otimes N &= T^{*3}N \oplus T^{*2}T \oplus N^2T^* \oplus \mathfrak{g}T^* \oplus NT \oplus T^*\\
T \otimes N^* &= TN^* \oplus T^*\\
(N \otimes N^*)^{\mathfrak{r}^c} \otimes T^* &= N^2T^* \oplus NT\\
(T \otimes T^*)^{\mathfrak{r}^c} \otimes T^* &= T^{*2}T \oplus T_2T \oplus \mathfrak{g}T^* \oplus NT \oplus T^*.
\end{aligned}$$

Here $\mathfrak{g}$ denotes the adjoint representation. See [**13**, **14**] for an explanation of T_2.

The decompositions above show that there are six components of F_3 on $\mathcal{F}^1$ and four when we restrict to $\mathcal{F}^n$.

We may choose our model such that e_1 is a highest weight vector (since it is in Base$|II_{X,x}|$). We may also have e^b be a highest weight vector for N^*.

The first component of $S^3T^* \otimes N$ is eliminated from F_3 either by Bertini (as it has highest weight vector r^b_{111}) or by comparing modules. The second and third components are also eliminated by Bertini. The last two components my be normalized away. Thus we are left with $\mathfrak{g}T^*$, which has highest weight coefficient $r^a_{11\bar{1}}$. To eliminate this module we examine F_4, which in particular, contains the equation

$$r^a_{111\beta}\omega^\beta_0 = r^a_{11\bar{1}}\omega^{\bar{1}}_1$$

but it is easy to check that $\omega^{\bar{1}}_1$ is independent of the semi-basic forms. The higher order invariants are safely left to the reader. q.e.d.

To compare with the $G(2,6)$ case in the standard model, we have $e_1 = e_{(13)}$,$e_{\bar{1}} = e_{(24)}$, $q^a = q^{34}$,$q^b = q^{56}$ etc....

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