Tôhoku Math. J.
48 (1996), 575–600

GRÖBNER BASES FOR D-MODULES ON A NON-SINGULAR AFFINE ALGEBRAIC VARIETY

TOSHINORI OAKU

(Received July 20, 1995, revised October 31, 1995)

Abstract. We consider an algebraic D-module on a non-singular affine algebraic variety from an algorithmic viewpoint. Our main purpose is to show that the method of Gröbner basis can be applied to concrete computation of invariants such as the characteristic variety of an algebraic D-module.

Introduction. The theory of D-modules, i.e., modules over the ring of differential operators, was first developed in the complex analytic category by M. Sato, Kashiwara, T. Kawai (cf. [Ka]), and for the Weyl algebra (i.e. the ring of differential operators with polynomial coefficients) by I. N. Bernstein (cf. [Bj]). As a generalization of the latter, the theory of algebraic D-modules has been developed by many authors as is presented, e.g., in [Bj], [Bo], [H], [TH]. The aim of this paper is to show that the notion and the algorithm of Gröbner basis by Buchberger ([Bu], [CLO], [BW]) can be effectively applied to actual computation of algebraic D-modules as was the case with the Weyl algebra.

Let V be a non-singular algebraic variety over the field of complex numbers. Then the sheaf of rings $\mathscr{D}_V$ of algebraic differential operators is defined on V as a sheaf of subrings of that of analytic differential operators $\mathscr{D}_V^{\mathrm{an}}$ on V as a complex manifold. A left coherent $\mathscr{D}_V$-module $\mathscr{M}$ corresponds to a system of algebraic linear differential equations on V. Our purpose is to present algorithmic methods for treating a coherent $\mathscr{D}_V$-module.

From the theoretical viewpoint, since V is locally isomorphic to an affine space as complex manifold, nothing new happens in this situation at least locally. However, from the viewpoint of computation, this passage from the affine space to an algebraic variety makes much difference because the local isomorphism stated above is not bi-regular in general and this breaks the effective computability enjoyed in the affine space by virtue of Gröbner bases for the Weyl algebra developed by Galligo, Takayama et al. (cf. [Ga], [C], [N], [T1], [O1], [O2]). One possibility is to use the so-called Kashiwara equivalence which states that the category of coherent $\mathscr{D}_V$-modules on V and that of coherent $\mathscr{D}_{C^n}$-modules with supports contained in V are equivalent (cf.

Partly supported by the Grants-in-Aid for Scientific Research on Priority Areas, the Ministry of Education, Science and Culture.

1991 *Mathematics Subject Classification*. Primary 14Q20; Secondary 16S32, 35A27.

[Bo], [H], [TH]). However this equivalence makes the structure of the module more complicated in general.

In this paper, we shall show that the method of Gröbner basis can be directly applied to ideals of (or, more generally, modules over) the ring of algebraic differential operators on V. As applications, we get algorithms for computing a free resolution, the characteristic variety and the singular locus of a coherent $\mathscr{D}_V$-module.

The contents of this paper are as follows: In Section 1, we review the definition of the ring $\mathscr{D}_V(V)$ of algebraic differential operators on a non-singular affine algebraic variety V of dimension d. This ring is generated (locally in the Zariski topology) by linearly independent derivations $\vartheta_1, \ldots, \vartheta_d$ and the regular functions on V (as multiplication operators). In Section 2, we see that if the derivations $\vartheta_1, \ldots, \vartheta_d$ have polynomial coefficients, then the theory and the algorithm (the Buchberger algorithm) of Gröbner bases can be applied to this ring. We can choose $\vartheta_1, \ldots, \vartheta_d$ so that they commute with each other. This commutativity enables us to apply the Leibniz rule to the product of two differential operators, which faciliates the actual computation as is shown in Section 3. Here one drawback is that these derivations have rational functions as coefficients in general, which prevents us from applying the Gröbner basis algorithm directly. We bypass this difficulty by embedding V into a higher dimensional affine space in Section 4. As an application, we present algorithms for computing the characteristic variety (in the algebraic sense) and the singular locus of a coherent $\mathscr{D}_V$-module in Section 5.

In Section A1, we state the relation between our method and the Kashiwara equivalence. The fact that the characteristic variety in the algebraic sense coincides with that in the analytic sense, which is more directly connected to the analytic theory of D-modules (cf. [Ka]), is proved in Section A2. This fact should be well-known, but an explicit proof seems lacking in the literature.

The author is grateful to the referee(s) for valuable comments, especially for suggesting the possible relation to the Kashiwara equivalence. He is also grateful to Prof. Takayama for pointing out that F.-O. Schreyer's method for computing syzygies (cf. [E]) also applies to rings of differential operators.

1. The ring of algebraic differential operators on an affine algebraic variety. In this section, we recall the definition of the ring of algebraic differential operators (cf. [Bj], [Bo]) and give its concrete expression suited to actual computation.

Let K be an algebraically closed field of characteristic 0 and let $x_1, \ldots, x_n$ be indeterminates. We denote by $K[x]=K[x_1, \ldots, x_n]$ and $K(x)=K(x_1, \ldots, x_n)$ the ring of polynomials and the field of rational functions respectively over K. Let V be an affine algebraic variety in K^n and $I(V)$ be the defining ideal of V. Then $\mathscr{O}_V(V):=K[x]/I(V)$ is an integral domain since V is irreducible by definition. We denote by $R(V)$ the quotient field of $\mathscr{O}_V(V)$.

We shall use the Zariski topology on V unless otherwise stated. For a (Zariski) open subset W of V, we denote by $\mathcal{O}_V(W)$ the set of rational functions on V which are regular on W. This defines a coherent sheaf of rings $\mathcal{O}_V$ on V and its stalk at $p \in V$ is denoted by $(\mathcal{O}_V)_p$. For an element $a(x)$ of $K(x)$ and an open subset W of V where $a(x)$ is regular (i.e., its denominator never vanishes on W), we denote by $a(\bar{x})$ its residue class in $\mathcal{O}_V(W) \subset R(V)$.

For an open subset W of V, let $\Theta(W)$ be the set of derivations of $\mathcal{O}_V(W)$ over K; i.e., $\Theta(W)$ is the set of K-linear mappings $\delta: \mathcal{O}_V(W) \to \mathcal{O}_V(W)$ satisfying $\delta(fg) = f\delta(g) + g\delta(f)$ for any $f, g \in \mathcal{O}_V(W)$. Note that $\Theta(W)$ is an $\mathcal{O}_V(W)$-module. Then the ring $\mathscr{D}_V(W)$ of algebraic differential operators on W is defined as the subring of the ring $\mathrm{Hom}_K(\mathcal{O}_V(W), \mathcal{O}_V(W))$ of K-linear homomorphisms of $\mathcal{O}_V(W)$ to itself which is generated by $\Theta(W)$ and $\mathcal{O}_V(W)$. Here, an element of $\mathcal{O}_V(W)$ operates on $\mathcal{O}_V(W)$ as a multiplication.

In the sequel, we shall express $\mathscr{D}_V(W)$ more concretely. Let $f_1, \ldots, f_s$ be a set of generators of $I(V)$. We denote by d the dimension of V. Then $R(V) \otimes_{\mathcal{O}_V(V)} \Theta(V)$ is a d-dimensional vector space over $R(V)$. Moreover, the rank of the matrix

$$\frac{\partial(f_1, \ldots, f_s)}{\partial(f_1, \ldots, f_n)} := \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_s/\partial x_1 & \cdots & \partial f_s/\partial x_n \end{pmatrix}$$

is equal to $n-d$ at any non-singular point of V.

Hence for each non-singular point p of V, we can choose an open neighborhood W of p, and indices $i_1, \ldots, i_{n-d}$ and $j_1, \ldots, j_{n-d}$ such that

$$\det \frac{\partial(f_{i_1}, \ldots, f_{i_{n-d}})}{\partial(x_{j_1}, \ldots, x_{j_{n-d}})} \neq 0$$

on W. For the sake of simplicity of the notation, we assume $i_1 = 1, \ldots, i_{n-d} = n-d$ and $j_1 = n-d+1, \ldots, j_{n-d} = n$. Then $x_1, \ldots, x_d$ constitute a regular system of parameters of the local ring $(\mathcal{O}_V)_q$ for any $q \in W$. Hence $x_1, \ldots, x_d$ serve as a local coordinate system of V viewed as a complex manifold if K is the field of complex numbers $\boldsymbol{C}$.

There exist $\vartheta_1, \ldots, \vartheta_d \in \Theta(W)$ which generate $\Theta(W)$ over $\mathcal{O}_V(W)$. For example, we can choose them as follows: In general, put

$$\vartheta = \sum_{j=1}^{n} a_j(\bar{x}) \frac{\partial}{\partial x_j}$$

with $a_j(\bar{x}) = \vartheta(\bar{x}_j) \in \mathcal{O}_V(W)$. Then ϑ belongs to $\Theta(W)$ if and only if

$$\sum_{j=1}^{n} a_j(\bar{x}) \frac{\partial f_i}{\partial x_j}(\bar{x}) = 0 \quad \text{on} \quad W \qquad (\forall i = 1, \ldots, n-d). \tag{1.1}$$

Hence $\vartheta \in \Theta(W)$ of the above form is uniquely determined by $a_1(\bar{x}), \ldots, a_d(\bar{x})$. Put

$$\begin{pmatrix} a_{d+1}^{(i)}(x) \\ \vdots \\ a_n^{(i)}(x) \end{pmatrix} := -\left(\frac{\partial(f_1, \dots, f_{n-d})}{\partial(x_{n-d+1}, \dots, x_n)}\right)^{-1} \frac{\partial(f_1, \dots, f_{n-d})}{\partial(x_1, \dots, x_d)} e_i \in K(x)^{n-d}$$

for $i=1, \dots, d$, where $e_1, \dots, e_d$ are the d-dimensional unit column vectors. Then we have

$$\vartheta_i := \frac{\partial}{\partial x_i} + \sum_{j=d+1}^{n} a_j^{(i)}(\bar{x}) \frac{\partial}{\partial x_j} \in \Theta(W)$$

in view of (1.1), and $\vartheta_1, \dots, \vartheta_d$ generate $\Theta(W)$ over $\mathcal{O}_V(W)$. Moreover, $\vartheta_1, \dots, \vartheta_d$ thus constructed commute with each other, i.e.,

$$[\vartheta_i, \vartheta_j] := \vartheta_i \vartheta_j - \vartheta_j \vartheta_i = 0 \qquad (i, j = 1, \dots, d).$$

This follows from the following lemma:

LEMMA 1.1. *In the notation above, define an open subset* $\tilde{W}$ *of* K^n *by*

$$\tilde{W} := \left\{ x \in K^n \,\middle|\, \det \frac{\partial(f_1, \dots, f_{n-d})}{\partial(x_{n-d+1}, \dots, x_n)}(x) \neq 0 \right\},$$

and elements $\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d$ *of* $\Theta(\tilde{W})$ *by*

$$\tilde{\vartheta}_i := \frac{\partial}{\partial x_i} + \sum_{j=d+1}^{n} a_j^{(i)}(x) \frac{\partial}{\partial x_j}, \tag{1.2}$$

where $a_j^{(i)}(x)$ *is defined above and regarded as an element of* $\mathcal{O}_{K^n}(\tilde{W}) \subset K(x)$. *Then as elements of* $\Theta(\tilde{W})$, *we have*

$$[\tilde{\vartheta}_i, \tilde{\vartheta}_j] := \tilde{\vartheta}_i \tilde{\vartheta}_j - \tilde{\vartheta}_j \tilde{\vartheta}_i = 0 \qquad (i, j = 1, \dots, d).$$

PROOF. Put

$$\mathcal{T} := \left\{ \tilde{\vartheta} = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j} \in \Theta(\tilde{W}) \,\middle|\, \sum_{j=1}^{n} a_j(x) \frac{\partial f_i}{\partial x_j} = 0 \ (i = 1, \dots, n-d) \right\}.$$

For $\tilde{\vartheta} \in \Theta(\tilde{W})$, we have $\tilde{\vartheta} \in \mathcal{T}$ if and only if $\tilde{\vartheta}(f_i) = 0$ for any $i = 1, \dots, n-d$. Then in view of the definitions above, we have $\tilde{\vartheta}_i \in \mathcal{T}$ for $i = 1, \dots, d$. Hence for $1 \leq i, j \leq d$, we get

$$[\tilde{\vartheta}_i, \tilde{\vartheta}_j](f_k) = \tilde{\vartheta}_i(\tilde{\vartheta}_j(f_k)) - \tilde{\vartheta}_j(\tilde{\vartheta}_i(f_k)) = 0 \qquad (k = 1, \dots, n-d).$$

This implies $[\tilde{\vartheta}_i, \tilde{\vartheta}_j] \in \mathcal{T}$. On the other hand, we have

$$[\tilde{\vartheta}_i, \tilde{\vartheta}_j] = \sum_{k=d+1}^{n} b_{ijk}(x) \frac{\partial}{\partial x_k}$$

with some $b_{ijk}(x) \in \mathcal{O}_{K^n}(\tilde{W})$. It follows that $[\tilde{\vartheta}_i, \tilde{\vartheta}_j] = 0$ since an element $\tilde{\vartheta}$ of $\mathcal{T}$ is uniquely determined by $\tilde{\vartheta}(x_1), \ldots, \tilde{\vartheta}(x_d)$. □

In the sequel, we fix $\vartheta_1, \ldots, \vartheta_d \in \Theta(W)$ which generate $\Theta(W)$ over $\mathcal{O}_V(W)$. (We do not assume that $\vartheta_1, \ldots, \vartheta_d$ commute with each other except in Proposition 1.2 below.) Then an arbitrary element P of $\mathcal{D}_V(W)$ can be written uniquely in the form

$$P = \sum_{\alpha \in \boldsymbol{N}^d} a_\alpha(\bar{x}) \vartheta^\alpha , \tag{1.3}$$

where $a_\alpha(\bar{x})$ are elements of $\mathcal{O}_V(W)$ which are zero except for a finite number of α's. Here we use the notation $\vartheta^\alpha := \vartheta_1^{\alpha_1} \cdots \vartheta_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \boldsymbol{N}^d$ with $\boldsymbol{N} := \{0, 1, 2, 3, \ldots\}$.

Let $\theta_1, \ldots, \theta_d$ be indeterminates. To P of the form (1.3), we associate its *total symbol* $P(\bar{x}, \theta)$ defined by

$$P(\bar{x}, \theta) := \sum_{\alpha \in \boldsymbol{N}^d} a_\alpha(\bar{x}) \theta^\alpha \in \mathcal{O}_V(W)[\theta]$$

with $\theta = (\theta_1, \ldots, \theta_d)$. The *order* of P is defined by $\max\{|\alpha| = \alpha_1 + \cdots + \alpha_d \mid a_\alpha(\bar{x}) \neq 0\}$, and if $m = \mathrm{ord}(P)$, the *principal symbol* of P (with respect to $\vartheta_1, \ldots, \vartheta_d$) is defined by

$$\sigma(P) = \sigma_m(P) := \sum_{|\alpha| = m} a_\alpha(\bar{x}) \theta^\alpha \in \mathcal{O}_V(W)[\theta] .$$

(If $P = 0$, we put $\mathrm{ord}(P) = -\infty$ and $\sigma(P) = 0$.) For $P, Q \in \mathcal{D}_V(W) \setminus \{0\}$, we have $\mathrm{ord}(PQ) = \mathrm{ord}(P) + \mathrm{ord}(Q)$ and $\sigma(PQ) = \sigma(P)\sigma(Q)$.

We write $\alpha! = \alpha_1! \ldots \alpha_n!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \boldsymbol{N}^n$.

PROPOSITION 1.2 (The Leibniz rule). *Assume that $\vartheta_1, \ldots, \vartheta_d \in \Theta(W)$ commute with each other. Let $P = \sum_{\alpha \in \boldsymbol{N}^d} a_\alpha(\bar{x}) \vartheta^\alpha$ and $Q = \sum_{\alpha \in \boldsymbol{N}^d} b_\alpha(\bar{x}) \vartheta^\alpha$ be two elements of $\mathcal{D}_V(W)$ and put $R = PQ$. Then the total symbol of R is given by*

$$R(\bar{x}, \theta) = \sum_{\alpha \in \boldsymbol{N}^d} \frac{1}{\alpha!} \vartheta^\alpha(Q(\bar{x}, \theta)) \frac{\partial^{|\alpha|}}{\partial \theta^\alpha} P(\bar{x}, \theta) , \tag{1.4}$$

where $\vartheta^\alpha(Q(\bar{x}, \theta))$ means the action of the differential operator ϑ^α on $Q(\bar{x}, \theta)$ viewed as a function of $\bar{x}$, hence it gives an element of $\mathcal{O}_V(W)[\theta]$.

PROOF. (1) First we prove (1.4) for $P = \vartheta_i^\nu$ and $Q = g(\bar{x}) \in \mathcal{O}_V(W)$ with $\nu \in \boldsymbol{N}$ by induction on ν. Put $R_\nu := PQ$ and suppose that (1.4) holds for ν. Then we have

$$R_\nu(\bar{x}, \theta) = \sum_{j=0}^{\nu} \frac{1}{j!} \nu(\nu-1) \cdots (\nu-j+1) \vartheta_i^j(g(\bar{x})) \theta_i^{\nu-j} .$$

Since $R_{\nu+1} = \vartheta_i R_\nu$, we get

$$\begin{aligned} R_{\nu+1}(\bar{x}, \theta) &= \sum_{j=0}^{\nu} \frac{1}{j!} \nu(\nu-1) \cdots (\nu-j+1)(\vartheta_i^{j+1}(g(\bar{x}))\theta_i^{\nu-j} + \vartheta_i^j(g(\bar{x}))\theta_i^{\nu-j+1}) \\ &= \sum_{j=0}^{\nu+1} \frac{1}{j!} (\nu+1)\nu(\nu-1) \cdots (\nu+1-j+1) \vartheta_i^j(g(\bar{x})) \theta_i^{\nu+1-j} . \end{aligned}$$

This proves (1.4) for any $\nu \in N$ since (1.4) holds for $\nu = 0$.

(2) By using step (1) repeatedly, we have (1.4) for $P = \vartheta^\nu$ with $\nu \in N^d$ and $Q = g(\bar{x}) \in \mathcal{O}_V(W)$.

(3) General case: Let P and Q be as in the statement of the proposition. Then we have $R = \sum_{\alpha,\beta \in N^d} a_\alpha(\vartheta^\alpha b_\beta)\vartheta^\beta$. By using (2), we know that the total symbol of $\vartheta^\alpha b_\beta$ is given by

$$\sum_{\gamma \in N^d} \frac{1}{\gamma!} \vartheta^\gamma(b_\beta(\bar{x})) \frac{\partial^{|\gamma|}}{\partial \theta^\gamma} \theta^\alpha .$$

Since $\vartheta^\alpha \vartheta^\beta = \vartheta^\beta \vartheta^\alpha$ holds by virtue of the commutativity assumption, we get

$$\begin{aligned} R(\bar{x}, \theta) &= \sum_{\alpha,\beta,\gamma} \frac{1}{\gamma!} a_\alpha(\bar{x}) \vartheta^\gamma(b_\beta(\bar{x})) \left(\frac{\partial^{|\gamma|}}{\partial \theta^\gamma} \theta^\alpha \right) \theta^\beta \\ &= \sum_\gamma \frac{1}{\gamma!} \vartheta^\gamma(Q(\bar{x}, \theta)) \frac{\partial^{|\gamma|}}{\partial \theta^\gamma} P(\bar{x}, \theta) . \end{aligned}$$

□

The sheaf $\mathcal{D}_V$ of rings of algebraic differential operators is defined on V so that the set of its sections $\Gamma(W, \mathcal{D}_V)$ over an open set W coincides with $\mathcal{D}_V(W)$. Note that $\mathcal{D}_V$ is a coherent sheaf of rings (cf. [Bj]). Let $\mathcal{M}$ be a left coherent $\mathcal{D}_V$-module (i.e. a sheaf of $\mathcal{D}_V$-modules) defined on an open subset W of V. Then by definition, for any point p of W, there exists a neighborhood $W' \subset W$ of p and an exact sequence

$$0 \longleftarrow \mathcal{M} \xleftarrow{\varphi} (\mathcal{D}_V)^{r_0} \xleftarrow{\psi} (\mathcal{D}_V)^{r_1}$$

of left $\mathcal{D}_V$-modules on W'. Put

$$u_i := \varphi(\underbrace{0, \ldots, \overset{(i)}{1}, \ldots, 0}_{r_0}) \in \Gamma(W', \mathcal{M}) \qquad (i = 1, \ldots, r_0) ,$$

$$P_i = (P_{i1}, \ldots, P_{ir_0}) := \psi(\underbrace{0, \ldots, \overset{(i)}{1}, \ldots, 0}_{r_1}) \in \mathcal{D}_V(W')^{r_0} \qquad (i = 1, \ldots, r_1) .$$

Then we have an isomorphism $\mathcal{M} \simeq (\mathcal{D}_V)^{r_0}/(\mathcal{D}_V P_1 + \cdots + \mathcal{D}_V P_{r_1})$ on W'. Moreover $u_1, \ldots, u_{r_0}$ satisfy

$$\sum_{j=1}^{r_0} P_{ij} u_j = 0 \qquad (i = 1, \ldots, r_1) ,$$

which is a concrete expression for $\mathcal{M}$ as a system of linear differential equations.

2. Gröbner bases for modules over a subring of the Weyl algebra. Let V be an affine algebraic variety in K^n of dimension d. Let $\vartheta_1, \dots, \vartheta_d$ be elements of $\Theta(V)$ which generate $R(V) \otimes_{\mathcal{O}_V(V)} \Theta(V)$ over $R(V)$. We do not assume that $\vartheta_1, \dots, \vartheta_d$ commute with each other. We denote by D_V the subring of $\mathcal{D}_V(V)$ generated by $\mathcal{O}_V(V)$ and $\vartheta_1, \dots, \vartheta_d$. Then D_V is also a subring of $\mathcal{D}_V(W)$ for any open subset W of V. Our purpose is to perform explicit calculation for ideals of (or modules over) D_V. For this purpose, we work in a subring A_V of the Weyl algebra A_n.

Write ϑ_i explicitly as

$$\vartheta_i = a_1^{(i)}(\bar{x}) \frac{\partial}{\partial x_1} + \cdots + a_n^{(i)}(\bar{x}) \frac{\partial}{\partial x_n}$$

with $a_j^{(i)}(\bar{x}) \in \mathcal{O}_V(V)$. Each $a_j^{(i)}(\bar{x})$ is the restriction to W of a polynomial $a_j^{(i)}(x) \in K[x]$. Put

$$\tilde{\vartheta}_i := a_1^{(i)}(x) \frac{\partial}{\partial x_1} + \cdots + a_n^{(i)}(x) \frac{\partial}{\partial x_n} \in \Theta(K^n) .$$

Let A_V be the subring of the Weyl algebra

$$A_n := K[x] \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle$$

generated by $K[x]$ and $\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d$; i.e.,

$$A_V := K[x] \langle \tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d \rangle \subset A_n .$$

Note that A_V is not defined uniquely by $\vartheta_1, \dots, \vartheta_d$ being dependent on their extensions $\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d$. The total symbol $P(\bar{x}, \theta)$ and the principal symbol $\sigma(P)$ are defined in the same way as in Section 1 as elements of $\mathcal{O}_V(V)[\theta]$ with $\theta = (\theta_1, \dots, \theta_d)$. Now put

$$\boldsymbol{I}(V)A_V := \boldsymbol{I}(V)\langle \tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d \rangle := \left\{ \sum_\alpha a_\alpha(x) \tilde{\vartheta}^\alpha \in A_V \,\middle|\, a_\alpha(x) \in \boldsymbol{I}(V) \right\} .$$

Then $\boldsymbol{I}(V)A_V$ is a two-sided ideal of A_V and D_V is isomorphic to the residue ring $A_V / \boldsymbol{I}(V)A_V$. Let us denote by ϖ the canonical surjective ring homomorphism of A_V to D_V. We also denote by ϖ the canonical A_V-homomorphism of $(A_V)^r$ to $(D_V)^r$. Then $\varpi^{-1}(N)$ is a left A_V-submodule of $(A_V)^r$ for a left D_V-submodule N of $(D_V)^r$. This defines a one-to-one correspondence between left D_V-submodules of $(D_V)^r$ and left A_V-submodules of $(A_V)^r$ containing $(\boldsymbol{I}(V)A_V)^r$.

Hence, in the sequel, we treat a left A_V-submodule N of $(A_V)^r$. It is easy to see that the notion of Gröbner basis can be applied to N (cf. [T]). We fix a total order $\prec$ of the set $\boldsymbol{N}^{d+n} \times \{1, \dots, r\}$ which satisfies the following conditions:

(O-1) $(\alpha, i) \prec (\beta, j)$ implies $(\alpha + \gamma, i) \prec (\beta + \gamma, j)$ for any $\alpha, \beta, \gamma \in \boldsymbol{N}^{d+n}$ and $i \in \{1, \dots, r\}$;

(O-2) $(0, i) = (0, \dots, 0, i) \preceq (\alpha, i)$ for any $\alpha \in \boldsymbol{N}^{d+n}$ and $i \in \{1, \dots, r\}$;

(O-3) For any $\alpha, \alpha' \in \boldsymbol{N}^d$, $\beta, \beta' \in \boldsymbol{N}^n$ and $i \in \{1, \dots, r\}$,

$$|\alpha|<|\alpha'|\Rightarrow(\alpha,\beta,i)\prec(\alpha',\beta',i).$$

The first two conditions imply that $\prec$ is a well-order. An element P of $(A_V)^r$ is written uniquely in the form

$$P=\sum_{j=1}^{r}\sum_{\alpha}a_{j\alpha}(x)\tilde{\vartheta}^{\alpha}e_j,$$

where $a_{j\alpha}(x)=\sum_{\beta\in N^n}a_{j\alpha\beta}x^{\beta}\in K[x]$, and

$$e_j:=(\underbrace{0,\ldots,\overset{(j)}{1},\ldots,0}_{r}).$$

Assume $P\neq 0$ and define the *set of exponents*, the *leading exponent*, the *leading coefficient*, the *leading term*, the *leading point* of P by

$$\begin{aligned}
\mathrm{exps}(P)&:=\{(\alpha,\beta,j)\in N^{d+n}\mid a_{j\alpha\beta}\neq 0\}\subset N^{d+n}\times\{1,\ldots,r\},\\
\mathrm{lexp}(P)&:=\max{}_{\prec}\,\mathrm{exps}(P)\in N^{d+n}\times\{1,\ldots,r\},\\
\mathrm{lcoef}(P)&:=a_{j\alpha\beta}\in K\qquad\text{with}\quad(\alpha,\beta,j)=\mathrm{lexp}(P),\\
\mathrm{lterm}(P)&:=a_{j\alpha\beta}x^{\beta}\tilde{\vartheta}^{\alpha}e_j\in(A_V)^r\qquad\text{with}\quad(\alpha,\beta,j)=\mathrm{lexp}(P),\\
\mathrm{lp}(P)&:=j\in\{1,\ldots,r\}\qquad\text{with}\quad(\alpha,\beta,j)=\mathrm{lexp}(P)
\end{aligned}$$

respectively, where $\max_{\prec}$ denotes taking the maximum in the order $\prec$. In particular, if $P\in A_V\setminus\{0\}$, we suppose $\mathrm{lexp}(P)\in N^{d+n}$ omitting the trivial index 1. For $(\alpha,i)\in N^{d+n}\times\{1,\ldots,r\}$ and $\beta\in N^{d+n}$, we write $(\alpha,i)+\beta=(\alpha+\beta,i)$.

LEMMA 2.1. *For $P\in(A_V)^r\setminus\{0\}$ and $Q\in A_V\setminus\{0\}$, we have* $\mathrm{lexp}(QP)=\mathrm{lexp}(P)+\mathrm{lexp}(Q)$.

PROOF. Write $P=(P_1,\ldots,P_r)$. In view of the condition (O-3), we have $\mathrm{lexp}(P)=\mathrm{lexp}((\sigma(P_1),\ldots,\sigma(P_r)))$ in general. Hence condition (O-1) and Proposition 1.4 imply

$$\begin{aligned}
\mathrm{lexp}(QP)&=\mathrm{lexp}((\sigma(QP_1),\ldots,\sigma(QP_r)))\\
&=\mathrm{lexp}((\sigma(Q)\sigma(P_1),\ldots,\sigma(Q)\sigma(P_r)))\\
&=\mathrm{lexp}(\sigma(P_1),\ldots,\sigma(P_r))+\mathrm{lexp}(\sigma(Q))\\
&=\mathrm{lexp}(P)+\mathrm{lexp}(Q).
\end{aligned}$$

□

For a subset S of $(A_V)^r$, the *set of leading exponents* of S is defined by

$$E(S):=\{\mathrm{lexp}(P)\mid P\in S\setminus\{0\}\}.$$

For a subset F of $N^{d+n}\times\{1,\ldots,r\}$, we put

$$F+N^{d+n}:=\{(\alpha+\beta,i)\mid(\alpha,i)\in F,\ \beta\in N^{d+n}\},$$

which is called the *monoideal* generated by F.

DEFINITION 2.2. Let N be a left A_V-submodule of $(A_V)^r$ and $\boldsymbol{G}$ a finite subset of N. Then $\boldsymbol{G}$ is called a *Gröbner basis* of N (with respect to the order $\prec$) if $E(N)=E(\boldsymbol{G})+\boldsymbol{N}^{d+n}$.

If N is a left A_V-submodule of $(A_V)^r$, then $E(N)+\boldsymbol{N}^{d+n}=E(N)$ holds; i.e. $E(N)$ is a *monoideal* in view of Lemma 2.1. Every monoideal is generated by a finite set (Dickson's lemma (cf. [CLO])).

For $(\alpha, i), (\beta, j)\in \boldsymbol{N}^{d+n}\times\{1,\dots,r\}$, the relation $(\alpha, i)\le(\beta, j)$ means both $i=j$ and $\beta-\alpha\in\boldsymbol{N}^{d+n}$. For a finite subset $\boldsymbol{G}$ of $(A_V)^r$ and an element P of $(A_V)^r$, the *reduction* (or division) of P by $\boldsymbol{G}$ is defined by the following algorithm:

ALGORITHM 2.3 (reduction).
Input: $P\in(A_V)^r$ and a finite set $\boldsymbol{G}\subset(A_V)^r$;
While ($P\neq 0$ and $\mathrm{lexp}(P)\in E(\boldsymbol{G})+\boldsymbol{N}^{d+n}$) {
  Choose $Q\in\boldsymbol{G}$ such that $\mathrm{lexp}(P)\ge\mathrm{lexp}(Q)$;
  $P:=P-(\mathrm{lcoef}(P)/\mathrm{lcoef}(Q))x^\beta\mathfrak{F}^\alpha Q$ with (α, β) satisfying
  $\mathrm{lexp}(P)=\mathrm{lexp}(Q)+(\alpha,\beta)$;
}
Return P;

This algorithm terminates since $\mathrm{lexp}(P)$ gets smaller with respect to the well-order $\prec$ in the execution of the algorithm. We note that the output of Algorithm 2.3 is not necessarily unique; it may depend on the choice of Q in the algorithm.

The following propositions follow from the definition of Gröbner basis, Dickson's lemma, and the reduction algorithm:

PROPOSITION 2.4. *If $\boldsymbol{G}$ is a Gröbner basis of a left A_V-submodule N of $(A_V)^r$, then $\boldsymbol{G}$ generates N over A_V.*

PROPOSITION 2.5. *Any left A_V-submodule of $(A_V)^r$ has a Gröbner basis, and hence is finitely generated. In particular, A_V is a Noetherian ring.*

PROPOSITION 2.6. *Let N, M be left A_V-submodules of $(A_V)^r$ such that $N\subset M$. Then $N=M$ if and only if $E(N)=E(M)$.*

In general, for vectors $\alpha=(\alpha_1,\dots,\alpha_m)$ and $\beta=(\beta_1,\dots,\beta_m)$ in $\boldsymbol{N}^m$, we put

$$\alpha\vee\beta:=(\max\{\alpha_1,\beta_1\},\dots,\max\{\alpha_m,\beta_m\})$$

and $(\alpha, i)\vee(\beta, i):=(\alpha\vee\beta, i)$.

DEFINITION 2.7. For $P, Q\in(A_V)^r$, put $\mathrm{lexp}(P)=(\alpha,\beta,i)\in\boldsymbol{N}^d\times\boldsymbol{N}^n\times\{1,\dots,r\}$ and $\mathrm{lexp}(Q)=(\alpha',\beta',j)$. Then the *S-operator* of P and Q is defined by

$$\mathrm{sp}(P,Q):=\mathrm{lcoef}(Q)x^{\beta\vee\beta'-\beta}\mathfrak{F}^{\alpha\vee\alpha'-\alpha}P-\mathrm{lcoef}(P)x^{\beta\vee\beta'-\beta'}\mathfrak{F}^{\alpha\vee\alpha'-\alpha'}Q$$

if $i=j$, and $\mathrm{sp}(P, Q):=0$ if $i \neq j$.

THEOREM 2.8 (Takayama [T1]). *Let* $\boldsymbol{G}=\{P_1, \ldots, P_s\}$ *be a finite subset of* $(A_V)^r$ *which generates a left* A_V*-submodule* N *of* $(A_V)^r$*. Then the following conditions are equivalent*:

(1) $\boldsymbol{G}$ *is a Gröbner basis of* N;

(2) *For every* $P \in N$, *its arbitrary reduction by* $\boldsymbol{G}$ *is zero*;

(3) *For any pair* $P, Q \in \boldsymbol{G}$ *such that* $\mathrm{lp}(P)=\mathrm{lp}(Q)$, *some reduction of* $\mathrm{sp}(P, Q)$ *by* $\boldsymbol{G}$ *becomes zero*;

(4) *For any* $i, j \in \{1, \ldots, s\}$ *such that* $i<j$ *and* $\mathrm{lp}(P_i)=\mathrm{lp}(P_j)$, *there exist* $Q_{ij1}, \ldots, Q_{ijs} \in A_V$ *so that* $\mathrm{sp}(P_i, P_j)=\sum_{k=1}^{s} Q_{ijk}P_k$ *and that* $Q_{ijk}=0$ *or* $\mathrm{lexp}(Q_{ijk}P_k) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$ *for each* k.

The condition (3) of this theorem provides the following Buchberger algorithm of computing a Gröbner basis from a given set of generators:

ALGORITHM 2.9 (Gröbner basis).
Input: a finite set $\boldsymbol{G} \subset (A_V)^r$;
Repeat
    $\boldsymbol{G}' := \boldsymbol{G}$;
    For each pair $(P, Q) \in \boldsymbol{G} \times \boldsymbol{G}$ such that $P \neq Q$ and $\mathrm{lp}(P)=\mathrm{lp}(Q)$ {
        Let R be a reduction of $\mathrm{sp}(P, Q)$ by $\boldsymbol{G}$;
        If $(R \neq 0)$ $\boldsymbol{G} := \boldsymbol{G} \cup (R\}$;
    }
Until $\boldsymbol{G}=\boldsymbol{G}'$;
Return $\boldsymbol{G}$;

This algorithm terminates since the monoideal $E(\boldsymbol{G})+N^{d+n}$ is strictly increasing during the execution of the algorithm, which would contradict the Noetherian property of monoideals if the algorithm did not terminate. The output of this algorithm is a Gröbner basis of the left A_V-submodule generated by $\boldsymbol{G}$ in view of Theorem 2.8.

A Gröbner basis immediately solves the membership problem:

PROPOSITION 2.10. *Let* $\boldsymbol{G}$ *be a Gröbner basis of a left* A_V*-submodule* N *of* $(A_V)^r$*. Then for an element* P *of* $(A_V)^r$, *the following three conditions are equivalent*:

(1) $P \in N$;

(2) *A reduction of* P *by* $\boldsymbol{G}$ *is zero*;

(3) *Any reduction of* P *by* $\boldsymbol{G}$ *is zero*.

Next, let us study the so-called syzygy module.

DEFINITION 2.11. For $P_1, \ldots, P_s \in (A_V)^r$, their (first) *syzygy module* is defined by

$$S(P_1, \ldots, P_s) := \left\{(Q_1, \ldots, Q_s) \in (A_V)^s \,\middle|\, \sum_{j=1}^{s} Q_j P_j = 0\right\}.$$

This is a left A_V-submodule of $(A_V)^s$.

The following theorem can be proved in the same way as its polynomial counterpart (cf. [BW], [CLO]). See also [O2] for results of this type for rings of differential operators.

THEOREM 2.12. *Let $G=\{P_1, \dots, P_s\}$ be a Gröbner basis of a left A_V-submodule N of $(A_V)^r$. Then for any $i, j \in \{1, \dots, s\}$ with $i \neq j$ and $\mathrm{lp}(P_i)=\mathrm{lp}(P_j)$, there exist $Q_{ij1}, \dots, Q_{ijs} \in A_V$ such that $\mathrm{sp}(P_i, P_j)=\sum_{k=1}^{s} Q_{ijk}P_k$ and that $Q_{ijk}=0$ or $\mathrm{lexp}(Q_{ijk}P_k) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$ for each k. (Such Q_{ijk} can be obtained by the reduction algorithm.) Put $\mathrm{lexp}(P_i)=(\alpha^{(i)}, \beta^{(i)}, v_i)$ and*

$$S_{ij} := \mathrm{lcoef}(P_i) x^{\beta^{(i)} \vee \beta^{(j)} - \beta^{(i)}} \tilde{\vartheta}^{\alpha^{(i)} \vee \alpha^{(j)} - \alpha^{(i)}},$$

$$V_{ij} := (\underbrace{0, \dots, \overset{(i)}{S_{ij}}, \dots, \overset{(j)}{-S_{ji}}, \dots, 0}_{s}) - (Q_{ij1}, \dots, Q_{ijs}) \in (A_V)^s$$

if $v_i = v_j$. Then the syzygy module $S(P_1, \dots, P_s)$ is generated by $V := \{V_{ij} \mid 1 \leq i < j \leq s, v_i = v_j\}$.

Note that the definitions and results of this section also apply, with trivial modification, to *right* A_V-submodules of $(A_V)^r$.

3. Symbol calculus of Gröbner bases. We use the same notation as in the preceding sections. In general, it would be difficult to choose commuting derivations $\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d \in \Theta(K^n)$ so that their restriction $\vartheta_1, \dots, \vartheta_d$ to W generate $\Theta(W)$. However, as we shall see later, we can take them so that $\vartheta_1, \dots, \vartheta_d$ commute with each other and generate $\Theta(W)$. Our purpose is to show that we can use the Leibniz rule for the computation of Gröbner basis in A_V.

Throughout this section, we assume that $\vartheta_1, \dots, \vartheta_d \in \Theta(V)$ commute with each other and have polynomial coefficients.

DEFINITION 3.1. We define another 'product' $R := P * Q$ of $P, Q \in A_V$ by

$$R(x, \theta) := \sum_{\alpha \in \mathbf{N}^d} \frac{1}{\alpha!} \tilde{\vartheta}^{\alpha}(Q(x, \theta)) \frac{\partial^{|\alpha|}}{\partial \theta^{\alpha}} P(x, \theta).$$

We call $P * Q$ the *$*$-product* of P and Q.

Note that the $*$-product is non-associative in general.

LEMMA 3.2. *For any $P, Q, R \in A_V$, we have*

$$P * Q - PQ \in I(V) A_V,$$

$$(P+Q) * R = P * R + Q * R, \quad P * (Q+R) = P * Q + P * R,$$

$$\sigma(P * Q) = \sigma(P)\sigma(Q).$$

PROOF. By the assumptions on $\tilde{\vartheta}_1, \dots, \tilde{\vartheta}_d$, Proposition 1.2 holds for elements of $\mathscr{D}_V(W)$, which contains $D_V = A_V/I(V)A_V$ as a subset. Hence the residue classes of $P*Q$ and of PQ coincide in D_V. This means $P*Q - PQ \in I(V)A_V$. The other identities follow immediately from the definition. □

LEMMA 3.3. *For $P \in (A_V)^r \setminus \{0\}$ and $Q \in A_V \setminus \{0\}$, we have*

$$\mathrm{lexp}(Q*P) = \mathrm{lexp}(P) + \mathrm{lexp}(Q) = \mathrm{lexp}(QP) .$$

Our purpose is to show that we can use the $*$-product in Algorithms 2.3 and 2.9 as long as G contains a Gröbner basis of $I(V)^r$.

In the sequel, let G_0 be a Gröbner basis of $I(V)^r$ as a $K[x]$-module with respect to the same order $\prec$. (Hence G_0 is a subset of $K[x]^r$.) Then in view of Theorem 2.8, G_0 is also a Gröbner basis of the left A_V-module $(I(V)A_V)^r$, and at the same time, G_0 is a Gröbner basis of $(I(V)A_V)^r$ as a right A_V-module. This fact follows from Theorem 2.8 since we can take Q_{ijk} as an element of $K[x]$ in the condition (4) of Theorem 2.8.

ALGORITHM 3.4 ($*$-reduction).
Input: $P \in (A_V)^r$ and a finite set $G \subset (A_V)^r$ containing G_0;
While ($P \neq 0$ and $\mathrm{lexp}(P) \in E(G) + N^{d+n}$) {
  Choose $Q \in G$ such that $\mathrm{lexp}(P) \geq \mathrm{lexp}(Q)$;
  Take $(\alpha, \beta) \in N^{d+n}$ such that $\mathrm{lexp}(P) = \mathrm{lexp}(Q) + (\alpha, \beta)$;
  If ($Q \notin G_0$) then
    $P := P - (\mathrm{lcoef}(P)/\mathrm{lcoef}(Q))(x^\beta \tilde{\vartheta}^\alpha) * Q$;
  If ($Q \in G_0$) then
    $P := P - (\mathrm{lcoef}(P)/\mathrm{lcoef}(Q))Q(x^\beta \tilde{\vartheta}^\alpha)$;
}
Return P;

We call the output of this algorithm a *$*$-reduction* of P by G, which is not necessarily determined uniquely.

PROPOSITION 3.5. *Algorithm 3.4 terminates in a finite number of steps. Let R be an output of Algorithm 3.4 with inputs P and $G = \{P_1, \dots, P_s\}$. Then there exist $Q_1, \dots, Q_s \in A_V$ so that*

$$P = Q_1 P_1 + \cdots + Q_s P_s + R$$

and that for each i, $Q_i = 0$ or $\mathrm{lexp}(P) \succeq \mathrm{lexp}(Q_i P_i)$. In particular, $P - R$ is contained in the left A_V-module generated by G.

PROOF. Let us denote P and Q (of the right hand side) at the k-th execution of the While loop in Algorithm 3.4 by $R^{(k)}$ and P_{i_k} respectively. Then there exists $B_k \in A_V$ such that

$$R^{(k+1)} = R^{(k)} - B_k * P_{i_k}, \quad \mathrm{lterm}(R^{(k)}) = \mathrm{lterm}(B_k P_{i_k})$$

if $P_{i_k} \notin G_0$; and

$$R^{(k+1)} = R^{(k)} - P_{i_k} B_k, \quad \mathrm{lterm}(R^{(k)}) = \mathrm{lterm}(P_{i_k} B_k)$$

if $P_{i_k} \in G_0$. Hence we have, in both cases, $\mathrm{lexp}(R^{(k+1)}) \prec \mathrm{lexp}(R^{(k)})$. It follows that the algorithm stops in finitely many steps because the order $\prec$ is a well-order.

Note that $R^{(k+1)} - R^{(k)} \in (I(V)A_V)^r$ holds in the latter case. Hence combining the rewriting expression for every k, we see that there exist $Q'_1, \ldots, Q'_s \in A_V$ and $R' \in (I(V)A_V)^r$ such that

$$P = Q'_1 * P_1 + \cdots + Q'_s * P_s + R' + R \tag{3.1}$$

and that for each i, $Q'_i = 0$ or $\mathrm{lexp}(P) \succeq \mathrm{lexp}(Q'_i P)$, and $\mathrm{lexp}(R') \preceq \mathrm{lexp}(P)$. We may assume $G_0 = \{P_{t+1}, \ldots, P_s\}$ with some $t < s$. Since G_0 is a Gröbner basis of $(I(V)A_V)^r$ as a left A_V-module, there exist $Q''_{t+1}, \ldots, Q''_s \in A_V$ such that

$$R' = Q''_{t+1} P_{t+1} + \cdots + Q''_s P_s \tag{3.2}$$

and that $Q''_i = 0$ or $\mathrm{lexp}(Q''_i P_i) \preceq \mathrm{lexp}(R')$. Combining (3.1), (3.2) and Lemma 3.2, we are done. □

Definition 3.6. For $P, Q \in (A_V)^r$, put $\mathrm{lexp}(P) = (\alpha, \beta, i) \in N^d \times N^n \times \{1, \ldots, r\}$ and $\mathrm{lexp}(Q) = (\alpha', \beta', j)$. Then the *S∗-operator* of P and Q is defined by

$$\mathrm{sp}*(P, Q) := \mathrm{lcoef}(Q)(x^{\beta \vee \beta' - \beta} \mathfrak{J}^{\alpha \vee \alpha' - \alpha}) * P - \mathrm{lcoef}(P)(x^{\beta \vee \beta' - \beta'} \mathfrak{J}^{\alpha \vee \alpha' - \alpha'}) * Q$$

if $i = j$, and $\mathrm{sp}*(P, Q) := 0$ if $i \neq j$.

The following lemma is an immediate consequence of the above definition and Lemmas 3.2 and 3.3:

Lemma 3.7. *Let* $P, Q \in (A_V)^r$. *We have* $\mathrm{sp}*(P, Q) - \mathrm{sp}(P, Q) \in (I(V)A_V)^r$. *Furthermore, if* $\mathrm{sp}*(P, Q) \neq 0$, *we have* $\mathrm{lexp}(\mathrm{sp}*(P, Q)) \prec \mathrm{lexp}(P) \vee \mathrm{lexp}(Q)$.

Algorithm 3.8 (Gröbner basis via symbol calculus).

```
Input: a finite set G ⊂ (A_V)^r containing a Gröbner basis G_0 of I(V)^r;
Repeat
    G' := G;
    For all P ∈ G and Q ∈ G \ G_0 such that P ≠ Q and lp(P) = lp(Q) {
        Let R be a *-reduction of sp*(P, Q) by G;
        If (R ≠ 0) then G := G ∪ {R};
    }
Until G = G';
Return G;
```

Theorem 3.9. *Algorithm* 3.8 *terminates in a finite number of steps, and its output* G *is a Gröbner basis of the* A_V*-submodule of* $(A_V)^r$ *generated by* G.

PROOF. The termination of the algorithm is an immediate consequence of Dicskon's lemma. Let $G=\{P_1, \dots, P_s\}$ be the output of Algorithm 3.8. Then by Proposition 3.5 and Lemma 3.7, for $1 \le i < j \le s$ such that $\mathrm{lp}(P_i)=\mathrm{lp}(P_j)$, there exist $Q_{ij1}, \dots, Q_{ijt} \in A_V$ so that

$$\mathrm{sp}*(P_i, P_j)=Q_{ij1}P_1+\cdots+Q_{ijt}P_t \tag{3.3}$$

and that $Q_{ijk}=0$ or $\mathrm{lexp}(Q_{ijk}P_k) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$. By using Lemma 3.7 again, we get

$$R:=\mathrm{sp}(P_i, P_j)-\mathrm{sp}*(P_i, P_j) \in (I(V)A_V)^r \tag{3.4}$$

and $\mathrm{lexp}(R) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$. Since G_0 is a Gröbner basis of $(I(V)A_V)^r$, there exist $Q'_{t+1}, \dots, Q'_s \in A_V$ such that

$$R=Q'_{t+1}P_{t+1}+\cdots+Q'_sP_s \tag{3.5}$$

and $\mathrm{lexp}(Q'_iP_i) \preceq \mathrm{lexp}(R)$, where we assume $G_0=\{P_{t+1}, \dots, P_s\}$ with $t<s$. Combining (3.3), (3.4), (3.5), we see that the condition (4) of Theorem 2.8 is satisfied. □

THEOREM 3.10. *Let $G=\{P_1, \dots, P_s\}$ be a finite subset of $(A_V)^r$. We assume that G contains a Gröbner basis $G_0=\{P_{t+1}, \dots, P_s\}$ of $I(V)^r$ with $1 \le t < s$. Suppose, for any $(i,j) \in I:=\{(i,j) \mid 1 \le i < j \le s, i \le t, v_i=v_j\}$, there exist $Q_{ij1}, \dots, Q_{ijt} \in A_V$ such that*

$$\mathrm{sp}*(P_i, P_j)-\sum_{k=1}^{t} Q_{ijk}*P_k \in (I(V)A_V)^r$$

and that $Q_{ijk}=0$ or $\mathrm{lexp}(Q_{ijk}*P_k) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$ for each k. (Such Q_{ijk} can be obtained through the $*$-reduction algorithm.) Put $\mathrm{lexp}(P_i)=(\alpha^{(i)}, \beta^{(i)}, v_i)$ and

$$S_{ij}:=\mathrm{lcoef}(P_j)x^{\beta^{(i)} \vee \beta^{(j)}-\beta^{(i)}}\mathfrak{J}^{\alpha^{(i)} \vee \alpha^{(j)}-\alpha^{(i)}},$$

$$V'_{ij}:=\begin{cases} (\underbrace{0, \dots, \overset{(i)}{S_{ij}}, \dots, -\overset{(j)}{S_{ji}}, \dots, 0}_{t})-(Q_{ij1}, \dots, Q_{ijt}) & \text{if } j \le t \\ (\underbrace{0, \dots, \overset{(i)}{S_{ij}} \dots, 0}_{t})-(Q_{ij1}, \dots, Q_{ijt}) & \text{if } j > t \end{cases}$$

for $(i,j) \in I$. Then the left A_V-module

$$N:=\{(Q_1, \dots, Q_t) \in (A_V)^t \mid Q_1P_1+\cdots+Q_tP_t \in (I(V)A_V)^r\}$$

is generated by $V':=\{V'_{ij} \mid (i,j) \in I\}$.

PROOF. In view of Proposition 3.5, there exist $Q_{ijk} \in A_V$ also for $(i,j) \notin I$ or $k>t$ so that $\mathrm{sp}(P_i, P_j)=\sum_{k=1}^{s} Q_{ijk}P_k$ and $\mathrm{lexp}(Q_{ijk}P_k) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$. Moreover, we may assume $Q_{ijk}=0$ *if* $i,j>t$ and $k \le t$. Suppose $(Q_1, \dots, Q_t) \in N$. Then there exist $Q_{t+1}, \dots, Q_s \in A_V$ so that $\sum_{k=1}^{s} Q_kP_k=0$. Let V_{ij} be as in Theorem 2.12. Then by

Theorem 2.12, there exist $C_{ij} \in A_V$ such that

$$(Q_1, \ldots, Q_t, Q_{t+1}, \ldots, Q_s) = \sum_{1 \le i < j \le s,\, v_i = v_j} C_{ij} V_{ij}.$$

From this we get $(Q_1, \ldots, Q_t) = \sum_{(i,j) \in I} C_{ij} V'_{ij}$. □

THEOREM 3.11. *In the same notation as in Theorems* 3.10 *and* 2.12, *let* $\prec_1$ *be a total order on* $N^{d+n} \times \{1, \ldots, s\}$ *defined by*

$$(\alpha, i) \prec_1 (\beta, j) \Longleftrightarrow \operatorname{lexp}(P_i) + \alpha \prec \operatorname{lexp}(P_j) + \beta$$
$$\text{or} \quad \operatorname{lexp}(P_i) + \alpha = \operatorname{lexp}(P_j) + \beta \ \text{and} \ i > j$$

for $\alpha, \beta \in N^{d+n}$ *and* $i, j \in \{1, \ldots, s\}$, *and let* $\prec'_1$ *be the restriction of* $\prec_1$ *to* $N^{d+n} \times \{1, \ldots, t\}$. *Then* $\prec'_1$ *satisfies the conditions* (O-1)–(O-3), *and* V' *is a Gröbner basis of* N *with respect to the order* $\prec'_1$.

PROOF. Since the theorem of Schreyer (cf. [E. Theorem 15.10]) applies to the situation of Theorem 2.12, V is a Gröbner basis of $S(P_1, \ldots, P_s)$ with respect to the order $\prec_1$. It is easy to see that

$$\operatorname{lexp}_{\prec'_1}(V'_{ij}) = \operatorname{lexp}_{\prec_1}(V_{ij}) = (\alpha^{(i)} \vee \alpha^{(j)} - \alpha^{(i)}, \beta^{(i)} \vee \beta^{(j)} - \beta^{(i)}, i)$$

for $(i, j) \in I$. Now assume $(Q_1, \ldots, Q_t) \in N$. Then in the proof of Theorem 3.10, we can choose $Q_{t+1}, \ldots, Q_s$ so that

$$\operatorname{lexp}(Q_j P_j) \preceq \max\{\operatorname{lexp}(Q_i P_i) \mid 1 \le i \le t\} \qquad (j = t+1, \ldots, s)$$

since G_0 is a Gröbner basis of $(I(V)A_V)^r$ with respect to the order $\prec$. This implies

$$\operatorname{lexp}_{\prec'_1}(Q_1, \ldots, Q_t) = \operatorname{lexp}_{\prec_1}(Q_1, \ldots, Q_t, Q_{t+1}, \ldots, Q_s)$$
$$\in (E_{\prec_1}(V) + N^{d+n}) \cap (N^{d+n} \times \{1, \ldots, t\}) = E_{\prec'_1}(V') + N^{d+n},$$

where $E_{\prec_1}$ denotes the set of the leading exponents with respect to the order $\prec_1$. □

4. Application of Gröbner basis to modules over the ring of algebraic differential operators. Let V be an affine algebraic variety in K^n. We use the same notation as in Section 1. In particular, we assume that $\vartheta_1, \ldots, \vartheta_d$ generate $\Theta(W)$ with an open subset W of V. We can assume that there exists a polynomial $f_0(x) \in K[x]$ so that $W = \{\bar{x} \in V \mid f_0(\bar{x}) \neq 0\}$. In fact, we can take, e.g.,

$$f_0(x) := \text{the square free part of } \det \frac{\partial(f_1, \ldots, f_{n-d})}{\partial(x_{n-d+1}, \ldots, x_n)},$$

in the notation of Section 1. Suppose that $\mathscr{M}$ is a coherent $\mathscr{D}_V$-module on W. Then by replacing W with its open subset if necessary, we may assume that

$$\mathscr{M}=(\mathscr{D}_V)^r/(\mathscr{D}_V P'_1+\cdots+\mathscr{D}_V P'_s)$$

with $P'_i \in \mathscr{D}_V(W)^r$.

In order to apply the arguments in Section 2, we need to take the derivations $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d$ so that their coefficients are polynomials. In what follows, we propose two different methods for this purpose. The first method (A) does not increase the number of variables, but the restrictions of derivations do not commute with each other in general. In the second method (B), we introduce an additional variable but we can take commuting derivations when restricted to an affine variety so that the arguments in Section 3, in particular, the Leibniz rule can be applied. We suppose that the method (B) has an advantage also from the viewpoint of complexity (see [T2, Proposition 1.2] for an estimate of the complexity of the Leibniz rule).

A. Direct method. Multiplying by the least common multiple of the denominators of the coefficients, we can take $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d \in \Theta(K^n)$ whose restrictions $\vartheta_1, \ldots, \vartheta_d$ to W generate $\Theta(W)$. Then the method of Section 2 can be directly applied with $A_V := K[x]\langle\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d\rangle$. We have an inclusion

$$D_V=A_V/I(V)A_V \subset \mathscr{D}_V(W)=\mathscr{O}_V(W)\langle\vartheta_1, \ldots, \vartheta_d\rangle$$

and the pojection $\varpi: A_V \to D_V$.

There exist $\nu_i \in N$ and $P_i \in (A_V)^r$ such that $f_0^{\nu_i} P'_i=\varpi(P_i) \in (D_V)^r$. Then we have an exact sequence

$$0 \longleftarrow \mathscr{M} \longleftarrow (\mathscr{D}_V)^r \overset{\psi}{\longleftarrow} (\mathscr{D}_V)^s$$

on W with

$$\psi(Q_1, \ldots, Q_s)=Q_1\varpi(P_1)+\cdots+Q_s\varpi(P_s)$$

since f_0 is invertible in $\mathscr{O}_V(W)$. Thus we can apply the argument of Section 2 for actual computation of $\mathscr{M}$.

B. Embedding method. We start with the derivations $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d$ given in Lemma 1.1, which commute with each other. Let f_0 be as above. Then the coefficients of $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d$ belong to the affine ring $K[x, f_0^{-1}]=\mathscr{O}_{K^n}(\tilde{W})$ with $\tilde{W}:=\{x \in K^n \mid f_0(x) \neq 0\}$. Define an affine algebraic variety W' in K^{n+1} by

$$W':=\{(x_1, \ldots, x_n, x_{n+1}) \in K^{n+1} \mid (x_1, \ldots, x_n) \in V, f_0(x)x_{n+1}-1=0\}$$

and define a map $\varphi: W \to W'$ by

$$\varphi(x_1, \ldots, x_n)=\left(x_1, \ldots, x_n, \frac{1}{f_0(x)}\right).$$

Then φ defines a bi-regular mapping of W to W'. In the notation of Lemma 1.1, there exists a polynomial $b_j^{(i)}(x, x_{n+1}) \in K[x, x_{n+1}]$ such that $a_j^{(i)}(x)=b_j^{(i)}(x, f_0(x)^{-1})$. For

$i=1, \ldots, d$, put

$$\tilde{\vartheta}_i' := \frac{\partial}{\partial x_i} + \sum_{j=d+1}^{n} b_j^{(i)}(x, x_{n+1}) \frac{\partial}{\partial x_j}$$
$$- x_{n+1}^2 \left(\frac{\partial f_0}{\partial x_i}(x) + \sum_{j=d+1}^{n} \frac{\partial f_0}{\partial x_j} b_j^{(i)}(x, x_{n+1}) \right) \frac{\partial}{\partial x_{n+1}} \in \Theta(K^{n+1}) .$$

LEMMA 4.1. *Let $\vartheta_i' \in \Theta(W')$ be the restriction of $\tilde{\vartheta}_i' \in \Theta(K^{n+1})$ defined above. Then $\vartheta_1', \ldots, \vartheta_d'$ generate $\Theta(W')$ over $\mathcal{O}_{W'}(W')$ and commute with each other.*

PROOF. We use the same notation as in Lemma 1.1. First note that

$$\det \frac{\partial(f_1, \ldots, f_{n-d}, x_{n+1} f_0 - 1)}{\partial(x_{n-d+1}, \ldots, x_n, x_{n+1})} = f_0 \det \frac{\partial(f_1, \ldots, f_{n-d})}{\partial(x_{n-d+1}, \ldots, x_n)} \neq 0$$

on W' by the assumption. It is easy to see that $\tilde{\vartheta}_i'(f_k)$ and $\tilde{\vartheta}_i'(x_{n+1} f_0(x) - 1)$ vanish on W' for $i = 1, \ldots, d$ and $k = 1, \ldots, n-d$. Then by the same reasoning as in the proof of Lemma 1.1, we see that $\vartheta_1', \ldots, \vartheta_d'$ generate $\Theta(W')$ and they commute with each other. □

The isomorphism φ induces a ring isomorphism

$$\varphi_* : \mathscr{D}_V(W) \xrightarrow{\sim} \mathscr{D}_{W'}(W') = D_{W'} = A_{W'}/I(W')A_{W'}$$

with $A_{W'} = \mathcal{O}_{W'}(W')\langle \tilde{\vartheta}_1', \ldots, \tilde{\vartheta}_d' \rangle$. More concretely, for an element $P = \sum_\alpha a_\alpha(\bar{x}) \vartheta^\alpha$ of $\mathscr{D}_V(W)$ with $a_\alpha(x) \in K[x]$, we have

$$\varphi_*(P) = \sum_\alpha b_\alpha(\bar{x}, \bar{x}_{n+1}) \vartheta'^\alpha \in D_{W'}$$

with $b_\alpha(x, x_{n+1}) \in K[x, x_{n+1}]$ which satisfies $b_\alpha(x, 1/f_0(x)) = a_\alpha(x)$, where $\vartheta_i \in \Theta(W)$ is the restriction of $\tilde{\vartheta}_i \in \Theta(K^n)$. We denote by $\varpi' : A_{W'} \to D_{W'}$ the canonical projection.

Now let $\mathscr{M}$ be a coherent $\mathscr{D}_V$-module on W. Then, by replacing W by its affine open subset if necessary, we may assume that there exist $P_1', \ldots, P_s' \in \mathscr{D}_V(W)^r$ so that

$$\mathscr{M} = (\mathscr{D}_V)^r / (\mathscr{D}_V P_1' + \cdots + \mathscr{D}_V P_s')$$

holds on W as a sheaf of left $\mathscr{D}_V$-modules.

Choose $P_i \in (A_{W'})^r$ such that $\varpi'(P_i) = \varphi_*(P_i')$ for $i = 1, \ldots, s$. We use the same notation as in Theorem 3.10 with n replaced by $n+1$ and V by W'. Then we may assume $P_1, \ldots, P_s$ satisfy the same condition as in Theorem 3.10. Define $D_{W'}$-homomorphisms ψ_0, ψ_1 by

$$\psi_0(U_1, \ldots, U_t) = \sum_{i=1}^{t} U_i \varpi'(P_i) \qquad \text{for} \quad (U_1, \ldots, U_t) \in (D_{W'})^t ,$$

$$\psi_1((U_{ij})_{(i,j) \in I}) = \sum_{(i,j) \in I} U_{ij} \varpi'(V_{ij}') \qquad \text{for} \quad (U_{ij})_{(i,j) \in I} \in (D_{W'})^{r_2}$$

with $r_2 := \sharp I$. Then applying Theorem 3.10, we have an exact sequence

$$0 \longleftarrow M \longleftarrow (D_{W'})^{r_0} \xleftarrow{\psi_0} (D_{W'})^{r_1} \xleftarrow{\psi_1} (D_{W'})^{r_2}$$

of left $D_{W'}$-modules, where $r_0 := r$, $r_1 := t$ and

$$M := (D_{W'})^r / (D_{W'}\varpi'(P_1) + \cdots + D_{W'}\varpi'(P_t)) .$$

In view of Theorem 3.11, $\boldsymbol{V}'$ is a Gröbner basis of $N_1 := \varpi'^{-1}(\ker \psi_0) \subset (A_{W'})^{r_1}$ with respect to the order $\prec_1'$. Let $\boldsymbol{G}_1$ be a Gröbner basis of $\boldsymbol{I}(W')^{r_2}$ with respect to $\prec_1'$. Then $\boldsymbol{V}' \cup \boldsymbol{G}_1$ is also a Gröbner basis of N_1 with respect to $\prec_1'$. Hence we can apply Theorems 3.10 and 3.11 again to $\boldsymbol{V}' \cup \boldsymbol{G}_1$ instead of $\boldsymbol{G}$. Thus we get a free resolution

$$0 \longleftarrow M \longleftarrow (D_{W'})^{r_0} \xleftarrow{\psi_0} (D_{W'})^{r_1} \xleftarrow{\psi_1} (D_{W'})^{r_2} \xleftarrow{\psi_2} (D_{W'})^{r_3} \longleftarrow \cdots$$

of M succesively only by $*$-reductions and the Gröbner basis computation in the polynomial ring. Since the stalk $(\mathscr{D}_{W'})_p$ is flat over $\mathscr{D}_{W'}(W') = D_{W'}$, this immediately gives a free resolution of $\varphi_* \mathscr{M}$ on W'. As an application, we obtain an algorithm for computing

$$\begin{aligned}\mathscr{E}xt^j_{\mathscr{D}_V}(\mathscr{M}, \mathscr{D}_V)_p &= \mathscr{E}xt^j_{\mathscr{D}_{W'}}(\varphi_*\mathscr{M}, \mathscr{D}_{W'})_{\varphi(p)} \\ &= \operatorname{Ext}^j_{D_{W'}}(M, D_{W'}) \otimes_{D_{W'}} (\mathscr{D}_{W'})_{\varphi(p)}\end{aligned}$$

for $p \in W \subset V$ by applying the *right* module version of Theorem 3.10 to the complex

$$0 \longrightarrow (D_{W'})^{r_0} \xrightarrow{(\psi_0)^*} (D_{W'})^{r_1} \xrightarrow{(\psi_1)^*} (D_{W'})^{r_2} \xrightarrow{(\psi_2)^*} (D_{W'})^{r_3} \longrightarrow \cdots .$$

5. Computation of the characteristic variety and the singular locus. We shall describe a method of computing the characteristic variety of a given coherent $\mathscr{D}_V$-module. We use the method B of the preceding section and retain the same notation. First let us recall the algebraic definition of the characteristic variety of a coherent $\mathscr{D}_V$-module $\mathscr{M}$ on an affine algebraic variety V. We assume that V is non-singular. For each integer k, let $\mathscr{D}_V(k)$ be the subsheaf of $\mathscr{D}_V$ consisting of the sections of $\mathscr{D}_V$ of order at most k. Let $\mathscr{M}$ be a left coherent $\mathscr{D}_V$-module on V. Then the *graded ring* $\operatorname{gr}(\mathscr{D}_V) := \bigoplus_{k \geq 0} \mathscr{D}_V(k)/\mathscr{D}_V(k-1)$ is a sheaf of commutative rings on V, which is locally isomorphic to $\mathscr{O}_V[\theta]$.

Suppose that with each integer k is associated a subsheaf $\mathscr{M}(k)$ of $\mathscr{M}$. Then the family $\{\mathscr{M}(k)\}_{k \in \boldsymbol{Z}}$ is called a *good filtration* of $\mathscr{M}$ if the following conditions are satisfied:

(1) Every $\mathscr{M}(k)$ is a locally finitely generated $\mathscr{O}_V$-module;

(2) $\mathscr{M}(k) \subset \mathscr{M}(k+1)$ for any k;

(3) There exists $k_0 \in \boldsymbol{Z}$ such that $\mathscr{M}(k_0) = 0$;

(4) $\bigcup_{k \in \boldsymbol{Z}} \mathscr{M}(k)_p = \mathscr{M}_p$ holds for any point $p \in V$;

(5) $\mathscr{D}_V(l)\mathscr{M}(k) \subset \mathscr{M}(k+l)$ for any $k, l \in \boldsymbol{Z}$;

(6) There exists $k_1 \in \boldsymbol{Z}$ such that $\mathscr{M}(k) = \mathscr{D}_V(k-k_0)\mathscr{M}(k_0)$ for any $k \geq k_1$.
Then the *graded module* $\mathrm{gr}(\mathscr{M}) := \bigoplus_{k \in \boldsymbol{Z}} D_V(k)/\mathscr{D}_V(k-1)$ associated with this filtration has a natural structure of $\mathrm{gr}(\mathscr{D}_V)$-module. Let us denote by T^*V the cotangent bundle of V and let $\pi: T^*V \to V$ be the projection. Then the (*algebraic*) *characteristic variety* $\mathrm{Char}(\mathscr{M})$ of $\mathscr{M}$ is, by definition, the support of the sheaf

$$\mu(\mathrm{gr}(\mathscr{M})) := \mathscr{O}_{T^*V} \otimes_{\pi^{-1}\mathrm{gr}(\mathscr{D}_V)} \pi^{-1}\mathrm{gr}(\mathscr{M}) .$$

This definition is independent of the choice of a good filtration of $\mathscr{M}$.

Now let $\mathscr{M}$ be a left coherent $\mathscr{D}_V$-module on V. We may assume that $\mathscr{M}$ is given explicitly by

$$\mathscr{M} = (\mathscr{D}_V)^r / (\mathscr{D}_V P_1 + \cdots + \mathscr{D}_V P_s)$$

on an affine open subset W of V with $P_1, \ldots, P_s \in \mathscr{D}_V(W)^r$. Let $u_1, \ldots, u_r$ be the residue classes of $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1) \in (\mathscr{D}_V)^r$ in $\mathscr{M}$ and put

$$\mathscr{M}(k) := \mathscr{D}_V(k)u_1 + \cdots + \mathscr{D}_V(k)u_r \qquad (k \in \boldsymbol{Z}) .$$

Then $\{\mathscr{M}(k)\}$ constitutes a good filtration of $\mathscr{M}$ on W. Let $\mathscr{N}$ be the left $\mathscr{D}_V$-submodule of $(\mathscr{D}_V)^r$ generated by $P_1, \ldots, P_s$. Let W', φ be as in the preceding section and choose $Q_i \in (A_{W'})^r$ so that $\varphi_*(P_i) = \varpi'(Q_i)$. Note that we have an isomorphism $T^*W \simeq W \times K^d$ as algebraic varieties given by

$$T^*W \ni (\bar{x}, \theta_1 dx_1 + \cdots + \theta_d dx_d) \longleftrightarrow (\bar{x}, \theta_1, \ldots, \theta_d) \in W \times K^d .$$

Here we impose the following conditions on the order $\prec$ in addition to (O-1)–(O-3) in Section 2 (where we should now replace n by $n+1$):

(O-4) $|\alpha| < |\alpha'|$ implies $(\alpha, \beta, i) \prec (\alpha', \beta', j)$ for any $\alpha, \alpha' \in \boldsymbol{N}^d$, $\beta, \beta' \in \boldsymbol{N}^{n+1}$, $i, j \in \{1, \ldots, r\}$;

(O-5) If $|\alpha| = |\alpha'|$ and $i < j$, then $(\alpha, \beta, i) \prec (\alpha', \beta', j)$.

For an element $P = (P_1, \ldots, P_r)$ of $(A_{W'})^r$ with $\mathrm{lp}(P) = \nu$, we put $\sigma(P)_\nu := \sigma(P_\nu)$.

THEOREM 5.1. *In the above notation, let* $\boldsymbol{G}$ *be a Gröbner basis of the left* $A_{W'}$*-submodule*

$$N := (\boldsymbol{I}(W')A_{W'})^r + A_{W'}Q_1 + \cdots + A_{W'}Q_s$$

of $(A_{W'})^r$ *with respect to the order* $\prec$ *satisfying* (O-1)–(O-5). *Then the characteristic variety of* $\mathscr{M}$ *is given by*

$$\mathrm{Char}(\mathscr{M}) = \bigcup_{\nu=1}^{r} \{(\bar{x}, \theta) \in W \times K^d \mid \sigma(P)_\nu(\varphi(x), \theta) = 0 \text{ for any } P \in \boldsymbol{G} \text{ such that } \mathrm{lp}(P) = \nu\} .$$

PROOF. We denote by $\bar{u}_1, \ldots, \bar{u}_r$ the residue classes in $\mathrm{gr}(\mathscr{M})$ of $u_1, \ldots, u_r$. Let p be an arbitrary point of W and define a $\mathrm{gr}(\mathscr{D}_V)_p$-submodule $\mathscr{L}_\nu$ of $\mathrm{gr}(\mathscr{M})_p$ by

$$\mathscr{L}_\nu := \mathrm{gr}(\mathscr{D}_V)_p \bar{u}_1 + \cdots + \mathrm{gr}(\mathscr{D}_V)_p \bar{u}_\nu$$

for $\nu=0, 1, \dots, r$. Then it is easy to see that

$$\mathrm{Char}(\mathscr{M}) \cap \pi^{-1}(p) = \bigcup_{\nu=1}^{r} \{(p, \theta) \mid \mu(\mathscr{L}_\nu/\mathscr{L}_{\nu-1})_{(p,\theta)} \neq 0\}$$

since μ is an exact functor. Put $\mathscr{I}_\nu := \{f \in \mathrm{gr}(\mathscr{D}_V)_p \mid f\bar{u}_\nu \in \mathscr{L}_{\nu-1}\}$. Then we have an isomorphism $\mathscr{L}_\nu/\mathscr{L}_{\nu-1} \simeq \mathrm{gr}(\mathscr{D}_V)_p/\mathscr{I}_\nu$. Hence, in order to prove the assertion of the theorem, it suffices to show that $\mathscr{I}_\nu$ is generated by

$$\bar{\boldsymbol{G}}_\nu := \{\sigma(\varphi_*^{-1}(\varpi'(P)))_\nu \mid P \in \boldsymbol{G}, \mathrm{lp}(P) = \nu\}.$$

Take $P = (P_1, \dots, P_r) \in \boldsymbol{G}$ such that $\mathrm{lp}(P) = \nu$ and put $m := \mathrm{ord}(P) = \max\{\mathrm{ord}(P_i) \mid i = 1, \dots, r\}$. Then we have

$$\sigma_m(\varphi_*^{-1}(\varpi'(P_1)))\bar{u}_1 + \cdots + \sigma_m(\varphi_*^{-1}(\varpi'(P_\nu)))\bar{u}_\nu = 0$$

since $\mathrm{ord}(P_i) < m$ for $i > \nu$ in view of the conditions (O-4) and (O-5). This implies, in particular, that $\sigma(\varphi_*^{-1}(\varpi'(P)))_\nu \in \mathscr{I}_\nu$.

Conversely, suppose $f \in \mathscr{I}_\nu \setminus \{0\}$. We may assume $f \in \mathscr{D}_V(m)_p/\mathscr{D}_V(m-1)_p$ for some $m \geq 0$. Then there exist $f_1, \dots, f_{\nu-1} \in \mathscr{D}_V(m)_p/\mathscr{D}_V(m-1)_p$ such that

$$f_1\bar{u}_1 + \cdots + f_{\nu-1}\bar{u}_{\nu-1} + f\bar{u}_\nu = 0$$

in $\mathrm{gr}(\mathscr{M})_p$. Hence there exists some $P = (P_1, \dots, P_r) \in \mathscr{N}_p$ of order m such that $\sigma_m(P_i) = f_i$ for $i = 1, \dots, \nu-1$, $\sigma_m(P_\nu) = f$, and $\sigma_m(P_i) = 0$ for $i > \nu$. Multiplying P by a suitable polynomial $a \in K[x]$ such that $a(p) \neq 0$ from the left, we may assume that there exists some $Q = (Q_1, \dots, Q_r) \in N$ such that $\varphi_*(P) = \varpi'(Q)$ and that $\mathrm{ord}(Q_i) = \mathrm{ord}(P_i)$ for $i = 1, \dots, r$. Hence $\mathrm{lp}(Q) = \nu$. Put $\boldsymbol{G} = \{G_1, \dots, G_t\}$. Then there exist $S_1, \dots, S_t \in \mathscr{D}_{W'}(W')$ such that

$$Q = S_1G_1 + \cdots + S_tG_t$$

and that $\mathrm{lexp}(S_iG_i) \preceq \mathrm{lexp}(Q)$ for $i = 1, \dots, t$. This implies, in view of (O-4) and (O-5), that $f = \sigma(\varphi_*^{-1}(\varpi'(Q_\nu)))$ belongs to the ideal of $\mathrm{gr}(\mathscr{D}_V)_p$ generated by $\bar{\boldsymbol{G}}_\nu$. □

For a coherent $\mathscr{D}_V$-module $\mathscr{M}$ defined on an open subset W of V, its *singular support* $\mathrm{Sing}(\mathscr{M})$ is defined by

$$\mathrm{Sing}(\mathscr{M}) := \pi(\mathrm{Char}(\mathscr{M}) \setminus T_W^*W),$$

where T_W^*W denotes the zero section of the cotangent bundle T^*W. Since the characteristic variety is homogeneous with respect to the fiber coordinates θ, the singular locus is an algebraic set in W.

THEOREM 5.2. *Under the same assumptions as in Theorem* 5.1, *let* $\boldsymbol{H}_{\nu,i}$ *be a Gröbner basis of the ideal of* $K[x, x_{n+1}, \theta]$ *generated by*

$$\{\mathrm{subst}(\sigma(Q)_\nu, \theta_i, 1) \mid Q \in \boldsymbol{G}, \mathrm{lp}(Q) = \nu\}$$

with respect to the order $\prec$ *restricted to* $N^{d+n+1} \times \{v\}$; *here* $\mathrm{subst}(f(x, \theta), \theta_i, 1)$ *denotes putting* $\theta_i = 1$ *in* f. *Then the singular locus of* $\mathcal{M}$ *is given by*

$$\mathrm{Sing}(\mathcal{M}) = \bigcup_{i=1}^{n} \bigcup_{v=1}^{r} \{\bar{x} \in W \mid f(\varphi(x)) = 0 \text{ for any } f \in \boldsymbol{H}_{v,i} \cap K[x, x_{n+1}]\} .$$

This theorem can be proved in the same way as Proposition 2 and Algorithm in [O1] by using the preceding theorem.

REMARK. The term order used in the computation of $\boldsymbol{H}_{v,i}$ may be any term order for eliminating $\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_d$ (cf. [BW], [CLO]).

EXAMPLE 5.3. Put $V := \{(x, y, z) \in K^3 \mid x^3 + y^3 + z^3 - 1 = 0\}$ and $W := \{(x, y, z) \in V \mid z \neq 0\}$. Then $\Theta(W)$ is generated by commuting derivations

$$\vartheta_x := \frac{\partial}{\partial x} - \frac{x^2}{z^2}\frac{\partial}{\partial z}, \qquad \vartheta_y := \frac{\partial}{\partial y} - \frac{y^2}{z^2}\frac{\partial}{\partial z}.$$

Put $W' := \{(x, y, z, t) \in V \times K \mid tz - 1 = 0\}$ and define $\tilde{\vartheta}'_x, \tilde{\vartheta}'_y \in \Theta(K^4)$ by

$$\tilde{\vartheta}'_x := \frac{\partial}{\partial x} - x^2 t^2 \frac{\partial}{\partial z} + x^2 t^4 \frac{\partial}{\partial t}, \qquad \tilde{\vartheta}'_y := \frac{\partial}{\partial y} - y^2 t^2 \frac{\partial}{\partial z} + y^2 t^4 \frac{\partial}{\partial t}.$$

Consider a $\mathscr{D}_V$-module $\mathcal{M} := \mathscr{D}_V / (\mathscr{D}_V P_1 + \mathscr{D}_V P_2)$ on W, where

$$P_1 := xz\vartheta_x + cx^3 - az, \qquad P_2 := yz\vartheta_y + cy^3 - bz \quad \in \mathscr{D}_V(W)$$

with constants $a, b, c \in K \setminus \boldsymbol{Z}$. Put

$$Q_1 := xz\tilde{\vartheta}'_x + cx^3 - az, \qquad Q_2 := yz\tilde{\vartheta}'_y + cy^3 - bz \quad \in A_{W'}$$

and let N be the left ideal of $A_{W'}$ generated by Q_1, Q_2 and $f_0 := tz - 1$, $f_1 := x^3 + y^3 + z^3 - 1$. Then we get $\boldsymbol{G} = \{f_0, f_1, G_1, G_2, G_3, G_4, G_5\}$ as a Gröbner basis of N (with respect to an order satisfying (O-1)–(O-5)), where

$$G_1 = y\tilde{\vartheta}'_y + cty^3 - b,$$

$$G_2 = x\tilde{\vartheta}'_x - cty^3 - cz^2 + ct - a,$$

$$G_3 = (y^3 + z^3 - 1)\tilde{\vartheta}'_x + x^2(cty^3 + cz^2 - ct + a),$$

$$G_4 = (-z + t^2)\tilde{\vartheta}'_x\tilde{\vartheta}'_y + y^2t^2(cty^3 - b)\tilde{\vartheta}'_x + x^2(-ct^3y^3 + ct^3 - at^2 - c)\tilde{\vartheta}'_y + cx^2y^2t^3(t^3 - 1),$$

$$G_5 = (z^2 - t)\tilde{\vartheta}'_x\tilde{\vartheta}'_y + ty^2(-cty^3 + b)\tilde{\vartheta}'_x + x^2(ct^2y^3 + cz - ct^2 + at)\tilde{\vartheta}'_y - cx^2y^2t^2(t^3 - 1).$$

Hence the characteristic variety and the singular locus of $\mathcal{M}$ are

$$\begin{aligned}\mathrm{Char}(\mathcal{M}) &= \{(x, y, z, \theta_x, \theta_y) \in W \times K^2 \mid x\theta_x = y\theta_y = (1/z^2 - z)\theta_x\theta_y \\ &= (y^3 + z^3 - 1)\theta_x = (1/z - z^2)\theta_x\theta_y = 0\} \\ &= \{\theta_x = \theta_y = 0\} \cup \{x = \theta_y = 0\} \cup \{y = \theta_x = 0\} \cup \{x = y = 0\}, \\ \mathrm{Sing}(\mathcal{M}) &= \{(x, y, z) \in W \mid xy = 0\}.\end{aligned}$$

This computation has been done by using a program written in a computer algebra language Risa/Asir. We acknowledge the assistance of T. Shimoyama and M. Noro at Fujitsu Laboratories Limited.

A1. The Kashiwara equivalence. Let V be a non-singular affine algebraic variety in K^n. Let us denote by $\iota: V \to K^n$ the natural embedding. We use the same notation as in Section 2. For a left coherent $\mathcal{D}_V$-module $\mathcal{M}$ on V, its direct image is defined by $\iota_+\mathcal{M} := \mathcal{D}_{K^n \leftarrow V} \otimes_{\mathcal{D}_V} \mathcal{M}$. The functor ι_+ induces an equivalence between the category of coherent left $\mathcal{D}_V$-modules on V and that of coherent left $\mathcal{D}_{K^n}$-modules whose supports (as sheaves on K^n) are contained in V (cf. [Bo], [H], [TH]).

PROPOSITION A.1. *Suppose that $\mathcal{M}$ is given by $\mathcal{M} = (\mathcal{D}_V)^r / (\mathcal{D}_V P_1 + \cdots + \mathcal{D}_V P_s)$ on W with $P_1, \ldots, P_s \in (D_V)^r$. Take $Q_j \in (A_V)^r$ so that $\varpi(Q_j) = P_j$. Then we have an isomorphism*

$$\iota_+\mathcal{M} \simeq \mathcal{D}_{K^n}^r / (\mathcal{D}_{K^n} Q_1 + \cdots + \mathcal{D}_{K^n} Q_s + \mathcal{D}_{K^n} I(V)^r)$$

as left $\mathcal{D}_{K^n}$-modules on an open neighborhood of W in K^n.

PROOF. We have an isomorphism $\mathcal{D}_{K^n \leftarrow V} \simeq \mathcal{D}_{K^n} / \mathcal{D}_{K^n} I(V)$ on W as $(\mathcal{D}_{K^n}, \mathcal{D}_V)$-bimodules. Define a sheaf $\mathcal{A}$ of subrings of $\mathcal{D}_{K^n}$ by $\mathcal{A} = \mathcal{O}_{K^n}\langle \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d \rangle$. Then $\mathcal{D}_{K^n} / \mathcal{D}_{K^n} I(V)$ has also a structure of $(\mathcal{D}_{K^n}, \mathcal{A})$-bimodule which is compatible with the natural ring homomorphism $\mathcal{A} \to \mathcal{D}_V \simeq \mathcal{A} / \mathcal{A} I(V)$. This ring homomorphism induces an isomorphism

$$(\mathcal{D}_{K^n} / \mathcal{D}_{K^n} I(V)) \otimes_{\mathcal{A}} (\mathcal{A}^r / \mathcal{J}) \xrightarrow{\simeq} (\mathcal{D}_{K^n} / \mathcal{D}_{K^n} I(V)) \otimes_{\mathcal{D}_V} \mathcal{M}$$

with $\mathcal{J} := \mathcal{A} Q_1 + \cdots + \mathcal{A} Q_s + \mathcal{A} I(V)^r$. On the other hand, the exact sequence

$$0 \longrightarrow \mathcal{D}_{K^n} I(V) \longrightarrow \mathcal{D}_{K^n} \longrightarrow \mathcal{D}_{K^n} / \mathcal{D}_{K^n} I(V) \longrightarrow 0$$

of $(\mathcal{D}_{K^n}, \mathcal{A})$-bimodules yields the exact sequence

$$0 = \mathcal{D}_{K^n} I(V) \otimes_{\mathcal{A}} (\mathcal{A}^r / \mathcal{J}) \longrightarrow \mathcal{D}_{K^n} \otimes_{\mathcal{A}} (\mathcal{A}^r / \mathcal{J}) \longrightarrow (\mathcal{D}_{K^n} / \mathcal{D}_{K^n} I(V)) \otimes_{\mathcal{A}} (\mathcal{A}^r / \mathcal{J}) \longrightarrow 0$$

of left $\mathcal{D}_{K^n}$-modules. Thus we get

$$\iota_+\mathcal{M} \simeq \mathcal{D}_{K^n} \otimes_{\mathcal{A}} (\mathcal{A}^r / \mathcal{J}) \simeq \mathcal{D}_{K^n}^r / (\mathcal{D}_{K^n} Q_1 + \cdots + \mathcal{D}_{K^n} Q_s + \mathcal{D}_{K^n} I(V)^r).$$

□

Hence it is possible to compute, e.g. the characteristic variety of $\mathcal{M}$ via the Gröbner basis computation for $\iota_+\mathcal{M}$ in $(A_n)^r$. However, as is verified by actual computation of Example 5.3, such computation yields more complicated results than our algorithm in general. This is probably caused by the fact that $A_n I(V)$ is not a two-sided ideal and, when the codimension of V is higher, the number of variables becomes larger.

A2. Coincidence of the algebraic and the analytic characteristic varieties. Let $V \subset \boldsymbol{C}^n$ be a non-singular algebraic variety over the field $\boldsymbol{C}$ of complex numbers and let $\mathcal{M}$ be a coherent $\mathcal{D}_V$-module on V. In the sequel, we use the usual topology on V instead of the Zariski topology. This does not affect our arguments below since the stalks of $\mathcal{D}_V$ with respect to these two topologies are isomorphic.

In view of the isomorphism φ_* defined in Section 4, we may assume that $\mathcal{M}$ is given by

$$\mathcal{M} = (\mathcal{D}_V)^r/(\mathcal{D}_V P'_1 + \cdots + \mathcal{D}_V P'_t)$$

on V with $P'_i \in \mathcal{D}_V(V)^r$.

Let $\mathcal{D}_V^{\mathrm{an}}$ be the sheaf of rings of *analytic* differential operators on the complex manifold V and put

$$\mathcal{M}^{\mathrm{an}} := \mathcal{D}_V^{\mathrm{an}} \otimes_{\mathcal{D}_V} \mathcal{M} = (\mathcal{D}_V^{\mathrm{an}})^r/(\mathcal{D}_V^{\mathrm{an}} P'_1 + \cdots + \mathcal{D}_V^{\mathrm{an}} P'_t) .$$

Let $u_1, \ldots, u_r$ be the residue classes in $\mathcal{M}$ of $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1) \in (\mathcal{D}_V)^r$ and put

$$\mathcal{M}(k) := \mathcal{D}_V(k)u_1 + \cdots + \mathcal{D}_V(k)u_r \subset \mathcal{M} ,$$

$$\mathcal{M}^{\mathrm{an}}(k) := \mathcal{D}_V^{\mathrm{an}}(k)(1 \otimes u_1) + \cdots + \mathcal{D}_V^{\mathrm{an}}(k)(1 \otimes u_r) \subset \mathcal{M}^{\mathrm{an}}$$

for each integer k, where $\mathcal{D}_V^{\mathrm{an}}(k)$ denotes the subsheaf of $\mathcal{D}_V^{\mathrm{an}}$ consisting of operators of order at most k. Define the associated graded modules by

$$\mathrm{gr}(\mathcal{M}) := \bigoplus_{k \geq 0} \mathcal{M}(k)/\mathcal{M}(k-1) ,$$

$$\mathrm{gr}(\mathcal{M}^{\mathrm{an}}) := \bigoplus_{k \geq 0} \mathcal{M}^{\mathrm{an}}(k)/\mathcal{M}^{\mathrm{an}}(k-1) .$$

Note that there is a natural $\mathrm{gr}(\mathcal{D}_V)$-module homomorphism $\mathrm{gr}(\mathcal{M}) \to \mathrm{gr}(\mathcal{M}^{\mathrm{an}})$ induced by the homomorphism $\mathcal{M} \to \mathcal{M}^{\mathrm{an}}$.

The *analytic characteristic variety* of $\mathcal{M}^{\mathrm{an}}$ is defined as the support of the sheaf

$$\mu(\mathrm{gr}(\mathcal{M}^{\mathrm{an}})) := \mathcal{O}_{T^*V}^{\mathrm{an}} \otimes_{\pi^{-1}\mathrm{gr}(\mathcal{D}_V^{\mathrm{an}})} \pi^{-1}\, \mathrm{gr}(\mathcal{M}^{\mathrm{an}})$$

with $\pi: T^*V \to V$ being the projection and $\mathcal{O}_{T^*V}^{\mathrm{an}}$ the sheaf of holomorphic functions on T^*V. Since $\mathcal{O}_{T^*V}^{\mathrm{an}}$ is faithfully flat over $\mathcal{O}_{T^*V}$, it suffices to prove the following proposition in order to show that the analytic and the algebraic characteristic varieties coincide.

PROPOSITION A.2. *Under the assumptions and notation above, we have an isomorphism*

$$\mathrm{gr}(\mathscr{M}^{\mathrm{an}}) \simeq \mathrm{gr}(\mathscr{D}_V^{\mathrm{an}}) \otimes_{\mathrm{gr}(\mathscr{D}_V)} \mathrm{gr}(\mathscr{M})$$

induced by the pairing $\mathrm{gr}(\mathscr{D}_V^{\mathrm{an}}) \times \mathrm{gr}(\mathscr{M}) \to \mathrm{gr}(\mathscr{M}^{\mathrm{an}})$.

PROOF. Put

$$\mathscr{N}(k) := \mathscr{D}_V(k)P_1 + \cdots + \mathscr{D}_V(k)P_t\,,$$
$$\mathscr{N}^{\mathrm{an}}(k) := \mathscr{D}_V^{\mathrm{an}}(k)P_1 + \cdots + \mathscr{D}_V^{\mathrm{an}}(k)P_t$$

and

$$\mathrm{gr}(\mathscr{N}) := \bigoplus_{k \geq 0} \mathscr{N}(k)/\mathscr{N}(k-1)\,, \qquad \mathrm{gr}(\mathscr{N}^{\mathrm{an}}) := \bigoplus_{k \geq 0} \mathscr{N}^{\mathrm{an}}(k)/\mathscr{N}^{\mathrm{an}}(k-1)\,.$$

Then we have short exact sequences

$$0 \longrightarrow \mathrm{gr}(\mathscr{N}) \longrightarrow \mathrm{gr}(\mathscr{D}_V)^r \longrightarrow \mathrm{gr}(\mathscr{M}) \longrightarrow 0\,,$$
$$0 \longrightarrow \mathrm{gr}(\mathscr{N}^{\mathrm{an}}) \longrightarrow \mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})^r \longrightarrow \mathrm{gr}(\mathscr{M}^{\mathrm{an}}) \longrightarrow 0\,.$$

Since $\mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})$ is flat over $\mathrm{gr}(\mathscr{D}_V)$, it suffices to show that the natural homomorphism

$$h\colon \mathrm{gr}(\mathscr{D}_V^{\mathrm{an}}) \otimes_{\mathrm{gr}(\mathscr{D}_V)} \mathrm{gr}(\mathscr{N}) \longrightarrow \mathrm{gr}(\mathscr{N}^{\mathrm{an}})$$

is an isomorphism. It follows from the above exact sequence that h is injective. Now choose $P_i \in (A_V)^r$ so that $\varpi(P_i) = P_i'$ for $i = 1, \ldots, t$ with $A_V = \boldsymbol{C}[x]\langle \tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d \rangle$. Here $\tilde{\vartheta}_1, \ldots, \tilde{\vartheta}_d \in \Theta(\boldsymbol{C}^n)$ are derivations whose restrictions $\vartheta_1, \ldots, \vartheta_d$ to $\mathcal{O}_V(V)$ are generators of $\Theta(V)$ commuting with each other.

We may assume that $\boldsymbol{G} = \{P_1, \ldots, P_t, P_{t+1}, \ldots, P_s\}$ is a Gröbner basis of

$$N := A_V P_1 + \cdots + A_V P_t + A_V P_{t+1} + \cdots + A_V P_s$$

with respect to an order satifying (O-1)–(O-5), and $\boldsymbol{G}_0 := \{P_{t+1}, \ldots, P_s\}$ is a Gröbner basis of $(\boldsymbol{I}(V)A_V)^r$ with respect to the same order. Using the same notation as in Theorem 3.10, put

$$s_{ij} := \mathrm{lcoef}(P_i) x^{\beta^{(i)} \vee \beta^{(j)} - \beta^{(i)}} \theta^{\alpha^{(i)} \vee \alpha^{(j)} - \alpha^{(i)}}\,,$$

$$v_{ij}' := \begin{cases} (\underbrace{0, \ldots, \overset{(i)}{s_{ij}}, \ldots, \overset{(j)}{-s_{ji}}, \ldots, 0}_{t}) - (\sigma_{m_{ij}-m_1}(Q_{ij1}), \ldots, \sigma_{m_{ij}-m_t}(Q_{ijt})) & \text{if } j \leq t \\[2ex] (\underbrace{0, \ldots, \overset{(i)}{s_{ij}} \ldots, 0}_{t}) - (\sigma_{m_{ij}-m_1}(Q_{ij1}), \ldots, \sigma_{m_{ij}-m_t}(Q_{ijt})) & \text{if } j > t \end{cases}$$

for $(i, j) \in \boldsymbol{I}$, where $m_i := |\alpha^{(i)}|$, $m_{ij} := |\alpha^{(i)} \vee \alpha^{(j)}|$. Then by the same argument as in the proof of Theorem 3.10, the $\mathrm{gr}(\mathscr{D}_V)$-module

$$\{(f_1, \ldots, f_t) \in (\mathrm{gr}(\mathscr{D}_V))^r \mid f_1\sigma(P_1') + \cdots + f_t\sigma(P_t') = 0\}$$

is generated by $\boldsymbol{v} := \{\varpi(v'_{ij}) \mid (i,j) \in I\}$. Hence we have an exact sequence

$$\mathrm{gr}(\mathscr{D}_V)^{r_2} \xrightarrow{\psi} \mathrm{gr}(\mathscr{D}_V)^t \xrightarrow{\varphi} \mathrm{gr}(\mathscr{D}_V)^r$$

of $\mathrm{gr}(\mathscr{D}_V)$-modules with homomorphisms φ and ψ defined by

$$\varphi((f_1, \dots, f_t)) = \sum_{k=1}^{t} f_k \sigma(P'_k), \qquad \psi((f_{ij})) = \sum_{(i,j)\in I} f_{ij} \varpi(v'_{ij}),$$

where $r_2 := \sharp I$. Since $\mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})$ is flat over $\mathrm{gr}(\mathscr{D}_V)$, we have also an exact sequence

$$\text{(A.1)} \qquad \mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})^{r_2} \xrightarrow{1\otimes\psi} \mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})^t \xrightarrow{1\otimes\varphi} \mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})^r .$$

Let p be an arbitrary point of V and let $[P]$ be the element of $\mathrm{gr}(\mathscr{N}^{\mathrm{an}})_p$ represented by $P \in \mathscr{N}^{\mathrm{an}}(k)_p \setminus \mathscr{N}^{\mathrm{an}}(k-1)_p$. Then there exist $U_1, \dots, U_t \in (\mathscr{D}_V^{\mathrm{an}})_p$ such that

$$\text{(A.2)} \qquad P = \sum_{i=1}^{t} U_i P'_i .$$

We claim that we can take U_i so that $\mathrm{ord}(U_i P'_i) \leq k$ for any $i = 1, \dots, t$. Assume $m := \max\{\mathrm{ord}(U_i P'_i) \mid i = 1, \dots, t\} > k$. Then we have

$$(1 \otimes \varphi)(\sigma_{m-m_1}(U_1), \dots, \sigma_{m-m_t}(U_t)) = 0 .$$

In view of the exact sequence (A.1), there exist $(f_{ij}) \in (\mathrm{gr}(\mathscr{D}_V^{\mathrm{an}})_p)^{r_2}$ such that

$$\sum_{(i,j)\in I} f_{ij} \varpi(v'_{ij}) = (\sigma_{m-m_1}(U_1), \dots, \sigma_{m-m_t}(U_t)) .$$

Here we may assume $f_{ij} \in \mathscr{D}_V^{\mathrm{an}}(m - m_{ij})_p / \mathscr{D}_V^{\mathrm{an}}(m - m_{ij} - 1)_p$. Let F_{ij} be an element of $\mathscr{D}_V^{\mathrm{an}}(m - m_{ij})_p$ such that $[F_{ij}] = f_{ij}$ and define $U'_1, \dots, U'_t \in (\mathscr{D}_V^{\mathrm{an}})_p$ by

$$(U'_1, \dots, U'_t) = (U_1, \dots, U_t) - \sum_{(i,j)\in I} F_{ij} \varpi(V'_{ij}) .$$

Then we get $P = \sum_{i=1}^{t} U'_i P'_i$ and $\mathrm{ord}(U'_i P'_i) \leq m - 1$. Hence by induction, we can choose U_i in (A.2) so that $\mathrm{ord}(U_i P'_i) \leq k$ for $i = 1, \dots, t$. This implies $[P] \in \mathrm{im}\, h$. □

References

[BW] T. Becker and V. Weispfenning, Gröbner bases, Springer-Verlag, Berlin, 1993.

[Bj] Björk, Rings of Differential Operators, North-Holland, Amsterdam, 1979.

[Bo] A. Borel et al., Algebraic D-modules, Academic Press, Boston, 1987.

[Bu] B. Buchberger, Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems, Aequationes Math. 4 (1970), 374–383.

[C] F. Castro, Calculs effectifs pour les idéaux d'opérateurs différentiels, Travaux en Cours, vol. 24, Hermann, Paris, 1987, pp. 1–19.

[CLO] D. Cox, J. Little and D. O'Shea, Ideals, Varieties, and Algorithms, Springer-Verlag, Berlin, 1992.

[E] D. EISENBUD, Commutative algebra with a view toward algebraic geometry, Springer-Verlag, New York, 1995.

[G] A. GALLIGO, Some algorithmic questions on ideals of differential operators, Lecture Notes in Comput. Sci., vol. 204, Springer-Verlag, Berlin, 1985, pp. 413–421.

[H] R. HOTTA, Introduction to $\mathscr{D}$-modules, Lecture Notes Series 1, Inst. of Math. Sci., Madras, 1987.

[KW] A. KANDRY-RODY AND V. WEISPFENNING, Non-commutative Gröbner bases in algebras of solvable type, J. Symbolic Computation 9 (1990), 1–26.

[Ka] M. KASHIWARA, Systems of microdifferential equations, Birkhäuser, Boston, 1983.

[M] B. MALGRANGE, Équations differentielles à coefficients polynomiaux, Birkhäuser, Boston, 1991.

[N] M. NOUMI, Wronskian determinants and the Gröbner representation of a linear differential equation, in Algebraic Analysis (M. Kashiwara, T. Kawai, eds.), Academic Press, Boston, 1988, pp. 549–569.

[O1] T. OAKU, Computation of the characteristic variety and the singular locus of a system of differential equations with polynomial coefficients, Japan J. Indust. Appl. Math. 11 (1994), 485–497.

[O2] T. OAKU, Gröbner bases and systems of linear partial differential equations, Sophia Kokyuroku in Mathematics, No. 38 (in Japanese), Department of Mathematics, Sophia University, Tokyo, 1994.

[T1] N. TAKAYAMA, Gröbner basis and the problem of contiguous relations, Japan J. Appl. Math 6 (1989), 147–160.

[T2] N. TAKAYAMA, An algorithm for finding recurrence relations of binomial sums and its complexity, to appear in J. Symbolic Computation 20 (1995), 637–651.

[TH] T. TANISAKI AND R. HOTTA, D-modules and algebraic groups (in Japanese), Springer-Verlag, Tokyo, 1995.

DEPARTMENT OF MATHEMATICAL SCIENCES
YOKOHAMA CITY UNIVERSITY
22-2 SETO, KANAZAWA-KU, YOKOHAMA 236
JAPAN