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GRÖBNER BASES, H–BASES AND INTERPOLATION

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Abstract. The paper is concerned with a construction for H–bases of polynomial ideals without relying on term orders. The main ingredient is a homogeneous reduction algorithm which orthogonalizes leading terms instead of completely canceling them. This allows for an extension of Buchberger's algorithm to construct these H–bases algorithmically. In addition, the close connection of this approach to minimal degree interpolation, and in particular to the least interpolation scheme due to de Boor and Ron, is pointed out.

1. Introduction

The concept of Gröbner bases, introduced by Buchberger [7] in 1965, has become an important ingredient for the treatment of various problems in computational algebra, see [9] for an extensive survey. This concept has also been extended to more general situations, like Gröbner bases of modules, for example, in [19]. However, all approaches related to Gröbner bases are fundamentally tied to *term orders*, which leads to asymmetry among the variables to be considered. On the other hand, the concept of H–bases, introduced long ago by Macaulay [14], is based solely on homogeneous terms of a polynomial. This paper gives an algorithmic approach to H–bases which works in terms of homogeneous polynomials only and is based on a reduction algorithm which *orthogonalizes* (homogeneous) leading terms instead of canceling them. In contrast to the situation of term orders, where the leading terms are only single monomials, cancellation is in general impossible for full homogeneous terms, but if it is possible, the orthogonalization is capable of doing that. Nevertheless, this generalized reduction is suitable for a characterization of H–bases by means of reduction of a basis of the module of syzygies. This will lead to a straightforward extension of Buchberger's algorithm for the generation of H–bases.

Buchberger's first intention for the introduction of Gröbner bases for an ideal $\mathcal{I}$ was to compute a multiplication table modulo the ideal, where the notion of reduction gave rise to a "natural" or "standard" basis for the vector space $\Pi/\mathcal{I}$. If $\mathcal{I}$ is a zero dimensional ideal (or, an ideal of finite codimension), i.e., $\mathcal{I} = \ker \Theta$ for some finite set Θ of linear functionals defined on Π, then we can ask for the associated interpolation problem. Any representation of $\Pi/\mathcal{I}$ is now an interpolation space (i.e., a finite dimensional subspace of Π where the interpolation problem is uniquely solvable), and one can ask again for "natural" or "standard" interpolation spaces.

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Indeed, we will find that the interpolation space induced by the reduction process is a well–known one: it is the least interpolation space, developed by de Boor and Ron [3]. In this context it is now possible by the general H–basis construction to find H–bases for the ideal that reflect some symmetries or geometric properties of the ideal in the interpolation space which will be destroyed by the more artificial preferences among variables which term orders induce.

The paper is organized as follows. After setting up the necessary notation in Section 2, the reduction algorithm will be presented in Section 3. In Section 4 the notion of an H–basis will be recalled and it will be shown that the reduction algorithm plays the same role for characterizing H–bases as is known for Gröbner bases. Finally, in Section 5, the connections to minimal degree interpolation will be pointed out.

2. Notation

For a field $\mathbb{K}$, we denote the ring of d–variate polynomials over $\mathbb{K}$ by

$$\Pi = \mathbb{K}[x] = \mathbb{K}\left[\xi_1, \dots, \xi_d\right],$$

where the number of variables d is fixed throughout this paper. We will use standard multi-index notation, writing, for $\alpha \in \mathbb{N}_0^d$ and $x = (\xi_1, \dots, \xi_d) \in \mathbb{K}^d$,

$$\alpha! = \alpha_1! \cdots \alpha_d!, \qquad x^\alpha = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d},$$

as well as

$$|\alpha| = \sum_{j=1}^d \alpha_j$$

for the *length* of a multi-index $\alpha \in \mathbb{N}_0^d$. Let $w \in \mathbb{N}^d$ be a *weight vector* of positive integers. This weight vector induces a notion of w–degree, δ_w, if we set

$$\delta_w\left(x^\alpha\right) = w \cdot \alpha = \sum_{j=1}^d w_j \alpha_j, \qquad \alpha \in \mathbb{N}_0^d,$$

for the monomials and use the straightforward extension

$$\delta_w(p) = \max\left\{\delta_w\left(x^\alpha\right) : p_\alpha \neq 0\right\}, \qquad p = \sum_{\alpha \in \mathbb{N}_0^d} p_\alpha x^\alpha.$$

By $\Pi_{n,w} \subset \Pi$ we denote the vector space of all polynomials of w–degree less than or equal to n, and by $\Pi_{n,w}^0 \subset \Pi_n$ we denote the vector space of all *homogeneous* polynomials of total degree exactly n. Using the *normalized monomials* as a convenient basis, we can write

$$\Pi_{n,w} = \left\{ \sum_{w\cdot\alpha \le n} c_\alpha \frac{x^\alpha}{\alpha!} : c_\alpha \in \mathbb{K} \right\}, \qquad \Pi_{n,w}^0 = \left\{ \sum_{w\cdot\alpha = n} c_\alpha \frac{x^\alpha}{\alpha!} : c_\alpha \in \mathbb{K} \right\}.$$

Moreover, we will write $\Lambda_w(p) \in \Pi^0_{\delta_w(p),w}$ for the *leading term* of p with respect to the grading induced by w, which is the unique homogeneous polynomial of w–degree $\delta_w(p)$ such that $\delta_w\left(p - \Lambda_w(p)\right) < \delta_w(p)$. In the special situation that $w = (1, \dots, 1)$, the above notation reduces to the total degree; in this case, we will omit the reference to w, i.e., $\delta(p)$ denotes the total degree of a polynomial and so on.

Let $\mathcal{P} \subset \Pi$ be any finite or infinite set of polynomials. Then we denote the ideal generated by $\mathcal{P}$ by

$$\langle \mathcal{P} \rangle = \langle p : p \in \mathcal{P} \rangle = \left\{ \sum_{p \in \mathcal{P}} q_p p : q_p \in \Pi,\, p \in \mathcal{P} \right\}.$$

Let $\mathcal{I} \subset \Pi$ be an ideal. Then

$$\Lambda_w(\mathcal{I}) := \{\Lambda_w(p) : p \in \mathcal{I}\}$$

is called the *w–homogeneous ideal* generated by $\mathcal{I}$.

3. A reduction algorithm

For $m \in \mathbb{N}$, an m–vector of polynomials $(p_1, \ldots, p_m) \in \Pi^m$ and $n \in \mathbb{N}_0$, we define the following vector space of homogeneous polynomials:

$$V_n(p_1, \ldots, p_m) = \left\{ \sum_{j=1}^m q_j \Lambda_w(p_j) : q_j \in \Pi^0_{n-\delta_w(p_j),w},\, j = 1, \ldots, m \right\} \subset \Pi^0_{n,w},$$

where we use the standard convention that $q_j = 0$ if $n < \delta_w(p_j)$. Moreover, let any inner product defined on Π be given, i.e., any (strictly) positive definite bilinear (or, if $\mathbb{K} = \mathbb{C}$, sesquilinear) form mapping $\Pi \times \Pi \to \mathbb{K}$. This inner product induces a notion of orthogonality, and therefore we can define the following decomposition into successive orthogonal complements:

$$\begin{aligned} W_n(p_1) &:= V_n(p_1), \\ W_n(p_1, \ldots, p_j) &:= V_n(p_1, \ldots, p_j) \ominus V_n(p_1, \ldots, p_{j-1}), \qquad j = 2, \ldots, m. \end{aligned}$$

Hence, there is the direct sum decomposition

$$V_n(p_1, \ldots, p_m) = \bigoplus_{j=1}^m W_n(p_1, \ldots, p_j).$$

Note that in general this decomposition depends on the order of $p_1, \ldots, p_m$ and that certain of the subspaces $W_n(p_1, \ldots, p_j)$ can be trivial, which will mean that p_j is redundant for the reduction process. The latter happens if and only if for any $q_j \in \Pi^0_{n-\delta_w(p_j),w}$ there exist $q_k \in \Pi^0_{n-\delta_w(p_k),w}$, $k = 1, \ldots, j-1$, such that we have the following syzygy of leading terms:

$$q_j \Lambda_w(p_j) = \sum_{k=1}^{j-1} q_k \Lambda_w(p_k);$$

in other words, p_j is *redundant* iff

$$\Lambda_w(p_j)\ \Pi^0_{n-\delta_w(p_j),w} \subseteq \langle \Lambda_w(p_k) : k = 1, \ldots, j-1 \rangle.$$

The main ingredient for what follows is a "nonlinear version" of Gaussian elimination or Gram–Schmidt orthogonalization which divides off ideal terms to greatest possible extent.

Algorithm 3.1 (Reduction)**.**
Given: $p \in \Pi$ and $(p_1, \ldots, p_m) \in \Pi^m$.

1. Set $f_{\delta_w(p)} = p$.
2. For $n = \delta_w(p), \delta_w(p) - 1, \ldots, 0$.

(a) (Successive orthogonal projection)
For $j = 1, 2, \dots, m$:
Determine $q_j^n \in W_n(p_1, \dots, p_j)$,

$$q_j^n = \sum_{k=1}^{j} q_{jk}^n \Lambda_w(p_k), \qquad q_{jk}^n \in \Pi^0_{n-\delta_w(p_k),w}, \tag{3.1}$$

such that

$$\Lambda_w(f_n) - \sum_{k=1}^{j} q_{jk}^n \Lambda_w(p_k) \perp W_n(p_1, \dots, p_j). \tag{3.2}$$

(b) Set

$$r_n := \Lambda_w(f_n) - \sum_{j=1}^{m} q_j^n = \Lambda_w(f_n) - \sum_{j=1}^{m} q_{jk}^n \Lambda_w(p_k). \tag{3.3}$$

(c) (Cancellation of leading term)
Set

$$f_{n-1} := f_n - r_n - \sum_{j=1}^{m} \sum_{k=1}^{j} q_{jk}^n p_k.$$

Result: Representation

$$p = \sum_{k=1}^{m} \left(\sum_{n=0}^{\delta_w(p)} \sum_{j=k}^{m} q_{jk}^n \right) p_k + \sum_{n=0}^{\delta_w(p)} r_n =: \sum_{k=1}^{m} q_k p_k + r, \tag{3.4}$$

where

$$\delta_w(q_k) + \delta_w(p_k) \le \delta_w(p) \qquad \text{and} \qquad r_n \perp V_n(p_1, \dots, p_m). \tag{3.5}$$

Definition 3.2. A polynomial $f \in \Pi$ is called *reduced* with respect to the vector of polynomials $(p_1, \dots, p_m)$ if each homogeneous term of f is reduced to zero; in other words, if we write

$$f = \sum_{j=0}^{\delta_w(f)} f_j, \qquad f_j \in \Pi^0_{j,w},\ j = 0, \dots, \delta_w(f),$$

then f is reduced if and only if

$$f_j \perp V_j(p_1, \dots, p_m), \qquad j = 0, \dots, \delta_w(f).$$

Remark 3.3. Since $V_n(p_1, \dots, p_m) = V_n\left(p_{\sigma(1)}, \dots, p_{\sigma(m)}\right)$ for any permutation σ of the numbers $\{1, \dots, m\}$, the question whether a polynomial is reduced or not is *independent* of the order of polynomials in the vector. If the remainder r is not zero, however, this remainder will in general depend on the order of the polynomials $p_1, \dots, p_m$.

Remark 3.4. It is also worthwhile to note here that the notion of a reduced polynomial depends on the inner product used in the direct sum decomposition of the reduction algorithm, and that different inner products will usually give different classes of reduced polynomials. We will make the inner product more specific when we consider the connection to least interpolation in a later section.

Hence, equation (3.4) can be interpreted as decomposing a polynomial into a part which lies in $\langle p_1, \dots, p_m \rangle$, the ideal generated by $(p_1, \dots, p_m)$, and a *reduced* "remainder" $r \in \Pi$. We will denote the reduced part of p with respect to the vector $(\mathcal{P}) = (p_1, \dots, p_m)$ by

$$p \rightarrow_{(p_1,\dots,p_m)} \qquad \text{or by} \qquad p \rightarrow_{(\mathcal{P})},$$

where the ordering of $(\mathcal{P})$ is arbitrary but fixed.

However, we still have to verify (3.5). This is an immediate consequence of the following observation on the intermediate polynomials of the reduction step.

Lemma 3.5. *The polynomials r_n, $n = 0, \dots, \delta_w(p)$, defined in* (3.3)*, satisfy equation* (3.5)*.*

Proof. We verify by induction on $j = 1, \dots, m$ that the polynomials q_j^n, defined in (3.1), have the property that

$$\Lambda_w (f_n) - \sum_{k=1}^{j} q_j^n \perp V_n (p_1, \dots, p_j);$$

Equation (3.5) is then the case $j = m$. Indeed, $j = 1$ follows readily from the definition of q_1^n. For $j > 1$ we use the induction hypothesis and the fact that

$$q_j^n \in W_n (p_1, \dots, p_j) \perp W_n (p_1, \dots, p_{j-1})$$

to conclude that

$$\Lambda_w (f_n) - \sum_{k=1}^{j} q_j^n \perp V_n (p_1, \dots, p_{j-1}).$$

Together with (3.2) and the identity

$$V_n (p_1, \dots, p_j) = V_n (p_1, \dots, p_{j-1}) \oplus W_n (p_1, \dots, p_j),$$

this advances the induction hypothesis. $\square$

4. H–bases as homogeneous Gröbner bases

We begin this section by recalling the notion of an H–basis, introduced by Macaulay [14].

Definition 4.1. A (finite) set $\mathcal{H} \subset \Pi$ is called an *H–basis* (or *Macaulay basis*) for the ideal $\mathcal{I}$ if for any $f \in \mathcal{I}$ there exist polynomials q_p, $p \in \mathcal{H}$, such that

$$f = \sum_{p \in \mathcal{H}}^{m} q_p p \qquad \text{and} \qquad \delta (q_p p) \le \delta(f), \quad p \in \mathcal{H}. \tag{4.1}$$

On the other hand, we have the following, well–known alternative description of H–bases.

Proposition 4.2. *A finite set $\mathcal{H} \subset \Pi$ is an H–basis for an ideal $\mathcal{I}$ if and only if*

$$\Lambda(\mathcal{I}) := \langle \Lambda(p) : p \in \mathcal{I} \rangle = \langle \Lambda(p) : p \in \mathcal{H} \rangle =: \langle \Lambda(\mathcal{H}) \rangle. \tag{4.2}$$

Clearly, this notion can easily be extended to arbitrary weight vectors $w \in \mathbb{N}^d$ and the grading they induce. Thus, we call $\mathcal{H} \subset \Pi$ an H–basis (with respect to w) for the ideal $\mathcal{I}$ if for any $f \in \mathcal{I}$ there exist polynomials $q_p \in \Pi$, $p \in \mathcal{H}$, such that

$$f = \sum_{p \in \mathcal{H}} q_p \, p, \qquad \delta_w \left(q_p p \right) \leq \delta_w(f), \quad p \in \mathcal{H},$$

or, equivalently, if $\Lambda_w(\mathcal{I}) = \langle \Lambda_w(\mathcal{H}) \rangle$.

We next show that H–bases are closely related to the idea of reduction introduced in Algorithm 3.1.

Theorem 4.3. *Let $\mathcal{H}$ be an H–basis for $\langle \mathcal{H} \rangle$. Suppose that $f \in \Pi$ can be written as*

$$f = \sum_{p \in \mathcal{H}} q_p p + r, \qquad q_p \in \Pi, \, p \in \mathcal{H},$$

for some reduced *polynomial $r \in \Pi$. Then*

$$r = p \rightarrow_{(\mathcal{H})} .$$

This immediately implies the following conclusion.

Corollary 4.4. *If $\mathcal{H}$ is an H–basis, then the reduced polynomial generated by the reduction algorithm is independent of the order of the elements in $\mathcal{H}$.*

Therefore, whenever $\mathcal{H}$ is an H–basis, we can simply speak of *reduction modulo* (the set) $\mathcal{H}$, which will be written as $\rightarrow_{\mathcal{H}}$.

Proof of Theorem 4.3. Suppose that $(\mathcal{H}) = (p_1, \ldots, p_m)$, and let $\tilde{q}_j$, $j = 1, \ldots, m$, and $\tilde{r} = p \rightarrow_{(p_1, \ldots, p_m)}$ be the coefficients and the remainder obtained by the reduction algorithm. Then,

$$f = \sum_{j=1}^{m} q_j p_j + r = \sum_{j=1}^{m} \tilde{q}_j p_j + \tilde{r},$$

or, in other words,

$$r - \tilde{r} = \sum_{j=1}^{m} \left(\tilde{q}_j - q_j \right) p_j \in \langle p_1, \ldots, p_m \rangle .$$

Set $g := r - \tilde{r}$ and assume that $g \neq 0$. Since r and $\tilde{r}$ are reduced, we have that

$$\Lambda_w(r) \perp V_{\delta_w(r)} \left(p_1, \ldots, p_m \right) \qquad \text{and} \qquad \Lambda_w(\tilde{r}) \perp V_{\delta_w(\tilde{r})} \left(p_1, \ldots, p_m \right) .$$

If $\delta_w(r) \neq \delta_w(\tilde{r})$, then either $\Lambda_w(g) = \Lambda_w(r)$ or $\Lambda_w(g) = \Lambda_w(\tilde{r})$, and therefore

$$\Lambda_w(g) \perp V_{\delta_w(g)} \left(p_1, \ldots, p_m \right) . \tag{4.3}$$

The same also follows if $\delta_w(r) = \delta_w(\tilde{r})$ and $\Lambda_w(r) \neq \Lambda_w(\tilde{r})$. In the remaining case, we continue with the polynomials $r - \Lambda_w(r)$ and $\tilde{r} - \Lambda_w(\tilde{r})$, which are still reduced but have strictly smaller degree. Hence, after a finite number of steps we must again arrive at (4.3), since we assumed that $g \neq 0$. However, since $g \in \langle p_1, \ldots, p_m \rangle$, we also obtain that

$$\Lambda_w(g) \in \langle \Lambda(\mathcal{H}) \rangle \cap \Pi^0_{\delta_w(g), w} = V_{\delta_w(g)} \left(p_1, \ldots, p_m \right) ,$$

which is a contradiction. Hence, $g = 0$ and therefore $r = \tilde{r} = f \rightarrow_{(\mathcal{H})}$. $\square$

Let us now investigate the relationship between H–bases and Gröbner bases in more detail. For that purpose let us recall that a finite set $\mathcal{G} \subset \Pi$ is called a *Gröbner basis* (with respect to the term order $\prec$) if

$$\Lambda_{\prec}\left(\langle \mathcal{G} \rangle\right) = \langle \Lambda_{\prec}\left(\mathcal{G}\right) \rangle , \tag{4.4}$$

where $\Lambda_{\prec}$ denotes the leading term according to the term order. Gröbner bases have been introduced by Buchberger in his thesis [7, 8] (supervised by Gröbner) and became a useful tool (not only) in Computer Algebra systems. An introduction to Gröbner bases can be found in [11]. It has, for example, been remarked in [17] that any Gröbner basis with respect to a term order which is subordinate with the partial ordering by the degree δ_w (i.e., $\delta_w(p) < \delta_w(q)$ implies $p \prec q$) is also an H–basis with respect to w. The classical example of such a term order for $w = (1, \dots, 1)$ is the graded lexicographical one.

In view of (4.2) and (4.4), H–bases are the homogeneous counterpart of Gröbner bases, without using term orders any more. Besides the striking simplicity of this relationship, it allows us to find a way to construct *term order free* H–bases by straightforwardly modifying Buchberger's algorithm. This is based on the following characterization of H–bases via reduction.

Theorem 4.5. *A finite set $\mathcal{H} \subset \Pi$ is an H–basis if and only if*

$$\delta_w \left(\sum_{p\in\mathcal{H}} q_p p \right) < \max_{p\in\mathcal{H}} \delta_w\left(q_p p\right) \quad \Rightarrow \quad \sum_{p\in\mathcal{H}} q_p p \to_{(\mathcal{H})} 0. \tag{4.5}$$

Proof. Let

$$g = \sum_{p\in\mathcal{H}} q_p p.$$

Since $g \in \langle \mathcal{H} \rangle$ and since the remainder of reduction is unique for H–bases by Theorem 4.3, the direction "$\Rightarrow$" is obvious.

To prove "$\Leftarrow$", we follow the argumentation from [18] and pick any $f \in \langle \mathcal{H} \rangle$ which can be written as

$$f = \sum_{p\in\mathcal{H}} q_p p, \qquad q_p \in \Pi,\ p \in \mathcal{H}.$$

If

$$\delta_w(f) = \max_{p\in\mathcal{H}} \delta_w\left(q_p p\right) = \max_{p\in\mathcal{H}} \left(\delta_w(q_p) + \delta_w(p)\right),$$

then $\Lambda_w(f) \in \langle \Lambda_w(\mathcal{H}) \rangle$, which is what we want. Suppose now that there is a cancellation of leading terms, which means that $\delta_w(f) < \max_{p\in\mathcal{H}}\left(\delta_w(q_p) + \delta_w(p)\right)$. Consequently, there is a finite subset $\mathcal{J} \subset \mathcal{H}$ such that

$$\sum_{p\in\mathcal{J}} \Lambda_w\left(q_p\right) \Lambda_w(p) = 0.$$

By the assumption (4.5) we have that

$$\sum_{p\in\mathcal{J}} \Lambda_w\left(q_p\right) p \to_{(\mathcal{H})} 0, \qquad \text{hence} \qquad \sum_{p\in\mathcal{J}} \Lambda_w\left(q_p\right) p = \sum_{p\in\mathcal{H}} g_p p,$$

for appropriate polynomials $g_p \in \Pi$, $p \in \mathcal{J}$, such that

$$\max_{p\in\mathcal{H}} \left(\delta_w(g_p) + \delta_w(p)\right) < \max_{p\in\mathcal{J}} \left(\delta_w(q_p) + \delta_w(p)\right).$$

Setting, in addition, $g_p = \Lambda_w(q_p)$, $p \in \mathcal{H} \setminus \mathcal{J}$, we obtain that

$$f = \sum_{p \in \mathcal{H}} \left(q_p - \Lambda_w(q_p) + g_p \right) p =: \sum_{p \in \mathcal{H}} \tilde{q}_p p,$$

where

$$\delta_w(p) \leq \max_{p \in \mathcal{H}} \delta_w \left(\tilde{q}_p \right) + \delta_w(p) < \max_{p \in \mathcal{H}} \delta_w \left(q_p \right) + \delta_w(p).$$

Since the maximal total degree of the representation is strictly reduced in any step, we arrive, after repeating this argument as long as necessary, at the case that $\delta_w(f) = \max_{p \in \mathcal{H}} \delta_w \left(q_p p \right)$ and therefore $\Lambda_w(f) \in \langle \Lambda_w(\mathcal{H}) \rangle$, which means that $\mathcal{H}$ is an H–basis. $\square$

The first statement in (4.5) means that the polynomials q_p, $p \in \mathcal{H}$, form a *syzygy of leading terms* of $\mathcal{H}$. Let us briefly recall the notion of a syzygy: given a finite set $\mathcal{P} \subset \Pi$, a $\mathcal{P}$–tuple $g = (g_p : p \in \mathcal{P}) \subset \Pi^{\mathcal{P}}$ is called a *syzygy* with respect to $\mathcal{P}$ if

$$\sum_{p \in \mathcal{P}} g_p p = 0.$$

Also, we denote by $S(\mathcal{P})$ the module of all syzygies with respect to $\mathcal{P}$. It is well–known (cf. [12]) that $S(\mathcal{P})$ is finite. This means that there is a finite generating set $G \subset S(\mathcal{P})$, most conveniently written as a matrix

$$G = \left[g_{j,p} : j = 1, \dots, M,\, p \in \mathcal{P} \right],$$

such that any syzygy $g \in S(\mathcal{P})$ can be written as

$$g = \left(\sum_{j=1}^{M} q_j g_{j,p} : p \in \mathcal{P} \right), \qquad q_j \in \Pi,\, j = 1, \dots, M.$$

Moreover, such a basis can be constructed effectively (see [9, Method 6.17]) by using a (reduced) Gröbner basis for the ideal $\langle \mathcal{P} \rangle$.

Next, let us record the fact that, instead of considering the reduction of all the syzygies in $S\left(\Lambda(\mathcal{H})\right)$, it suffices to check a basis only. The proof of this result is almost obvious.

Corollary 4.6. *Let $\mathcal{H} \subset \Pi$ be a finite set of polynomials and let G be a basis of $S\left(\Lambda(\mathcal{H})\right)$. Then $\mathcal{H}$ is an H–basis if and only if*

$$\sum_{p \in \mathcal{P}} g_p p \to_{(\mathcal{H})} 0, \qquad g \in G.$$

This allows us to reformulate Buchberger's algorithm for the construction of H–bases without term orders.

Algorithm 4.7. Given: finite set $\mathcal{H} \subset \Pi$.

1. Construct a basis G for $S\left(\Lambda(\mathcal{H})\right)$.
2. For $g \in G$
   (a) Compute
   $$h = \sum_{p \in \mathcal{H}} g_p p \to_{(\mathcal{H})} .$$
   (b) If $h \neq 0$, set $\mathcal{H} := \mathcal{H} \cup \{h\}$ and continue at 1.

Result: H–basis $\mathcal{H}$.

The proof that this algorithm terminates (a consequence of the *ascending chain condition*) and produces an H–basis (which follows from Corollary 4.6) is exactly the same as for Gröbner bases with respect to a term order.

Remark 4.8. Clearly, Algorithm 4.7 is far from optimal efficiency. The "standard method" for the computation of a basis of the module of syzygies from [9] requires the computation of a reduced Gröbner basis for $\langle \Lambda_w(\mathcal{H}) \rangle$. Therefore a fast and simple method for this task would be crucial for performance. At least some simple possibilities for the improvement of the algorithm should be given here.

1. There is a lot of redundancy in computing the basis G in step 1 "from scratch" each time. As already remarked, the method sketched in [9] uses a reduced Gröbner basis for $\langle \Lambda_w(\mathcal{H}) \rangle$. Since in any step of the iteration process the set of polynomials $\mathcal{H}$ is almost the same, except that one more polynomial is added, the original Gröbner basis can be re–used to a great extent.
2. As Möller pointed out in [18], it is also possible to re–use the earlier bases of syzygies in variants of Buchberger's algorithm: a basis for $S\left(\Lambda(\mathcal{H} \cup \{h\})\right)$ is given by
$$\{(g,0) : g \in S\left(\Lambda(\mathcal{H})\right)\} \cup G',$$
where G' is some basis for the syzygies involving the additional element h.
3. In view of the simplicity of the reduction algorithm from the previous section, it may be beneficial to have the elements of $\mathcal{H}$ mutually reduced, i.e., $p \to_{\mathcal{H}\setminus\{p\}} 0$, $p \in \mathcal{H}$. First, this removes redundancies and therefore keeps the number of elements in $\mathcal{H}$ as small as possible; and, second, it makes the orthogonal projections simpler and more efficient. Adding one more h (which is already reduced with respect to $\mathcal{H}$) then only requires us to set $p := p \to_{\{h\}}$, $p \in \mathcal{H}$.

Nevertheless, the effort to compute H–bases even in the above way may pay off since, in contrast to Gröbner bases, H–bases are able to preserve symmetries. A simple example is the ideal generated by the two ellipses

$$p_1(x,y) = ax^2 + y^2 - 1, \qquad p_2(x,y) = x^2 + ay^2 - 1, \qquad a > 1, \tag{4.6}$$

which are easily seen to be an H–basis with respect to the total degree grading: since the module of syzygies of leading terms is generated by $\left(x^2+ay^2, -\left(ax^2+y^2\right)\right)$; any syzygy of leading terms is of the form

$$\begin{aligned} & q\left(x^2+ay^2\right) p_1(x,y) - q\left(ax^2+y^2\right) p_2(x,y) \\ = \;& q\left(ax^2+y^2\right) - q\left(x^2+ay^2\right) = q\left(p_1(x,y) - p_2(x,y)\right), \qquad q \in \Pi, \end{aligned}$$

and thus reduces to zero. A *reduced* H–basis with respect to the inner product in (5.4) is given, for example, by

$$\begin{aligned} f_1(x,y) &= \frac{p_1+p_2}{2} = \frac{a+1}{2}x^2 + \frac{a+1}{2}y^2 - 1 \\ f_2(x,y) &= \frac{p_1-p_2}{2} = \frac{a-1}{2}x^2 - \frac{a-1}{2}y^2. \end{aligned}$$

This basis still captures the symmetry of the problem to a great extent; any Gröbner basis, on the other hand, will destroy the symmetry of these polynomials by enforcing a term order.

It is a well–known fact (cf. [11]) that the *reduced Gröbner basis* for an ideal is unique. This is no longer true for H–bases, even if we require all elements of the basis to be mutually reduced. The details of this observation and a characterization of all "normalizations" which map reduced H–bases into reduced H–bases of the same ideal can be found in [20].

It has also been pointed out in [20] how H–bases can be used to replace Gröbner bases for the computation of common zeros of polynomials, i.e., solutions of systems of polynomial equations, for example by Stetter's method (cf. [24]), where the problem of solving nonlinear equations is reduced to an eigenvalue problem (see also [21]). Here, H–bases can be used to overcome *representation singularities*, which are known from [25] to cause severe numerical problems, as is shown by the following modification of the above example. For $\phi \in [0, 2\pi)$, let $R_\phi \in \mathbb{R}^{2\times 2}$ denote the rotation matrix with respect to the angle ϕ and consider the polynomials

$$p_{\phi,j} := p_j \left(R_\phi^{-1} \cdot \right), \qquad j = 1, 2, \tag{4.7}$$

which again form an H–basis for the ideal $\mathcal{I}_\phi$ they generate. Clearly, this H–basis depends continuously on ϕ. The Gröbner bases for $\mathcal{I}_\phi$, however, depend only on whether ϕ is a multiple of $\pi/2$ or not; hence they show a *discontinuous* behavior at the values $\phi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. "Close" to these cases, the quotient between the moduli of the largest and the smallest nonzero coefficient of at least one element of the Gröbner basis grows to infinity. As shown in [25], this effect makes the nonlinear system almost unsolvable when done in finite precision arithmetic.

5. Interpolation

The motivation for this paper was a close relationship between Gröbner bases for zero dimensional ideals (or, ideals of finite codimension) and associated interpolation spaces as shown in [10, 15]. In the context of *minimal degree interpolation spaces* with minimal monomials these results were partially rediscovered and partially extended in [22, 23]. More precisely, if $\mathcal{G}$ denotes the unique reduced Gröbner basis of a zero dimensional ideal with respect to a certain term order, then the associated minimal degree interpolation space (defined by means of interpolation) is nothing but the standard representation for $\Pi / \langle \mathcal{G} \rangle$ induced by the Gröbner basis, and the processes of interpolation and reduction are equivalent.

Let us recall some terminology on polynomial interpolation. A finite set of linear functionals $\Theta \subset \Pi'$ is said to be an *ideal interpolation scheme* (cf. [1]) if

$$\ker \Theta = \{ p \in \Pi : \theta(p) = 0, \ \theta \in \Theta \} \subset \Pi$$

forms an ideal in Π. In this case the components of the primary decomposition of $\ker \Theta$ correspond to interpolation of a D–invariant set of partial differential operators (see [4, 16]) which may be viewed as a natural extension of Hermite interpolation. A finite dimensional subspace $\mathcal{P} \subset \Pi$ is called an *interpolation space* with respect to Θ if for any $q \in \Pi$ there is a *unique* $p \in \mathcal{P}$ such that $\Theta(p) = \Theta(q)$, i.e., $\theta(p) = \theta(q)$, $\theta \in \Theta$.

A subspace $\mathcal{P} \subset \Pi$ is called a *minimal degree interpolation space* if it is an interpolation space such that the *interpolation operator* $L_{\Theta,\mathcal{P}} : \Pi \to \mathcal{P}$ is *w–degree reducing*, i.e.,

$$\delta_w \left(L_{\Theta,\mathcal{P}} q \right) \le \delta_w(q), \qquad q \in \Pi.$$

Minimal degree interpolation is treated, for example, in [6, 22]. Given any Θ, there is usually a multitude of minimal degree interpolation spaces, so that choosing a "proper" one, or at least a unique one, requires some extra efforts. However, any H–basis defines a minimal degree interpolation space, as is easy to see now.

Theorem 5.1. *Let $\Theta \subset \Pi'$ be an ideal interpolation scheme and let $\mathcal{H}$ be an H–basis for* $\ker \Theta$*. Then $\mathcal{P}_{\mathcal{H}} = \Pi \to_{\mathcal{H}}$ is a minimal degree interpolation space with the associated interpolation operator*

$$L_{\Theta,\mathcal{P}_{\mathcal{H}}} q = q \to_{\mathcal{H}}, \qquad q \in \Pi. \tag{5.1}$$

Proof. Since, for any $q \in \Pi$, we have that $q - (q \to_{\mathcal{H}}) \in \langle \mathcal{H} \rangle = \ker \Theta$, and therefore

$$\Theta (q \to_{\mathcal{H}}) = \Theta (q - q + q \to_{\mathcal{H}}) = \Theta (q) - \Theta (q - q \to_{\mathcal{H}}) = \Theta(q),$$

it follows that the space $\mathcal{P}_{\mathcal{H}}$ is an interpolation space; and since the definition of the reduction process implies that $\delta_w (q \to_{\mathcal{H}}) \le \delta_w(q)$, it is degree reducing and therefore of minimal degree. □

We first remark that Theorem 5.1 allows for *implicit interpolation* where the interpolation conditions Θ are only given by its dual, the ideal $\ker \Theta$. It is interesting that the oldest approach to multivariate polynomial interpolation, pursued by Kronecker [13] in 1865, starts with exactly this assumption.

Moreover,[1] in the case $w = (1, \dots, 1)$ the "simplest" inner product, which is, for

$$p = \sum_{|\alpha| \le \delta_w(p)} p_\alpha \, x^\alpha, \qquad q = \sum_{|\alpha| \le \delta_w(q)} q_\alpha \, x^\alpha$$

defined as

$$(p, q)_* = \sum_{\alpha \in \mathbb{N}_0^d} p_\alpha \, q_\alpha, \tag{5.2}$$

yields Macaulay's *inverse systems* (cf. [12, p. 174]) as the interpolation space. Conversely, the reduction approach from the previous section therefore gives an algorithm to compute this inverse system by using reduction instead of the systems of equations give in [12], which also assume the knowledge of an H–basis.

The *least interpolation* space, introduced by de Boor and Ron in [3] (see also [5, 6] for further investigations and algorithmic aspects), is given as

$$\mathcal{P}_l = \bigcap_{q \in \ker \Theta} \Lambda(q)(D). \tag{5.3}$$

The name "least interpolation" stems from representing linear functionals in Π' as formal power series and defining the *least term* of a power series (the "counterpiece" of the leading term) as the nonzero homogeneous term of least total degree in the power series. Then $\mathcal{P}_l$ is the vector space spanned by the least terms of the power series representation of $\theta \in \Theta$. Actually, this is how de Boor and Ron introduced least interpolation, but for our purposes here, the equivalent description by (5.3) is more convenient. Again, the extension of this notion to arbitrary weight vectors is straightforward.

To relate reduction with respect to a particular H–basis with least interpolation, we have to specify the inner product. For the sake of simplicity we will restrict ourselves to $\mathbb{K} = \mathbb{R}$; the case $\mathbb{K} = \mathbb{C}$ can be easily obtained by inserting complex conjugation in a straightforward way.

The (canonical) inner product we have in mind here and which has also been used in [4, 6] is, for $p, q \in \Pi$, to choose

$$(p, q) = (p(D)q)\,(0), \tag{5.4}$$

[1] I am grateful to one of the referees for pointing out this connection to me.

where for any polynomial $p \in \Pi$ we let

$$p(D) = p\left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_d}\right)$$

denote the associated constant coefficient partial differential operator which is obtained by replacing the powers x^α in p by $\partial^{|\alpha|}/\partial x^\alpha$. If we write

$$p = \sum_{|\alpha| \le \delta_w(p)} p_\alpha \frac{x^\alpha}{\alpha!}, \qquad q = \sum_{|\alpha| \le \delta_w(q)} q_\alpha \frac{x^\alpha}{\alpha!},$$

then

$$(p, q) = \sum_{|\alpha| \in \mathbb{N}_0^d} \frac{p_\alpha q_\alpha}{\alpha!}.$$

Remark 5.2. Note that there is a subtle but important difference between the two inner products (5.2) and (5.4), because the latter one admits the useful identity $(pq, f) = (p, q(D)f)$, which immediately implies that, whenever a polynomial p is reduced with respect to a H–basis $\mathcal{H}$, then all derivatives of p are reduced as well, and hence the space $\Pi \to_{\mathcal{H}}$ is closed under differentiation. This property, which also follows directly from (5.3), has been observed by de Boor and Ron, who first observed and proved the *D–invariance* of the least interpolation space.

Nevertheless, D–invariance relies on the choice of the inner product and is *not* satisfied by Macaulay's inverse systems, as the following simple example shows: the inverse system with respect to the bivariate (reduced) H–basis

$$\mathcal{H} = \left\{ x + y, x^3 - x^2 y + x y^2 - y^3 \right\}$$

is easily seen to be

$$\operatorname{span} \left\{ 1, x - y, x^2 - xy + y^2 \right\};$$

however, the polynomial

$$\frac{\partial}{\partial x}\left(x^2 - xy + y^2\right) = 2x - y$$

does not belong to that linear space.

The following result finally connects the notion of reduced polynomials with respect to the inner product $(\cdot,\cdot)$ to least interpolation.

Proposition 5.3. *Let $\mathcal{H} \subset \Pi$ be an H–basis for $\langle \mathcal{H} \rangle$. Then a polynomial $q \in \Pi$ is reduced with respect to the inner product $(\cdot,\cdot)$ from* (5.4) *if and only if*

$$q \in \bigcap_{p \in \mathcal{H}} \ker \Lambda_w(p)(D) = \bigcap_{p \in \langle \mathcal{H} \rangle} \ker \Lambda_w(p)(D). \tag{5.5}$$

Proof. Let us first assume that q is w–homogeneous, i.e.,

$$q = \sum_{w \cdot \alpha = \delta_w(q)} q_\alpha \frac{x^\alpha}{\alpha!}.$$

For any $p \in \Pi$, if $\delta_w(p) > \delta_w(q)$, then $\Lambda_w(p)(D)q = 0$, while in the case $\delta_w(p) \leq \delta_w(q)$ we have that

$$\begin{aligned}\Lambda_w(p)(D)q &= \sum_{w\cdot\alpha=\delta_w(p)} \frac{p_\alpha}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha} \sum_{w\cdot\beta=\delta_w(q)} \frac{q_\beta}{\beta!}x^\beta \\ &= \sum_{w\cdot\alpha=\delta_w(p)} \sum_{w\cdot\beta=\delta_w(q)} \frac{p_\alpha q_\beta}{\alpha!\beta!}\frac{\beta!}{(\beta-\alpha)!}x^{\beta-\alpha} \\ &= \sum_{w\cdot\alpha=\delta_w(p)} \sum_{w\cdot\beta=\delta_w(q)} \frac{p_\alpha q_\beta}{\alpha!(\beta-\alpha)!}x^{\beta-\alpha} \\ &= \sum_{w\cdot\gamma=\delta_w(q)-\delta_w(p)} \sum_{\beta-\alpha=\gamma} \frac{p_\alpha q_\beta}{\alpha!\gamma!}x^\gamma .\end{aligned}$$

Now, we have $\Lambda_w(p)(D)q = 0$ if and only if, for any $\gamma \in \mathbb{N}_0^d$ such that $w \cdot \gamma = \delta_w(q) - \delta_w(p)$,

$$\begin{aligned}0 &= \sum_{\beta-\alpha=\gamma} \frac{p_\alpha q_\beta}{\alpha!\gamma!} = \sum_{w\cdot\beta=\delta_w(p)} \frac{p_{\beta-\gamma} q_\beta}{(\beta-\gamma)!\gamma!} \\ &= \sum_{w\cdot\beta=\delta_w(p)} \frac{1}{\beta!}\left(\sum_{\alpha+\mu=\beta} \frac{\beta!}{\alpha!\mu!}p_\alpha \delta_{\mu,\gamma}\right) q_\beta = \left(x^\gamma \Lambda_w(p), q\right).\end{aligned}$$

Since the set

$$\{x^\gamma p : w\cdot\gamma = \delta_w(q) - \delta_w(p),\, p \in \mathcal{H}\} \subset \Pi^0_{\delta_w(q),w}$$

generates $V_{\delta_w(q)}(\mathcal{H})$, we conclude from the linearity of the differential operators that

$$q \in \bigcap_{p\in\mathcal{H}}^{m} \ker \Lambda_w(p)(D) \qquad \Leftrightarrow \qquad q \in V_{\delta_w(q)}(\mathcal{H}),$$

i.e., if and only if q is reduced. Moreover, if q is not a homogeneous polynomial, then we apply the above argumentation to all the homogeneous terms of q separately.

Finally, we show that

$$\bigcap_{p\in\mathcal{H}} \ker \Lambda_w(p)(D) = \bigcap_{p\in\langle\mathcal{H}\rangle} \ker \Lambda_w(p)(D).$$

Indeed, the inclusion $\subset$ is trivial, while for the converse we observe that for $p \in \mathcal{H}$, $f \in \Pi$ and $q \in \ker \Lambda_w(p)(D)$ we have

$$\Lambda_w(fp)(D)q = \left(\Lambda_w(f)(D)\Lambda_w(p)(D)\right)q = \Lambda_w(f)(D)\left(\Lambda_w(p)(D)q\right) = 0.$$

□

Corollary 5.4. *Let Θ be an ideal interpolation scheme and let $\mathcal{H}$ be an H–basis for $\ker\Theta$ with respect to $(\cdot,\cdot)$. Then $\mathcal{P}_l = \Pi \to_{\mathcal{H}}$ and*

$$L_{\Theta,\mathcal{P}_l} q = q \to_{\mathcal{H}}, \qquad q \in \Pi. \tag{5.6}$$

Proof. Due to Proposition 5.3, Theorem 5.1 and (5.3), all we have to do is to verify (5.6). For that purpose we first note that both $L_{\Theta,\mathcal{P}_l}$ and $\rightarrow_{\mathcal{H}}$ are projections $\Pi \rightarrow \mathcal{P}_l$. Moreover, pick any $q \in \Pi$ which can be written as

$$q = \sum_{p \in \mathcal{H}} q_p p + q \rightarrow_{\mathcal{H}},$$

with the *unique* remainder $q \rightarrow_{\mathcal{H}}$, because $\mathcal{H}$ is an H–basis. Then,

$$L_{\Theta,\mathcal{P}_l} q = L_{\Theta,\mathcal{P}_l} \left(\sum_{p \in \mathcal{H}} q_p p \right) + L_{\Theta,\mathcal{P}_l} \left(q \rightarrow_{\mathcal{H}} \right) = L_{\Theta,\mathcal{P}_l} \left(q \rightarrow_{\mathcal{H}} \right) = q \rightarrow_{\mathcal{H}},$$

which proves (5.6). $\square$

The above identification allows us to switch between interpolation and reduction. For example, if the set Θ is given, either as a finite set of points and associated differential operators or as power series representations, then it is possible to compute the H–basis (see [2]) or the Gröbner basis (see [22, 23]) by means of Gauss elimination (or, equivalently, a Gram–Schmidt orthogonalization process), i.e., by using only methods from *linear* algebra. Conversely, if the points are given implicitly, for example as common zeros of some orthogonal or quasi–orthogonal polynomials, then, in order to interpolate a polynomial, it is not necessary to find these points, using reduction instead after having computed an H–basis for the ideal.

Let us finally give some examples where the space $\Pi \rightarrow_{\mathcal{H}}$ inherits some appealing geometric properties from least interpolation which are unavailable by using Gröbner bases with respect to a term order. For that purpose, we restrict ourselves to the case of *Lagrange interpolation*, i.e., all the functionals are point evaluations.

Example 5.5. Let $\Theta = \left(\delta_{x_j} : j = 0, \dots, N \right)$

1. Suppose that all the points x_j lie on a straight line, i.e., $x_j - x_k = \lambda_{jk} a$, $a \in \mathbb{K}^d$, $\lambda_{jk} \in \mathbb{K}$, $j, k = 0, \dots, n$. Let $\mathcal{G}_\prec$ denote the Gröbner basis with respect to the term order $\prec$; then $\Pi \rightarrow_{\mathcal{G}_\prec}$ is spanned by $\{1, \xi_k, \xi_k^2, \dots, \xi_k^N\}$, where k is the index of the $\prec$–minimal nonzero component of a. On the other hand, avoiding the artificial ordering, $\Pi \prec \mathcal{H}$ is spanned by
$$\left\{ (a, x)^k : k = 0, \dots, N \right\}.$$
Setting, for simplicity, $d = 2$, it is easy to see that the H–basis for $\mathcal{I}_\Theta$ is
$$\mathcal{H} = \left\{ \left(a^\perp, x\right), (a, x)^{N+1} - q \right\}, \qquad q \in \mathcal{P}_\Theta, \left(a^\perp, a\right) = 0.$$
It is worth mentioning that, in contrast to the Gröbner basis, the H-basis depends *continuity* of affine transformations of the line on which points lie, avoiding the so-called *representation singularities* which are known for Gröbner bases, cf. [25].
2. Suppose that $N \geq (d+1)(d+2)/2$ is sufficiently large and that the points x_j lie on some sphere in $\mathbb{K}^d$. Then, the polynomial $\xi_1^2 + \cdots + \xi_d^2$ belongs to $\ker \Theta$ and, as it is often desired when working on the sphere, all the reduced polynomials are harmonic, i.e.,
$$\Pi \rightarrow_{\mathcal{H}} \subset \ker \Delta, \qquad \Delta = \sum_{k=1}^d \frac{\partial^2}{\xi_k^2}.$$

This is not the case if we choose, for example, reduction with respect to the graded lexicographical Gröbner basis.

3. Returning to the previous example from (4.6), it is easy to see that in this case the interpolation space is spanned by $\{1, x, y, xy\}$; however, since the common zeros of the two ellipses form a square whose edges are parallel to the coordinate axes, the same interpolation space would also by obtained by reducing modulo *any* Gröbner basis.
4. In the case that the interpolation points are the common zero of the polynomials $p_{\phi,j}$, $j = 1, 2$, a straightforward computation shows that the least interpolation space is spanned by

$$\left\{1, x, y, \frac{\sin 2\phi}{2}\left(x^2 - y^2\right) + (\cos 2\phi)\, xy\right\}$$

and therefore again depends continuously on the rotation angle. The interpolation space generated by reduction modulo a lexicographical Gröbner basis with, say, $x \prec y$, however, is "almost always" (except when $\phi \in \left\{\frac{k\pi}{2} : k = 0, 1, 2, 3\right\}$) spanned by $\left\{1, x, x^2, x^3\right\}$ and has discontinuities which make the interpolation problem poorly conditioned close to the singularities.

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