GRÖBNER METHODS FOR REPRESENTATIONS OF COMBINATORIAL CATEGORIES

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Abstract. Given a category $\mathcal{C}$ of a combinatorial nature, we study the following fundamental question: how does the combinatorial behavior of $\mathcal{C}$ affect the algebraic behavior of representations of $\mathcal{C}$? We prove two general results. The first gives a combinatorial criterion for representations of $\mathcal{C}$ to admit a theory of Gröbner bases. From this, we obtain a criterion for noetherianity of representations. The second gives a combinatorial criterion for a general "rationality" result for Hilbert series of representations of $\mathcal{C}$. This criterion connects to the theory of formal languages, and makes essential use of results on the generating functions of languages, such as the transfer-matrix method and the Chomsky–Schützenberger theorem.

Our work is motivated by recent work in the literature on representations of various specific categories. Our general criteria recover many of the results on these categories that had been proved by ad hoc means, and often yield cleaner proofs and stronger statements. For example: we give a new, more robust, proof that FI-modules (originally introduced by Church–Ellenberg–Farb), and a family of natural generalizations, are noetherian; we give an easy proof of a generalization of the Lannes–Schwartz artinian conjecture from the study of generic representation theory of finite fields; we significantly improve the theory of Δ-modules, introduced by Snowden in connection to syzygies of Segre embeddings; and we establish fundamental properties of twisted commutative algebras in positive characteristic.

Contents

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1. Introduction

Informally, a **combinatorial category** is a category whose objects are finite sets, possibly with extra structure, and whose morphisms are functions, possibly with extra structure. A **representation** of such a category over a ring $\mathbf{k}$ is a functor to the category of $\mathbf{k}$-modules. Typically, a representation can be thought of as a sequence of representations of certain finite groups together with transition maps satisfying certain conditions. Some examples of interest include:

- The category **FI** of finite sets with injections. A representation of this category can be thought of as a sequence $(M_n)_{n \geq 0}$, where M_n is a representation of the symmetric group S_n, together with transition maps $M_n \to M_{n+1}$ satisfying certain compatibilities. This category was studied in [CEF, CEFN], where many examples of representations occurring in algebra and topology are discussed, and from a different point of view in [SS1].
- Variants of **FI**. In [Sn], modules over twisted commutative algebras are studied; these can be viewed (in certain cases) as representations of a category $\mathbf{FI}_d$ generalizing **FI**. In [Wi] analogs of **FI** for other classical Weyl groups are studied. In [WG] the category **FA** of finite sets with all maps is studied.
- The category $\mathbf{FS}_G$ of finite sets with G-surjections, G being a finite group (see §11.1.2 for the definition). Really, it is the opposite category that is of interest. A representation of $\mathbf{FS}_G^{\mathrm{op}}$ can be thought of as a sequence $(M_n)_{n \geq 0}$, where M_n is a representation of the wreath product $S_n \wr G$, together with transition maps $M_n \to M_{n+1}$ satisfying certain compatibilities (quite different from those in the **FI** case). As we show below, the theory of $\mathbf{FS}_G^{\mathrm{op}}$ representations, with G a symmetric group, is essentially equivalent to the theory of Δ-modules studied in [Sn].
- The category $\mathbf{VA}_{\mathbf{F}_q}$ of finite-dimensional vector spaces over a finite field $\mathbf{F}_q$ with all linear maps. Representations of this category (in particular when $\mathbf{k} = \mathbf{F}_q$) have been studied in relation to algebraic K-theory, rational cohomology, and the Steenrod algebra, see [K4] for a survey and additional references.

The referenced works use a variety of methods, often ad hoc, to study representations. However, one is struck by the fact that many of the results appear to be quite similar. For instance, each proves (or conjectures) a noetherianity result. This suggests that there are general principles at play, and leads to the subject of our paper:

> **Main Problem.** Find practical combinatorial criteria for categories that imply interesting algebraic properties of their representations.

We give solutions to this problem for the algebraic properties of noetherianity and rationality of Hilbert series. Our criteria easily recover and strengthen most known results, and allow us to resolve some open questions. Without a doubt, they will be applicable to many categories not yet considered.

In the remainder of the introduction, we summarize our results and applications in more detail, and indicate some interesting open problems. See §1.6 for a guide for the paper.

1.1. **Noetherianity.** We say that a representation of the category $\mathcal{C}$ is **noetherian** if any subrepresentation is finitely generated; for the definition of "finitely generated," see §4.1. We say that the category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ of all representations of $\mathcal{C}$ is **noetherian** if all finitely generated representations are noetherian. This is a fundamental property, and has played a crucial role in all applications. The first main theoretical result of this paper is a combinatorial criterion on $\mathcal{C}$ that ensures $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, for any left-noetherian ring $\mathbf{k}$. We now explain this criterion, and the motivation behind it.

We start by recalling a combinatorial proof of the Hilbert basis theorem. For simplicity of exposition, we show that $\mathbf{k}[x_1, \dots, x_n]$ is a noetherian ring when $\mathbf{k}$ is a field. Pick an admissible order on the monomials, i.e., a well-order compatible with multiplication. Using the order, we can define initial ideals, and reduce the study of the ascending chain condition to monomial ideals. Now, the set of monomial ideals is naturally in bijection with the set of ideals in the poset $\mathbf{N}^r$. Thus noetherianity of the ring $\mathbf{k}[x_1, \dots, x_n]$ follows from noetherianity of the poset $\mathbf{N}^r$ (Dixon's lemma), which is a simple combinatorial exercise: given infinitely many vectors $v_1, v_2, \dots$ in $\mathbf{N}^r$ one must show that $v_i \le v_j$ for some $i \neq j$, where $\le$ means coordinate-by-coordinate comparison.

To apply this method to $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ we must first make sense of what "monomials" are. Given an object x of $\mathcal{C}$, define a representation P_x of $\mathcal{C}$ by $P_x(y) = \mathbf{k}[\mathrm{Hom}(x, y)]$, i.e., $P_x(y)$ is the free $\mathbf{k}$-module with basis $\mathrm{Hom}(x, y)$. We call P_x the **principal projective** at x. These representations take the place of free modules; in fact, one should think of P_x as the free representation with one generator of degree x. Given a morphism $f \colon x \to y$ in $\mathcal{C}$, there is a corresponding element e_f of $P_x(y)$. A **monomial** is an element of $P_x(y)$ of the form λe_f, where λ is a non-zero element of $\mathbf{k}$. A subrepresentation M of P_x is **monomial** if $M(y)$ is spanned by the monomials it contains, for all $y \in \mathcal{C}$.

We now carry over the combinatorial proof of the Hilbert basis theorem. For simplicity of exposition, we assume that $\mathbf{k}$ is a field and $\mathcal{C}$ is directed, i.e., if $f : x \to x$ is an endomorphism in $\mathcal{C}$ then $f = \mathrm{id}_x$. For an object x of $\mathcal{C}$, we write $|\mathcal{C}_x|$ for the set of isomorphism classes of morphisms $x \to y$. This can be thought of as the set of monomials in P_x. Suppose that $\mathcal{C}$ satisfies the following condition:

(G1) For each $x \in \mathcal{C}$, the set $|\mathcal{C}_x|$ admits an admissible order $\prec$, that is, a well-order compatible with post-composition, i.e., $f \prec f'$ implies $gf \prec gf'$ for all g.

Given a subrepresentation M of a principal projective P_x, we can use $\prec$ to define the **initial subrepresentation** $\mathrm{init}(M)$. This allows us to reduce the study of the ascending chain condition for subrepresentations of P_x to monomial subrepresentations. Monomial subrepresentations are naturally in bijection with ideals in the poset $|\mathcal{C}_x|$, where the order is defined by $f \le g$ if $g = hf$ for some h. (Note: $\prec$ and $\le$ are two different orders on $|\mathcal{C}_x|$; the former is chosen, while the latter is canonical.) We now assume that $\mathcal{C}$ satisfies one further condition:

(G2) For each $x \in \mathcal{C}$, the poset $|\mathcal{C}_x|$ is noetherian.

Given this, we see that ascending chains of monomial subrepresentations stabilize, and we conclude that P_x is a noetherian representation. One easily deduces from this that $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is a noetherian category.

The above discussion motivates one of the main definitions in this paper:

Definition 1.1.1. A directed category $\mathcal{C}$ is **Gröbner** if (G1) and (G2) hold. □

We can summarize the previous paragraph as: if $\mathcal{C}$ is a directed Gröbner category then $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian. In the body of the paper, we give a slightly more sophisticated definition of Gröbner that does not require directed. However, it still precludes non-trivial automorphisms. Unfortunately, many of the categories of primary interest do have non-trivial automorphisms, and therefore cannot be Gröbner. This motivates a weakening of the above definition:

Definition 1.1.2. A category $\mathcal{C}$ is **quasi-Gröbner** if there is a Gröbner category $\mathcal{C}'$ and an essentially surjective functor $\mathcal{C}' \to \mathcal{C}$ satisfying property (F) (see Definition 4.2.1). □

Property (F) is a finiteness condition that intuitively means $\mathcal{C}'$ locally has finite index in $\mathcal{C}$. Our main combinatorial criterion for noetherianity is the following theorem:

Theorem 1.1.3. *If $\mathcal{C}$ is a quasi-Gröbner category then $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, for any left-noetherian ring $\mathbf{k}$.*

This is a solution of an instance of the Main Problem: "(quasi-)Gröbner" is a purely combinatorial condition on $\mathcal{C}$, which can be checked easily in practice, and the above theorem connects it to an important algebraic property of representations.

Example 1.1.4. Recall that **FI** is the category whose objects are finite sets and whose morphisms are injections. The automorphism groups in this category are non-trivial: they are symmetric groups. Thus **FI** is not a Gröbner category. Define **OI** to be the category whose objects are totally ordered finite sets and whose morphisms are order-preserving injections. This category is directed. We show that **OI** is a Gröbner category; this is a special case of Theorem 7.1.1. There is a natural functor $\mathbf{OI} \to \mathbf{FI}$ given by forgetting the total order. This functor is essentially surjective and satisfies property (F), so **FI** is a quasi-Gröbner category. In particular, we see that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI})$ is noetherian for any left-noetherian ring $\mathbf{k}$. See Remark 7.1.4 for the history of this result and its generalizations. □

The above example is a typical application of the theory of Gröbner categories: The main category of interest (in this case **FI**) has automorphisms, and is therefore not Gröbner. One therefore adds extra structure (e.g., a total order) to obtain a more rigid category. One then shows that this rigidified category is Gröbner, which usually comes down to an explicit combinatorial problem (in this case, Dixon's lemma). Finally, one deduces that the original category is quasi-Gröbner, which is usually quite easy.

1.2. **Hilbert series.** A **norm** on a category $\mathcal{C}$ is a function $\nu \colon |\mathcal{C}| \to \mathbf{N}$, where $|\mathcal{C}|$ is the set of isomorphism classes of $\mathcal{C}$. Suppose that $\mathcal{C}$ is equipped with a norm and M is a representation of $\mathcal{C}$ over a field $\mathbf{k}$. We then define the **Hilbert series** of M by

$$\mathrm{H}_M(t) = \sum_{x \in |\mathcal{C}|} \dim_{\mathbf{k}} M(x) \cdot t^{\nu(x)},$$

when this makes sense, i.e., when the coefficient of t^n is finite for all n. The second main theoretical result of this paper is a combinatorial condition on $(\mathcal{C}, \nu)$ that ensures $\mathrm{H}_M(t)$ has a particular form, for any finitely generated representation M.

The key idea is to connect to the theory of formal languages. Let Σ be a finite set (an alphabet), and let $\Sigma^\star$ denote the set of all finite words in Σ (i.e., the free monoid generated by Σ). A **language** on Σ is a subset of $\Sigma^\star$. Given a language $\mathcal{L}$, we define its **Hilbert series** by

$$\mathrm{H}_{\mathcal{L}}(t) = \sum_{w \in \mathcal{L}} t^{\ell(w)},$$

where $\ell(w)$ is the length of the word w. There are many known results on Hilbert series of languages: for example, regular languages have rational Hilbert series and unambiguous context-free languages have algebraic Hilbert series (Chomsky–Schützenberger theorem). We review these results, and establish some new ones, in §3.

Let $\mathcal{P}$ be a class of languages. A **$\mathcal{P}$-lingual structure** on $\mathcal{C}$ at x consists of a finite alphabet Σ and an injection $i\colon |\mathcal{C}_x| \to \Sigma^\star$ such that the following two conditions hold: (1) for any $f\colon x \to y$ in $|\mathcal{C}_x|$ we have $\nu(y) = \ell(i(f))$; and (2) if S is a poset ideal of $|\mathcal{C}_x|$ then the language $i(S)$ is of class $\mathcal{P}$. The first condition essentially means that we can interpret monomials of P_x as words in some alphabet in such a way that their norm agrees with the length of the word. We say that $\mathcal{C}$ is **$\mathcal{P}$-lingual** if it admits a $\mathcal{P}$-lingual structure at every object. A special case of our main result on Hilbert series is then:

Theorem 1.2.1. *Suppose that $\mathcal{C}$ is a $\mathcal{P}$-lingual Gröbner category and let M be a finitely generated representation of $\mathcal{C}$. Then H_M is a $\mathbf{Z}$-linear combination of series of the form $\mathrm{H}_{\mathcal{L}}$, where each $\mathcal{L}$ is a language of class $\mathcal{P}$.*

This too is a solution of an instance of the Main Problem: "$\mathcal{P}$-lingual" is a purely combinatorial condition on $(\mathcal{C}, \nu)$, which can be easily checked in practice, and the above theorem connects it to an important algebraic property of representations.

Example 1.2.2. Define a norm on $\mathbf{OI}$ by $\nu(x) = \#x$. We show that $(\mathbf{OI}, \nu)$ is $\mathcal{P}$-lingual, where $\mathcal{P}$ is the class of "ordered languages" introduced in §3.3. Let us briefly indicate the main idea. Let x be an object of $\mathbf{OI}$ of cardinality k. Then a morphism $x \to [n]$ in $\mathbf{OI}$ can be recorded by marking k elements of $[n]$. We can think of such a marking as a word of length n in the alphabet $\Sigma = \{0, 1\}$, where 0 indicates a marked spot and 1 an unmarked spot. We have thus defined an injection $i\colon |\mathbf{OI}_x| \to \Sigma^*$. Showing that this defines a $\mathcal{P}$-lingual structure at x is routine; we refer to Theorem 7.1.1 for the details.

As a consequence of the above result, we show that if M is a finitely generated $\mathbf{FI}$-module then $\mathrm{H}_M(t)$ is of the form $\frac{p(t)}{(1-t)^k}$ where $p(t)$ is a polynomial and $k \geq 0$ is an integer. In particular, the function $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial. See Remark 7.1.7 for the history of this result and its generalizations. □

Remark 1.2.3. In the body of the paper, we allow norms to take values in $\mathbf{N}^r$, which leads to multivariate Hilbert series. □

1.3. **Applications.** We apply our theory to prove a large number of results about categories of interest. We mention three of these results here.

1.3.1. *Lannes–Schwartz artinian conjecture.* This conjecture, which first appears in print as [K2, Conjecture 3.12], is equivalent to the statement that $\mathrm{Rep}_{\mathbf{F}_q}(\mathbf{VA}_{\mathbf{F}_q})$ is noetherian. (The conjecture asserts the dual statement that the principal injectives are artinian.) Some previous work on this conjecture appears in [Dj1, Dj2, Dj3, K3, Po1, Po2, Po3].

The conjecture is a special case of our result (Corollary 8.3.6) that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VA}_R)$ is noetherian for any left-noetherian ring $\mathbf{k}$ and finite commutative ring R. One of the original motivations for this conjecture is that it implies that Ext modules between finitely generated functors are finite-dimensional (equivalently every finitely generated functor has a resolution by finitely generated projectives — this was previously only known for the restricted class of "finite" functors). See [FFSS] for some general results and calculations for these Ext groups. A similar (but distinct) proof of this conjecture appears in [PS].

1.3.2. *Syzygies of Segre embeddings.* In [Sn], the p-syzygies of all Segre embeddings (with any number of factors of any dimensions, but with p fixed) are assembled into a single algebraic structure called a Δ-module. The two main results of [Sn] state that, in characteristic 0, this structure is finitely generated and has a rational Hilbert series. Informally, these results mean that the p-syzygies of Segre embeddings admit a finite description.

We improve the results of [Sn] in three ways. First, we show that the main theorems continue to hold in positive characteristic. It is not clear if one would expect this a priori, since the syzygies of the Segre embeddings are known to behave differently in positive characteristic (see, for example, [Has] for the case of determinantal ideals, which for 2×2 determinants are special cases of Segre embeddings). On the other hand, while we only work with Δ-modules over fields, one can work over $\mathbf{Z}$ and show that for any given syzygy module, the type of torsion that appears is bounded.

Second, we greatly improve the rationality result for the Hilbert series and give an affirmative answer to [Sn, Question 5]. Finally, we remove a technical assumption from [Sn]: that paper only dealt with "small" Δ-modules, whereas our methods handle all finitely generated Δ-modules. This is useful for technical reasons: for instance, it shows that a finitely generated Δ-module admits a resolution by finitely generated projective Δ-modules.

1.3.3. *Twisted commutative algebras in positive characteristic.* A twisted commutative algebra (tca) is a graded algebra on which the symmetric group S_n acts on the nth graded piece in such a way that the multiplication is commutative up to a "twist" by the symmetric group. In characteristic 0, one can use Schur–Weyl duality to describe tca's in terms of large commutative algebras equipped with an action of $\mathbf{GL}(\infty)$, and we have fruitfully exploited this to obtain many results [Sn, SS2, SS1, SS3]. This method is inapplicable in positive characteristic, and consequently we know much less about tca's there. In [CEFN], the univariate tca $\mathbf{k}\langle x\rangle$ was analyzed, for any ring $\mathbf{k}$, and certain fundamental results (such as noetherianity) were established. Here we establish many of the same results for the multivariate tca $\mathbf{k}\langle x_1, \dots, x_d\rangle$. See §7.3 for details.

1.4. **Relation to previous work.**

- The idea of reducing the problem of showing that some algebraic structure is noetherian to showing that some poset is noetherian has been used before in different contexts. We highlight [Co] for an example in universal algebra and [Hi] for examples in abstract algebra.
- Many of the categories that we are interested in come with nontrivial automorphism groups, which interfere with the application of Gröbner basis techniques. The first step in our proofs is to define a certain subcategory which does not have automorphisms and to apply Gröbner methods there. A similar idea of "breaking symmetry" was used in [DK1] to develop a theory of Gröbner bases for symmetric operads by passing to the weaker structure of shuffle operads. This idea is used in [KP] to study Hilbert series of operads with well-behaved Gröbner bases.
- A related topic (and one that serves as motivation for us) is the notion of "noetherianity up to symmetry" in multilinear algebra and algebraic statistics. We point to [DE, DrK, HM, HS] for some applications of noetherianity and to [Dr] for a survey and further references.

  An important topic that comes up is equivariant Gröbner bases for polynomial rings: the setup is a monoid acting on a polynomial ring in infinitely many variables and the problem is to develop a Gröbner basis theory that makes use of the monoid action. One main difference between this and our work can, loosely speaking, be summarized by saying that we consider Gröbner bases for all projective (free) modules, and not just ideals.

  While preparing this article, we discovered that an essentially equivalent version of Proposition 8.2.1 is proven in the proof of [DrK, Proposition 7.5].

1.5. **Open problems.** We close the introduction with some open problems.

1.5.1. *Krull dimension.* There is a notion of Krull dimension for abelian categories [Ga, Ch. IV] generalizing that for commutative rings. If $\mathbf{k}$ is a field of characteristic 0, then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ has Krull dimension 0 [WG], $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI})$ has Krull dimension 1 [SS1, Corollary 2.2.6], and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{OI})$ has infinite Krull dimension (easy). These results hold in positive characteristic as well: $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ is handled in Theorem 8.4.4, $\mathrm{Rep}_{\mathbf{k}}(\mathbf{OI})$ remains easy, and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI})$ is an as yet unpublished result of ours.

One would like a combinatorial method to compute the Krull dimension of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$. In Proposition 5.2.11 we give a criterion for dimension 0, but it is probably far from optimal. For higher dimension, we have some partial results, but none that apply to the main categories of interest. One difficulty is that there is not an obvious way to reduce to Gröbner categories, as the above examples indicate (**FI** and **OI** have very different Krull dimensions).

1.5.2. *Enhanced Hilbert series.* Our definition of the Hilbert series of $M \in \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ only records the dimension of $M(x)$, for each object x. One could attempt to improve this by recording the representation of $\mathrm{Aut}(x)$ on $M(x)$. We can formalize this problem as follows. Suppose R is a ring and for each $x \in |\mathcal{C}|$ we have an additive function $\mu_x \colon \mathcal{R}_{\mathbf{k}}(\mathrm{Aut}(x)) \to R$ ($\mathcal{R}_{\mathbf{k}}$ denotes the Grothendieck group). We define the **enhanced Hilbert series** of M by

$$\widetilde{\mathrm{H}}_M = \sum_{x \in |\mathcal{C}|} \mu_x([M(x)]),$$

where $[M(x)]$ is the class of $M(x)$ in $\mathcal{R}_{\mathbf{k}}(\mathrm{Aut}(x))$, when this sum makes sense. For example, if $\mathcal{C} = \mathbf{FI}$ and $\mathbf{k}$ has characteristic 0, we define maps μ_x to $R = \mathbf{Q}[\![t_1, t_2, \ldots]\!]$ in [SS1, §5.1], and prove a sort of rationality result there. We can now prove the analogous result for $\mathbf{FI}_d$-modules as well. In this paper, we define an enhanced Hilbert series for $\mathbf{FS}_G^{\mathrm{op}}$-modules, and prove a rationality result (see §11.2). Is it possible to prove a general rationality result?

1.5.3. *Minimal resolutions and Poincaré series.* Suppose $\mathcal{C}$ is a weakly directed category (i.e., any self-map is invertible) and $\mathbf{k}$ is a field. Suppose furthermore that the automorphism groups in $\mathcal{C}$ are finite and have order invertible in $\mathbf{k}$. There is then a notion of minimal projective resolution for representations of $\mathcal{C}$. One would like to understand the nature of these resolutions combinatorially.

We now give a slightly more specific problem. For a representation M of $\mathcal{C}$, let $\Psi(M)$ be the representation of $\mathcal{C}$ defined as follows: $\Psi(M)(x)$ is the quotient of $M(x)$ by the images of all maps $M(y) \to M(x)$ induced by non-isomorphisms $y \to x$ in $\mathcal{C}$. One can think of $\Psi(M)$ as analogous to tensoring M with the residue field in the case of modules over an augmented algebra. The left-derived functors of Ψ exist, and one can read off from $\mathrm{L}^i\Psi(M)$ the ith projective in the minimal resolution of M. Assuming $\mathcal{C}$ is normed, define the **Poincaré series** of M by

$$\mathrm{P}_M(\mathbf{t}, q) = \sum_{i=0}^{\infty} \mathrm{H}_{\mathrm{L}^i\Psi(M)}(\mathbf{t})(-q)^i.$$

This contains strictly more information than the Hilbert series, but is much more subtle since it does not factor through the Grothendieck group. What can one say about the form of this series? We proved a rationality theorem for Poincaré series of **FI**-modules in [SS1, §6.7], when $\mathbf{k}$ has characteristic 0, and can now generalize this result to $\mathbf{FI}_d$-modules (still in characteristic 0). Preliminary computations with $\mathbf{VI}_{\mathbf{F}_q}$-modules (defined in §8.3) have seen theta series come into play; it is not clear yet if there is a deeper meaning to this.

1.5.4. *Noetherianity results for other categories.* There are combinatorial categories which are expected to be noetherian, but do not fall into our framework. For example: Let $\mathcal{C}$ be the category whose objects are finite sets, and where a morphism $S \to T$ is an injection $f \colon S \to T$ together with a perfect matching on $T \setminus f(S)$. When $\mathbf{k}$ has characteristic 0, $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is equivalent to the category of $\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{k}^\infty))$-modules with a compatible polynomial action of $\mathbf{GL}_\infty(\mathbf{k})$. See also [SS3, §4.2] for a connection between $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ (where $\mathcal{C}$ is called (db)) and the stable representation theory of the orthogonal group.

We expect $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, but cannot prove it. We note that passing to the directed category $\mathcal{D}$ where the objects are ordered finite sets and the maps $f \colon S \to T$ are order-preserving does not work since the principal projective $P_\varnothing$ is not noetherian: if $M_n \colon \varnothing \to [2n]$ is the perfect matching on $\{1, \dots, 2n\}$ consisting of the edges $(i, i+3)$ where $i = 1, 3, \dots, 2n-3$ and the edge $(2n-1, 2)$, then $M_3, M_4, M_5, \dots$ are pairwise incomparable. Is there a way to extend the scope of the methods of this paper to include these categories?

1.5.5. *Coherence.* The category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is **coherent** if the kernel of any map of finitely generated projective representations is finitely generated. This is a weaker property than noetherianity, and should therefore be easier to prove. We have some partial combinatorial results on coherence, but none that apply in cases of interest. We would be especially interested in a criterion that applies to the category mentioned in §1.5.4.

1.5.6. *Optimal results for Hilbert series of $\mathbf{FS}_G^{\mathrm{op}}$-modules.* Let G be a finite group whose order is invertible in the field $\mathbf{k}$, and let M be a finitely generated $\mathbf{FS}_G^{\mathrm{op}}$-module. Consider the minimal subfield F of $\mathbf{C}$ with the following property: $\mathrm{H}_M(\mathbf{t})$ can be written in the form $f(\mathbf{t})/g(\mathbf{t})$ where $f \in F[\mathbf{t}]$ and $g(\mathbf{t})$ factors as $\prod(1-\lambda_i)$ where each λ_i is a linear combination of the $\mathbf{t}$ with coefficients in the ring of integers $\mathcal{O}_F$.

When $\mathbf{k}$ is algebraically closed, we prove $F \subseteq \mathbf{Q}(\zeta_N)$, where N is the exponent of G. When G is the symmetric group, we show (Corollary 11.3.4) that $F = \mathbf{Q}$. It would be interesting to determine F in general. This is related to the question of finding optimal good collections of subgroups of G in the sense of §11.2, although it is probably necessary to find an alternative approach.

1.5.7. *Hilbert series of ideals in permutation posets.* There are many algebraic structures not mentioned in this paper which lead to interesting combinatorial problems. As an example, consider the poset $\mathfrak{S}$ of all finite permutations, i.e., the disjoint union of all finite symmetric groups. Represent a permutation σ in one-line notation: $\sigma(1)\sigma(2)\cdots\sigma(n)$. Say that $\tau \le \sigma$ if there is a consecutive subword $\sigma(i)\sigma(i+1)\cdots\sigma(i+r-1)\sigma(i+r)$ which gives the same permutation as τ, i.e., $\sigma(j) > \sigma(j')$ if and only if $\tau_j > \tau_{j'}$ for all $j \ne j'$. (In the literature, τ is a **consecutive pattern** in σ.) If we drop the condition "consecutive," then this becomes the poset of pattern containment (see [B, §7.2.3]) which is known to have infinite anti-chains ([B, Theorem 7.35]) and hence the same is true for the weaker relation of consecutive pattern containment. But we can still ask about the behavior of Hilbert series of finitely generated ideals in this poset. For example, are they always D-finite (see [St2, §6.4] for the definition)? (This is asked for the pattern containment poset in [B, Conjecture 5.8].) The corresponding algebraic setup is monomial ideals in the free shuffle algebra (see [Ro, Example 2.2(b)] or [DK2, §2.2, Example 2]).

1.5.8. *A reconstruction problem.* Let G and H be finite groups and let $\mathbf{k}$ be an algebraically closed field. Suppose $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_G^{\mathrm{op}})$ and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_H^{\mathrm{op}})$ are equivalent as $\mathbf{k}$-linear abelian categories. Are G and H isomorphic?

1.6. **Guide for the paper.** Here we provide a roadmap to the contents of this paper. We have divided the paper into two parts: theory and applications.

The first part on theory begins with background results on noetherian posets §2 and formal languages §3. The material on posets is standard and we have included proofs to make it self-contained. This is essential for our applications to noetherian conditions on representations of categories which come later. The section on formal languages is a mixture of review and some new material on ordered and quasi-ordered languages. This material is essential for our applications to rationality properties of Hilbert series of finitely generated representations.

In §4 we introduce basic terminology and properties of representations of categories and functors between them. We state criteria for noetherian conditions on representations which will be further developed in later sections. The next section §5 introduces and develops the main topic of this paper: Gröbner bases for representations of categories. We give a criterion for categories to admit a Gröbner basis theory and relate noetherian properties of representations to those of posets. We encapsulate the key properties into the notion of (quasi-)Gröbner categories. The final theory section §6 is concerned with Hilbert series of representations of categories. Here we introduce the notion of lingual structures on categories and connect properties of Hilbert series with formal languages. We are most concerned with when the Hilbert series is a rational or algebraic function, and refinements of those results.

The second part of the paper is concerned with applications of the theory developed in the first part. The first section §7 is about categories of finite sets and injective functions of different kinds. This has two sources of motivation: the theory of **FI**-modules [CEF] and the theory of twisted commutative algebras [SS2]. We recover and strengthen known results on noetherianity and Hilbert series for these categories and related ones.

§8 is about categories of finite sets and surjective functions. These categories are much more complicated than their injective counterparts. A significant application of the results here is the proof of the Lannes–Schwartz artinian conjecture (discussed in §1.3.1). The proof mainly relies on Proposition 4.2.6, Theorem 8.1.1 (the proof of which is in §8.2), and Theorem 8.1.2, so the reader mainly interested in the proof of this conjecture need only read those results and their minor preliminaries.

The next main topic of the paper in §9 is applications to Δ-modules (introduced by the second author in [Sn]). We prove that finitely generated Δ-modules are noetherian over any field, thus significantly improving the results of [Sn] where it is only shown in characteristic 0 under a further "smallness" assumption. The main result needed for the noetherian condition is contained in the section on surjections mentioned above. For Hilbert series of Δ-modules, we affirmatively resolve and strengthen [Sn, §6, Question 5] by proving a stronger rationality result.

In §11, we study G-equivariant versions of the category of finite sets with injections or surjections for G a finite group. From the perspective that those categories are about representations of symmetric groups, these categories can be thought of as a wreath product generalization. We originally developed these results for applications to Δ-modules (see

Remark 11.1.12), but that approach is no longer needed. We keep these sections for independent interest. In fact, the case $G = \mathbf{Z}/2\mathbf{Z}$ appears in the literature [Wi]. The technical results needed to study Hilbert series is developed in §10, which deals with weighted sets and surjections (and can be thought of as the case when G is abelian). We highlight §11.3 which contains a group theory problem which we think is of interest in its own right.

Finally, we end with some examples that illustrate features of the theory not seen in the other examples. All of the categories studied so far have the property that their finitely generated representations have rational Hilbert series. In §12, we study a category related to the category of finite sets and injections such that finitely generated representations have algebraic (but non-rational) Hilbert series. In §13 we study two examples of categories with infinite hom sets (all of the other examples we have studied have finite hom sets) and prove that finitely generated representations are noetherian: a linear-algebraic category built out of upper unitriangular integer matrices, and the category of finite sets and G-injections when G is a polycyclic-by-finite group.

1.7. **Notation and quick reference.** We list the categories that we study (see the indicated paragraph for the definition of the category):

(1) $\mathbf{OI}_d$, $\mathbf{FI}_d$ §7.1
(2) $\mathbf{FA}$ §7.4
(3) $\mathbf{OS}^{\mathrm{op}}$, $\mathbf{FS}^{\mathrm{op}}$ §8.1
(4) $\mathbf{FA}^{\mathrm{op}}$ §8.4
(5) $\mathbf{VI}_R$, $\mathbf{VA}_R$ §8.3
(6) $\mathbf{OWS}_\Lambda^{\mathrm{op}}$, $\mathbf{FWS}_\Lambda^{\mathrm{op}}$ §10.1
(7) $\mathbf{FI}_G$, $\mathbf{FA}_G$ §11.1.1
(8) $\mathbf{FS}_G^{\mathrm{op}}$ §11.2
(9) $\mathscr{B}_{\mathbf{Z}}^+$, $\mathscr{B}_{\mathbf{Z}}$ §13

For each category $\mathcal{C}$ above, we show that $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian when $\mathbf{k}$ is left-noetherian. Except for $\mathscr{B}_{\mathbf{Z}}$, all categories are shown to be quasi-Gröbner. We also prove results about Hilbert series for most of these categories.

Finally, we list some commonly used notation:

- If Σ is a set, let $\Sigma^\star$ denote the set of words in Σ, i.e., the free monoid generated by Σ. For $w \in X^\star$, let $\ell(w)$ denote the length of the word.
- For a non-negative integer $n \geq 0$, set $[n] = \{1, \dots, n\}$, with the convention $[0] = \varnothing$.
- Let $\mathbf{n}$ be an element of $\mathbf{N}^r$. We write $|\mathbf{n}|$ for the sum of the coordinate of $\mathbf{n}$ and $\mathbf{n}!$ for $n_1! \cdots n_r!$. We let $[\mathbf{n}]$ be the tuple $([n_1], \dots, [n_r])$ of finite sets. Given variables $t_1, \dots, t_r$, we let $\mathbf{t}^{\mathbf{n}}$ be the monomial $t_1^{n_1} \cdots t_r^{n_r}$.
- For each positive integer N, we fix a primitive Nth root of unity ζ_N.
- $\widehat{\mathrm{Sym}}$ denotes the completion of Sym with respect to the homogeneous maximal ideal, i.e., $\widehat{\mathrm{Sym}}(V) = \prod_{n \geq 0} \mathrm{Sym}^n(V)$.
- Let G be a finite group. The Grothendieck group of all, resp. projective, finitely generated $\mathbf{k}[G]$-modules is denoted $\mathcal{R}_{\mathbf{k}}(G)$, resp. $\mathcal{P}_{\mathbf{k}}(G)$.

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Part 1. Theory

2. Partially ordered sets

In this section, we state some basic definitions and properties of noetherian posets. This section can be skipped and referred back to as necessary since it serves a technical role only.

Let X be a poset. We say that X satisfies the **ascending chain condition** (ACC) if every ascending chain in X stabilizes, i.e., given $x_1 \le x_2 \le \cdots$ in X we have $x_i = x_{i+1}$ for i sufficiently large. The **descending chain condition** (DCC) is defined similarly. An **anti-chain** in X is a sequence $x_1, x_2, \ldots$ such that $x_i \nleq x_j$ for all $i \ne j$. An **ideal** in X is a subset I of X such that $x \in I$ and $x \le y$ implies $y \in I$. We write $\mathcal{I}(X)$ for the set of ideals of X, given the structure of a poset by inclusion. For $x \in X$, the **principal ideal** generated by x is $\{y \mid y \ge x\}$. An ideal is **finitely generated** if it is a finite union of principal ideals. The following result is standard.

Proposition 2.1. *The following conditions on X are equivalent:*

(a) *The poset X satisfies DCC and has no infinite anti-chains.*
(b) *Given a sequence $x_1, x_2, \ldots$ in X, there exists $i < j$ such that $x_i \le x_j$.*
(c) *The poset $\mathcal{I}(X)$ satisfies ACC.*
(d) *Every ideal of X is finitely generated.*

Definition 2.2. The poset X is **noetherian** if the above conditions are satisfied. □

Remark 2.3. Where we say "X is noetherian," one often sees "$\le$ is a well-quasi-order" in the literature. Similarly, where we say "X satisfies DCC" one sees "$\le$ is well-founded." □

Proposition 2.4. *Let X be a noetherian poset and let $x_1, x_2, \ldots$ be a sequence in X. Then there exists an infinite sequence of indices $i_1 < i_2 < \cdots$ such that $x_{i_1} \le x_{i_2} \le \cdots$.*

Proof. Let I be the set of indices such that $i \in I$ and $j > i$ implies that $x_i \nleq x_j$. If I is infinite, then there is $i < i'$ with $i, i' \in I$ such that $x_i \le x_{i'}$ by definition of noetherian and hence contradicts the definition of I. So I is finite; let i_1 be any number larger than all elements of I. Then by definition of I, we can find $x_{i_1} \le x_{i_2} \le \cdots$. □

Proposition 2.5. *Let X and Y be noetherian posets. Then $X \times Y$ is noetherian.*

Proof. Let $(x_1, y_1), (x_2, y_2), \ldots$ be an infinite sequence in $X \times Y$. Since X is noetherian, there exists $i_1 < i_2 < \cdots$ such that $x_{i_1} \le x_{i_2} \le \cdots$ (Proposition 2.4). Since Y is noetherian, there exists $i_j < i_{j'}$ such that $y_{i_j} \le y_{i_{j'}}$, and hence $(x_{i_j}, y_{i_j}) \le (x_{i_{j'}}, y_{i_{j'}})$. □

Let X and Y be posets and let $f \colon X \to Y$ be a function. We say that f is **order-preserving** if $x \le x'$ implies $f(x) \le f(x')$. We say that f is **strictly order-preserving** if $x \le x'$ is equivalent to $f(x) \le f(x')$. Suppose f is strictly order-preserving. Then it is necessarily injective; indeed, if $f(x) = f(x')$ then $f(x) \le f(x')$ and $f(x') \le f(x)$, and so $x \le x'$ and $x' \le x$, and so $x = x'$. We can thus regard X as a subset of Y with the induced order. In particular, if Y is noetherian then so is X.

Let $\mathcal{F} = \mathcal{F}(X, Y)$ be the set of all order-preserving functions $f \colon X \to Y$. We partially order $\mathcal{F}$ by $f \le g$ if $f(x) \le g(x)$ for all $x \in X$.

Proposition 2.6. *We have the following.*

(a) *If X is noetherian and Y satisfies ACC then $\mathcal{F}$ satisfies ACC.*
(b) *If $\mathcal{F}$ satisfies ACC and X is non-empty then Y satisfies ACC.*
(c) *If $\mathcal{F}$ satisfies ACC and Y has two distinct comparable elements then X is noetherian.*

Proof. (a) Suppose X is noetherian and $\mathcal{F}$ does not satisfy ACC. Let $f_1 < f_2 < \cdots$ be an ascending chain in $\mathcal{F}$. For each i, choose $x_i \in X$ such that $f_i(x_i) < f_{i+1}(x_i)$. Since X is noetherian, by passing to a subsequence we can assume $x_1 \le x_2 \le \cdots$ (Proposition 2.4). Let $y_i = f_i(x_i)$. Then

$$y_i = f_i(x_i) < f_{i+1}(x_i) \le f_{i+1}(x_{i+1}) = y_{i+1},$$

and so $y_1 < y_2 < \cdots$ shows that Y does not satisfy ACC.

(b) Now suppose $\mathcal{F}$ satisfies ACC and X is non-empty. Then Y embeds into $\mathcal{F}$ as the set of constant functions, and so Y satisfies ACC.

(c) Finally, suppose $\mathcal{F}$ satisfies ACC and Y contains elements $y_1 < y_2$. Given an ideal I of X, define $\chi_I \in \mathcal{F}$ by

$$\chi_I(x) = \begin{cases} y_2 & x \in I \\ y_1 & x \notin I \end{cases}.$$

Then $I \mapsto \chi_I$ defines an embedding of $\mathcal{I}(X)$ into $\mathcal{F}$, and so $\mathcal{I}(X)$ satisfies ACC, and so X is noetherian. □

Finally, we end with Higman's lemma. Given a poset X, let $X^\star$ be the set of finite words $x_1 \cdots x_n$ with $x_i \in X$. We define $x_1 \cdots x_n \le x'_1 \cdots x'_m$ if there exist $1 \le i_1 < \cdots < i_n \le m$ such that $x_j \le x'_{i_j}$ for $j = 1, \dots, n$.

Theorem 2.7 (Higman [Hi])**.** *If X is a noetherian poset, then the same is true for $X^\star$.*

Proof. Suppose that $X^\star$ is not noetherian. We use Nash-Williams' theory of minimal bad sequences [NW] to get a contradiction. A sequence $w_1, w_2, \dots$ of elements in $X^\star$ is bad if $w_i \not\le w_j$ for all $i < j$. We pick a bad sequence minimal in the following sense: for all $i \ge 1$, among all bad sequences beginning with $w_1, \dots, w_{i-1}$ (this is the empty sequence for $i = 1$), $\ell(w_i)$ is as small as possible. Let $x_i \in X$ be the first element of w_i and let v_i be the subword of w_i obtained by removing x_i. By Proposition 2.4, there is an infinite sequence $i_1 < i_2 < \cdots$ such that $x_{i_1} \le x_{i_2} \le \cdots$. Then $w_1, w_2, \dots, w_{i_1-1}, v_{i_1}, v_{i_2}, \dots$ is a bad sequence because $v_{i_j} \le w_{i_j}$ for all j, and $v_{i_j} \le v_{i_{j'}}$ would imply that $w_{i_j} \le w_{i_{j'}}$. It is smaller than our minimal bad sequence, so we have reached a contradiction. Hence $X^\star$ is noetherian. □

3. Formal languages

In this section we give basic definitions and results on formal languages. In particular, we define a few classes of formal languages (regular, ordered, quasi-ordered, unambiguous context-free) which will be used in this paper along with results on their generating functions. We believe that the material in §§3.3, 3.4 on ordered and quasi-ordered languages is new. The rest of the material is standard.

3.1. **Generalities.** Fix a finite set Σ (which we also call an **alphabet**). A **language** on Σ is a subset of $\Sigma^\star$. Let $\mathcal{L}$ and $\mathcal{L}'$ be two languages. The **union** of $\mathcal{L}$ and $\mathcal{L}'$, denoted $\mathcal{L} \cup \mathcal{L}'$, is simply their union as subsets of $\Sigma^\star$. The **concatenation** of $\mathcal{L}$ and $\mathcal{L}'$, denoted $\mathcal{L}\mathcal{L}'$, is the language consisting of all words of the form ww' with $w \in \mathcal{L}$ and $w' \in \mathcal{L}'$. The **Kleene star** of $\mathcal{L}$, denoted $\mathcal{L}^\star$, is the language consisting of words of the form $w_1 \cdots w_n$ with $w_1, \dots, w_n \in \mathcal{L}$, i.e., the submonoid (under concatenation) of $\Sigma^\star$ generated by $\mathcal{L}$.

Let Σ be an alphabet. A **norm** on Σ is a monoid homomorphism $\nu \colon \Sigma^\star \to \mathbf{N}^I$, for some finite set I, such that elements of Σ map to basis vectors. (This condition could be omitted, but is convenient for us.) Obviously, specifying a norm is equivalent to giving a function $\Sigma \to I$. By a norm on a language $\mathcal{L}$ over Σ, we mean a function $\mathcal{L} \to \mathbf{N}^I$ which is the restriction of a norm on Σ. Every language admits a canonical norm over $\mathbf{N}$, namely, the length function $\ell\colon \mathcal{L} \to \mathbf{N}$. We say that a norm ν is **universal** if the map $\Sigma \to I$ is injective. The concept is important for the following reason: if ν is a universal norm with values in $\mathbf{N}^I$, and ν' is some other norm with values in $\mathbf{N}^J$, then there is a function $f\colon I \to J$ such that $\nu'(w) = f_*(\nu(w))$, where $f_*\colon \mathbf{N}^I \to \mathbf{N}^J$ is the homomorphism induced by f.

Let $\mathcal{L}$ be a language equipped with a norm ν with values in $\mathbf{N}^I$. Let $\mathbf{t} = (t_i)_{i \in I}$ be indeterminates. The **Hilbert series** of $\mathcal{L}$ (with respect to ν) is

$$\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t}) = \sum_{w \in \mathcal{L}} \mathbf{t}^{\nu(w)},$$

when this makes sense (i.e., the coefficient of $\mathbf{t}^{\mathbf{n}}$ is finite for all $\mathbf{n}$). The coefficient of $\mathbf{t}^{\mathbf{n}}$ in $\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t})$ is the number of words of norm $\mathbf{n}$. We omit the norm ν from the notation when it is not needed.

There are two special cases of interest. When $\nu = \ell$, the coefficient of t^n in $\mathrm{H}_{\mathcal{L}}$ counts the number of words of length n in $\mathcal{L}$. This series is often called the **generating function** of $\mathcal{L}$ in the literature. When ν is a universal norm, we say that the series is a **universal Hilbert series**. Any Hilbert series of $\mathcal{L}$ (under any norm) is a homomorphic image of a universal Hilbert series of $\mathcal{L}$.

3.2. **Regular languages.** The set of **regular languages** on Σ is the smallest set of languages on Σ containing the empty language and the singleton languages $\{c\}$ (for $c \in \Sigma$), and closed under finite union, concatenation, and Kleene star.

A **deterministic finite-state automata** (DFA) for the alphabet Σ is a tuple $(Q, T, \sigma, \mathcal{F})$ consisting of:

- A finite set Q, the set of **states**.
- A function $T\colon \Sigma \times Q \to Q$, the **transition table**.
- A state $\sigma \in Q$, the **initial state**.
- A subset $\mathcal{F} \subset Q$, the set of **final states**.

Fix a DFA. We write $\alpha \xrightarrow{c} \beta$ to indicate $T(c, \alpha) = \beta$. Given a word $w = w_1 \cdots w_n$, we write $\alpha \xrightarrow{w} \beta$ if there are states $\alpha = \alpha_0, \dots, \alpha_n = \beta$ such that

$$\alpha_0 \xrightarrow{w_1} \alpha_1 \xrightarrow{w_2} \cdots \xrightarrow{w_{n-1}} \alpha_{n-1} \xrightarrow{w_n} \alpha_n.$$

Intuitively, we think of $\alpha \xrightarrow{w} \beta$ as saying that the DFA starts in state α, reads the word w, and ends in the state β. We say that the DFA **accepts** the word w if $\sigma \xrightarrow{w} \tau$ with τ a final state. The set of accepted words is called the language **recognized** by the DFA.

The following result is standard; see [HU, Ch. 2], for example.

Theorem 3.2.1. *A language is regular if and only if it is recognized by some DFA.*

The following result is also standard, but we include a proof since it is short and we will need to refer to it later. We refer to [St1, Theorem 4.7.2] for a more detailed treatment.

Theorem 3.2.2. *If $\mathcal{L}$ is regular then $\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t})$ is a rational function of $\mathbf{t}$, for any norm ν.*

Proof. Choose a DFA recognizing $\mathcal{L}$. Let A be the $Q \times Q$ matrix given by

$$A_{\alpha,\beta} = \sum_{\substack{\alpha \xrightarrow{w} \beta \\ \ell(w)=1}} \mathbf{t}^{\nu(w)}.$$

A simple computation shows that

$$(A^k)_{\alpha,\beta} = \sum_{\substack{\alpha \xrightarrow{w} \beta \\ \ell(w)=k}} \mathbf{t}^{\nu(w)}.$$

Thus if $\tau_1, \ldots, \tau_s$ are the final states then

$$\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t}) = \sum_{i=1}^{s} \sum_{k=0}^{\infty} (A^k)_{\sigma,\tau_i} = \sum_{i=1}^{s} \frac{(-1)^{\sigma+\tau_i} \det(1 - A : \tau_i, \sigma)}{\det(1-A)},$$

where the notation $(1 - A : \tau_i, \sigma)$ means that we remove the τ_ith row and σth column from $1 - A$ (for this to make sense, we pick a bijection between Q and the set $\{1, \ldots, |Q|\}$). □

Example 3.2.3. Consider the language $\mathcal{L} = \{0, 11\}^\star$ on the alphabet $X = \{0, 1\}$. Let $\mathcal{L}_n$ be the words of length n in $\mathcal{L}$. There are maps $\mathcal{L}_{n-2} \to \mathcal{L}_n$ and $\mathcal{L}_{n-1} \to \mathcal{L}_n$, given by appending 11 and 0, respectively. Together, these define a bijection $\mathcal{L}_n \cong \mathcal{L}_{n-1} \amalg \mathcal{L}_{n-2}$. It follows that $\#\mathcal{L}_n$ is the nth Fibonacci number, and so

$$\mathrm{H}_{\mathcal{L},\ell}(t) = \frac{1}{1 - t - t^2}.$$ □

3.3. **Ordered languages.** The set of **ordered languages** on Σ is the smallest set of languages on Σ that contains the singleton languages and the languages $\Pi^\star$, for $\Pi \subseteq \Sigma$, and that is closed under finite union and concatenation. We have not found this class of languages considered in the literature.

A DFA is **ordered** if the following condition holds: if α and β are states and there exists words u and w with $\alpha \xrightarrow{u} \beta$ and $\beta \xrightarrow{w} \alpha$, then $\alpha = \beta$. The states of an ordered DFA admit a natural partial order by $\alpha \le \beta$ if there exists a word w with $\alpha \xrightarrow{w} \beta$.

We prove two theorems about ordered languages. The first is the following.

Theorem 3.3.1. *A language is ordered if and only if it is recognized by some ordered DFA.*

Proof. Suppose $\mathcal{L}$ is an ordered language. Then we can write $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$, where each $\mathcal{L}_i$ is a concatenation of singleton languages and languages of the form $\Pi^\star$ with $\Pi \subset \Sigma$. By Lemmas 3.3.4 and 3.3.5 below, each $\mathcal{L}_i$ is recognized by an ordered DFA. Thus by Lemma 3.3.2 below, $\mathcal{L}$ is recognized by an ordered DFA.

The converse is proved in Lemma 3.3.6. □

Lemma 3.3.2. *Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are languages recognized by ordered DFA's. Then $\mathcal{L} \cup \mathcal{L}'$ is recognized by an ordered DFA.*

Proof. Suppose $(Q, T, \sigma, \mathcal{F})$ is an ordered DFA recognizing $\mathcal{L}$ and $(Q', T', \sigma', \mathcal{F}')$ is an ordered DFA recognizing $\mathcal{L}'$. Let $Q'' = Q \times Q'$. Let $T'' \colon \Sigma \times Q'' \to Q''$ be defined in the obvious manner, i.e.,

$$T''(c, (\alpha, \alpha')) = (T(c, \alpha), T'(c, \alpha')).$$

Let $\sigma'' = (\sigma, \sigma')$, and let

$$\mathcal{F}'' = (\mathcal{F} \times Q') \cup (Q \times \mathcal{F}').$$

Then $(Q'', T'', \sigma'', \mathcal{F}'')$ is a DFA recognizing $\mathcal{L} \cup \mathcal{L}'$. Note that $(\alpha, \alpha') \xrightarrow{w} (\beta, \beta')$ if and only if $\alpha \xrightarrow{w} \beta$ and $\alpha' \xrightarrow{w} \beta'$, and so this DFA is ordered. □

Lemma 3.3.3. *Suppose that $\mathcal{L}$ is a non-empty language recognized by an ordered DFA. Then we can write $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$ where each $\mathcal{L}_n$ is recognized by an ordered DFA with a single final state.*

Proof. Suppose $\mathcal{L}$ is recognized by the ordered DFA $(Q, T, \sigma, \mathcal{F})$. Enumerate $\mathcal{F}$ as $\{\tau_1, \dots, \tau_n\}$. Let $\mathcal{L}_i$ be the set of words $w \in \mathcal{L}$ for which $\sigma \xrightarrow{w} \tau_i$. Then $\mathcal{L}$ is clearly the union of the $\mathcal{L}_i$. Moreover, $\mathcal{L}_i$ is recognized by the ordered DFA $(Q, T, \sigma, \{\tau_i\})$. □

Lemma 3.3.4. *Let $\mathcal{L}$ be a language recognized by an ordered DFA and let $\mathcal{L}' = \{c\}$ be a singleton language. Then the concatentation $\mathcal{L}\mathcal{L}'$ is recognized by an ordered DFA.*

Proof. Suppose first that $\mathcal{L}$ is recognized by the ordered DFA $(Q, T, \sigma, \mathcal{F})$, where $\mathcal{F} = \{\tau\}$ has a single element. If $T(c, \tau) = \tau$, then nothing needs to be changed; indeed, in this case, $\mathcal{L} = \mathcal{L}\mathcal{L}'$. Suppose then that $T(c, \tau) = \rho \neq \tau$. Let $Q' = Q \amalg \{\tau'\}$, and define a transition function T' as follows. First, $T'(b, \alpha) = T(b, \alpha)$ if $\alpha \in Q$, unless $\alpha = \tau$ and $b = c$. We define $T'(c, \tau) = \tau'$. Finally, we define $T'(b, \tau') = \rho$, for any b. One easily verifies that $(Q', T', \sigma, \{\tau'\})$ is an ordered DFA recognizing $\mathcal{L}\mathcal{L}'$.

We now treat the general case. If $\mathcal{L}$ is the empty language, the result is easy, so assume this is not the case. Write $\mathcal{L} = \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_n$, where each $\mathcal{L}_i$ is recognized by an ordered DFA with a single final state, which is possible by Lemma 3.3.3. Then $\mathcal{L}\mathcal{L}' = (\mathcal{L}_1\mathcal{L}') \cup \cdots \cup (\mathcal{L}_n\mathcal{L}')$. Each $\mathcal{L}_i\mathcal{L}'$ is recognized by an ordered DFA by the previous paragraph. Thus $\mathcal{L}\mathcal{L}'$ is as well, by Lemma 3.3.2. □

Lemma 3.3.5. *Let $\mathcal{L}$ be a language recognized by an ordered DFA and let $\mathcal{L}' = \Pi^\star$ for some subset Π of Σ. Then the concatentation $\mathcal{L}\mathcal{L}'$ is recognized by an ordered DFA.*

Proof. Suppose first that $\mathcal{L}$ is accepted by the ordered DFA $(Q, T, \sigma, \mathcal{F})$, where $\mathcal{F} = \{\tau\}$. Let $Q' = Q \amalg \{\tau', \rho\}$, and define a transition function T' as follows. First, $T'(c, \alpha) = T(c, \alpha)$ for $\alpha \in Q \setminus \{\tau\}$. Let Δ be the set of elements $c \in \Sigma$ such that $T(c, \tau) = \tau$. We define

$$T'(c, \tau) = \begin{cases} \tau & \text{if } c \in \Delta \\ \tau' & \text{if } c \in \Pi \setminus \Delta \\ \rho & \text{if } c \notin \Pi \cup \Delta \end{cases}, \qquad T'(c, \tau') = \begin{cases} \tau' & \text{if } c \in \Pi \\ \rho & \text{if } c \notin \Pi \end{cases}.$$

Finally, we define $T'(c, \rho) = \rho$ for all $c \in \Sigma$. One easily verifies that $(Q', T', \sigma, \{\tau, \tau'\})$ is an ordered DFA accepting $\mathcal{L}\mathcal{L}'$. The deduction of the general case from this special case proceeds exactly as the corresponding argument in the proof of Lemma 3.3.4. □

Lemma 3.3.6. *The language recognized by an ordered automata is ordered.*

Proof. Let $(Q, T, \sigma, \mathcal{F})$ be an ordered automata, and let $\mathcal{L}$ be the language it recognizes. We show that $\mathcal{L}$ is ordered, following the proof of [HU, Thm. 2.4]. Enumerate the states Q as $\{\alpha_1, \dots, \alpha_s\}$ in such a way that if $\alpha_i \leq \alpha_j$ then $i \leq j$. Let $\mathcal{L}_{i,j}^k$ be the set of words $w = w_1 \cdots w_n$ such that

$$\alpha_i = \beta_0 \xrightarrow{w_1} \beta_1 \xrightarrow{w_2} \cdots \xrightarrow{w_n} \beta_n = \alpha_j,$$

with

$$\beta_1, \dots, \beta_{n-1} \in \{\alpha_1, \dots, \alpha_k\}.$$

In other words, $w \in \mathcal{L}_{i,j}^k$ if it induces a transition from α_i to α_j via intermediate states of the form α_ℓ with $\ell \leq k$. We prove by induction on k that each $\mathcal{L}_{i,j}^k$ is an ordered language.

To begin with, when $k = 0$ no intermediate states are allowed (i.e., $n = 1$), and so $\mathcal{L}^k_{i,j}$ is a subset of Σ, and therefore an ordered language. For $k \ge 1$, we have

$$\mathcal{L}^k_{i,j} = \mathcal{L}^{k-1}_{i,k}(\mathcal{L}^{k-1}_{k,k})^\star \mathcal{L}^{k-1}_{k,j} \cup \mathcal{L}^{k-1}_{i,j}.$$

Since there is no way to transition from α_k to α_i with $i < k$, any word in $\mathcal{L}^{k-1}_{k,k}$ must have length 1. Thus $\mathcal{L}^{k-1}_{k,k}$ is a subset of Σ, and so $(\mathcal{L}^{k-1}_{k,k})^\star$ is an ordered language. The above formula then establishes inductively that $\mathcal{L}^k_{i,j}$ is ordered for all k.

Let $\mathcal{F} = \{\alpha_j\}_{j \in J}$ be the set of final states, and let $\sigma = \alpha_i$ be the initial state. Then $\mathcal{L} = \bigcup_{j \in J} \mathcal{L}^s_{i,j}$, and is therefore an ordered language. □

The following is our second main result about ordered languages. To state it, we need one piece of terminology: we say that a subset Π of Σ is **repeatable** with respect to some language $\mathcal{L}$ if there exist w and w' in $\Sigma^\star$ such that $w\Pi^\star w' \subset \mathcal{L}$.

Theorem 3.3.7. *Suppose $\mathcal{L}$ is an ordered language equipped with a norm. Let $\Pi_1, \dots, \Pi_r$ be the repeatable subsets of Σ, and let $\lambda_i = \sum_{c \in \Pi_i} \mathbf{t}^{\nu(c)}$, an $\mathbf{N}$-linear combination of the t's. Then $\mathrm{H}_\mathcal{L}(\mathbf{t}) = f(\mathbf{t})/g(\mathbf{t})$ where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials, and $g(\mathbf{t})$ factors as $\prod_{i=1}^r (1 - \lambda_i)^{e_i}$ where $e_i \ge 0$.*

Proof. Choose an ordered DFA recognizing $\mathcal{L}$. We can assume that every state is $\ge$ the initial state. Call a state α **prefinal** if there exists a final state τ such that $\alpha \le \tau$. Enumerate the states as $\alpha_1, \dots, \alpha_s$ such that the following conditions hold: (1) α_1 is the initial state; (2) if $\alpha_i \le \alpha_j$ then $i \le j$; and (3) there exists n such that $\{\alpha_1, \dots, \alpha_n\}$ is the set of prefinal states. Let A be as in the proof of Theorem 3.2.2, thought of as an $s \times s$ matrix. Then A is upper-triangular. Furthermore, the diagonal entry at (i, i) is $\sum_{c \in \Pi} \mathbf{t}^{\nu(c)}$, where Π is the set of letters c which induce a transition from α_i to itself. It is clear that if α_i is prefinal then the set Π is one of the Π_j's, and so the diagonal entry at (i, i) is one of the λ_j's. If $\alpha_{t_1}, \dots, \alpha_{t_s}$ are the final states then, as in the proof of Theorem 3.2.2, we have

$$\mathrm{H}_\mathcal{L}(\mathbf{T}) = \sum_{i=0}^{s} \sum_{k=0}^{\infty} (A^k)_{1,t_s}.$$

It is clear that this is of the stated form. □

Corollary 3.3.8. *Suppose $\mathcal{L}$ is an ordered language. Then $\mathrm{H}_{\mathcal{L},\ell}(t)$ can be written in the form $f(t)/g(t)$ where $f(t)$ and $g(t)$ are polynomials, and $g(t)$ factors as $\prod_{a=1}^r (1 - at)^{e(a)}$ where $e(a) \ge 0$ and r is the cardinality of the largest repeatable subset of Σ with respect to $\mathcal{L}$.*

Remark 3.3.9. The above result can be used to show that a language is not ordered. For example, the language considered in Example 3.2.3 is not ordered. □

We now give a slight variant of the above theorem that will be convenient for applications. Let $\mathcal{L}$ be a language on Σ equipped with a norm ν with values in $\mathbf{N}^I$. A **subpartition** of I is an ordered collection $I_\bullet = (I_1, \dots, I_r)$ of disjoint subsets of I (i.e., a partition of a subset of I). A language $\mathcal{L}$ is **adapted** to a subpartition $I_\bullet$ of I if there exists an integer N with the following property: every word in $\mathcal{L}$ can be written in the form $w_1 \cdots w_r$ where all but at most N letters of w_j belong to $\nu^{-1}(I_j)$.

Theorem 3.3.10. *Suppose that $\mathcal{L}$ is an ordered language equipped with a norm ν valued in $\mathbf{N}^I$ adapted to a subpartition $\{I_1, \dots, I_r\}$. Then $\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t})$ can be written in the form $f(\mathbf{t})/g(\mathbf{t})$ where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials, and $g(\mathbf{t})$ factors as $\prod_{k=1}^n (1-\alpha_k)$, where for each k there* *exists an index j such that α_k is a linear combination of the t_i, for $i \in I_j$, with non-negative coefficients.*

Proof. It is clear that every repeatable set is contained in one of the sets $\nu^{-1}(I_j)$. □

3.4. **Quasi-ordered languages.** Let Λ be a finite abelian group and let $\varphi \colon \Sigma \to \Lambda$ be a function. Extend φ to a monoid homomorphism on $\Sigma^\star$. Given a subset S of Λ, let $\Sigma^\star_{\varphi,S}$ be the set of all words $w \in \Sigma^\star$ for which $\varphi(w) \in S$. We say that a language $\mathcal{L} \subset \Sigma^\star$ is a **congruence language** if it is of the form $\Sigma^\star_{\varphi,S}$ for some Λ, φ and S. The **modulus** of a congruence language is the exponent of the group Λ. (Recall that the **exponent** of a group is the least common multiple of the orders of all elements in the group.)

Let $F(\mathbf{t})$ be a power series in variables $\mathbf{t} = (t_1, \ldots, t_r)$. An N**-cyclotomic translate** of F is a series of the form $F(\zeta_1 t_1, \ldots, \zeta_r t_r)$, where $\zeta_1, \ldots, \zeta_r$ are Nth roots of unity.

Lemma 3.4.1. *Let Λ be a finite abelian group of exponent N, let S be a subset of Λ, and let $\psi \colon \mathbf{Z}^r_{\geq 0} \to \Lambda$ be a monoid homomorphism. Suppose that $F(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^r} a_\mathbf{n} \mathbf{t}^\mathbf{n}$ is a power series over $\mathbf{C}$. Let $G(\mathbf{t}) = \sum a_\mathbf{n} \mathbf{t}^\mathbf{n}$, where the sum is extended over those $\mathbf{n} \in \mathbf{N}$ for which $\psi(\mathbf{n}) \in S$. Then G is a $\mathbf{Q}(\zeta_N)$-linear combination of N-cyclotomic translates of F.*

Proof. We have $G(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^r} \chi(\psi(\mathbf{n})) a_\mathbf{n} \mathbf{t}^\mathbf{n}$, where $\chi \colon \Lambda \to \{0, 1\}$ is the characteristic function of S. To obtain the result, simply express χ as a $\mathbf{Q}(\zeta_N)$-linear combination of characters of Λ. □

Proposition 3.4.2. *Let $\mathcal{L}$ be a language on Σ equipped with a universal norm ν with values in $\mathbf{N}^I$, let $\mathcal{K}$ be a congruence language on Σ of modulus N, and let $\mathcal{L}' = \mathcal{L} \cap \mathcal{K}$. Then $\mathrm{H}_{\mathcal{L}',\nu}(\mathbf{t})$ is a $\mathbf{Q}(\zeta_N)$-linear combination of N-cyclotomic translates of $\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t})$.*

Proof. Choose $\varphi \colon \Sigma \to \Lambda$ and $S \subset \Lambda$ so that $\mathcal{K} = \Sigma^\star_{\varphi,S}$. Since ν is universal, the map $\varphi \colon \Sigma^\star \to \mathcal{L}$ can be factored as $\psi \circ \nu$, where $\psi \colon \mathbf{N}^I \to \Lambda$ is a monoid homomorphism. Thus if $\mathrm{H}_\mathcal{L}(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^I} a_\mathbf{n} \mathbf{t}^\mathbf{n}$, then $\mathrm{H}_{\mathcal{L}'}(\mathbf{t})$ is obtained by simply discarding the terms for which $\psi(\mathbf{n}) \notin S$. The result now follows from Lemma 3.4.1. □

A **quasi-ordered language** (of modulus N) is the intersection of an ordered language and a congruence language (of modulus N). Quasi-ordered languages are regular. The class of quasi-ordered languages is not closed under unions, intersections, or concatenations.

Our main result on quasi-ordered languages is the following theorem.

Theorem 3.4.3. *Let $\mathcal{L}$ be a quasi-ordered language of modulus N equipped with a norm valued in $\mathbf{N}^I$ adapted to a subpartition $\{I_1, \ldots, I_r\}$. Then $\mathrm{H}_\mathcal{L}(\mathbf{t})$ can be written in the form $f(\mathbf{t})/g(\mathbf{t})$, where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials with coefficients in $\mathbf{Q}(\zeta_N)$, and $g(\mathbf{t})$ factors as $\prod_{i=1}^n (1 - \lambda_k)$, where for each k there exists a j such that λ_k is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i, for $i \in I_j$.*

Proof. This follows immediately from Theorem 3.3.10 and Proposition 3.4.2. □

Definition 3.4.4. Let $N \geq 1$ be an integer. We say that a series $h \in \mathbf{Q}[\![t_1, \ldots, t_r]\!]$ is of class $\mathcal{K}_N$ if it can be expressed in the form $f(\mathbf{t})/g(\mathbf{t})$ where f and g are polynomials in the t_i with coefficients in $\mathbf{Q}(\zeta_N)$ and g factors as $\prod_{k=1}^n (1 - \lambda_i)$, where λ_i is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i.

There is also a coordinate-free version of the definition. Suppose Ξ is a finite free $\mathbf{Z}$-module and $f \in \widehat{\mathrm{Sym}}(\Xi_\mathbf{Q})$. We say that f is $\mathcal{K}_N$ if there is a $\mathbf{Z}$-basis $t_1, \ldots, t_r$ of Ξ so that f is $\mathcal{K}_N$

as a series in the t_i. This is independent of the choice of basis. We drop the N from the notation if it is irrelevant. □

Lemma 3.4.5. *Let $i\colon \Xi \to \Xi'$ be a split injection of finite free $\mathbf{Z}$-modules, and let f be a series in $\widehat{\mathrm{Sym}}(\Xi_{\mathbf{Q}})$. Suppose that $i(f) \in \widehat{\mathrm{Sym}}(\Xi'_{\mathbf{Q}})$ is $\mathcal{K}_N$. Then f is $\mathcal{K}_N$.*

Proof. Let $j\colon \Xi' \to \Xi$ be a splitting of i. Then $f = j(i(f))$. Since j clearly takes $\mathcal{K}_N$ functions to $\mathcal{K}_N$ functions, it follows that f is $\mathcal{K}_N$. □

3.5. **Unambiguous context-free languages.** Let Σ be a finite alphabet, which we also call **terminal** symbols. Let N be another finite set, disjoint from Σ, which we call the **non-terminal** symbols. A **production rule** is $n \to P(n)$ where $n \in N$ and $P(n)$ is a word in $\Sigma \cup N$. A **context-free grammar** is a tuple $(\Sigma, N, \mathcal{P}, n_0)$, where Σ and N are as above, $\mathcal{P}$ is a finite set of production rules, and n_0 is a distinguished element of N. Given such a grammar, a word w in Σ is a **valid n-expression** if there is a production rule $n \to P(n)$, where $P(n) = c_1 \cdots c_r$, and a decomposition $w = w_1 \cdots w_r$ (with $r > 1$), such that w_i is a valid c_i-expression if c_i is non-terminal, and $w_i = c_i$ if c_i is terminal. The language **recognized** by a grammar is the set of valid n_0-expressions, and a language of this form is called a **context-free language**. A grammar is **unambiguous** if for each valid n-expression w the production rule $n \to P(n)$ and decomposition of w above is unique. An **unambiguous context-free language** is one defined by such a grammar. See [St2, Definition 6.6.4] for an alternative description.

Theorem 3.5.1 (Chomsky–Schützenberger [CS])**.** *Let $\mathcal{L}$ be an unambiguous context-free language equipped with a norm ν. Then $\mathrm{H}_{\mathcal{L},\nu}(\mathbf{t})$ is an algebraic function.*

Proof. Let $\mathcal{L}$ be a context-free language and pick a grammar that recognizes $\mathcal{L}$. Consider the modified Hilbert series $\sum_{w \in \mathcal{L}} a_w \mathbf{t}^{\nu(w)}$ where ν is the universal norm and a_w is the number of ways that w can be built using the production rules. This is an algebraic function [St2, Theorem 6.6.10]. If $\mathcal{L}$ is unambiguous, then we can pick a grammar so that $a_w = 1$ for all $w \in \mathcal{L}$, and we obtain the usual Hilbert series. □

4. Representations of categories

This section introduces the main topic of this paper: representations of categories. Our goal is to lay out the main definitions and basic properties of representations and functors between categories of representations and to state some criteria for representations to be noetherian. More specifically, definitions are given in §4.1, properties of functors between categories are in §4.2, and tensor products of representations are discussed in §4.3.

4.1. **Basic definitions and results.** Let $\mathcal{C}$ be a category. If $\mathcal{C}$ is essentially small, we denote by $|\mathcal{C}|$ the set of isomorphism classes in $\mathcal{C}$. For an object x of $\mathcal{C}$, we let $\mathcal{C}_x$ be the category of morphisms from x; thus the objects of $\mathcal{C}_x$ are morphisms $x \to y$ (with y variable), and the morphisms in $\mathcal{C}_x$ are the obvious commutative triangles. We say that $\mathcal{C}$ is **directed** if every self-map in $\mathcal{C}$ is the identity. If $\mathcal{C}$ is directed then so is $\mathcal{C}_x$, for any x. If $\mathcal{C}$ is essentially small and directed, then $|\mathcal{C}|$ is naturally a poset by defining $x \le y$ if there exists a morphism $x \to y$. We say that $\mathcal{C}$ is Hom-finite if all Hom sets are finite.

Fix a nonzero ring $\mathbf{k}$ (not necessarily commutative) and let $\mathrm{Mod}_{\mathbf{k}}$ denote the category of left $\mathbf{k}$-modules. A **representation** of $\mathcal{C}$ (or a **$\mathcal{C}$-module**) over $\mathbf{k}$ is a functor $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$. A map of $\mathcal{C}$-modules is a natural transformation. We write $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ for the category of representations, which is abelian. Let M be a representation of $\mathcal{C}$. By an **element** of M we mean an element of $M(x)$ for some object x of $\mathcal{C}$. Given any set S of elements of M, there is a smallest subrepresentation of M containing S; we call this the subrepresentation **generated by** S. We say that M is **finitely generated** if it is generated by a finite set of elements. For a morphism $f: x \to y$ in $\mathcal{C}$, we typically write f_* for the given map of $\mathbf{k}$-modules $M(x) \to M(y)$.

Let x be an object of $\mathcal{C}$. Define a representation P_x of $\mathcal{C}$ by $P_x(y) = \mathbf{k}[\mathrm{Hom}(x, y)]$, i.e., $P_x(y)$ is the free left $\mathbf{k}$-module with basis $\mathrm{Hom}(x, y)$. For a morphism $f: x \to y$, we write e_f for the corresponding element of $P_x(y)$. If M is another representation then $\mathrm{Hom}(P_x, M) = M(x)$. This shows that $\mathrm{Hom}(P_x, -)$ is an exact functor, and so P_x is a projective object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$. We call it the **principal projective** at x. One easily sees that an object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is finitely generated if and only if it is a quotient of a finite direct sum of principal projective objects.

An object of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every ascending chain of subobjects stabilizes; this is equivalent to every subrepresentation being finitely generated. The category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every finitely generated object in it is.

Proposition 4.1.1. *The category* $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ *is noetherian if and only if every principal projective is noetherian.*

Proof. Obviously, if $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian then so is every principal projective. Conversely, suppose every principal projective is noetherian. Let M be a finitely generated object. Then M is a quotient of a finite direct sum P of principal projectives. Since noetherianity is preserved under finite direct sums, P is noetherian. And since noetherianity descends to quotients, M is noetherian. This completes the proof. □

4.2. **Pullback functors.** Let $\Phi: \mathcal{C} \to \mathcal{C}'$ be a functor. There is then a pullback functor on representations $\Phi^*: \mathrm{Rep}_{\mathbf{k}}(\mathcal{C}') \to \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$. In this section, we study how Φ^* interacts with finiteness conditions. The following definition is of central importance:

Definition 4.2.1. We say that Φ satisfies **property (F)** if the following condition holds: given any object x of $\mathcal{C}'$ there exist finitely many objects $y_1, \dots, y_n$ of $\mathcal{C}$ and morphisms $f_i: x \to \Phi(y_i)$ in $\mathcal{C}'$ such that for any object y of $\mathcal{C}'$ and any morphism $f: x \to \Phi(y)$ in $\mathcal{C}'$, there exists a morphism $g: y_i \to y$ in $\mathcal{C}$ such that $f = \Phi(g) \circ f_i$. □

Proposition 4.2.2. *Suppose* $\Phi: \mathcal{C}_1 \to \mathcal{C}_2$ *and* $\Psi: \mathcal{C}_2 \to \mathcal{C}_3$ *satisfy property (F). Then the composition* $\Psi \circ \Phi$ *satisfies property (F).*

Proof. Let x be an object of $\mathcal{C}_3$. Let $f_i: x \to \Psi(y_i)$, for i in a finite set I, be the morphisms in $\mathcal{C}_3$ provided by property (F). For each $i \in I$, let $g_{i,j}: y_i \to \Phi(z_{i,j})$, for j in a finite set J_i, be the morphisms in $\mathcal{C}_2$ provided by property (F). Let $h_{i,j}: x \to \Psi(\Phi(z_{i,j}))$ be the composition $\Psi(g_{i,j}) \circ f_i$. Suppose we are given a morphism $h: x \to \Psi(\Phi(z))$. Then we can write $h = \Psi(h') \circ f_i$ for some $i \in I$, where $h': y_i \to \Phi(z)$. We can then write $h' = \Phi(h'') \circ g_{i,j}$ for some $j \in J_i$, where $h'': z_{i,j} \to z$. Thus $h = \Psi(\Phi(h'')) \circ h_{i,j}$, and so $\Psi \circ \Phi$ satisfies property (F). □

Proposition 4.2.3. *Let* $\Phi: \mathcal{C}_1 \to \mathcal{C}_2$ *and* $\Psi: \mathcal{C}_2 \to \mathcal{C}_3$ *be functors. Suppose that* Φ *is essentially surjective and* $\Psi \circ \Phi$ *satisfies property (F). Then* Ψ *satisfies property (F).*

Proof. Let x be an object of $\mathcal{C}_3$, and let $f_i: x \to \Psi(\Phi(z_i))$ be the morphisms in $\mathcal{C}_3$ provided by property (F). Let $y_i = \Phi(z_i)$. Suppose $f: x \to \Psi(y)$ is a morphism in $\mathcal{C}_3$. Since Φ is essentially surjective, we can assume $y = \Phi(z)$. Thus there is a map $g\colon z_i \to z$ in $\mathcal{C}_1$ such that $f = \Phi(\Psi(g)) \circ f_i$. Letting $g' = \Phi(g)$, we obtain $f = \Phi(g') \circ f_i$, which shows that Ψ satisfies property (F). □

Proposition 4.2.4. *Suppose $\Phi\colon \mathcal{C} \to \mathcal{C}'$ satisfies property (F). Then Φ^* takes finitely generated objects of $\operatorname{Rep}_{\mathbf{k}}(\mathcal{C}')$ to finitely generated objects of $\operatorname{Rep}_{\mathbf{k}}(\mathcal{C})$.*

Proof. It suffices to show that Φ^* takes principal projectives to finitely generated representations. Thus let P_x be the principal projective of $\operatorname{Rep}_{\mathbf{k}}(\mathcal{C}')$ at an object x. Note that $\Phi^*(P_x)(y)$ has for a basis the elements e_f for $f \in \operatorname{Hom}_{\mathcal{C}'}(x, \Phi(y))$. Let $f_i\colon x \to \Phi(y_i)$ be as in the definition of property (F). We claim that the e_{f_i} generate $\Phi^*(P_x)$. Indeed, given any $f\colon x \to y$, we have $f = \Phi(g) f_i$ for some $g\colon y_i \to y$ in $\mathcal{C}$, and so $e_f = g_*(e_{f_i})$. Thus every e_f belongs to the submodule generated by the e_{f_i}, which proves the proposition. □

Proposition 4.2.5. *Suppose that $\Phi\colon \mathcal{C} \to \mathcal{C}'$ is an essentially surjective functor. Let M be an object of $\operatorname{Rep}_{\mathbf{k}}(\mathcal{C}')$ such that $\Phi^*(M)$ is finitely generated (resp. noetherian). Then M is finitely generated (resp. noetherian).*

Proof. Let S be a set of elements of $\Phi^*(M)$. Let S' be the correspond set of elements of M. (Thus if S contains $m \in \Phi^*(M)(y)$ then S' contains $m \in M(\Phi(y))$.) If N is a subrepresentation of M containing S' then $\Phi^*(N)$ is a subrepresentation of $\Phi^*(M)$ containing S. It follows that if N (resp. N') is the subrepresentation of M (resp. $\Phi^*(M)$) generated by S' (resp. S), then $N' \subset \Phi^*(N)$. Thus if S generates $\Phi^*(M)$ then $\Phi^*(N) = \Phi^*(M)$, which implies $N = M$ since Φ is essentially surjective, i.e., S generates M. In particular, if $\Phi^*(M)$ is finitely generated then so is M.

Now suppose that $\Phi^*(M)$ is noetherian. Given a subrepresentation N of M, we obtain a subrepresentation $\Phi^*(N)$ of $\Phi^*(M)$. Since $\Phi^*(M)$ is noetherian, it follows that $\Phi^*(N)$ is finitely generated. Thus N is finitely generated, and so M is noetherian. □

Proposition 4.2.6. *Let $\Phi\colon \mathcal{C} \to \mathcal{C}'$ be an essentially surjective functor satisfying property (F) and suppose $\operatorname{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian. Then $\operatorname{Rep}_{\mathbf{k}}(\mathcal{C}')$ is noetherian.*

Proof. Let M be a finitely generated representation of $\mathcal{C}'$. Then $\Phi^*(M)$ is finitely generated by Proposition 4.2.4, and therefore noetherian, and so M is noetherian by Proposition 4.2.5. □

4.3. **Tensor products.** Suppose $\mathbf{k}$ is a commutative ring. Given representations M and N of categories $\mathcal{C}$ and $\mathcal{D}$ over $\mathbf{k}$, we define their **external tensor product**, denoted $M \boxtimes N$, to be the representation of $\mathcal{C} \times \mathcal{D}$ given by $(x, y) \mapsto M(x) \otimes_{\mathbf{k}} N(y)$. One easily sees that if M and N are finitely generated then so is $M \boxtimes N$. If $\mathcal{C} = \mathcal{D}$ then we define the **pointwise tensor product** of M and N, denoted $M \odot N$, to be the representation of $\mathcal{C}$ given by $x \mapsto M(x) \otimes_{\mathbf{k}} N(x)$. The two tensor products are related by the identity $M \odot N = \Delta^*(M \boxtimes N)$, where $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the diagonal functor.

We say that a category $\mathcal{C}$ satisfies **property (F)** if the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ satisfies property (F). Explicitly, this means that for every pair of objects x and x', we can find objects $y_1, \dots, y_n$ and morphisms $f_i\colon x \to y_i$ and $f'_i\colon x' \to y_i$ such that given any morphisms $f\colon x \to y$ and $f'\colon x' \to y$ there exists a morphism $g\colon y_i \to y$, for some i, such that $f = g \circ f_i$ and $f' = g \circ f'_i$.

The following result follows immediately from the above discussion and Proposition 4.2.4.

Proposition 4.3.1. *Suppose $\mathcal{C}$ satisfies property (F). Then for any commutative ring $\mathbf{k}$, the pointwise tensor product of finitely generated representations is finitely generated,*

5. Noetherianity and Gröbner categories

This section introduces another main topic of this paper: Gröbner bases for representations of categories. Definitions and basic properties of Gröbner bases and initial submodules are given in §5.1 and we state a Gröbner-theoretic approach to proving the noetherian property. In §5.2, we introduce the notions of Gröbner and quasi-Gröbner categories, which are those categories for which the formalism of the first section can be applied. Nearly all the categories that we study in this paper are (quasi-)Gröbner.

5.1. **Gröbner bases.** Let $\mathcal{C}$ be an essentially small category and let **Set** be the category of sets. Fix a functor $S\colon \mathcal{C} \to \mathbf{Set}$, and let $P = \mathbf{k}[S]$, i.e., $P(x)$ is the free $\mathbf{k}$-module on the set $S(x)$. The purpose of this section is to develop a theory of Gröbner bases for P, and use this theory to give a combinatorial criterion for P to be noetherian.

We begin by defining a partially ordered set $|S|$ associated to S, which is one of the main combinatorial objects of interest. We say that a subfunctor of S is **principal** if it is generated by a single element. (Here we use "element" and "generated" as with representations of $\mathcal{C}$.) We define $|S|$ to be the set of principal subfunctors of S, partially ordered by reverse inclusion. We can describe this poset more concretely as follows, at least when $\mathcal{C}$ is small. Let $\widetilde{S}$ be the set of all elements of S, i.e., $\widetilde{S} = \bigcup_{x \in \mathcal{C}} S(x)$. Given $f \in S(x)$ and $g \in S(y)$, define $f \le g$ if there exists $h\colon x \to y$ with $h_*(f) = g$. Then $\le$ defines a quasi-order on $\widetilde{S}$, i.e., an order which is transitive and reflexive, but not necessarily anti-symmetric. Define an equivalence relation on $\widetilde{S}$ by $f \sim g$ if $f \le g$ and $g \le f$. The poset $|S|$ is the quotient of $\widetilde{S}$ by $\sim$, with the induced partial order.

Given $f \in S(x)$, we write e_f for the corresponding element of $P(x)$. We say that an element of P is a **monomial** if it is of the form λe_f for some $\lambda \in \mathbf{k}$ and $f \in S(x)$. A subrepresentation M of P is **monomial** if $M(x)$ is spanned by the monomials it contains, for all objects x.

We now classify the monomial subrepresentations of P in terms of $|S|$. Given $f \in \widetilde{S}$, let $I_M(f)$ be the set of all $\lambda \in \mathbf{k}$ such that λe_f belongs to M. Then $I_M(f)$ is an ideal of $\mathbf{k}$. If $f \le g$ then $I_M(f) \subset I_M(g)$. Indeed, suppose $f \in S(x)$ and $g \in S(y)$, and let $h\colon x \to y$ satisfy $h_*(f) = g$. If $\lambda \in I_M(f)$ then λe_f belongs to $M(x)$, and so $h_*(\lambda e_f) = \lambda e_g$ belongs to $M(y)$, and so λ belongs to $I_M(g)$. In particular, $I_M(f) = I_M(g)$ if $f \sim g$.

Let $\mathcal{I}(\mathbf{k})$ be the poset of left-ideals in $\mathbf{k}$ and let $\mathcal{M}(P)$ be the poset of monomial subrepresentations of P. Given $M \in \mathcal{M}(P)$, we have constructed an order-preserving function $I_M\colon |S| \to \mathcal{I}(\mathbf{k})$, i.e., an element of $\mathcal{F}(|S|, \mathcal{I}(\mathbf{k}))$ (see §2).

Proposition 5.1.1. *The map $I\colon \mathcal{M}(P) \to \mathcal{F}(|S|, \mathcal{I}(\mathbf{k}))$ is an isomorphism of posets.*

Proof. Suppose that for every $f \in |S|$ we have a left-ideal $I(f)$ of $\mathbf{k}$ such that for $f \le g$ we have $I(f) \subseteq I(g)$. We then define a monomial subrepresentation M of P by $M(x) = \sum_{f \in S(x)} I(f) e_f$. This defines a function $\mathcal{F}(|S|, \mathcal{I}(\mathbf{k})) \to \mathcal{M}(P)$, which is clearly inverse to I. It is clear from the constructions that I and its inverse are order-preserving, and so I is an isomorphism of posets. □

Corollary 5.1.2. *The following are equivalent (assuming P is non-zero).*

(a) *Every monomial subrepresentation of P is finitely generated.*

(b) *The poset* $\mathcal{M}(P)$ *satisfies ACC.*
(c) *The poset* $|S|$ *is noetherian and* $\mathbf{k}$ *is left-noetherian.*

Proof. The equivalence of (a) and (b) is standard, while the equivalence of (b) and (c) follows from Propositions 5.1.1 and 2.6. (Note: $|S|$ is non-empty if $P \neq 0$, and $\mathcal{I}(\mathbf{k})$ contains two distinct comparable elements, namely the zero and unit ideals.) □

To connect arbitrary subrepresentations of P to monomial subrepresentations, we need a theory of monomial orders. Let **WO** be the category whose objects are well-ordered sets and whose morphisms are strictly order-preserving functions. There is a forgetful functor $\mathbf{WO} \to \mathbf{Set}$. An **ordering** on S is a lifting of S to **WO**. More concretely, an ordering on S is a choice of well-order on $S(x)$, for each $x \in \mathcal{C}$, such that for every morphism $x \to y$ in $\mathcal{C}$ the induced map $S(x) \to S(y)$ is strictly order-preserving. We write $\preceq$ for an ordering. We say that S is **orderable** if it admits an ordering.

Suppose $\preceq$ is an ordering on S. Given non-zero $\alpha = \sum_{f \in S(x)} \lambda_f e_f$ in $P(x)$, we define the **initial term** of α, denoted $\mathrm{init}(\alpha)$, to be $\lambda_g e_g$, where $g = \max_{\preceq}\{f \mid \lambda_f \neq 0\}$. The **initial variable** of α, denoted $\mathrm{init}_0(\alpha)$, is g. Now let M be a subrepresentation of P. We define the **initial subrepresentation** of M, denoted $\mathrm{init}(M)$, as follows: $\mathrm{init}(M)(x)$ is the $\mathbf{k}$-span of the elements $\mathrm{init}(\alpha)$ for non-zero $\alpha \in M(x)$. The name is justified by the following result.

Proposition 5.1.3. *Notation as above,* $\mathrm{init}(M)$ *is a subrepresentation of* M*.*

Proof. Let $\alpha = \sum_{i=1}^n \lambda_i e_{f_i}$ be an element of $M(x)$, ordered so that $f_i \prec f_1$ for all $i > 1$. Thus $\mathrm{init}(\alpha) = \lambda_1 e_{f_1}$. Let $g \colon x \to y$ be a morphism. Then $g_*(\alpha) = \sum_{i=1}^n \lambda_i e_{g_*(f_i)}$. Since $g_* \colon S(x) \to S(y)$ is strictly order-preserving, we have $g_*(f_i) \prec g_*(f_1)$ for all $i > 1$. Thus $\mathrm{init}(g_*(\alpha)) = \lambda_1 e_{gf_1}$, or, in other words, $\mathrm{init}(g_*(\alpha)) = g_*(\mathrm{init}(\alpha))$. This shows that g_* maps $\mathrm{init}(M)(x)$ into $\mathrm{init}(M)(y)$, which completes the proof. □

Proposition 5.1.4. *Suppose* $N \subseteq M$ *are subrepresentations of* P *and* $\mathrm{init}(N) = \mathrm{init}(M)$*. Then* $M = N$*.*

Proof. Assume that $M(x) \neq N(x)$ for some object x of $\mathcal{C}$. Let $K \subset S(x)$ be the set of all elements which appear as the initial variable of some element of $M(x)$ not belonging to $N(x)$. Then K is non-empty, and therefore has a minimal element f with respect to $\preceq$. Let α be an element of $M(x)$ not belonging to $N(x)$ with $\mathrm{init}_0(\alpha) = f$. By assumption, there exists $\beta \in N$ with $\mathrm{init}(\alpha) = \mathrm{init}(\beta)$. But then $\alpha - \beta$ is also an element of $M(x)$ not belonging to $N(x)$, and $\mathrm{init}_0(\alpha - \beta) \prec \mathrm{init}_0(\alpha)$, a contradiction. Thus $M = N$. □

Let M be a subrepresentation of P. A set $\mathfrak{G}$ of elements of M is a **Gröbner basis** of M if $\{\mathrm{init}(\alpha) \mid \alpha \in \mathfrak{G}\}$ generates $\mathrm{init}(M)$. Note that M has a finite Gröbner basis if and only if $\mathrm{init}(M)$ is finitely generated. As usual, we have:

Proposition 5.1.5. *Let* $\mathfrak{G}$ *be a Gröbner basis of* M*. Then* $\mathfrak{G}$ *generates* M*.*

Proof. Let $N \subseteq M$ be the subrepresentation generated by $\mathfrak{G}$. Then $\mathrm{init}(N)$ contains $\mathrm{init}(\alpha)$ for all $\alpha \in \mathfrak{G}$, and so $\mathrm{init}(N) = \mathrm{init}(M)$. Thus $M = N$ by Proposition 5.1.4. □

We now come to our main result:

Theorem 5.1.6. *Suppose* $\mathbf{k}$ *is left-noetherian,* S *is orderable, and* $|S|$ *is noetherian. Then every subrepresentation of* P *has a finite Gröbner basis. In particular,* P *is a noetherian object of* $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$*.*

Proof. Let $M \subseteq P$ be a subrepresentation. By Lemma 5.1.2, $\mathrm{init}(M)$ is finitely generated, so M has a finite Gröbner basis. Then M is finitely generated by Proposition 5.1.5, so P is noetherian. □

Remark 5.1.7. We have not touched on the important topic of algorithms for Gröbner bases, which is one of the most attractive features of the theory! See [Ei, Chapter 15] for a self-contained exposition of this theory for the classical situation of modules over polynomial rings. Two important results are Buchberger's criterion using S-pairs for determining if a set of elements is actually a Gröbner basis, and Schreyer's extension of this idea to calculate free resolutions. These ideas can be extended to our settings and will be developed in future work. There is one important difference to mention: for monomials in a polynomial ring, the LCM is well-defined, but in our setting, one has to replace the LCM of two morphisms with a set of morphisms in general. □

5.2. **Gröbner categories.** Let $\mathcal{C}$ be an essentially small category. For an object x, let $S_x \colon \mathcal{C} \to \mathbf{Set}$ be the functor given by $S_x(y) = \mathrm{Hom}(x, y)$. Note that $P_x = \mathbf{k}[S_x]$.

Definition 5.2.1. We say that $\mathcal{C}$ is **Gröbner** if, for all objects x, the functor S_x is orderable and the poset $|S_x|$ is noetherian. We say that $\mathcal{C}$ is **quasi-Gröbner** if there exists a Gröbner category $\mathcal{C}'$ and an essentially surjective functor $\mathcal{C}' \to \mathcal{C}$ satisfying property (F). □

The following theorem is one of the two main theoretical results of this paper. It connects the purely combinatorial condition "(quasi-)Gröbner" with the algebraic condition "noetherian" for representations.

Theorem 5.2.2. *Let $\mathcal{C}$ be a quasi-Gröbner category. Then for any left-noetherian ring $\mathbf{k}$, the category $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian.*

Proof. First suppose that $\mathcal{C}$ is a Gröbner category. Then every principal projective of $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, by Theorem 5.1.6, and so $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian by Proposition 4.1.1.

Now suppose that $\mathcal{C}$ is quasi-Gröbner, and let $\Phi \colon \mathcal{C}' \to \mathcal{C}$ be an essentially surjective functor satisfying property (F), with $\mathcal{C}'$ Gröbner. Then $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}')$ is noetherian, by the previous paragraph, and so $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian by Proposition 4.2.6. □

Remark 5.2.3. If the functor S_x is orderable, then the group $\mathrm{Aut}(x) = S_x(x)$ admits a well-order compatible with the group operation, and is therefore trivial. Thus, in a Gröbner category, there are no non-trivial automorphisms. □

The definition of Gröbner is rather abstract. We now give a more concrete reformulation when $\mathcal{C}$ is a directed category, which is the version we will apply in practice. Let x be an object of $\mathcal{C}$. An **admissible order** on $|\mathcal{C}_x|$ is a well-order $\preceq$ satisfying the following additional condition: given $f, f' \in \mathrm{Hom}(x, y)$ with $f \prec f'$ and $g \in \mathrm{Hom}(y, z)$, we have $gf \prec gf'$.

Proposition 5.2.4. *Let $\mathcal{C}$ be a directed category. Then $\mathcal{C}$ is Gröbner if and only if for all x the poset $|\mathcal{C}_x|$ is noetherian and admits an admissible order.*

Proof. It suffices to treat the case where $\mathcal{C}$ is small. Let x be an object of $\mathcal{C}$. The sets $\mathrm{Ob}(\mathcal{C}_x)$ and $\widetilde{S}_x$ are equal: both are the set of all morphisms $x \to y$. In $|\mathcal{C}_x|$, two morphisms f and g are identified if $g = hf$ for some isomorphism h. In $|S_x|$, two morphisms f and g are identified if there are morphisms h and h' such that $g = hf$ and $f = h'g$. Since $\mathcal{C}$ is directed, h and h' are necessarily isomorphisms. So $|\mathcal{C}_x|$ and $|S_x|$ are the same quotient of $\mathrm{Ob}(\mathcal{C}_x) = \widetilde{S}_x$. The orders on each are defined in the same way, and thus the two are isomorphic posets. Thus $|\mathcal{C}_x|$ is noetherian if and only if $|S_x|$ is.

Now let $\preceq$ be an admissible order on $|\mathcal{C}_x|$. Since $\mathcal{C}$ is directed, the natural map $S_x(y) \to |\mathcal{C}_x|$ is an injection. We define a well-ordering on $S_x(y)$ by restricting $\preceq$ to it. One readily verifies that this defines an ordering of S_x.

Finally, suppose that $\preceq$ is an ordering on S_x. Let $\mathcal{C}_0$ be a set of isomorphism class representatives for $\mathcal{C}$. Since $\mathcal{C}$ is directed, the natural map $\coprod_{y \in \mathcal{C}_0} S_x(y) \to |\mathcal{C}_x|$ is a bijection. Choose an arbitrary well-ordering $\preceq$ on $\mathcal{C}_0$. Define a well-order $\preceq$ on $\coprod_{y \in \mathcal{C}_0} S_x(y)$ as follows. If $f \colon x \to y$ and $g \colon x \to z$ then $f \preceq g$ if $y \prec z$, or $y = z$ and $f \prec g$ as elements of $S_x(y)$. One easily verifies that this induces an admissible order on $|\mathcal{C}_x|$. □

In the remainder of this section, we establish some formal properties of (quasi-)Gröbner categories.

Proposition 5.2.5. *The cartesian product of two (quasi-) Gröbner categories is (quasi-) Gröbner.*

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be Gröbner categories. Given $x \in \mathcal{C}$ and $y \in \mathcal{D}$, we have $S_{(x,y)} = S_x \times S_y$. (Here we write S_x for the pullback to $\mathcal{C} \times \mathcal{D}$ of the functor S_x on $\mathcal{C}$.) Given well-ordered sets X and Y, the set $X \times Y$ is well-ordered via $(x, y) \preceq (x', y')$ if $x \prec x'$, or $x = x'$ and $y \preceq y'$. This construction defines a functor $\mathbf{WO} \times \mathbf{WO} \to \mathbf{WO}$ lifting the product on $\mathbf{Set}$. Since S_x and S_y are orderable, it follows that $S_{(x,y)}$ is as well. The poset $|S_{(x,y)}| = |S_x| \times |S_y|$ is noetherian by Proposition 2.5. We conclude that $\mathcal{C} \times \mathcal{D}$ is Gröbner.

Now suppose that $\mathcal{C}$ and $\mathcal{D}$ are quasi-Gröbner. Let $\Phi \colon \mathcal{C}' \to \mathcal{C}$ and $\Psi \colon \mathcal{D}' \to \mathcal{D}$ be essentially surjective functors satisfying property (F), with $\mathcal{C}'$ and $\mathcal{D}'$ Gröbner. Then $\Phi \times \Psi \colon \mathcal{C}' \times \mathcal{D}' \to \mathcal{C} \times \mathcal{D}$ is essentially surjective and satisfies property (F). As $\mathcal{C}' \times \mathcal{D}'$ is Gröbner, by the first paragraph, we find that $\mathcal{C} \times \mathcal{D}$ is quasi-Gröbner. □

Proposition 5.2.6. *Suppose that $\Phi \colon \mathcal{C}' \to \mathcal{C}$ is an essentially surjective functor satisfying property (F) and $\mathcal{C}'$ is quasi-Gröbner. Then $\mathcal{C}$ is quasi-Gröbner.*

Proof. Let $\Psi \colon \mathcal{C}'' \to \mathcal{C}'$ be an essentially surjective functor satisfying property (F), with $\mathcal{C}''$ Gröbner. Then $\Phi \circ \Psi$ is essentially surjective and satisfies property (F) by Proposition 4.2.2, which completes the proof. □

Definition 5.2.7. We say that a functor $\Phi \colon \mathcal{C}' \to \mathcal{C}$ satisfies **property (S)** if the following condition holds: if $f \colon x \to y$ and $g \colon x \to z$ are morphisms in $\mathcal{C}'$ and there exists a morphism $\widetilde{h} \colon \Phi(y) \to \Phi(z)$ such that $\Phi(g) = \widetilde{h}\Phi(f)$ then there exists a morphism $h \colon y \to z$ such that $g = hf$. A subcategory $\mathcal{C}' \subset \mathcal{C}$ satisfies property (S) if the inclusion functor does. □

Proposition 5.2.8. *Let $\Phi \colon \mathcal{C}' \to \mathcal{C}$ be a faithful functor satisfying property (S) and suppose $\mathcal{C}$ is Gröbner. Then $\mathcal{C}'$ is Gröbner.*

Proof. Let x be an object of $\mathcal{C}'$. We first claim that the natural map $i \colon |S_x| \to |S_{\Phi(x)}|$ induced by Φ is strictly order-preserving. Indeed, let $f \colon x \to y$ and $g \colon x \to z$ be elements of $|S_x|$ such that $i(f) \le i(g)$. Then there exists $\widetilde{h} \colon \Phi(y) \to \Phi(z)$ such that $\widetilde{h}\Phi(f) = \Phi(g)$. By property (S), there exists $h \colon y \to z$ such that $hf = g$. Thus $f \le g$, establishing the claim. It follows from this, and the noetherianity of $|S_{\Phi(x)}|$, that $|S_x|$ is noetherian. Finally, an ordering on $S_{\Phi(x)}$ obviously induces one on $S_{\Phi(x)}|_{\mathcal{C}'}$, and this restricts to one on S_x. (Note that S_x is a subfunctor of $S_{\Phi(x)}|_{\mathcal{C}'}$ since Φ is faithful.) □

We end with a simple combinatorial criterion for $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ to have Krull dimension 0. Our criterion is far from necessary, but applies in at least one case of interest (see Theorem 8.4.4). Assume that $\mathbf{k}$ is a field. Given $M \in \mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$, let $M^\vee \in \mathrm{Rep}_{\mathbf{k}}(\mathcal{C}^{\mathrm{op}})$ be defined by $M^\vee(x) = \mathrm{Hom}_{\mathbf{k}}(M(x), \mathbf{k})$.

Definition 5.2.9. We say that $\mathcal{C}$ satisfies **property (D)** if the following condition holds: for every object x of $\mathcal{C}$ there exist finitely many objects $\{y_i\}_{i \in I}$ and finite subsets S_i of $\mathrm{Hom}(x, y_i)$ such that if $f \colon x \to z$ is any morphism then there exists $i \in I$ and a morphism $g \colon z \to y_i$ such that $g_*^{-1}(S_i) = \{f\}$, where g_* is the map $\mathrm{Hom}(x, z) \to \mathrm{Hom}(x, y_i)$. □

Proposition 5.2.10. *Suppose that $\mathcal{C}$ is* Hom-*finite and satisfies property (D), and $\mathbf{k}$ is a field. Then for any object x, the representation $P_x^\vee$ of $\mathcal{C}^{\mathrm{op}}$ is finitely generated.*

Proof. The space $P_x(y)$ has for a basis the elements e_f. We let e_f^* be the dual basis of $P_x^\vee(y)$. We note that the e_f^* generate $P_x^\vee$ as a representation. Let y_i and S_i be as in the definition of property (D). For $i \in I$, define $\varphi_i \in \mathbf{k}[\mathrm{Hom}(x, y_i)]^*$ by $\varphi_i = \sum_{f \in S_i} e_f^*$. We claim that the φ_i generate $P_x^\vee$. To see this, let $f \colon x \to y$ be an arbitrary morphism and let $g \colon y \to y_i$ satisfy $g_*^{-1}(S_i) = \{f\}$. To avoid confusion, let $g' \colon y_i \to y$ be the morphism g in $\mathcal{C}^{\mathrm{op}}$. Then $g'_*(\varphi_i) = g^*(\varphi_i) = e_f^*$, which completes the proof. □

Proposition 5.2.11. *Assume that $\mathbf{k}$ is a field. Suppose that $\mathcal{C}$ is* Hom-*finite and satisfies property (D), and that $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are both quasi-Gröbner. Then $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C})$ has Krull dimension 0, that is, every finitely generated representation of $\mathcal{C}$ has finite length.*

Proof. Let M be a finitely generated representation of $\mathcal{C}$. Since $\mathcal{C}$ is quasi-Gröbner, M is noetherian. It suffices to show that M is also artinian, so let $M \supseteq M_1 \supseteq M_2 \supseteq \cdots$ be a descending chain. Choose a surjection $P \to M$, where P is a finite direct sum of principal projectives. Then $M^\vee$ is a subrepresentation of $P^\vee$, and is therefore noetherian since $P^\vee$ is finitely generated (Proposition 5.2.10) and $\mathrm{Rep}_{\mathbf{k}}(\mathcal{C}^{\mathrm{op}})$ is noetherian. Let K_i be the kernel of the surjection $M^\vee \to M_i^\vee$ dual to the inclusion $M_i \subseteq M$. Then we have an ascending chain $K_1 \subseteq K_2 \subseteq \cdots \subseteq M^\vee$. This chain stabilizes by the noetherian property, so the same is true for the descending chain M_i, and hence M is artinian. □

6. Hilbert series and lingual categories

In this section we use the theory of formal languages (discussed in §3) together with the Gröbner techniques in §5 to study Hilbert series of representations of quasi-Gröbner categories. Our goal is to introduce norms, Hilbert series, lingual structures, and lingual categories and to prove a few basic properties about lingual categories.

6.1. **Normed categories and Hilbert series.** Let $\mathcal{C}$ be an essentially small category. A **norm** on $\mathcal{C}$ is a function $\nu \colon |\mathcal{C}| \to \mathbf{N}^I$, where I is a finite set. A **normed category** is a category equipped with a norm. Fix a category $\mathcal{C}$ with a norm ν with values in $\mathbf{N}^I$. As in §3.1, we let $\{t_i\}_{i \in I}$ be indeterminates. Let M be a representation of $\mathcal{C}$ over a field $\mathbf{k}$. We define the **Hilbert series** of M as

$$\mathrm{H}_{M,\nu}(\mathbf{t}) = \sum_{x \in |\mathcal{C}|} \dim_{\mathbf{k}} M(x) \cdot \mathbf{t}^{\nu(x)},$$

when this makes sense. We omit the norm ν from the notation when possible.

Remark 6.1.1. Suppose $\nu\colon |\mathcal{C}| \to \mathbf{N}^I$ has finite fibers, M is finitely generated, and $\mathcal{C}$ is Hom-finite. Then the Hilbert series of M makes sense, i.e., the coefficient of $\mathbf{t}^{\mathbf{n}}$ is finite for any $\mathbf{n} \in \mathbf{N}^I$. □

6.2. **Lingual structures.** We return to the set-up of §5.1: let $S\colon \mathcal{C} \to \mathbf{Set}$ be a functor, and let $P = \mathbf{k}[S]$. However, we now also assume that $\mathcal{C}$ is directed and normed over $\mathbf{N}^I$. We define a norm on $|S|$ as follows: given $f \in |S|$, let $\widetilde{f} \in S(x)$ be a lift, and put $\nu(f) = \nu(x)$. This is well-defined because $\mathcal{C}$ is ordered: if $\widetilde{f}' \in S(y)$ is a second lift then x and y are necessarily isomorphic. Let $\mathcal{P}$ be a class of languages (e.g., regular languages).

Definition 6.2.1. A **lingual structure** on $|S|$ is a pair (Σ, i) consisting of a finite alphabet Σ normed over $\mathbf{N}^I$ and an injection $i\colon |S| \to \Sigma^\star$ compatible with the norms, i.e., such that $\nu(i(f)) = \nu(f)$. A **$\mathcal{P}$-lingual structure** is a lingual structure satisfying the following additional condition: for every poset ideal J of $|S|$, the language $i(J)$ is of class $\mathcal{P}$. □

Theorem 6.2.2. *Suppose $\mathcal{C}$ is directed, S is orderable, $|S|$ is noetherian, and $|S|$ admits a $\mathcal{P}$-lingual structure. Let M be a subrepresentation of P. Then $\mathrm{H}_M(\mathbf{t})$ is of the form $\mathrm{H}_{\mathcal{L}}(\mathbf{t})$, where $\mathcal{L}$ is a language of class $\mathcal{P}$.*

Proof. Choose an ordering on S and let N be the initial subrepresentation of M. Then $N(x)$ is an associated graded of $M(x)$, for any object x, and so the two have the same dimension. Thus M and N have the same Hilbert series. Let $J \subset |S|$ be the set of elements f for which e_f belongs to N. Then N and the language $i(J)$ have the same Hilbert series. □

Remark 6.2.3. We abbreviate "ordered," "quasi-ordered of modulus N," "regular," and "unambiguous context-free" to O, QO_N, R, and UCF. Thus an R-lingual structure is one for which the language $i(I)$ is always regular. □

Remark 6.2.4. If $|S|$ admits a lingual structure then $S(x)$ is finite for all x. Indeed, given $\mathbf{n} \in \mathbf{N}^I$, let $\Sigma^\star_{\mathbf{n}}$ denote the set of words of norm $\mathbf{n}$; this is a finite set since all such words have length $|\mathbf{n}|$. Since i maps $S(x)$ injectively into $\Sigma^\star_{\nu(x)}$, we see that $S(x)$ is finite. □

Remark 6.2.5. Let $I_\bullet$ be a subpartition of I. We say that a lingual structure on $|S|$ is **adapted** to $I_\bullet$ if for every poset ideal J the language $i(J)$ is adapted to $I_\bullet$. Clearly then, the language $\mathcal{L}$ of Theorem 6.2.2 is adapted to $I_\bullet$. □

6.3. **Lingual categories.** Recall that S_x denotes the functor $\mathrm{Hom}_{\mathcal{C}}(x, -)$.

Definition 6.3.1. A directed normed category $\mathcal{C}$ is **$\mathcal{P}$-lingual** if $|S_x|$ admits a $\mathcal{P}$-lingual structure for all objects x. □

The following theorem is the second main theoretical result of this paper. It connects the purely combinatorial condition "$\mathcal{P}$-lingual" with the algebraic invariant "Hilbert series."

Theorem 6.3.2. *Let $\mathcal{C}$ be a $\mathcal{P}$-lingual Gröbner category and let M be a finitely generated representation of $\mathcal{C}$. Then $\mathrm{H}_M(\mathbf{t})$ is a $\mathbf{Z}$-linear combination of series of the form $\mathrm{H}_{\mathcal{L}}(\mathbf{t})$ where $\mathcal{L}$ is a language of class $\mathcal{P}$. In particular,*

- *If $\mathcal{P} = \mathrm{R}$, then $\mathrm{H}_M(\mathbf{t})$ is a rational function of the t_i.*
- *If $\mathcal{P} = \mathrm{O}$, then $\mathrm{H}_M(\mathbf{t})$ is a rational function $f(\mathbf{t})/g(\mathbf{t})$, where $g(\mathbf{t})$ factors as $\prod_{j=1}^n (1 - \lambda_j)$ and each λ_j is a $\mathbf{N}$-linear combination of the t_i.*
- *If $\mathcal{P} = \mathrm{QO}_N$, then $\mathrm{H}_M(\mathbf{t})$ is $\mathcal{K}_N$, i.e., is a rational function $f(\mathbf{t})/g(\mathbf{t})$, where $f(\mathbf{t})$ and $g(\mathbf{t})$ are polynomials with coefficients in $\mathbf{Q}(\zeta_N)$ and $g(\mathbf{t})$ factors as $\prod_{j=1}^n (1 - \lambda_j)$ and each λ_j is a $\mathbf{Z}[\zeta_N]$-linear combination of the t_i.*

- *If* $\mathcal{P} = \mathrm{UCF}$, *then* $\mathrm{H}_M(\mathbf{t})$ *is an algebraic function of the* t_i.

Proof. If M is a subrepresentation of a principal projective then the first statement follows from Theorem 6.2.2. As such representations span the Grothendieck group of finitely generated representations, the first statement holds.

The remaining statements follow from the various results about Hilbert series of $\mathcal{P}$-languages: in the regular case, Theorem 3.2.2; in the ordered case, Theorem 3.3.7; in the quasi-ordered case, Theorem 3.4.3; and in the unambiguous context-free case, the relevant result is Theorem 3.5.1. □

Remark 6.3.3. Let $I_\bullet$ be a subpartition of I. We say that $\mathcal{C}$ admits a $\mathcal{P}$-lingual structure adapted to $I_\bullet$ if each $|S_x|$ does. In this case, we can be more precise about the factorization of the denominator in the ordered and quasi-ordered cases. □

Remark 6.3.4. Another rich class of generating functions is the D-finite functions (see [St2, §6.4] for the definition). The above result suggests that there might be a property $\mathcal{P}$ of languages coming from a natural class of Gröbner categories which yields D-finite generating functions. See [MZ] for some ideas on such a class of languages. □

In the remainder of this section, we investigate the compatibility of the $\mathcal{P}$-lingual condition with products. We first must define the product of normed categories. Suppose $\mathcal{C}_1$ is normed over $\mathbf{N}^{I_1}$ and $\mathcal{C}_2$ is normed over $\mathbf{N}^{I_2}$. We give $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ the structure of a normed category over $\mathbf{N}^I$, where $I = I_1 \amalg I_2$, by $\nu(x, y) = \nu(x) + \nu(y)$. Here we have identified $|\mathcal{C}|$ with $|\mathcal{C}_1| \times |\mathcal{C}_2|$. We call $\mathcal{C}$, with this norm, the **product** of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proposition 6.3.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be $\mathcal{P}$-lingual normed categories. Suppose that the posets $|\mathcal{C}_{1,x}|$ and $|\mathcal{C}_{2,y}|$ are noetherian for all x and y and that $\mathcal{P}$ is stable under finite unions and concatenations of languages on disjoint alphabets. Then $\mathcal{C}_1 \times \mathcal{C}_2$ is also $\mathcal{P}$-lingual.*

Proof. Keep the notation from the previous paragraph. Let x be an object of $\mathcal{C}_1$, and let $i_1 \colon |\mathcal{C}_{1,x}| \to \Sigma_1^\star$ be a $\mathcal{P}$-lingual structure at x. Similarly, let y be an object of $\mathcal{C}_2$, and let $i_2 \colon |\mathcal{C}_{2,y}| \to \Sigma_2^\star$ be a $\mathcal{P}$-lingual structure at y. Let $\Sigma = \Sigma_1 \amalg \Sigma_2$, normed over $\mathbf{N}^I$ in the obvious manner. Then the following diagram commutes:

$$\begin{array}{ccccccc} |\mathcal{C}_{(x,y)}| & = & |\mathcal{C}_{1,x}| \times |\mathcal{C}_{2,y}| & \xrightarrow{i_1 \times i_2} & \Sigma_1^\star \times \Sigma_2^\star & \longrightarrow & \Sigma^\star \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbf{N}^{I_1} \oplus \mathbf{N}^{I_2} & = & \mathbf{N}^{I_1} \oplus \mathbf{N}^{I_2} & = & \mathbf{N}^I \end{array}$$

The top right map is concatenation of words. We let $i \colon |\mathcal{C}_{(x,y)}| \to \Sigma^*$ be the composition along the first line, which is clearly injective. We claim that this is a $\mathcal{P}$-lingual structure on $|\mathcal{C}_{(x,y)}|$. The commutativity of the above diagram shows that it is a lingual structure. Now suppose S is an ideal of $|\mathcal{C}_{(x,y)}|$. Since this poset is noetherian (Proposition 2.5), it is a finite union of principal ideals $S_1, \dots, S_n$. Each S_j is of the form $T_j \times T'_j$, where T_j is an ideal of $\mathcal{C}_{1,x}$ and T'_j is an ideal of $\mathcal{C}_{2,y}$. By assumption, the languages $i_1(T_j)$ and $i_2(T'_j)$ satisfy property $\mathcal{P}$. Regarding $i_1(T_j)$ and $i_2(T'_j)$ as languages over Σ, the language $i(S_j)$ is their concatentation, and therefore satisfies $\mathcal{P}$. Finally, $i(S)$ satisfies $\mathcal{P}$, as it is the union of the $i(S_j)$. □

Remark 6.3.6. It is clear that the $\mathcal{P}$-lingual structure on constructed above is $\mathcal{C}_1 \times \mathcal{C}_2$ adapted to the partition $I_1 \amalg I_2$. In fact, if $\mathcal{C}_1$ admits a $\mathcal{P}$-lingual structure adapted to a subpartition $(J_1, \ldots, J_r)$, and $\mathcal{C}_2$ one adapted to $(J'_1, \ldots, J'_s)$, then $\mathcal{C}_1 \times \mathcal{C}_2$ admits one adapted to $(J_1, \ldots, J_r, J'_1, \ldots, J'_s)$. □

The above result does not directly apply when $\mathcal{P} = \mathrm{QO}$, as this is not stable under unions (but it is stable under concatenations of languages on disjoint alphabets). However, we have the following work-around that applies in our cases of interest.

A normed category $\mathcal{C}$ is **strongly $\mathbf{QO}_N$-lingual** if for each object x there exists a lingual structure $i\colon |S_x| \to \Sigma^\star$ and a congruence language $\mathcal{K}$ on Σ such that for every poset ideal I of $|S_x|$, the language $i(I)$ is the intersection of an ordered language with $\mathcal{K}$. (If we drop the adjective "strongly," then the congruence language $\mathcal{K}$ is allowed to depend on the ideal I.) We have the following variant of Proposition 6.3.5:

Proposition 6.3.7. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be strongly QO_N-lingual normed categories. Suppose the posets $|\mathcal{C}_{1,x}|$ and $|\mathcal{C}_{2,y}|$ are noetherian for all x and y. Then $\mathcal{C}_1 \times \mathcal{C}_2$ is also strongly QO_N-lingual.*

Proof. We just indicate the necessary changes to the proof of Proposition 6.3.5. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the given congruence languages of modulus N on Σ_1 and Σ_2, regarded as languages on Σ. Write $\mathcal{K}_i = (\Sigma_i^\star)_{\varphi_i, S_i}$, where $\varphi_i\colon \Sigma_i \to \Lambda_i$. Let $\varphi\colon \Sigma \to \Lambda_1 \oplus \Lambda_2$ be the map defined by $\varphi(x) = (\varphi_1(x), 0)$ for $x \in \Sigma_1$, and $\varphi(x) = (0, \varphi_2(x))$ for $x \in \Sigma_2$, let $S = S_1 \times S_2$, and let $\mathcal{K} = \Sigma^\star_{\varphi,S}$. Then $\mathcal{K}$ is a congruence language on Σ of modulus N and has the following property: if $\mathcal{L}_1$ and $\mathcal{L}_2$ are any languages on Σ_1 and Σ_2 then $(\mathcal{L}_1 \cap \mathcal{K}_1)(\mathcal{L}_2 \cap \mathcal{K}_2) = \mathcal{L}_1\mathcal{L}_2 \cap \mathcal{K}$. Let I be a poset ideal of $|\mathcal{C}_{(x,y)}|$ which is a union of principal ideals $S_1 = T_1 \times T'_1, \ldots, S_n = T_n \times T'_n$. Then $i_1(T_j)$ is the intersection of an ordered language with $\mathcal{K}_1$, while $i_2(T'_j)$ is the intersection of an ordered language with $\mathcal{K}_2$. It follows that $i(S_j)$ is the intersection of an ordered language with $\mathcal{K}$. Thus $i(I)$ is the intersection of an ordered language with $\mathcal{K}$, completing the proof. □

Part 2. Applications

7. Categories of injections

In this first applications section, we study the category **FI** of finite sets and injective functions along with generalizations and variations. The main results are listed in §7.1, proofs are in §7.2, applications to twisted commutative algebras are in §7.3, and complementary results (one such is on the cohomology of configuration spaces of disconnected spaces) are listed in §7.4.

7.1. The categories $\mathbf{OI}_d$ and $\mathbf{FI}_d$.

Let d be a positive integer. Define $\mathbf{FI}_d$ to be the following category. The objects are finite sets. Given two finite sets S and T, a morphism $S \to T$ is a pair (f, g) where $f\colon S \to T$ is an injection and g is a d-coloring of the complement of the image of f, i.e., a function $T \setminus f(S) \to \{1, \ldots, d\}$. Define $\mathbf{OI}_d$ to be the ordered version of $\mathbf{FI}_d$: its objects are totally ordered finite sets and its morphisms are pairs (f, g) with f an order-preserving injection (no condition is placed on g). We norm $\mathbf{FI}_d$ and $\mathbf{OI}_d$ over $\mathbf{N}$ by $\nu(x) = \#x$.

Our main result about $\mathbf{OI}_d$ is the following theorem.

Theorem 7.1.1. *The category $\mathbf{OI}_d$ is* O*-lingual and Gröbner.*

We defer the proof of this theorem to the next section, and use it now to study $\mathbf{FI}_d$. Let

$$\Phi \colon \mathbf{OI}_d \to \mathbf{FI}_d$$

be the natural forgetful functor.

Theorem 7.1.2. *The category $\mathbf{FI}_d$ is quasi-Gröbner.*

Proof. We claim that Φ satisfies property (F). Let $x = [n]$ be a given object of $\mathbf{FI}_d$. If y is any totally ordered set, then any morphism $f \colon x \to y$ can be factored as $x \xrightarrow{\sigma} x \xrightarrow{f'} y$, where σ is a permutation and f' is order-preserving. It follows that we can take $y_1, \dots, y_{n!}$ to all be $[n]$, and $f_i \colon x \to \Phi(y_i)$ to be the ith element of the symmetric group S_n (under any enumeration). This establishes the claim. Since $\mathbf{OI}_d$ is Gröbner, this shows that $\mathbf{FI}_d$ is quasi-Gröbner. ☐

Corollary 7.1.3. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$ is noetherian.*

Remark 7.1.4. This result was first proved (in a slightly different language) for $\mathbf{k}$ a field of characteristic 0 in [Sn, Thm. 2.3], though the essential idea goes back to Weyl; see also [SS2, Proposition 9.2.1]. It was reproved independently for $d = 1$ (and $\mathbf{k}$ a field of characteristic 0) in [CEF]. It was then proved for $d = 1$ and $\mathbf{k}$ an arbitrary commutative ring in [CEFN] (although, as they note, commutativity is probably not necessary for their methods). The result is new if $d > 1$ and $\mathbf{k}$ is not a field of characteristic 0.

The main advantage our approach has over that of [CEFN] is that it can be generalized to other situations. For example, it is remarked after [CEFN, Proposition 2.12] that the techniques there cannot handle linear analogues of $\mathbf{FI}$; we handle these in Theorem 8.3.1. ☐

Corollary 7.1.5. *Let M be a finitely generated $\mathbf{FI}_d$-representation over a field $\mathbf{k}$. Then the Hilbert series*

$$\mathrm{H}_M(t) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}} M([n]) \cdot t^n$$

is of the form $f(t)/g(t)$, where $f(t)$ and $g(t)$ are polynomials and $g(t)$ factors as $\prod_{j=1}^{d}(1-jt)^{e_j}$ where $e_j \ge 0$. In particular, if $d = 1$ then $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial.

Proof. The Hilbert series of M agrees with the Hilbert series of $\Phi^*(M)$, where Φ is as above. The statement follows from Theorems 7.1.1 and 6.3.2 except that it only guarantees that $g(t) = \prod_{j=1}^{r}(1-jt)^{e_j}$ for some r. But, as we will see in §7.2, for each set $x = [n]$, the lingual structure on $|S_x|$ (in the notation of §6.3) is built on the set $\Sigma = \{0, \dots, d\}$ and $\{1, \dots, d\}$ is the largest repeatable subset, so we can refine this statement using Corollary 3.3.8. ☐

Remark 7.1.6. Equivalently, one can say that the exponential Hilbert series

$$\mathrm{H}'_M(t) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}} M([n]) \cdot \frac{t^n}{n!}$$

is a polynomial in t and e^t. ☐

Remark 7.1.7. This result has a history similar to Corollary 7.1.3. It was first proved for $\mathrm{char}(\mathbf{k}) = 0$ in [Sn, Thm. 3.1]. It was then independently reproved for $d = 1$ and $\mathrm{char}(\mathbf{k}) = 0$ in [CEF]. In this case, refinements and deeper properties were discovered for these Hilbert series in [SS1] (see for example [SS1, §6.8]). It was then proved for $d = 1$ and any $\mathbf{k}$ in [CEFN]. The result is new if $d > 1$ and $\mathbf{k}$ has positive characteristic. ☐

7.2. **Proof of Theorem 7.1.1.** It is clear that $\mathcal{C} = \mathbf{OI}_d$ is directed. Let n be a non-negative integer, and regard $x = [n]$ as an object of $\mathcal{C}$. Let $\Sigma = \{0, \dots, d\}$, and let $\mathcal{L}$ be the language on Σ in which 0 appears exactly n times. Partially order $\mathcal{L}$ using the subsequence order, i.e., if $s\colon [i] \to \Sigma$ and $t\colon [j] \to \Sigma$ are words then $s \le t$ if there exists $I \subseteq [j]$ such that $s = t|_I$.

Lemma 7.2.1. *The poset $\mathcal{L}$ is noetherian and every ideal is an ordered language.*

Proof. Noetherianity is an immediate consequence of Higman's lemma (Theorem 2.7). Let $w = w_1 \cdots w_n$ be a word in $\mathcal{L}$. Then the ideal generated by w is the language

$$\Pi^\star w_1 \Pi^\star w_2 \cdots \Pi^\star w_n \Pi^\star,$$

where $\Pi = \Sigma \setminus \{0\}$, and is therefore ordered. As every ideal is a finite union of principal ideals, and a finite union of ordered languages is ordered, the result follows. □

Pick $(f, g) \in \mathrm{Hom}_{\mathcal{C}}([n], [m])$. Define $h\colon [m] \to \Sigma$ to be the function which is 0 on the image of f, and equal to g on the complement of the image of f. One can recover (f, g) from h since f is required to be order-preserving and injective. This construction therefore defines an isomorphism of posets $i\colon |\mathcal{C}_x| \to \mathcal{L}$. It follows that $|\mathcal{C}_x|$ is noetherian. Furthermore, the lexicographic order on $\mathcal{L}$ (using the standard order on Σ) is easily verified to restrict to an admissible order on $|\mathcal{C}_x|$. Thus $\mathcal{C}$ is Gröbner. For $f \in |\mathcal{C}_x|$ we have $|f| = \ell(i(f))$, and so (Σ, i) is a lingual structure on $|\mathcal{C}_x|$ if we norm words in $\Sigma^\star$ by their length. Finally, an ideal of $|\mathcal{C}_x|$ gives an ordered language over Σ, and so this is an O-lingual structure.

Remark 7.2.2. The results about $\mathbf{OI}_1$ can be made more transparent with the following observation: the set of order-increasing injections $f\colon [n] \to [m]$ is naturally in bijection with monomials in $x_0, \dots, x_n$ of degree $m - n$ by assigning the monomial $\mathbf{m}_f = \prod_{i=0}^n x_i^{f(i+1)-f(i)-1}$ using the convention $f(0) = 0$ and $f(n+1) = m+1$. Given $g\colon [n] \to [m']$, there is a morphism $h\colon [m] \to [m']$ with $g = hf$ if and only if $\mathbf{m}_f$ divides $\mathbf{m}_g$. Thus the monomial submodules of P_n are in bijection with monomial ideals in the polynomial ring $\mathbf{k}[x_0, \dots, x_n]$. The statements about noetherianity and Hilbert series follow immediately. □

7.3. **Twisted commutative algebras.** We now discuss the relationship between $\mathbf{FI}_d$ and certain variants of commutative algebras called twisted commutative algebras. We begin by recalling the definition:

Definition 7.3.1. A **twisted commutative algebra** (tca) over a commutative ring $\mathbf{k}$ is an associative and unital graded $\mathbf{k}$-algebra $A = \bigoplus_{n=0}^\infty A_n$ equipped with an action of S_n on A_n such that:

(a) the multiplication map $A_n \otimes A_m \to A_{n+m}$ is $(S_n \times S_m)$-equivariant (where we use the standard embedding $S_n \times S_m \subset S_{n+m}$ for the action on A_{n+m}); and
(b) given $x \in A_n$ and $y \in A_m$ we have $xy = (yx)^\tau$, where $\tau = \tau_{m,n} \in S_{n+m}$ is defined by

$$\tau(i) = \begin{cases} i+n & \text{if } 1 \le i \le m, \\ i-m & \text{if } m+1 \le i \le n+m. \end{cases}$$

This is the "twisted commutativity" condition. □

Definition 7.3.2. A **module** over a tca A is a graded A-module $M = \bigoplus_{n=0}^\infty M_n$ (in the ordinary sense) equipped with an action of S_n on M_n such that the multiplication map $A_n \otimes M_m \to M_{n+m}$ is $(S_n \times S_m)$-equivariant. □

Example 7.3.3. Let $x_1, \ldots, x_d$ be indeterminates, each regarded as having degree 1. We define a tca $A = \mathbf{k}\langle x_1, \ldots, x_d \rangle$, the **polynomial tca** in the x_i, as follows. As a graded $\mathbf{k}$-algebra, A is just the non-commutative polynomial ring in the x_i. The S_n-action on A_n is the obvious one: on monomials it is given by $\sigma(x_{i_1} \cdots x_{i_n}) = x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n)}}$. □

We now give a more abstract way to define tca's, which clarifies some constructions. Define a representation of S_* over $\mathbf{k}$ to be a sequence $M = (M_n)_{n \ge 0}$, where M_n is a representation of S_n over $\mathbf{k}$. Given S_*-representations M and N, we define an S_*-representation $M \otimes N$ by

$$(M \otimes N)_n = \bigoplus_{i+j=n} \operatorname{Ind}_{S_i \times S_j}^{S_n} (M_i \otimes_{\mathbf{k}} N_j).$$

There is a symmetry of the tensor product obtained by switching the order of M and N and conjugating $S_i \times S_j$ to $S_j \times S_i$ in S_n via $\tau_{i,j}$. This gives the category $\operatorname{Rep}_{\mathbf{k}}(S_*)$ of S_*-representations the structure of a symmetric abelian tensor category. The general notions of commutative algebra and module in such a category specialize to tca's and their modules.

Example 7.3.4. For an integer $n \ge 0$ let $\mathbf{k}\langle n \rangle$ be the S_*-representation which is the regular representation in degree n and 0 in all other degrees. The tca $\mathbf{k}\langle x_1, \ldots, x_d \rangle$ can then be identified with the symmetric algebra on the object $\mathbf{k}\langle 1 \rangle^{\oplus d}$. The symmetric algebra on $\mathbf{k}\langle n \rangle$ is poorly understood for $n > 1$ (although some results are known for $n = 2$). □

If $\mathbf{k}$ is a field of characteristic 0 then $\operatorname{Rep}_{\mathbf{k}}(S_*)$ is equivalent to the category of polynomial representations of $\mathbf{GL}(\infty)$ over $\mathbf{k}$, as symmetric abelian tensor categories. Under this equivalence, tca's correspond to commutative associative unital $\mathbf{k}$-algebras equipped with a polynomial action of $\mathbf{GL}(\infty)$. For example, the polynomial tca $\mathbf{k}\langle x \rangle$ corresponds to the polynomial ring $\mathbf{k}[x_1, x_2, \ldots]$ in infinitely many variables, equipped with its usual action of $\mathbf{GL}(\infty)$. This point of view has been exploited by us [Sn, SS2, SS1, SS3] to establish properties of tca's in characteristic 0.

There is an obvious notion of generation for a set of elements in a tca or a module over a tca. We say that a tca is **noetherian** if every submodule of a finitely generated module is again finitely generated.

Proposition 7.3.5. *Let $A = \mathbf{k}\langle x_1, \ldots, x_d \rangle$. Then Mod_A is equivalent to $\operatorname{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$.*

Proof. Pick $M \in \operatorname{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$ and let $M_m = M([m])$. Then M_m has an action of S_m. To define $A_n \otimes M_m \to M_{n+m}$, it is enough to define how $x_{i_1} \cdots x_{i_n}$ acts by multiplication for each $(i_1, \ldots, i_n)$. Let $(f, g) \colon [m] \to [n+m]$ be the morphism in $\mathbf{FI}_d$ where $f \colon [m] \to [n+m]$ is the injection $i \mapsto n+i$ and $g \colon [n] \to [d]$ is the function $g(j) = i_j$ and let the multiplication map be the induced map $M_{(f,g)} \colon M_m \to M_{n+m}$. By definition this is $(S_n \times S_m)$-equivariant and associativity follows from associativity of composition of morphisms in $\mathbf{FI}_d$.

It is easy to reverse this process: given an A-module M, we get a functor defined on the full subcategory of $\mathbf{FI}_d$ on objects of the form $[n]$ (note that every morphism $[m] \to [n+m]$ is a composition of the injections we defined above with an automorphism of $[n+m]$). To extend this to all of $\mathbf{FI}_d$, pick a total ordering on each finite set to identify it with $[n]$. The two functors we have defined are quasi-inverse to each other. □

Remark 7.3.6. In particular, the category of $\mathbf{FI}$-modules studied in [CEF] and [CEFN] is equivalent to the category of modules over the univariate tca $\mathbf{k}\langle x \rangle$. □

Corollary 7.3.7. *If $\mathbf{k}$ is noetherian then any tca over $\mathbf{k}$ finitely generated in degree 1 is noetherian.*

Proof. A tca finitely generated in degree 1 is a quotient of $\mathbf{k}\langle x_1, \ldots, x_d\rangle$ for some d, and noetherianity passes to quotients. $\square$

Remark 7.3.8. The same techniques also prove that a twisted graded-commutative algebra finitely generated in degree 1 is noetherian. $\square$

Suppose $\mathbf{k}$ is a vector space. We define the **Hilbert series** of a module M over a tca by

$$\mathrm{H}'_M(t) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}} M([n]) \frac{t^n}{n!}.$$

As the equivalence in Proposition 7.3.5 is clearly compatible with Hilbert series, we obtain:

Corollary 7.3.9. *Let M be a finitely generated $\mathbf{k}\langle x_1, \ldots, x_d\rangle$-module. Then $\mathrm{H}'_M(t) \in \mathbf{Q}[t, e^t]$.*

7.4. **Additional results.**

Proposition 7.4.1. *The category $\mathbf{FI}_1$ satisfies property (F) (see Definition 4.2.1)*

Proof. Let x and x' be sets, and consider maps $f\colon x \to y$ and $f'\colon x' \to y$. Let $\overline{y}$ be the union of the images of f and f'. Let $g\colon \overline{y} \to y$ be the inclusion, and write $\overline{f}$ for the map $x \to \overline{y}$ induced by f, and similarly for $\overline{f}'$. Then $f = g \circ \overline{f}$ and $f' = g \circ \overline{f}'$. Since $\#\overline{y} \le \#x + \#x'$, it follows that there are only finitely many choices for $(\overline{y}, \overline{f}, \overline{f}')$, up to isomorphism. This completes the proof. $\square$

Corollary 7.4.2. *For any commutative ring $\mathbf{k}$, the pointwise tensor product of finitely generated representations of $\mathbf{FI}_1$ is finitely generated.*

Proof. This follows from Proposition 4.3.1. $\square$

Remark 7.4.3. This reproves [CEF, Proposition 2.61]. $\square$

Remark 7.4.4. The above corollary is false for $\mathbf{FI}_d$ for $d > 1$. It follows that these categories do not satisfy property (F). To see this, consider $M = P_1 \odot P_1$. Then $\dim_{\mathbf{k}} M([n]) = (d^2)^n$, so the Hilbert series is $\mathrm{H}_M(t) = (1 - d^2 n)^{-1}$. By Corollary 7.1.5, the Hilbert series of a finitely generated $\mathbf{FI}_d$-module cannot have this form if $d > 1$. $\square$

Example 7.4.5 (A variant of $\mathbf{FI}_d$)**.** Let $\mathbf{FI}_d^{\le 1}$ be the category whose objects are finite sets, and where a morphism $S \to T$ is a pair (f, m) where $f\colon S \to T$ is an injection and m is a monomial in commuting variables $x_1, \ldots, x_d$ of total degree $\#(T \setminus S)$. Consider the functor $\Phi\colon \mathbf{FI}_d \to \mathbf{FI}_d^{\le 1}$ taking a morphism (f, g) to (f, m), where $m = x_1^{e_1} \cdots x_d^{e_d}$ with $e_i = \#g^{-1}(i)$. One easily sees that Φ is essentially surjective and satisfies property (F), and so $\mathbf{FI}_d^{\le 1}$ is quasi-Gröbner. Equipped with the obvious norm over $\mathbf{N}$, the ordered category $\mathbf{OI}_d^{\le 1}$ is O-lingual. In fact, one can show that if M is a finitely generated $\mathbf{FI}_d^{\le 1}$ module then $\mathrm{H}'_M(t)$ is of the form $p(t)e^t + q(t)$ with $p, q \in \mathbf{Q}[t]$. Thus $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial.

In the language of tca's, $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_d^{\le 1})$ is the category of modules over the tca whose underlying graded ring is the commutative polynomial ring $\mathbf{k}[x_1, \ldots, x_d]$ and where the S_n actions are trivial. If $\mathbf{k}$ is a field of characteristic 0, then the associated $\mathbf{GL}(\infty)$-algebra is the coordinate ring of the rank ≤ 1 matrices (Segre variety) in $\mathbf{k}^d \otimes \mathbf{k}^\infty$ where $\mathbf{GL}(\infty)$ only acts on the $\mathbf{k}^\infty$ factor. $\square$

The next results concern the category $\mathbf{FA}$ of finite sets, with all functions as morphisms.

Theorem 7.4.6. *The category $\mathbf{FA}$ is quasi-Gröbner.*

Proof. Let $\Phi \colon \mathbf{FI}_1 \to \mathbf{FA}$ be the inclusion functor. We claim that Φ satisfies property (F). Let x be a given set. Let $f_i \colon x \to y_i$ be representatives for the isomorphism classes of surjective maps from x; these are finite in number. Then any morphism $f \colon x \to y$ factors as $f = g \circ f_i$ for some i and some injective map $g \colon y_i \to y$. This establishes the claim. Since $\mathbf{FI}_1$ is quasi-Gröbner, the result now follows from Proposition 5.2.6. □

Corollary 7.4.7. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ *is noetherian.*

We greatly improve this result in Theorem 8.4.4, using our results on $\mathbf{FS}^{\mathrm{op}}$.

Suppose $\mathbf{k}$ is a field. By what we have shown, the Hilbert series of a finitely generated $\mathbf{FA}$-module is of the form $f(t)/(1-t)^d$ for some polynomial f and $d \geq 0$. Equivalently, the function $n \mapsto \dim_{\mathbf{k}} M([n])$ is eventually polynomial. In fact, one can do better:

Theorem 7.4.8. *If* $\mathbf{k}$ *is a field and* M *is a finitely generated* $\mathbf{FA}$*-module, then the function* $n \mapsto \dim_{\mathbf{k}} M([n])$ *agrees with a polynomial for all* $n > 0$*. Equivalently, the Hilbert series of* M *is of the form* $f(t)/(1-t)^d$ *where* $\deg f \leq d$*.*

Proof. Let $\mathbf{FA}^\circ$ be the category of nonempty finite sets, so that every $\mathbf{FA}$-module gives an $\mathbf{FA}^\circ$-module by restriction. Introduce an operator Σ on $\mathbf{FA}^\circ$-modules by $(\Sigma M)(S) = M(S \amalg \{*\})$. It is easy to see that ΣM is a finitely generated $\mathbf{FA}^\circ$-module if the same is true for M. For every nonempty set S, the inclusion $S \to S \amalg \{*\}$ can be split, and so the map $M(S) \to (\Sigma M)(S)$ is an inclusion for all S. Define $\Delta M = \Sigma M / M$. Since Σ is exact, it follows that $\Delta M \to \Delta M'$ is a surjection if $M \to M'$ is a surjection.

Define $h_M(n) = \dim_{\mathbf{k}} M([n])$. We recall that a function $f \colon \mathbf{Z}_{>0} \to \mathbf{Z}$ is a polynomial of degree $\leq d$ if and only if the function g defined by $g(n) = f(n+1) - f(n)$ is a polynomial of degree $\leq d-1$. Since $h_{\Delta M}(n) = h_M(n+1) - h_M(n)$, we get that h_M is a polynomial function of degree $\leq d$ if and only if $\Delta^{d+1} M = 0$. Pick a surjection $P \to M \to 0$ with P a direct sum of principal projectives P_S. Since $h_{P_S}(n) = n^{|S|}$ is a polynomial function, P is annihilated by some power of Δ (in fact, $\Delta P_S \cong \bigoplus_{T \subsetneq S} P_T$ where the sum is over all proper subsets of S). Since Δ preserves surjections, the same is true for M, and so $h_M(n)$ is a polynomial. □

Remark 7.4.9. The category of representations $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ is studied in [WG] in the case that $\mathbf{k}$ is a field of characteristic 0, and Theorems 7.4.8 and 8.4.4 are proved.

Set-valued functors on $\mathbf{FA}$ are studied in [Do], and the functions $n \mapsto |F([n])|$ that arise are characterized. □

Remark 7.4.10. The analogue of [K1, Proposition 4.10] for $\mathbf{FA}$ holds with the same proof, i.e., if $\deg h_M(n) = r$, then the lattice of $\mathbf{FA}$-submodules of M is isomorphic to the lattice of $\mathbf{k}[\mathrm{End}([r])]$-submodules of $M([r])$. So one can prove Theorem 8.4.4 from Theorem 7.4.8. We will give a different proof using Gröbner methods. □

The paper [CEF] gives many examples of $\mathbf{FI}_1$-modules appearing "in nature." We now give some examples of $\mathbf{FI}_d$-modules. First a general construction. Let $M_1, \dots, M_d$ be $\mathbf{FI}_1$-modules over a commutative ring $\mathbf{k}$. We define $N = M_1 \otimes \cdots \otimes M_d$ to be the following $\mathbf{FI}_d$-module: on sets S, it is defined by

$$N(S) = \bigoplus_{S = S_1 \amalg \cdots \amalg S_d} M_1(S_1) \otimes \cdots \otimes M_d(S_d).$$

Given a morphism $(f \colon S \to T, g \colon T \setminus f(S) \to [d])$ in $\mathbf{FI}_d$, let $T_i = g^{-1}(i)$ and define $N(S) \to N(T)$ to be the sum of the maps

$$M_1(S_1) \otimes \cdots \otimes M_d(S_d) \to M_1(S_1 \amalg T_1) \otimes \cdots \otimes M_d(S_d \amalg T_d).$$

It is clear that if $M_1, \ldots, M_d$ are finitely generated then the same is true for N. Furthermore, there is an obvious generalization where we allow each M_i to be a $\mathbf{FI}_{n_i}$-module and N is a $\mathbf{FI}_{n_1+\cdots+n_d}$-module.

Example 7.4.11. For simplicity, we assume that $\mathbf{k}$ is a field. For a topological space Y and finite set S, let $\mathrm{Conf}_S(Y)$ be the space of injective functions $S \to Y$. For any $r \geq 0$, we get an $\mathbf{FI}_1$-module $S \mapsto \mathrm{H}^r(\mathrm{Conf}_S(Y); \mathbf{k})$, which we denote by $\mathrm{H}^r(\mathrm{Conf}(Y))$. Let $X_1, \ldots, X_d$ be connected topological spaces, and let X be their disjoint union. The Künneth theorem gives

$$\mathrm{H}^r(\mathrm{Conf}_S(X); \mathbf{k}) = \bigoplus_{p_1+\cdots+p_d=r} \bigoplus_{S=S_1 \amalg \cdots \amalg S_d} \mathrm{H}^{p_1}(\mathrm{Conf}_{S_1}(X_1); \mathbf{k}) \otimes \cdots \otimes \mathrm{H}^{p_d}(\mathrm{Conf}_{S_d}(X_d); \mathbf{k}).$$

In particular, we can write

$$\mathrm{H}^r(\mathrm{Conf}(X)) = \bigoplus_{p_1+\cdots+p_d=r} \mathrm{H}^{p_1}(\mathrm{Conf}(X_1)) \otimes \cdots \otimes \mathrm{H}^{p_d}(\mathrm{Conf}(X_d)),$$

so that we can endow $\mathrm{H}^r(\mathrm{Conf}(X))$ with the structure of an $\mathbf{FI}_d$-module. Under reasonable hypotheses on the X_i, each $\mathrm{H}^r(\mathrm{Conf}(X_i))$ is a finitely generated $\mathbf{FI}_1$-module (see [CEFN, Theorem E]), and so $\mathrm{H}^r(\mathrm{Conf}(X))$ will be finitely generated as an $\mathbf{FI}_d$-module. □

8. Categories of surjections

In this section, we study $\mathbf{FS}^{\mathrm{op}}$, the (opposite of the) category of finite sets with surjective functions and variations. The ideas used here are similar to those used in the previous section, though the category of surjective functions behaves quite differently from the category of injective functions. The main results are stated in §8.1 and the proofs are in §8.2. The results on $\mathbf{FS}^{\mathrm{op}}$ are powerful enough to prove the Lannes–Schwartz artinian conjecture from the generic representation theory of finite fields; this is done in §8.3. Finally, we give some complementary results in §8.4: one such result is that finitely generated representations over a field of the category $\mathbf{FA}$ of finite sets with all functions have finite length.

8.1. The categories OS and FS.

Define $\mathbf{FS}$ to be the category whose objects are non-empty finite sets and whose morphisms are surjections of finite sets. We define an ordered version $\mathbf{OS}$ of this category as follows. The objects are totally ordered finite sets. A morphism $S \to T$ in $\mathbf{OS}$ is a surjective map $f \colon S \to T$ such that for all $i < j$ in T we have $\min f^{-1}(i) < \min f^{-1}(j)$. We call such maps **ordered surjections**. We norm $\mathbf{FS}$ and $\mathbf{OS}$ over $\mathbf{N}$ by $\nu(x) = \#x$.

Our main result about $\mathbf{OS}$ is the following theorem.

Theorem 8.1.1. *The category* $\mathbf{OS}^{\mathrm{op}}$ *is* O*-lingual and Gröbner.*

Again, we defer the proof to the next section and use the result to study $\mathbf{FS}$. Let

$$\Phi \colon \mathbf{OS}^{\mathrm{op}} \to \mathbf{FS}^{\mathrm{op}}$$

be the natural forgetful functor.

Theorem 8.1.2. *The category* $\mathbf{FS}^{\mathrm{op}}$ *is quasi-Gröbner.*

Proof. As in the proof of Theorem 7.1.2, one can show that Φ satisfies property (F). Now use Proposition 5.2.6. □

Corollary 8.1.3. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}})$ *is noetherian.*

Corollary 8.1.4. *Let M be a finitely generated $\mathbf{FS}^{\mathrm{op}}$-representation over a field $\mathbf{k}$. Then the Hilbert series*

$$\mathrm{H}_M(t) = \sum_{n=0}^{\infty} \dim_{\mathbf{k}} M([n]) \cdot t^n$$

has the form $f(t)/g(t)$ where $f(t)$ and $g(t)$ are polynomials, and $g(t)$ factors as $\prod_{j=1}^{r}(1-jt)^{e_j}$ for some r and $e_j \geq 0$. If M is generated in degree $\leq d$, then we can take $r = d$.

Proof. The Hilbert series of M agrees with the Hilbert series of $\Phi^*(M)$, where Φ is as above. The result then follows from Theorems 8.1.1 and 6.3.2. To prove the last statement, we note that it follows from §8.2 that for a finite set $x = [n]$, the lingual structure on $|S_x|$ (in the notation of §6.3) is built on the set $\{1, \ldots, n\}$. Now use Corollary 3.3.8. □

Remark 8.1.5. Using partial fraction decomposition, a function $f(t)/g(t)$ where $g(t) = \prod_{j=1}^{r}(1-jt)^{e_j}$ can be written as a sum $\sum_{j=1}^{r} f_j(t)/(1-jt)^{e_j}$ for some polynomials $f_j(t) \in \mathbf{Q}[t]$. So Corollary 8.1.4 says that if M is a finitely generated $\mathbf{FS}^{\mathrm{op}}$-module over a field $\mathbf{k}$, then there exist polynomials $p_1, \ldots, p_r$ so that the function $n \mapsto \dim_{\mathbf{k}} M([n])$ agrees with $\sum_{j=1}^{r} p_j(n) j^n$ for $n \gg 0$. □

Example 8.1.6. Let $a \geq 0$ be an integer, and let M be the $\mathbf{FS}^{\mathrm{op}}$-module defined by $M(y) = \mathbf{k}[\mathrm{Hom}_{\mathbf{FA}}(y, [a])]$. (Recall that the morphisms in $\mathbf{FA}$ are all functions.) One readily verifies that M is finitely generated. Clearly, $M([n])$ has dimension a^n, and so $\mathrm{H}_M(t) = (1-at)^{-1}$. Thus the quantity r appearing in the corollary can be arbitrarily large, in contrast to what happens for $\mathbf{FI}_d$. □

Remark 8.1.7. Let Γ be the category of finite sets with a basepoint and basepoint-preserving functions. It is shown in [Pi] that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}})$ is equivalent to the category of functors $\Gamma^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{k}}$ that send the one-point set to the zero module. □

We record the following more general form of the corollary for later use.

Corollary 8.1.8. *Let M be a finitely generated $(\mathbf{FS}^{\mathrm{op}})^r$-representation over a field $\mathbf{k}$. Then the Hilbert series*

$$\mathrm{H}_M(t) = \sum_{\mathbf{n} \in \mathbf{N}^r} \dim_{\mathbf{k}} M([\mathbf{n}]) \cdot \mathbf{t}^{\mathbf{n}}$$

is a rational function $f(\mathbf{t})/g(\mathbf{t})$, where $g(\mathbf{t})$ factors as $\prod_{i=1}^{r}\prod_{j=1}^{\infty}(1-jt_i)^{e_{i,j}}$, where all but finitely many $e_{i,j}$ are 0. Equivalently, the exponential Hilbert series

$$\mathrm{H}'_M(t) = \sum_{\mathbf{n} \in \mathbf{N}^r} \dim_{\mathbf{k}} M([\mathbf{n}]) \cdot \frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!},$$

is a polynomial in the t_i and the e^{t_i}. (Here $\mathbf{n}! = n_1! \cdots n_r!$.)

Proof. The category $(\mathbf{OS}^{\mathrm{op}})^r$ is normed over $[r]$, and, as explained in Remark 6.3.6, it has an O-lingual structure adapted to the partition $[r] = \{1\} \amalg \cdots \amalg \{r\}$. It follows that the Hilbert series of a finitely generated $(\mathbf{OS}^{\mathrm{op}})^r$-module is of the form $f(\mathbf{t})/g(\mathbf{t})$ where f and g are polynomials, and $g(\mathbf{t})$ factors as $\prod_{k=1}^{N}(1-\lambda_k)$, where each λ_k is a positive integral multiple of some t_i. This proves the statement about H_M. The exponential version follows from simple algebra. □

8.2. **Proof of Theorem 8.1.1.** Let Σ be a finite set. Let $s\colon [n] \to \Sigma$ and $t\colon [m] \to \Sigma$ be two elements of $\Sigma^\star$. We define $s \le t$ if there exists an order-preserving injection $\varphi\colon [n] \to [m]$ such that for all $i \in [n]$ we have $s_i = t_{\varphi(i)}$, and for all $j \in [m]$ there exists $j' \le j$ in the image of φ with $t_j = t_{j'}$. This defines a partial order on $\Sigma^\star$.

Proposition 8.2.1. *The poset $\Sigma^\star$ is noetherian and every ideal is an ordered language.*

Proof. Suppose that $\Sigma^\star$ is not noetherian. We use the notion of bad sequences from the proof of Theorem 2.7.

Given $x \in \Sigma^\star$ call a value of x *exceptional* if it appears exactly once. If x has a non-exceptional value, then let $m(x)$ denote the index, counting from the end, of the first non-exceptional value. Note that $m(x) \le \#\Sigma$. Also, $\ell(x) > \#\Sigma$ implies that x has some non-exceptional value, so $m(x_i)$ is defined for $i \gg 0$. (Note: $\ell(x_i) \to \infty$, as there are only finitely many sequences of a given length.) So we can find an infinite subsequence of $x_1, x_2, \dots$ on which m is defined and is constant, say equal to m_0. We can find a further infinite subsequence where the value in the position m_0 is constant. Call this subsequence $x_{i_1}, x_{i_2}, \dots$ and let y_{i_j} be the sequence x_{i_j} with the m_0th position (counted from the end) deleted. By construction, the sequence $x_1, \dots, x_{i_1-1}, y_{i_1}, y_{i_2}, \dots$ is not bad and so some pair is comparable. Note that $y_{i_j} \le x_{i_j}$, so x_i and y_{i_j} are incomparable, and so we have $j < k$ such that $y_{i_j} \le y_{i_k}$. But this implies $x_{i_j} \le x_{i_k}$, contradicting that $x_1, x_2, \dots$ forms a bad sequence. Thus $\Sigma^\star$ is noetherian.

We now show that a poset ideal of $\Sigma^\star$ is an ordered language. It suffices to treat the case of principal ideals. For this, simply note that the principal ideal generated by $w = w_1 \cdots w_n$ is the language

$$w_1 \Pi_1^\star w_2 \Pi_2^\star \cdots w_n \Pi_n^\star,$$

where $\Pi_i = \{w_1, \dots, w_i\}$, which is clearly ordered. □

Proof of Theorem 8.1.1. It is clear that $\mathcal{C} = \mathbf{OS}^{\mathrm{op}}$ is directed. Let n be a non-negative integer, and regard $x = [n]$ as an object of $\mathcal{C}$. A morphism $f\colon [m] \to [n]$ can be regarded as a word of length m in the alphabet $\Sigma = [n]$. In this way, we have an injective map $i\colon |\mathcal{C}_x| \to \Sigma^\star$. This map is strictly order-preserving with respect to the order on $\Sigma^\star$ defined above, and so Proposition 8.2.1 implies that $|\mathcal{C}_x|$ is noetherian. The lexicographic order on words induces an admissible order on $|\mathcal{C}_x|$. For $f \in |\mathcal{C}_x|$ we have $m = \nu(f) = \ell(i(f))$, and so we have a lingual structure on $|\mathcal{C}_x|$ if we norm words in $\Sigma^\star$ by their length. Finally, an ideal of $|\mathcal{C}_x|$ gives an ordered language over Σ by Proposition 8.2.1, and so this is an O-lingual structure. □

8.3. **Linear categories.** Let R be a finite commutative ring. A linear map between free R-modules is **splittable** if the image is a direct summand. Let $\mathbf{VA}_R$ (resp. $\mathbf{VI}_R$) be the category whose objects are finite rank free R-modules and whose morphisms are splittable maps (resp. splittable injections).

Theorem 8.3.1. *The category $\mathbf{VI}_R$ is quasi-Gröbner.*

Proof. Define a functor $\Phi\colon \mathbf{FS}^{\mathrm{op}} \to \mathbf{VI}_R$ by $S \mapsto \mathrm{Hom}_R(R[S], R) = R[S]^*$. It is clear that Φ is essentially surjective. We claim that Φ satisfies property (F). Fix $U \in \mathbf{VI}_R$. Pick a finite set S and a splittable injection $f\colon U \to R[S]^*$. Dualize this to get a surjection $R[S] \to U^*$. Letting $T \subseteq U^*$ be the image of S under this map, the map factorizes as $R[S] \to R[T] \to U^*$ where the first map comes from a surjective function $S \to T$. So we can take $y_1, y_2, \dots$ to be the set of subsets of U^* which span U^* as an R-module (there are finitely many of them since R is finite) and $f_i \colon U \to R[T]^*$ to be the dual of the natural map $R[T] \to U^*$. This establishes the claim. Since $\mathbf{FS}^{\mathrm{op}}$ is quasi-Gröbner (Theorem 8.1.2), the theorem follows from Proposition 5.2.6. □

Remark 8.3.2. The above idea of using the functor Φ in the above proof was communicated to us by Aurélien Djament after a first version of these results was circulated. The original version of the above proof involved working with a version of $\mathbf{VI}_R$ consisting of spaces with ordered bases and "upper-triangular" linear maps. This idea is no longer needed to prove the desired properties for $\mathbf{VI}_R$, but is still needed in [PS] to prove noetherianity of a related category $\mathbf{VIC}_R$ (splittable maps plus a choice of splitting). □

Corollary 8.3.3. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VI}_R)$ *is noetherian.*

Corollary 8.3.4. *Let* M *be a finitely generated* $\mathbf{VI}_R$*-representation over a field* $\mathbf{k}$*. Then the Hilbert series*

$$\mathrm{H}_M(t) = \sum_{n=0}^{\infty} \dim M_{\mathbf{k}}([n]) \cdot t^n$$

is a rational function of the form $f(t)/g(t)$ *where* $g(t)$ *factors as* $\prod_{j=1}^{r}(1-jt)^{e_r}$*, for some* r*.*

Theorem 8.3.5. *The category* $\mathbf{VA}_R$ *is quasi-Gröbner.*

Proof. Let $\Phi \colon \mathbf{VI}_R \to \mathbf{VA}_R$ be the inclusion functor. As in the proof of Theorem 7.4.6, one can show that Φ satisfies property (F). □

Corollary 8.3.6. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VA}_R)$ *is noetherian.*

Remark 8.3.7. When $R = \mathbf{k}$ is a finite field, Corollary 8.3.6 proves the Lannes–Schwartz artinian conjecture [K2, Conjecture 3.12]. This is also a consequence of the results in [PS], and so a similar, but distinct, proof of the artinian conjecture appears there as well. □

Remark 8.3.8. When R is a finite field and $\mathrm{char}(R)$ is invertible in $\mathbf{k}$, the category $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VA}_R)$ is equivalent to the category of functors $\coprod_{n \geq 0} \mathbf{GL}_n(R) \to \mathrm{Mod}_{\mathbf{k}}$ [K5]. □

Remark 8.3.9. While preparing this article, we learned that Corollary 8.3.3 is proven in [GL] in the special case that R is a field and $\mathbf{k}$ is a field of characteristic 0. □

Remark 8.3.10. Consider the category $\mathrm{Rep}_{\mathbf{F}_q}(\mathbf{VA}_{\mathbf{F}_q})$. Since $\mathbf{VA}_{\mathbf{F}_q}$ is like a linear version of $\mathbf{FA}$, it is tempting to find a linear analogue of Theorem 8.4.4 below, but this is not possible. An explicit description of the submodule lattice of $P_{\mathbf{F}_q}^{\vee}$ is given in [K4, Theorem 6.4]. In particular, one can read off from this description that the Krull dimension of $P_{\mathbf{F}_q}$ is 1. It is conjectured that the Krull dimension of P_W is $\dim(W)$ in general [K4, Conjecture 6.8]. □

8.4. **Additional results.** Recall that $\mathbf{FA}$ is the category of finite sets.

Theorem 8.4.1. *The category* $\mathbf{FA}^{\mathrm{op}}$ *is quasi-Gröbner.*

Proof. The proof is similar to that of Theorem 7.4.6; the role of $\mathbf{FI}_1$ is played by $\mathbf{FS}^{\mathrm{op}}$. □

Corollary 8.4.2. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA}^{\mathrm{op}})$ *is noetherian.*

Proposition 8.4.3. *The category* $\mathbf{FA}$ *satisfies property (D) (see Definition 5.2.9).*

Proof. Let x be a given finite set. Let $y = 2^x$ be the power set of x, and let $S \subset \mathrm{Hom}(x, y)$ be the set of maps f for which $a \in f(a)$ for all $a \in x$. Let $f : x \to z$ be given. Define $g : z \to y$ by $g(a) = f^{-1}(a)$. If $f' : x \to z$ is an arbitrary map then $g(f'(a)) = f^{-1}(f'(a))$ contains a if and only if $f(a) = f'(a)$. Thus $g_*(f')$ belongs to S if and only if $f = f'$. We conclude that $g_*^{-1}(S) = \{f\}$, which establishes the proposition. □

We can now give our improvement of Corollary 7.4.7:

Theorem 8.4.4. *If* $\mathbf{k}$ *is a field, then every finitely generated* $\mathbf{FA}$*-module is artinian, and so has a finite composition series. In other words,* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA})$ *has Krull dimension* 0.

Proof. This follows from the criterion of Proposition 5.2.11, together with the fact that $\mathbf{FA}$ is quasi-Gröbner (Theorem 7.4.6), $\mathbf{FA}^{\mathrm{op}}$ is quasi-Gröbner (Theorem 8.4.1), and $\mathbf{FA}$ satisfies property (D) (Proposition 8.4.3). □

Remark 8.4.5. It is not true that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA}^{\mathrm{op}})$ is artinian. The principal projective P_S in $\mathbf{FA}$ grows like a polynomial of degree $|S|$, i.e., $|\mathrm{Hom}_{\mathbf{FA}}(S, T)| = |T|^{|S|}$, but the principal projectives in $\mathbf{FA}^{\mathrm{op}}$ grow like exponential functions, so there is no chance for the duals of projective objects in $\mathbf{FA}^{\mathrm{op}}$ to be finitely generated $\mathbf{FA}$-modules (and so the projective objects of $\mathbf{FA}^{\mathrm{op}}$ have infinite descending chains of submodules). In particular, $\mathbf{FA}^{\mathrm{op}}$ does not satisfy property (D). □

Remark 8.4.6. Given a functor from $\mathbf{FA}^{\mathrm{op}}$ to finite sets, we can construct a functor to vector spaces by applying the free module construction. A characterization, which is attributed to George Bergman, of the Hilbert functions that can arise in this way is given in [Do, Theorem 1.3] and [Pa, Corollary 4.4]. Note that there is no guarantee that the resulting $\mathbf{FA}^{\mathrm{op}}$-modules are finitely generated. □

Proposition 8.4.7. *The category* $\mathbf{FS}^{\mathrm{op}}$ *satisfies property (F).*

Proof. Let x_1 and x_2 be given finite sets. Given surjections $f_1 : y \to x_1$ and $f_2 : y \to x_2$, let y' be the image of y in $x_1 \times x_2$, and let $g : y \to y'$ be the quotient map. Then $f_i = p_i \circ g$, where $p_i : x_1 \times x_2 \to x_i$ is the projection map. Since there are only finitely many choices for (y', p_1, p_2) up to isomorphism, the result follows. □

Corollary 8.4.8. *For any commutative ring* $\mathbf{k}$*, the pointwise tensor product of finitely generated representations of* $\mathbf{FS}^{\mathrm{op}}$ *is finitely generated.*

We record the following results for later use.

Proposition 8.4.9. *The functor* $\Phi : \mathbf{FS}^{\mathrm{op}} \times \mathbf{FS}^{\mathrm{op}} \to \mathbf{FS}^{\mathrm{op}}$ *given by disjoint union satisfies property (F).*

Proof. Pick a finite set S. Let $f : T_1 \amalg T_2 \to S$ be a surjection. Then this can be factored as $T_1 \amalg T_2 \to f(T_1) \amalg f(T_2) \to S$ where the first map is the image of a morphism $(T_1, T_2) \to (f(T_1), f(T_2))$ under Φ^{op}. So for the $y_1, y_2, \dots$ in the definition of property (F), we take the pairs (T, T') of subsets of S whose union is all of S. □

Proposition 8.4.10. *Let* S *be a finite set and let* M *be the* $\mathbf{FS}^{\mathrm{op}}$*-module defined by* $M(T) = \mathbf{k}[\mathrm{Hom}_{\mathbf{FA}}(T, S)]$*. Then* M *is finitely generated, and hence noetherian.*

Proof. Let $S_1, \dots, S_n$ be the subsets of S. Then $M(T) = \bigoplus_{i=1}^n \mathbf{k}[\mathrm{Hom}_{\mathbf{FS}}(T, S_i)]$. The ith summand is exactly the principal projective at S_i. □

9. Δ-modules

We now apply our methods to the theory of Δ-modules, originally introduced by the second author in [Sn]. In §9.1, we recall the definitions, the results of [Sn], and state our new results. In §9.2, we explain how our results yield new information on syzygies of Segre embeddings. The remainder of §9 is devoted to the proofs.

9.1. **Background and statement of results.** A **Δ-module** is a sequence $(F_n)_{n \ge 0}$, where F_n is an S_n-equivariant polynomial functor $\mathrm{Vec}^n \to \mathrm{Vec}$, equipped with transition maps

$$F_n(V_1, \dots, V_{n-1}, V_n \otimes V_{n+1}) \to F_{n+1}(V_1, \dots, V_{n+1}) \tag{9.1.1}$$

satisfying certain compatibilities. (See §9.3 below for background on polynomial functors.) To state these compatibilities succinctly, it is convenient to introduce an auxiliary category Vec^Δ. Objects of this category are finite collections of vector spaces $\{V_i\}_{i \in I}$. A morphism $\{V_i\}_{i \in I} \to \{W_j\}_{j \in J}$ consists of a surjection $f: J \to I$ together with, for each $i \in S$, a linear map $\eta_i: V_i \to \bigotimes_{f(j)=i} W_j$. A Δ-module is then a polynomial functor $F: \mathrm{Vec}^\Delta \to \mathrm{Vec}$, where polynomial means that the functor $F_n: \mathrm{Vec}^n \to \mathrm{Vec}$ given by $(V_1, \dots, V_n) \mapsto F(\{V_i\}_{i \in [n]})$ is polynomial for each n. We write Mod_Δ for the category of Δ-modules.

Let F be a Δ-module. By an **element** of F, we mean an element of $F(\{V_i\}_{i \in I})$ for some object $\{V_i\}$ of Vec^Δ. Given a set S of elements of F, there is a smallest Δ-submodule of F containing S. We call it the submodule of F **generated** by S. We say that F is **finitely generated** if it is generated by a finite set of elements. We say that F is **noetherian** if every Δ-submodule of F is finitely generated.

Let Λ be the ring of symmetric functions (see [St2, Chapter 7] for background). Given a finite length polynomial representation V of $\mathbf{GL}(\infty)^n$ over $\mathbf{k}$, we define its character $\widetilde{\mathrm{ch}}(V) \in \Lambda^{\otimes n}$ by

$$\widetilde{\mathrm{ch}}(V) = \sum_{\lambda_1, \dots, \lambda_n} \dim_{\mathbf{k}}(V_{\lambda_1, \dots, \lambda_n}) \cdot m_{\lambda_1} \otimes \cdots \otimes m_{\lambda_n},$$

where the sum is over all tuples of partitions, $V_{\lambda_1, \dots, \lambda_n}$ denotes the $(\lambda_1, \dots, \lambda_n)$ weight space of V, and m_λ denotes the monomial symmetric function indexed by λ. We let $\mathrm{ch}(V)$ be the image of $\widetilde{\mathrm{ch}}(V)$ in $\mathrm{Sym}^n(\Lambda)$, a polynomial of degree n in the m_λ. If V is an S_n-equivariant representation then $\widetilde{\mathrm{ch}}(V)$ belongs to $(\Lambda^{\otimes n})^{S_n}$, and so passing to $\mathrm{ch}(V)$ does not lose information. We extend the above definition to polynomial functors by evaluating on $\mathbf{k}^\infty$. Precisely, if $F: \mathrm{Vec}^n \to \mathrm{Vec}$ is a finite length polynomial functor, then we define $\mathrm{ch}(F)$ to be $\mathrm{ch}(F(\mathbf{k}^\infty, \dots, \mathbf{k}^\infty))$.

Suppose now that $F = (F_n)_{n \ge 0}$ is a Δ-module, and each F_n has finite length. The **Hilbert series** of F is

$$\mathrm{H}'_F(\mathbf{t}) = \sum_{n=0}^{\infty} \frac{\mathrm{ch}(F_n)}{n!}.$$

This is an element of $\widehat{\mathrm{Sym}}(\Lambda_{\mathbf{Q}}) \cong \mathbf{Q}[\![\mathbf{m}]\!]$. We also define a non-exponential version:

$$\mathrm{H}_F(\mathbf{t}) = \sum_{n=0}^{\infty} \mathrm{ch}(F_n).$$

If F is finitely generated, then each F_n is finite length; moreover, only finitely many of the m_λ appear in $\mathrm{H}_F(\mathbf{t})$.

In [Sn], a Δ-module is called "small" if it appears as a submodule of a Δ-module (F_n) generated by F_1, and the following abstract result is proved:

Theorem 9.1.2 (Snowden)**.** *Suppose* $\mathbf{k}$ *has characteristic* 0 *and* F *is a finitely generated small* Δ*-module. Then* F *is noetherian and* $\mathrm{H}_F(\mathbf{t})$ *is a rational function of the* m_λ*.*

Our main result, which we prove below, is the following:

Theorem 9.1.3. *Let* F *be a finitely generated* Δ*-module. Then* F *is noetherian and* $\mathrm{H}'_F(\mathbf{t})$ *is a polynomial in the* m_λ *and the* e^{m_λ}*.*

This result improves Theorem 9.1.2 in three ways:

(1) there is no restriction on the characteristic of $\mathbf{k}$;
(2) there is no restriction to small Δ-modules; and
(3) the Hilbert series statement is significantly stronger.

To elaborate on point (3), our result can be rephrased as: $\mathrm{H}_F(\mathbf{t})$ is of the form $f(\mathbf{t})/g(\mathbf{t})$, where f and g are polynomials in the m_λ and $g = \prod_{i=1}^n (1-\lambda_i)$, where each λ_i is a non-negative integer multiple of some m_λ. Thus it gives very precise information on the form of the denominator of $\mathrm{H}_F(\mathbf{t})$. This stronger result on Hilbert series answers [Sn, Question 5] affirmatively, and even goes beyond what is asked there. We remark that this question is phrased in terms of Schur functions rather than monomial symmetric functions, but the two of them are related by a change of basis with integer coefficients [St2, Corollary 7.10.6].

9.2. **Applications.** The motivating example of a Δ-module comes from the study of syzygies of the Segre embedding. Define the **Segre product** of two graded $\mathbf{k}$-algebras A and B to be the graded $\mathbf{k}$-algebra $A \boxtimes B$ given by

$$(A \boxtimes B)_n = A_n \otimes_{\mathbf{k}} B_n.$$

Given finite dimensional $\mathbf{k}$-vector spaces $V_1, \dots, V_n$, put

$$R(V_1, \dots, V_n) = \mathrm{Sym}(V_1) \boxtimes \cdots \boxtimes \mathrm{Sym}(V_n), \qquad S(V_1, \dots, V_n) = \mathrm{Sym}(V_1 \otimes \cdots \otimes V_n).$$

Then R is an S-algebra, and the corresponding map on Proj's corresponds to the Segre embedding

$$i(V_1, \dots, V_n) \colon \mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) \to \mathbf{P}(V_1 \otimes \cdots \otimes V_n).$$

Fix an integer $p \geq 0$, and define

$$F_n(V_1, \dots, V_n) = \mathrm{Tor}_p^{S(V_1,\dots,V_n)}(R(V_1, \dots, V_n), \mathbf{k}). \tag{9.2.1}$$

This is the space of p-syzygies of the Segre. The factorization

$$i(V_1, \dots, V_{n+1}) = i(V_1, \dots, V_{n-1}, V_n \otimes V_{n+1}) \circ (\mathrm{id} \times i(V_n, V_{n+1})),$$

combined with general properties of Tor, yield transition maps as in (9.1.1). In this way, the sequence $F = (F_n)_{n \geq 0}$ naturally has the structure of a Δ-module.

Theorem 9.2.2. *For every* $p \geq 0$ *and field* $\mathbf{k}$*, the* Δ*-module* F *defined by* (9.2.1) *is finitely generated and* $\mathrm{H}'_F(\mathbf{t})$ *is a polynomial in the* m_λ *and the* e^{m_λ}*.*

Proof. The Tor modules are naturally $\mathbf{Z}_{\geq 0}$-graded and are concentrated between degrees p and $2p$: to see this, we first note that the Segre variety has a quadratic Gröbner basis and the dimensions of these Tor modules is bounded from above by those of the initial ideal. The Taylor resolution of a monomial ideal gives the desired bounds [Ei, Exercise 17.11].

Using the Koszul resolution of $\mathbf{k}$ as an S-module, each graded piece of F can be realized as the homology of a complex of finitely generated Δ-modules. So the result follows from Theorem 9.1.3. $\square$

The finite generation statement means that there are finitely many p-syzygies of Segres that generate all p-syzygies of all Segres under the action of general linear groups, symmetric groups, and the maps (9.1.1), i.e., pullbacks along Segre embeddings. The statement about Hilbert series means that, with a single polynomial, one can store all of the characters of the $\mathbf{GL}$ actions on spaces of p-syzygies. We refer to the introduction of [Sn] for a more detailed account.

Remark 9.2.3. The syzygies of many varieties related to the Segre, such as secant and tangential varieties of the Segre, also admit the structure of a Δ-module. See [Sn, §4]. In fact, the argument in Theorem 9.2.2 almost goes through: one can show that each graded piece of each Tor module is finitely generated and has a rational Hilbert series, but there are few general results about these Tor modules being concentrated in finitely many degrees.

The results that we know of are in characteristic 0: [Ra] proves that the ideal of the secant variety of the Segre is generated by cubic equations, and [OR] proves that the tangential variety of the Segre is generated by equations of degree ≤ 4. $\square$

9.3. **Polynomial functors.** We now recall some background on polynomial functors. This material is standard (and first appeared in [FS]), but we could not find a convenient reference for multivariate functors, so we give a compact self-contained treatment here. Fix a field $\mathbf{k}$ and let Vec be the category of finite dimensional vector spaces over $\mathbf{k}$.

A **(strict) polynomial functor** $F\colon \mathrm{Vec} \to \mathrm{Vec}$ of degree d is a functor such that for all V and W, the map

$$\mathrm{Hom}_{\mathbf{k}}(V, W) \to \mathrm{Hom}_{\mathbf{k}}(F(V), F(W))$$

is given by homogeneous polynomial functions of degree d.

Lemma 9.3.1. *Let F be a polynomial functor of degree d taking values in finite-dimensional vector spaces. The function $n \mapsto \dim_{\mathbf{k}} F(\mathbf{k}^n)$ is weakly increasing and is a polynomial of degree $\le d$. In particular, if $F(\mathbf{k}^d) = 0$, then $F = 0$.*

Proof. This is similar to the proof of Theorem 7.4.8. Define $h_F(n) = \dim_{\mathbf{k}} F(\mathbf{k}^n)$. Define ΣF by $V \mapsto F(V \oplus \mathbf{k})$. There is a natural splitting $\Sigma F = F \oplus \Delta F$, so $h_{\Delta F}(n) = h_F(n+1) - h_F(n)$, and $h_F(n)$ is a weakly increasing function. It follows from the definition that ΔF is a polynomial functor of degree $< d$, so $\Delta^{d+1} F = 0$. Thus $h_{\Delta F}(n)$ is a polynomial of degree $\le d$. For the last sentence, if $F(\mathbf{k}^d) = 0$, then the above says that $h_F(n) = 0$ for $n = 0, \dots, d$. Since $h_F(n)$ is a polynomial of degree $\le d$, it must be identically 0. $\square$

A polynomial functor is a direct sum of polynomial functors of degree d for various d. Recall that a map of finite-dimensional vector spaces $U \to U'$ that is polynomial of degree d can be written as an element of $\mathrm{Sym}^d(U^*) \otimes U'$, or equivalently, as a linear map $\mathrm{D}^d(U) \to U'$, where D is the divided power functor. Hence we can rephrase the definition as follows. Let $\mathrm{D}^d(\mathrm{Vec})$ be the category whose objects are finite-dimensional $\mathbf{k}$-vector spaces and

$$\mathrm{Hom}_{\mathrm{D}^d(\mathrm{Vec})}(U, U') = \mathrm{D}^d \, \mathrm{Hom}_{\mathbf{k}}(U, U').$$

Then a polynomial functor of degree d is a linear functor $\mathrm{D}^d(\mathrm{Vec}) \to \mathrm{Vec}$.

We also extend this definition to functors with multiple arguments. A **(strict) polynomial functor** $F\colon \mathrm{Vec}^n \to \mathrm{Vec}$ of total degree d is a functor such that for all $V_1, \dots, V_n$ and $W_1, \dots, W_n$, the map

$$\mathrm{Hom}_{\mathbf{k}}(V_1, W_1) \times \cdots \times \mathrm{Hom}_{\mathbf{k}}(V_n, W_n) \to \mathrm{Hom}_{\mathbf{k}}(F(V_1, \dots, V_n), F(W_1, \dots, W_n))$$

is given by homogeneous polynomial functions of total degree d. A polynomial functor is a direct sum of polynomial functors of degree d for various d. Using the fact that

$$\mathrm{D}^d(U_1 \oplus \cdots \oplus U_n) = \bigoplus_{d_1+\cdots+d_n=d} \mathrm{D}^{d_1}(U_1) \otimes \cdots \otimes \mathrm{D}^{d_n}(U_n),$$

we can define polynomial functors of multidegree $\mathbf{d} = (d_1, \dots, d_n)$, and as above, we define a category $\mathrm{D}^{\mathbf{d}}(\mathrm{Vec})$ whose objects are n-tuples of vector spaces $\underline{V} = (V_1, \dots, V_n)$ and morphisms are

$$\mathrm{Hom}_{\mathrm{D}^{\mathbf{d}}(\mathrm{Vec})}(\underline{V}, \underline{W}) = \mathrm{D}^{d_1}(\mathrm{Hom}_{\mathbf{k}}(V_1, W_1)) \otimes \cdots \otimes \mathrm{D}^{d_n}(\mathrm{Hom}_{\mathbf{k}}(V_n, W_n)).$$

So polynomial functors of multidegree $\mathbf{d}$ are equivalent to linear functors $\mathrm{D}^{\mathbf{d}}(\mathrm{Vec}) \to \mathrm{Vec}$.

Given vector spaces $\underline{U} = (U_1, \dots, U_n)$, define a polynomial functor $P_{\underline{U}}$ by

$$P_{\underline{U}}(\underline{V}) = \mathrm{Hom}_{\mathrm{D}^{\mathbf{d}}(\mathrm{Vec})}(\underline{U}, \underline{V}).$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_s)$, set $\mathrm{D}^\lambda(V) = \mathrm{D}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{D}^{\lambda_s}(V)$. For a sequence of partitions $\Lambda = (\lambda^1, \dots, \lambda^n)$, define $\mathrm{D}^\Lambda \in \mathrm{Vec}^n \to \mathrm{Vec}$ by

$$\mathrm{D}^\Lambda(V_1, \dots, V_r) = \mathrm{D}^{\lambda^1}(V_1) \otimes \cdots \otimes \mathrm{D}^{\lambda^n}(V_n).$$

Proposition 9.3.2. *The* D^Λ *are projective generators for the category of polynomial functors* $\mathrm{Vec}^n \to \mathrm{Vec}$*.*

Proof. Every polynomial functor $\mathrm{Vec}^n \to \mathrm{Vec}$ naturally breaks up as a direct sum of polynomial functors of a given multidegree, so it suffices to fix a multidegree $\mathbf{d}$.

The functor $P_{\underline{U}}$ is projective by Yoneda's lemma. When $n = 1$, there is a natural map $F(U) \otimes P_U \to F$ whose image F' satisfies $F'(U) = F(U)$. So if $\dim(U) \ge d$, then $F/F' = 0$ by Lemma 9.3.1, and hence P_U is a projective generator in this case. Similarly, $P_{\underline{U}}$ is a projective generator if $\dim(U_i) \ge d_i$ for all i. Since $P_{\underline{U}}$ is isomorphic to a direct sum of functors of the form D^Λ, they are also projective generators. □

9.4. **Proof of Theorem 9.1.3: noetherianity.** Let N be a positive integer and define a functor $\Phi_N\colon \mathbf{FS}^{\mathrm{op}} \to \mathrm{Vec}^\Delta$ as follows. Given a finite set S, we let $\Phi_N(S)$ be the family $\{V_i\}_{i\in S}$ where $V_i = \mathbf{k}^N$ for all i. Given a surjection $f\colon S \to T$, we let $\Phi_N(f)$ be the map $\Phi_N(T) \to \Phi_N(S)$ that is f on index sets, and where the linear map $\eta_t\colon \mathbf{k}^N \to \bigotimes_{f(s)=t} \mathbf{k}^N$ takes the basis vector e_i to the basis vector $\bigotimes_{f(s)=t} e_i$. We thus obtain a functor $\Phi_N^*\colon \mathrm{Mod}_\Delta \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}})$.

Let $\mathbf{d} = (d_1, \dots, d_r)$ be a tuple of positive integers. Define a Δ-module $\mathcal{T}^{\mathbf{d}}$ by

$$\mathcal{T}^{\mathbf{d}}(\{V_i\}_{i\in I}) = \bigoplus_{\alpha\colon I \twoheadrightarrow [r]} \bigotimes_{i\in I} V_i^{\otimes d_{\alpha(i)}},$$

where the sum is over all surjections α from I to $[r]$.

We now proceed to show that $\mathcal{T}^{\mathbf{d}}$ is a noetherian Δ-module and that every finitely generated Δ-module is a subquotient of a finite direct sum of $\mathcal{T}^{\mathbf{d}}$'s. This will show that finitely generated Δ-modules are noetherian.

Lemma 9.4.1. *For all* N*, the representation* $\Phi_N^*(\mathcal{T}^{\mathbf{d}})$ *of* $\mathbf{FS}^{\mathrm{op}}$ *is noetherian.*

Proof. Put $M = \Phi_N^*(\mathcal{T}^{\mathbf{d}})$. We have

$$M(S) = \bigoplus_{\alpha \colon S \twoheadrightarrow [r]} \bigotimes_{i \in S} (\mathbf{k}^N)^{\otimes d_{\alpha(i)}}.$$

Let $\mathcal{F}(d)$ be the set of all functions $[d] \to [N]$. This set naturally indexes a basis of $(\mathbf{k}^N)^{\otimes d}$. The space $M(S)$ has a basis indexed by pairs (α, β), where α is a surjection $S \to [r]$ and β is a function assigning to each element $i \in S$ an element of $\mathcal{F}(d_{\alpha(i)})$. Write $e_{(\alpha,\beta)}$ for the basis vector corresponding to (α, β). This basis behaves well with respect to functoriality: if $f \colon T \to S$ is a surjection and $f^* \colon M(S) \to M(T)$ is the induced map then $f^*(e_{(\alpha,\beta)}) = e_{(f^*(\alpha), f^*(\beta))}$. A pair (α, β) defines a function $S \to [r] \times \mathcal{F}$, where $\mathcal{F}$ is the disjoint union of $\mathcal{F}(d_1), \dots, \mathcal{F}(d_r)$. The previous remark shows that M is a subrepresentation of the representation defined by $T \mapsto \mathbf{k}[\mathrm{Hom}_{\mathbf{FA}}(T, [r] \times \mathcal{F})]$. This functor is noetherian (Proposition 8.4.10), so the result follows. □

Proposition 9.4.2. *The* Δ*-module* $\mathcal{T}^{\mathbf{d}}$ *is noetherian.*

Proof. Let M be a submodule of $\mathcal{T}^{\mathbf{d}}$. We claim that M is generated by its restriction to the subcategory $\Phi_N(\mathbf{FS}^{\mathrm{op}})$, where $N = \max(d_i)$. To see this, pick an element $v \in M_n(\mathbf{k}^{s_1}, \dots, \mathbf{k}^{s_n})$ for some s_i. Then v is a linear combination of weight vectors, so it suffices to consider the case where v is a weight vector. The point is that any weight vector in $\mathcal{T}^{\mathbf{d}}(\mathbf{k}^{s_1}, \dots, \mathbf{k}^{s_n})$ can use at most N basis vectors from each space. Thus there is an element $g \in \prod_{i=1}^n \mathbf{GL}(s_i)$ (in fact, we just need each factor to be a permutation matrix) such that $gv \in \mathcal{T}^{\mathbf{d}}(\mathbf{k}^N, \dots, \mathbf{k}^N)$. This proves the claim.

The lemma now follows easily. Indeed, since M is generated by its restriction to the subcategory $\Phi_N(\mathbf{FS}^{\mathrm{op}})$, it follows that if $M \subset M'$ is a proper containment of submodules of $\mathcal{T}^{\mathbf{d}}$ then $\Phi_N^*(M) \subset \Phi_N^*(M')$ is also a proper containment. Thus a strictly ascending chain in $\mathcal{T}^{\mathbf{d}}$ would give one in $\Phi_N(\mathcal{T}^{\mathbf{d}})$. Since the latter is noetherian, no such chain exists. Thus $\mathcal{T}^{\mathbf{d}}$ is noetherian. □

We now start on the second part of our plan. Let T be the category whose objects are sequences $F = (F_n)_{n \ge 0}$ where F_n is a polynomial functor $\mathrm{Vec}^n \to \mathrm{Vec}$, with no extra data. Given an object F of this category, we define a Δ-module $\Gamma(F)$ by

$$\Gamma(F)(\{V_i\}_{i \in I}) = \bigoplus_{n=0}^{|I|} \bigoplus_{\alpha \colon I \twoheadrightarrow [n]} F_n(U_1, \dots, U_n),$$

where $U_j = \bigotimes_{\alpha(i)=j} V_i$ and the second sum is over all surjective functions $\alpha \colon I \to [n]$.

Proposition 9.4.3. *We have the following:*

(a) Γ *is the left adjoint of the forgetful functor* $\mathrm{Mod}_\Delta \to \mathrm{T}$.
(b) Γ *is an exact functor.*

Proof. (a) Pick $G \in \mathrm{Mod}_\Delta$ and $F \in \mathrm{T}$ and a morphism $f \colon F \to G$ in T. Suppose that $\{V_i\}_{i \in I}$ is an object of Mod_Δ and $\alpha \colon I \to [n]$ is a surjection; let U_j be as above. We have maps

$$F_n(U_1, \dots, U_n) \to G_n(U_1, \dots, U_n) \to G(\{V_i\}),$$

where the first is f and the second uses functoriality of G with respect to the morphism $\{U_j\}_{j \in [n]} \to \{V_i\}_{i \in I}$ in Vec^Δ corresponding to α (and the identity maps). Summing over all choices of α, we obtain a map $(\Gamma F)(\{V_i\}) \to G(\{V_i\})$, and thus a map of Δ-modules $\Gamma(F) \to G$.

(b) A Δ-module is a functor $\mathrm{Vec}^{\Delta} \to \mathrm{Vec}$, so exactness can be checked pointwise. Similarly, exactness of a sequence of objects in T can be checked pointwise since they are functors on $\coprod_{n \geq 0} \mathrm{Vec}^n$. So let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in T and pick $\{V_i\}_{i \in I} \in \mathrm{Vec}^{\Delta}$. The sequence

$$(\Phi F_1)(\{V_i\}) \to (\Phi F_2)(\{V_i\}) \to (\Phi F_3)(\{V_i\})$$

is the direct sum of sequences of the form

$$F_1(U_1, \dots, U_n) \to F_2(U_1, \dots, U_n) \to F_3(U_1, \dots, U_n)$$

for surjective functions $\alpha \colon I \twoheadrightarrow [n]$. The latter are exact by assumption, so our sequence of interest is also exact. □

Define $\mathcal{D}^{\Lambda}$ to be the Δ-module $\Gamma(\mathrm{D}^{\Lambda})$, where D^{Λ} (defined in §9.1) is regarded as an object of T concentrated in degree r.

Proposition 9.4.4. *The $\mathcal{D}^{\Lambda}$ are projective generators for* Mod_{Δ}.

Proof. By Proposition 9.3.2, the functors D^{Λ} are projective generators for T. Since Φ is the left adjoint of an exact functor, and D^{Λ} is projective, it follows that $\mathcal{D}^{\Lambda}$ is projective. Given a Δ-module M and an element $x \in M$, there is a map $\mathrm{D}^{\Lambda} \to M$ in T whose image contains x, and therefore a map of Δ-modules $\mathrm{D}^{\Lambda} \to M$ whose image contains x. Thus the $\mathcal{D}^{\Lambda}$ are generators. □

Corollary 9.4.5. *A finitely generated Δ-module is a subquotient of a finite direct sum of $\mathcal{T}^{\mathbf{d}}$'s.*

Proof. Given $\mathbf{d} = (d_1, \dots, d_r)$, let $\mathrm{T}^{\mathbf{d}} \colon \mathrm{Vec}^r \to \mathrm{Vec}$ be the functor given by

$$\mathrm{T}^{\mathbf{d}}(V_1, \dots, V_r) = V_1^{\otimes d_1} \otimes \cdots \otimes V_r^{\otimes d_r}.$$

Then $\mathcal{T}^{\mathbf{d}} = \Gamma(\mathrm{T}^{\mathbf{d}})$. Since D^{Λ} is a subobject of $\mathrm{T}^{\mathbf{d}}$, where $d_i = |\lambda^i|$, it follows that $\mathcal{D}^{\Lambda}$ is a submodule of $\mathcal{T}^{\mathbf{d}}$. As every finitely generated Δ-module is a quotient of a finite sum of $\mathcal{D}^{\Lambda}$'s, the result follows. □

The above corollary and Proposition 9.4.2 prove the noetherianity statement in Theorem 9.1.3.

9.5. **Proof of Theorem 9.1.3: Hilbert series.** By Corollary 9.4.5, the Grothendieck group of finitely generated Δ-modules is spanned by the classes of submodules of $\mathcal{T}^{\mathbf{d}}$. Therefore, it suffices to analyze the Hilbert series of a submodule M of $\mathcal{T}^{\mathbf{d}}$.

Let $N = \max(d_i)$, and write $\Phi = \Phi_N$. The space $\Phi^*(M)(S)$ has an action of the group $\mathbf{GL}(N)^S$ that we exploit. Let $\lambda_1, \dots, \lambda_n$ be the partitions of size at most N. These are the only partitions that appear as weights in $(\mathbf{k}^N)^{\otimes d_i}$. Let $(S_1, \dots, S_n)$ be a tuple of finite (possibly empty) sets and let S be their disjoint union. Define $\Psi(M)(S_1, \dots, S_n)$ to be the subspace of $\Phi^*(M)(S)$ where the torus in the ith copy of $\mathbf{GL}(N)$ acts by weight λ_j, where $i \in S_j$. It is clear that if $f_i \colon S_i \to T_i$ are surjections, and $f \colon S \to T$ is their disjoint union, then the map $f^* \colon \Phi^*(M)(T) \to \Phi^*(M)(S)$ carries $\Psi(M)(T_1, \dots, T_n)$ into $\Psi(M)(S_1, \dots, S_n)$. Thus $\Psi(M)$ is a representation of $(\mathbf{FS}^{\mathrm{op}})^n$. If M' is a subrepresentation of $\Psi(M)$ then $S \mapsto \bigoplus_{S = S_1 \amalg \cdots \amalg S_n} M'(S_1, \dots, S_n)$ is an $\mathbf{FS}^{\mathrm{op}}$-submodule of $\Phi^*(M)$. Since $\Phi^*(M)$ is noetherian, it follows that $\Psi(M)$ is noetherian. In particular, $\Psi(M)$ is finitely generated.

Let $e_1, \dots, e_n$ be non-negative integers summing to e. The coefficient of $m_{\lambda_1}^{e_1} \cdots m_{\lambda_n}^{e_n}$ in $\mathrm{H}'_M(\mathbf{t})$ is then

$$\frac{1}{e!} \sum_{\mu_1, \dots, \mu_e} \dim_{\mathbf{k}}(V_{\mu_1, \dots, \mu_e}),$$

where $V = M_e(\mathbf{k}^N, \dots, \mathbf{k}^N)$ and the μ_j are partitions such that exactly e_i of them are equal to λ_i for each i. Now, no matter what the μ_j's are, the space $V_{\mu_1,\dots,\mu_e}$ is isomorphic to $\Psi(M)([e_1], \dots, [e_n])$. Thus we see that the coefficient is equal to

$$\frac{1}{e_1! \cdots e_n!} \dim_{\mathbf{k}}(\Psi(M)([e_1], \dots, [e_n])).$$

It follows that $\mathrm{H}'_M(\mathbf{t})$ is equal to $\mathrm{H}'_{\Psi(M)}(\mathbf{t})$, and so the theorem follows from Corollary 8.1.8.

10. Categories of weighted surjections

In this section we study a generalization of the category of finite sets with surjective functions by considering weighted sets. This is mostly preparatory material for the next section.

10.1. **The categories $\mathbf{OWS}_\Lambda$ and $\mathbf{FWS}_\Lambda$.** Let Λ be a finite abelian group. A **weighting** on a finite set S is a function $\varphi \colon S \to \Lambda$. A **weighted set** is a set equipped with a weighting; we write φ_S to denote the weighting. Suppose φ is a weighting on S, and let $f \colon S \to T$ be a map of sets. We define $f_*(\varphi)$ to be the weighting on T given by $f_*(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$. A **map** of weighted sets $S \to T$ is a surjective function $f \colon S \to T$ such that $f_*(\varphi_S) = \varphi_T$. We let $\mathbf{FWS}_\Lambda$ denote the category of weighted sets.

As usual, we require an ordered version of the category as well. Let $\mathbf{OWS}_\Lambda$ be the following category. The objects are totally ordered weighted sets. The order and weighting are not required to interact in any way. The morphisms are ordered maps of weighted sets, i.e., a morphism $S \to T$ is a surjective function $f \colon S \to T$ such that $f_*(\varphi_S) = \varphi_T$, and for all $x < y$ in T we have $\min f^{-1}(x) < \min f^{-1}(y)$. We define a norm on $\mathbf{OWS}_\Lambda$ as follows. Enumerate Λ as $\lambda_1, \dots, \lambda_r$, so that we can identify $\mathbf{N}^\Lambda$ with $\mathbf{N}^r$. Then $\nu(S) = (n_1, \dots, n_r)$, where $n_i = \#\varphi_S^{-1}(\lambda_i)$.

Our main result about $\mathbf{OWS}_\Lambda$ is:

Theorem 10.1.1. *The category $\mathbf{OWS}_\Lambda^{\mathrm{op}}$ is Gröbner and strongly QO_N-lingual, where N is the exponent of Λ.*

We prove this in the next section, and now use it to study $\mathbf{FWS}_\Lambda$.

Theorem 10.1.2. *The category $\mathbf{FWS}_\Lambda^{\mathrm{op}}$ is quasi-Gröbner.*

Proof. The forgetful functor $\Phi \colon \mathbf{OWS}_\Lambda \to \mathbf{FWS}_\Lambda$ is easily seen to satisfy property (F), and so the result follows from Theorem 10.1.1. ☐

Corollary 10.1.3. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FWS}_\Lambda^{\mathrm{op}})$ is noetherian.*

Let M be a finitely generated representation of $\mathbf{FWS}_{\Lambda_1}^{\mathrm{op}} \times \cdots \times \mathbf{FWS}_{\Lambda_r}^{\mathrm{op}}$. Enumerate Λ_i as $\{\lambda_{i,j}\}$, and let $t_{i,j}$ be a formal variable corresponding to $\lambda_{i,j}$. Given $\mathbf{n} \in \mathbf{N}^{\#\Lambda_i}$, let $[\mathbf{n}]$ be the Λ_i-weighted set where n_j elements have weight $\lambda_{i,j}$. When $\mathbf{k}$ is a field, define the Hilbert series of M by

$$\mathrm{H}_M(\mathbf{t}) = \sum_{\mathbf{n}(1), \dots, \mathbf{n}(r)} C_{\mathbf{n}(1)} \cdots C_{\mathbf{n}(r)} \dim_{\mathbf{k}} M([\mathbf{n}(1)], \dots, [\mathbf{n}(r)]) \cdot \mathbf{t}_1^{\mathbf{n}(1)} \cdots \mathbf{t}_r^{\mathbf{n}(r)},$$

where for $\mathbf{n} \in \mathbf{N}^k$ we write $C_\mathbf{n}$ for the multinomial coefficient

$$C_\mathbf{n} = \frac{|\mathbf{n}|!}{\mathbf{n}!} = \frac{|\mathbf{n}|!}{n_1! \cdots n_k!}.$$

The following theorem is the main result we need in our applications, and follows immediately from Theorem 10.1.1 and the functor Φ used in the proof of the previous theorem. (Note: the reason the multinomial coefficient appears is that there are $C_\mathbf{n}$ isomorphism classes in $\mathbf{OWS}_{\Lambda_i}$ which map to the isomorphism class of $[\mathbf{n}]$ in $\mathbf{FWS}_{\Lambda_i}$.)

Theorem 10.1.4. *In the above situation,* $\mathrm{H}_M(\mathbf{t})$ *is* $\mathcal{K}_N$ *(see Definition 3.4.4).*

10.2. **Proof of Theorem 10.1.1.** Fix a finite set L and let $\Sigma = L \times \Lambda$. Given $a \in L$ and $\alpha \in \Lambda$, we write $\frac{a}{\alpha}$ for the corresponding element of Σ. We denote elements of $\Sigma^\star$ by $\frac{s}{\sigma}$, where $s \in L^\star$ and $\sigma \in \Lambda^\star$ are words of equal length. For $a \in L$, we define $w_a \colon \Sigma^\star \to \Lambda$ by

$$w_a\left(\frac{s_1 \cdots s_n}{\sigma_1 \cdots \sigma_n}\right) = \sum_{s_i = a} \sigma_i.$$

We let $\mathbf{w} \colon \Sigma^\star \to \Lambda^L$ be $(w_a)_{a \in L}$. For $\theta \in \Lambda^L$, we let $\mathcal{K}_\theta$ be the set of all $\frac{s}{\sigma} \in \Sigma^\star$ with $\mathbf{w}(\frac{s}{\sigma}) = \theta$. This is a congruence language of modulus N (the exponent of Λ).

We now define a partial order on $\Sigma^\star$. Let $\frac{s}{\sigma} \colon [n] \to \Sigma$ and $\frac{t}{\tau} \colon [m] \to \Sigma$ be two words. Define $\frac{s}{\sigma} \le \frac{t}{\tau}$ if there exists an ordered surjection $f \colon [m] \to [n]$ such that $t = f^*(s)$ and $\sigma = f_*(\tau)$. Note that if $\frac{s}{\sigma} \le \frac{t}{\tau}$ then $\mathbf{w}(\frac{s}{\sigma}) = \mathbf{w}(\frac{t}{\tau})$.

Lemma 10.2.1. *The poset* $\Sigma^\star$ *is noetherian.*

Proof. We modify the proof of Proposition 8.2.1, and use the notion of exceptional value defined there (applied to the numerator of an element of $\Sigma^\star$). Suppose that $\Sigma^\star$ is not noetherian and choose a minimal bad sequence $x_1 = \frac{s_1}{\sigma_1}, x_2 = \frac{s_2}{\sigma_2}, \ldots$.

We can find an infinite subsequence $x_{i_1}, x_{i_2}, \ldots$ such that $m(s_{i_j})$ is defined and is constant, say equal to m_0 (assume we pick the subsequence maximal with this property, i.e., we do not leave out any of the x_i which satisfy the property). Let $w_{i_j} = \frac{t_{i_j}}{\tau_{i_j}}$ be the subword of x_{i_j} of elements (except for the first instance) whose numerator is m_0. For $j \gg 0$, we can find a subword w'_{i_j} of w_{i_j} such that the sum of the elements in the denominator sum to 0. These subwords cannot form a bad sequence, or else, denoting by k the first j where w'_{i_j} exists, the sequence $x_1, x_2, \ldots, x_{i_k - 1}, w'_{i_k}, \ldots$ is bad (we cannot have $w'_{i_{k'}} \ge x_i$ or else the numerator of x_i is the constant word m_0 and its denominator sums to 0 which means it would be one of the w_{i_j}) and contradicts minimality. So we pick a further subsequence of the x_{i_j} so that the w'_{i_j} form a weakly increasing sequence.

Let $y_{i_j} = \frac{s'_{i_j}}{\sigma'_{i_j}}$ be the result of removing the subword w'_{i_j} from $\frac{s_{i_j}}{\sigma_{i_j}}$. Then $y_{i_j} \le x_{i_j}$ by construction of w'_{i_j}. If $x_1, \ldots, x_{i_1 - 1}, y_{i_1}, y_{i_2}, \ldots$ is a bad sequence, it violates minimality of $x_1, x_2, \ldots$, so we must have $y_{i_j} \le y_{i_k}$ for some $j < k$ ($x_i \le y_{i_j}$ implies $x_i \le x_{i_j}$ from what we just said). This inequality is witnessed by an ordered surjection $f \colon [\ell(y_{i_k})] \to [\ell(y_{i_j})]$, i.e., $s'_{i_k} = f^*(s'_{i_j})$ and $\sigma'_{i_j} = f_*(\sigma'_{i_k})$. We also have an inequality $w'_{i_j} \le w'_{i_k}$ so we can take the disjoint union of the two ordered surjections to get a surjection (which is still ordered because we removed the initial instance of m_0 from w_{i_j} above) that witnesses the inequality $x_{i_j} \le x_{i_k}$. This is a contradiction, so we conclude that minimal bad sequences do not exist and that $\Sigma^\star$ is noetherian. $\square$

Let $\frac{s}{\sigma} = \frac{s_1 \cdots s_n}{\sigma_1 \cdots \sigma_n}$ be a word in $\mathcal{K}_\theta$. Put

$$\Pi_i = \left\{ \tfrac{s_1}{*}, \cdots, \tfrac{s_i}{*} \right\},$$

where $*$ means any element of Λ. Define a language $\mathcal{L}(\frac{s}{\sigma})$ by

$$\mathcal{L}(\tfrac{s}{\sigma}) = (\tfrac{s_1}{\sigma_1})\Pi_1^\star \cdots (\tfrac{s_n}{\sigma_n})\Pi_n^\star.$$

It is clear that $\mathcal{L}(\frac{s}{\sigma})$ is an ordered language.

Lemma 10.2.2. *If $\frac{t}{\tau} \in \mathcal{L}(\frac{s}{\sigma}) \cap \mathcal{K}_\theta$ then $\frac{s}{\sigma} \le \frac{t}{\tau}$.*

Proof. Let $\frac{t}{\tau} \colon I \to \Sigma^\star$ in $\mathcal{L}(\frac{s}{\sigma}) \cap \mathcal{K}_\theta$ be given. Write $\frac{t}{\tau} = (\frac{s_1}{\sigma_1}) w_1 \cdots (\frac{s_n}{\sigma_n}) w_n$, with $w_i \in \Pi_i^\star$. Let $J \subset I$ be the indices occurring in the words $w_1, \ldots, w_n$ and let K be the complement of J, so that $\frac{t}{\tau}|_K = \frac{s}{\sigma}$. We now define a map $f \colon I \to [n]$. On K, we let f be the unique order-preserving bijection. For $a \in \{s_1, \ldots, s_n\}$, let $r(a) \in [n]$ be minimal so that $s_{r(a)} = a$. Now define f on J by $f(j) = r(t_j)$. It is clear that f is an ordered surjection and that $f^*(s) = t$. Since $\mathbf{w}(\frac{s}{\sigma}) = \mathbf{w}(\frac{t}{\tau}) = \theta$, it follows that $\mathbf{w}(\frac{t}{\tau}|_J) = 0$. From the way we defined f, it follows that $f_*(\tau|_J) = 0$. Thus $f_*(\tau) = \sigma$, which completes the proof. □

We say that a word $\sigma_1 \cdots \sigma_n \in \Lambda^\star$ is **minimal** if no non-empty subsequence of $\sigma_2 \cdots \sigma_n$ sums to 0. Note that we started with the second index. As any sufficiently long sequence in $\Lambda^\star$ contains a subsequence summing to 0, there are only finitely many minimal words. Let $\frac{s}{\sigma}$ in $\mathcal{K}_\theta$ be given. We say that $\frac{t}{\tau} \colon [m] \to \Sigma^\star$ is **minimal** over $\frac{s}{\sigma} \colon [n] \to \Sigma^\star$ if there is an ordered surjection $f \colon [m] \to [n]$ such that $t = f^*(s)$ and $\sigma = f_*(\tau)$ and for every $i \in [n]$ the word $\tau|_{f^{-1}(i)}$ is minimal. If $\frac{t}{\tau}$ is minimal over $\frac{s}{\sigma}$ then the length of $\frac{t}{\tau}$ is bounded, so there are only finitely many such minimal words.

Lemma 10.2.3. *Let $\frac{s}{\sigma} \le \frac{r}{\rho}$ be words in $\mathcal{K}_\theta$. Then there exists $\frac{t}{\tau}$ minimal over $\frac{s}{\sigma}$ such that $\frac{r}{\rho} \in \mathcal{L}(\frac{t}{\tau})$.*

Proof. Let $[n]$ and $[m]$ be the index sets of $\frac{s}{\sigma}$ and $\frac{r}{\rho}$, and choose a witness $f \colon [m] \to [n]$ to $\frac{s}{\sigma} \le \frac{r}{\rho}$. Let $I \subset [m]$ be the set of elements of the form $\min f^{-1}(i)$ for $i \in [n]$. Let $K \subset [m]$ be minimal subject to $I \subset K$ and $f_*(\rho|_K) = \sigma$. Then $\rho|_{f^{-1}(i) \cap K}$ is minimal for all $i \in [n]$. Indeed, if it were not then we could discard a subsequence summing to 0 and make K smaller. We thus see that $\frac{t}{\tau} = \frac{r}{\rho}|_K$ is minimal over $\frac{s}{\sigma}$. If $i \in [m] \setminus K$ then there exists $j < i$ in I with $t_i = t_j$, and so $\frac{r}{\rho} \in \mathcal{L}(\frac{t}{\tau})$. □

Lemma 10.2.4. *Every poset ideal of $\mathcal{K}_\theta$ is of the form $\mathcal{L} \cap \mathcal{K}_\theta$, where $\mathcal{L}$ is an ordered language on Σ.*

Proof. It suffices to treat the case of a principal ideal. Thus consider the ideal S generated by $\frac{s}{\sigma} \in \mathcal{K}_\theta$. Let $\frac{t_i}{\tau_i}$ for $1 \le i \le n$ be the words minimal over $\frac{s}{\sigma}$, and let $\mathcal{L} = \bigcup_{i=1}^n \mathcal{L}(\frac{t_i}{\tau_i})$. Then $\mathcal{L}$ is an ordered language, by construction. If $\frac{r}{\rho} \in \mathcal{L} \cap \mathcal{K}_\theta$ then $\frac{r}{\rho} \in \mathcal{L}(\frac{t_i}{\tau_i}) \cap \mathcal{K}_\theta$ for some i, and so $\frac{s}{\sigma} \le \frac{t_i}{\tau_i} \le \frac{r}{\rho}$ by Lemma 10.2.2, and so $\frac{r}{\rho} \in S$. Conversely, suppose $\frac{r}{\rho} \in S$. Then $\frac{r}{\rho} \in \mathcal{L}(\frac{t_i}{\tau_i})$ for some i by Lemma 10.2.3, and of course $\frac{r}{\rho} \in \mathcal{K}_\theta$, and so $\frac{r}{\rho} \in \mathcal{L} \cap \mathcal{K}_\theta$. □

Proof of Theorem 10.1.1. The category $\mathcal{C} = \mathbf{OWS}_\Lambda^{\mathrm{op}}$ is clearly directed. Let $x = ([n], \theta)$ be an object of $\mathcal{C}$. We apply the above theory with $L = [n]$. Suppose $f \colon x \to y$ is a map in $\mathcal{C}$, with $y = ([m], \varphi)$; note that this means that f is a surjection $[m] \to [n]$. We define a word $[m] \to \Sigma^\star$ by mapping $i \in [m]$ to $(f(i), \varphi(i))$. Obviously, one can reconstruct f from this word, and so this defines an injection $i\colon |\mathcal{C}_x| \to \Sigma^\star$. In fact, the image lands in $\mathcal{K}_\theta$. It is clear from the definition of the order on $\Sigma^\star$ that i is strictly order-preserving. Thus $|\mathcal{C}_x|$ is noetherian by Lemma 10.2.1. Lexicographic order on $\Sigma^\star$ induces an admissible order on $|\mathcal{C}_x|$. This follows since $\Sigma^\star$ is a subposet of the partial ordering on $\Sigma^\star$ defined in §8.2. Finally, since i maps ideals to ideals, we see that it gives a strong QO_N-lingual structure on $|\mathcal{C}_x|$ by Lemma 10.2.4. □

11. Categories of G-surjections

In this section, we study categories of functions that are decorated by a finite group. In §11.1.2 we give the definitions. The injective version of the category has a simple structure which is summarized in §11.1.1. The rest of the section is devoted to studying the surjective version: the noetherian property is deduced in §11.1.2 and results on Hilbert series are stated in §11.2. The rest of the section is devoted to proving the Hilbert series results. These results essentially recover our results on Δ-modules when the group is the symmetric group, and in general represent a significant generalization.

11.1. **Categories of G-maps and their representations.** Let G be a group. A **G-map** $S \to T$ between finite sets S and T is a pair (f, σ) consisting of a function $f\colon S \to T$ and a function $\sigma\colon S \to G$. Given G-maps $(f,\sigma)\colon S \to T$ and $(g,\tau)\colon T \to U$, their **composition** is the G-map $(h,\eta)\colon S \to U$ with $h = g \circ f$ and $\eta(x) = \sigma(x)\tau(f(x))$, the product taken in G. In this way, we have a category $\mathbf{FA}_G$ whose objects are finite sets and whose morphisms are G-maps. The automorphism group of $[n]$ in $\mathbf{FA}_G$ is $S_n \wr G = S_n \ltimes G^n$. Let $\mathbf{FS}_G$ (resp. $\mathbf{FI}_G$) denote the subcategory of $\mathbf{FA}_G$ containing all objects but only those morphisms (f,σ) with f surjective (resp. injective). In this section, we will always assume that G is finite.

11.1.1. *Representations of* $\mathbf{FI}_G$. We have the following basic property about representations of $\mathbf{FI}_G$:

Proposition 11.1.1. *There are natural functors* $\mathbf{FA} \to \mathbf{FA}_G$ *and* $\mathbf{FI} \to \mathbf{FI}_G$ *that satisfy property (F).*

Proof. Define a functor $\Phi\colon \mathbf{FA} \to \mathbf{FA}_G$ that sends a set to itself and a function $f\colon S \to T$ to $(f, 1)$ where $1\colon S \to G$ is the constant map sending every element to the identity of G. To see that Φ satisfies property (F), pick a set x of size n, set $y_1, \dots, y_{n^{|G|}}$ all equal to x and let $f_1, \dots, f_{n^{|G|}}$ correspond to all automorphisms of x in $\mathbf{FA}_G$ under some enumeration.

For the second functor, note that Φ restricts to a functor $\mathbf{FI} \to \mathbf{FI}_G$. □

Corollary 11.1.2. *The categories* $\mathbf{FA}_G$ *and* $\mathbf{FI}_G$ *are quasi-Gröbner.*

Proof. This follows from Proposition 5.2.6 and Theorems 7.4.6, 7.1.2. □

Corollary 11.1.3. *If* $\mathbf{k}$ *is left-noetherian then* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FA}_G)$ *and* $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ *are noetherian.*

Corollary 13.2.5 improves this result by allowing G to be any polycyclic-by-finite group.

Remark 11.1.4. Define a category $\mathbf{FI}_{d,G}$ of finite sets whose morphisms are pairs (f,σ) where f is a decorated injection as in the definition of $\mathbf{FI}_d$ and σ is as in the definition of $\mathbf{FI}_G$. As above, there is a natural functor $\mathbf{FI}_d \to \mathbf{FI}_{d,G}$ satisfying property (F). □

The category $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ only depends on $\mathrm{Rep}_{\mathbf{k}}(G)$ as an abelian category equipped with the extra structure of the invariants functor $\mathrm{Rep}_{\mathbf{k}}(G) \to \mathrm{Mod}_{\mathbf{k}}$. (See Proposition 13.2.3 for a precise statement.) Thus $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ "sees" very little of G. In good characteristic, we can be more explicit. Let **FB** be the groupoid of finite sets (maps are bijections).

Proposition 11.1.5. *Suppose* $\mathbf{k}$ *is a field in which the order of* G *is invertible. Then representations of* $\mathbf{FI}_G$ *are equivalent to representations of* $\mathbf{FI} \times \mathbf{FB}^r$*, where* r *is the number of non-trivial irreducible representations of* G *over* $\mathbf{k}$*.*

Proof. Let $V_1, \ldots, V_r$ be the non-trivial irreducible representations of G, and let V_0 be the trivial representation. Suppose M is an $\mathbf{FI}_G$-module. We can then decompose $M(S)$ into isotypic pieces for the action of G^S:

$$M(S) = \bigoplus_{S = S_0 \amalg \cdots \amalg S_r} N(S_0, \ldots, S_r) \otimes (V_0^{\boxtimes S_0} \boxtimes \cdots \boxtimes V_r^{\boxtimes S_r}), \tag{11.1.6}$$

where N is a multiplicity space. Suppose now that $f \colon S \to T$ is an injection. To build a morphism in $\mathbf{FI}_G$ we must also choose a function $\sigma \colon S \to G$. However, if σ and σ' are two choices then (f, σ) and (f, σ') differ by an element of $\mathrm{Aut}(T)$, namely an automorphism of the form (id_T, τ) where τ restricts to $\sigma'\sigma^{-1}$ on S. Thus it suffices to record the action of $(f, 1)$. Note that if $\tau \colon T \to G$ restricts to 1 on S then $(\mathrm{id}_T, \tau)(f, 1) = (f, 1)$. It follows that $(f, 1)$ must map $M(S)$ into the $G^{T \setminus S}$-invariants of $M(T)$. In other words, under the above decomposition, $(f, 1)$ induces a linear map

$$N(S_0, S_1, \ldots, S_r) \to N(S_0 \amalg (T \setminus S), S_1, \ldots, S_r).$$

Thus, associated to M we have built a representation N of $\mathbf{FI} \times \mathbf{FB}^r$. The above discussion makes clear that no information is lost in passing from M to N, and so this is a fully faithful construction. The inverse construction is defined by the formula (11.1.6). ∎

By the proposition, an $\mathbf{FI}_G$-module can be thought of as a sequence $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbf{N}^r}$, where each $M_{\mathbf{n}}$ is an **FI**-module equipped with an action of $S_{\mathbf{n}}$. There are no transition maps, so in a finitely generated $\mathbf{FI}_G$-module, all but finitely many of the $M_{\mathbf{n}}$ are zero. Thus, at least in good characteristic, $\mathbf{FI}_G$-modules are not much different from **FI**-modules, and essentially any result about **FI**-modules (e.g., noetherianity) carries over to $\mathbf{FI}_G$-modules.

Remark 11.1.7. The category $\mathbf{FI}_{\mathbf{Z}/2\mathbf{Z}}$ is equivalent to the category $\mathbf{FI}_{\mathrm{BC}}$ defined in [Wi, Defn. 1.2]. It is possible to define and prove properties about modified versions of our categories to include her category $\mathbf{FI}_{\mathrm{D}}$, but since we will not have any use for this, we leave the modifications to the reader. □

11.1.2. *Representations of* $\mathbf{FS}_G^{\mathrm{op}}$. The situation with $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_G^{\mathrm{op}})$ is completely different: since the diagonal map on G appears in $\mathbf{FS}_G$, representations of $\mathbf{FS}_G^{\mathrm{op}}$ "know" about the tensor structure on $\mathrm{Rep}_{\mathbf{k}}(G)$ (when $\mathbf{k}$ is commutative), and therefore "see" a lot of G. It seems plausible that one can recover G from the abelian category $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_G^{\mathrm{op}})$, see §1.5.8.

The remainder of §11 is devoted to studying representations of $\mathbf{FS}_G^{\mathrm{op}}$, and products of categories of this form. Actually, the bulk of this section is devoted to the study of Hilbert series; noetherianity is easy, given what we have already proved:

Proposition 11.1.8. *There is a natural functor* $\mathbf{FS}^{\mathrm{op}} \to \mathbf{FS}_G^{\mathrm{op}}$ *that satisfies property (F).*

Proof. Let $\Phi \colon \mathbf{FS} \to \mathbf{FS}_G$ be the functor taking a function $f \colon S \to T$ to the G-function $(f, \sigma) \colon S \to T$ where $\sigma = 1$. The "natural functor" in the statement of the proposition is Φ^{op}. Let $x \in \mathbf{FS}_G$ be given. Say that a morphism $(f, \sigma) \colon y \to x$ is minimal if the function $(f, \sigma) \colon y \to x \times G$ is injective. There are finitely many minimal maps up to isomorphism.

Now consider a map $(f, \sigma)\colon y \to x$ in $\mathbf{FS}_G$. Define an equivalence relation on y by $a \sim b$ if $f(a) = f(b)$ and $\sigma(a) = \sigma(b)$, and let $g\colon y \to y'$ be the quotient. Then the induced map $(f', \sigma)\colon y' \to x$ is minimal. Furthermore, $(f, \sigma) = (g, 1)(f', \sigma) = \Phi(g)(f', \sigma)$. Flipping all the arrows, we see that Φ^{op} satisfies property (F). □

Given a finite collection $\underline{G} = (G_i)_{i \in I}$ of finite groups, we write $\mathbf{FS}_{\underline{G}}$ for the product category $\prod_{i \in I} \mathbf{FS}_{G_i}$.

Corollary 11.1.9. *The category $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ is quasi-Gröbner.*

Proof. This follows from Propositions 5.2.5, 5.2.6 and Theorem 8.1.2. □

Corollary 11.1.10. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}})$ is noetherian.*

We now give an interesting source of examples of $\mathbf{FS}_G^{\mathrm{op}}$-modules, which provides motivation for the general study of $\mathbf{FS}_G^{\mathrm{op}}$-modules.

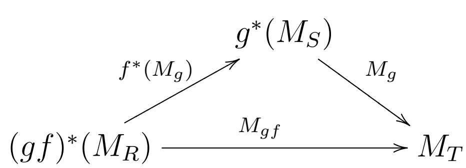
Example 11.1.11 (Segre products of simplicial complexes)**.** Let X and Y be simplicial complexes on finite vertex sets X_0 and Y_0. Define a simplicial complex $X * Y$ on the vertex set $X_0 \times Y_0$ as follows. Let $p_1\colon X_0 \times Y_0 \to X_0$ be the projection map, and similarly define p_2. Then $S \subset X_0 \times Y_0$ is a simplex if and only if $p_1(S)$ and $p_2(S)$ are simplices of X and Y and have the same cardinality as S. We call $X * Y$ the **Segre product** of X and Y. It is functorial for maps of simplicial complexes. It is not a topological construction, and depends in an essential way on the simplicial structure.

Fix a finite simplicial complex X, equipped with an action of a group G. The diagonal map $X_0 \to X_0 \times X_0$ induces a map of simplicial complexes $X \to X * X$. We thus obtain a functor from $\mathbf{FS}_G^{\mathrm{op}}$ to the category of simplicial complexes by $S \mapsto X^{*S}$. Fixing i, we obtain a representation M_i of $\mathbf{FS}_G^{\mathrm{op}}$ by $S \mapsto \mathrm{H}_i(X^{*S}; \mathbf{k})$. It is not difficult to directly show that $S \mapsto \mathrm{C}_i(X^{*S}; \mathbf{k})$ is a finitely generated representation of $\mathbf{FS}_G^{\mathrm{op}}$, where C_i denotes the space of simplicial i-chains. Thus by Corollary 11.1.10, M_i is a finitely generated representation of $\mathbf{FS}_G^{\mathrm{op}}$. Theorem 11.2.1 below gives information about the Hilbert series of M_i.

The case where X is just a single d-simplex is already extremely complicated and interesting, and is closely related to syzygies of the Segre embedding. □

We close this section by connecting $\mathbf{FS}_G^{\mathrm{op}}$-modules to Δ-modules.

Remark 11.1.12 (Generalized Δ-modules)**.** Let $\mathcal{A}$ be an abelian category equipped with a symmetric "cotensor" structure, i.e., a functor $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, and analogous data opposite to that of a tensor structure. (Here we are using the Deligne tensor product of abelian categories [De].) Given a surjection $f\colon T \to S$ of finite sets, there is an induced functor $f^*\colon \mathcal{A}^{\otimes S} \to \mathcal{A}^{\otimes T}$ by cotensoring along the fibers of f. A **Δ-module** over $\mathcal{A}$ is a rule M that assigns to each finite set S an object M_S of $\mathcal{A}^{\otimes S}$ and to each surjection $f\colon T \to S$ of finite sets a morphism $M_f\colon f^*(M_S) \to M_T$, such that if $f\colon T \to S$ and $g\colon S \to R$ are surjections, then the diagram



commutes. There are two main examples relevant to this paper:

- Let $\mathcal{A}$ be the category of polynomial functors Vec $\to$ Vec. Then $\mathcal{A}^{\otimes 2}$ is identified with the category of polynomial functors $\mathrm{Vec}^2 \to \mathrm{Vec}$. There is a comultiplication $\mathcal{A} \to \mathcal{A}^{\otimes 2}$ taking a functor F to the functor $(U, V) \mapsto F(U \otimes V)$, and this gives $\mathcal{A}$ the structure of a symmetric cotensor category. Δ-modules over $\mathcal{A}$ are Δ-modules as defined in §9.
- Let $\mathcal{A}$ be the category of representations of a finite group G. Then $\mathcal{A}^{\otimes 2}$ is identified with the category of representations of $G \times G$. There is a comultiplication $\mathcal{A} \to \mathcal{A}^{\otimes 2}$ taking a representation V of G to the representation $\mathrm{Ind}_G^{G\times G}(V)$ of $G \times G$, where G is included in $G \times G$ via the diagonal map. This gives $\mathcal{A}$ the structure of a symmetric cotensor category. Δ-modules over $\mathcal{A}$ are representations of $\mathbf{FS}_G^{\mathrm{op}}$.

If $n!$ is invertible in the base field then the category of polynomial functors of degree $\le n$ is equivalent, as a cotensor category, to the category $\prod_{k=0}^n \mathrm{Rep}(S_n)$. We thus find that Δ-modules of degree at most n (in the sense of §9) coincide with representations of $\prod_{k=0}^n \mathbf{FS}_{S_k}^{\mathrm{op}}$. Thus our results on $\mathbf{FS}_G^{\mathrm{op}}$ can be loosely viewed as a generalization of our results on Δ-modules ("loosely" because in bad characteristic the results are independent of each other). It seems possible that our results could generalize to Δ-modules over any "finite" abelian cotensor category. □

11.2. **Hilbert series.** Let M be a finitely generated representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ over a field $\mathbf{k}$. Let $\mathbf{n} \in \mathbf{N}^I$, and write $[\mathbf{n}]$ for $([n_i])_{i \in I}$. Then $M([\mathbf{n}])$ is a finite dimensional representation of $\underline{G}^{\mathbf{n}}$. Let $[M]_{\mathbf{n}}$ denote the image of the class of this representation under the map

$$\mathcal{R}_{\mathbf{k}}(\underline{G}^{\mathbf{n}}) = \bigotimes_{i \in I} \mathcal{R}_{\mathbf{k}}(G_i)^{\otimes n_i} \to \mathrm{Sym}^{|\mathbf{n}|}(\mathcal{R}_{\mathbf{k}}(\underline{G})),$$

where

$$\mathcal{R}_{\mathbf{k}}(\underline{G}) = \bigoplus_{i \in I} \mathcal{R}_{\mathbf{k}}(G_i).$$

Note that one can recover the isomorphism class of $M([\mathbf{n}])$ as a representation of $\underline{G}^{\mathbf{n}}$ from $[M]_{\mathbf{n}}$ due to the $S_{\mathbf{n}}$-equivariance. If $\{L_{i,j}\}$ are the irreducible representations of the G_i, then $[M]_{\mathbf{n}}$ can be thought of as a polynomial in corresponding variables $\{t_{i,j}\}$. Define the **Hilbert series** of M by

$$\mathrm{H}_M(\mathbf{t}) = \sum_{\mathbf{n} \in \mathbf{N}^I} [M]_{\mathbf{n}}.$$

This is an element of the ring $\widehat{\mathrm{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{G})_{\mathbf{Q}}) \cong \mathbf{Q}[\![t_{i,j}]\!]$. This definition does not fit into our framework of Hilbert series of normed categories, though it can be seen as an enhanced Hilbert series, as discussed in §1.5.2.

The following is a simplified version of our main theorem on Hilbert series. Recall the definition of $\mathcal{K}_N$ from Definition 3.4.4.

Theorem 11.2.1. *Let M be a finitely generated representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ over an algebraically closed field $\mathbf{k}$. Then $\mathrm{H}_M(\mathbf{t})$ is a rational function of the $\mathbf{t}$. In fact, it is a $\mathcal{K}_N$ function of the $\mathbf{t}$, where N is the least common multiple of the exponents of the G_i.*

Stating the full result requires some additional notions. Let G be a finite group and let $\mathbf{k}$ be an arbitrary field. Let $\{H_j\}_{j \in J}$ be a collection of subgroups whose orders are invertible in $\mathbf{k}$, and let H_j^{ab} be the abelianization of H_j. There is a functor $\mathrm{Rep}_{\mathbf{k}}(H_j) \to \mathrm{Rep}_{\mathbf{k}}(H_j^{\mathrm{ab}})$ given by taking coinvariants under $[H_j, H_j]$. This functor is exact since the order of H_j is invertible in $\mathbf{k}$, and thus induces a homomorphism $\mathcal{R}_{\mathbf{k}}(H_j) \to \mathcal{R}_{\mathbf{k}}(H_j^{\mathrm{ab}})$. There are also homomorphisms $\mathcal{R}_{\mathbf{k}}(G) \to \mathcal{R}_{\mathbf{k}}(H_j)$ given by restriction. We say that the family $\{H_j\}$ is **good** if the composite

$$\mathcal{R}_{\mathbf{k}}(G) \to \bigoplus_{j \in J} \mathcal{R}_{\mathbf{k}}(H_j) \to \bigoplus_{j \in J} \mathcal{R}_{\mathbf{k}}(H_j^{\mathrm{ab}})$$

is a split injection (i.e., an injection with torsion-free cokernel). We say that G is **N-good** if it admits a good family $\{H_j\}$ such that the exponent of each H_j^{ab} divides N. We say that a family $\underline{G}$ of finite groups is N-good if each member is. These notions depend on $\mathbf{k}$.

The following is our main theorem on Hilbert series. The proof is given in §11.4.

Theorem 11.2.2. *Suppose that $\underline{G}$ is N-good and $\mathbf{k}$ contains the Nth roots of unity. Let M be a finitely generated $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$-module over $\mathbf{k}$. Then $\mathrm{H}_M(\mathbf{t})$ is a $\mathcal{K}_N$ function of the $t_{i,j}$.*

Using Brauer's theorem, we show that over an algebraically closed field, every group is N-good for some N (Proposition 11.3.2), and so Theorem 11.2.1 follows from Theorem 11.2.2. We show that symmetric groups are 2-good if $n!$ is invertible in $\mathbf{k}$, which essentially recovers our results on Hilbert series of Δ-modules in good characteristic. For general groups, we know little about N. Finding some results could be an interesting group-theory problem; see §1.5.6.

Example 11.2.3. Let G be a finite group and let $\{V_i\}_{i \in I}$ be the set of irreducible representations of G over $\mathbf{C}$. Define an $\mathbf{FS}_G^{\mathrm{op}}$-module M_i by

$$M_i(S) = \mathrm{Ind}_G^{G^S}(V_i),$$

where $G \to G^S$ is the diagonal map. Let C be the set of conjugacy classes in G, χ_i be the character of V_i, and t_i be an indeterminate corresponding to V_i. A computation similar to that in [Sn, Lem. 5.7] gives

$$\mathrm{H}_{M_i}(\mathbf{t}) = \frac{1}{\#G} \sum_{c \in C} \frac{\#c \cdot \chi_i(c)}{1 - (\sum_{j \in I} \chi_j(c) t_j)}.$$

This is a $\mathcal{K}_N$ function of the t_i, as predicted by Theorem 11.2.1, where N is such that all characters of G take values in $\mathbf{Q}(\zeta_N)$. □

11.3. **Group theory.** Let $p = \mathrm{char}(\mathbf{k})$. If $p = 0$ then every group has order prime to p, and the only p-group is the trivial group. We say that a collection $\{H_i\}_{i \in I}$ of subgroups of G is a **covering** if the map on Grothendieck groups

$$\mathcal{R}_{\mathbf{k}}(G) \to \bigoplus_{i \in I} \mathcal{R}_{\mathbf{k}}(H_i)$$

is a split injection. Recall that if ℓ is a prime, then an **ℓ-elementary group** is one that is the direct product of an ℓ-group and a cyclic group of order prime to ℓ. An **elementary group** is a group which is ℓ-elementary for some prime ℓ.

Lemma 11.3.1. *The following result holds over any field $\mathbf{k}$:*

(a) *Let $\{H_i\}_{i \in I}$ be a covering of G, and for each i let $\{K_j\}_{j \in J_i}$ be a covering of H_i, and let $J = \amalg_{i \in I} J_i$. Then $\{K_j\}_{j \in J}$ is a covering of G.*
(b) *Let $\{H_i\}_{i \in I}$ be a covering of G, and suppose each H_i is N-good. Then G is N-good.*
(c) *Suppose that H is a p-elementary group and write $H = H_1 \times H_2$, where H_1 is cyclic of order prime to p and H_2 is a p-group. Then $\{H_1\}$ is a covering of H.*

The following hold if $\mathbf{k}$ *is algebraically closed:*

(d) *The collection of elementary subgroups* $\{H_i\}_{i \in I}$ *of* G *is a covering of* G.
(e) *Let* H *be a group of order prime to* p. *Then* H *is* N*-good for some* N.

Proof. (a) and (b) are clear.

(c) The only simple $\mathbf{k}[H_2]$-module is trivial, so $\mathcal{R}_{\mathbf{k}}(H) \to \mathcal{R}_{\mathbf{k}}(H_1)$ is an isomorphism.

(d) Let $\alpha \colon \mathcal{R}_{\mathbf{k}}(G) \to \bigoplus \mathcal{R}_{\mathbf{k}}(H_i)$ be the restriction map. Let $\mathcal{P}_{\mathbf{k}}(G)$ be the Grothendieck group of finite-dimensional projective $\mathbf{k}[G]$-modules. The map $\mathcal{R}_{\mathbf{k}}(G) \times \mathcal{P}_{\mathbf{k}}(G) \to \mathbf{Z}$ given by $(V, W) \mapsto \dim_{\mathbf{k}} \operatorname{Hom}_G(V, W)$ is a perfect pairing [Se, §14.5]. Combining this with Frobenius reciprocity, it follows that the dual of α can be identified with the induction map $\bigoplus \mathcal{P}_{\mathbf{k}}(H_i) \to \mathcal{P}_{\mathbf{k}}(G)$. This map is surjective by Brauer's theorem [Se, §17.2, Thm. 39]. Since the dual of α is surjective, it follows that α is a split injection, which proves the claim.

(e) Indeed, arguing with duals and Frobenius reciprocity again, it is enough to find subgroups $\{K_i\}_{i \in I}$ of H such that the induction map $\bigoplus_{i \in I} \mathcal{R}_{\mathbf{k}}(K_i^{\mathrm{ab}}) \to \mathcal{R}_{\mathbf{k}}(H)$ is surjective. (Note that $\mathcal{R}_{\mathbf{k}} = \mathcal{P}_{\mathbf{k}}$ for groups of order prime to p.) This follows from Brauer's theorem [Se, §10.5, Thm. 20]. □

Proposition 11.3.2. *Suppose* $\mathbf{k}$ *is algebraically closed and* G *is a finite group. Then* G *is* N*-good for some* N.

Proof. By parts (a), (c), and (d) of Lemma 11.3.1, G has a covering by its subgroups of order prime to p. For each of these groups, its set of subgroups is good by part (e). Now finish by applying (b). □

We now construct a good collection of subgroups for the symmetric group S_n in good characteristic. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\sum_i \lambda_i = n$, let $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_n}$ be the corresponding Young subgroup of S_n.

Proposition 11.3.3. *Suppose* $n!$ *is invertible in* $\mathbf{k}$. *Then* $\{S_\lambda\}$ *is a good collection of subgroups of the symmetric group* S_n.

Proof. Under the assumption on $\operatorname{char}(\mathbf{k})$, the representations of S_n are semisimple. Using Frobenius reciprocity, the restriction map on representation rings is dual to induction. We claim that each irreducible character of S_n is a $\mathbf{Z}$-linear combination of the permutation representations of S_n/S_λ (this implies $\{S_\lambda\}$ is a good collection of subgroups). Recall that the irreducible representations of S_n are indexed by partitions of n (we will denote them $\mathbf{M}_\lambda$). Also, recall the dominance order on partitions: $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i. An immediate consequence of Pieri's rule [SS2, (2.10)] is that the permutation representation S_n/S_λ contains $\mathbf{M}_\lambda$ with multiplicity 1 and the remaining representations $\mathbf{M}_\mu$ that appear satisfy $\mu \geq \lambda$. This proves the claim. □

Corollary 11.3.4. *Let* $\mathbf{k}$ *be a field in which* $n!$ *is invertible. Then* S_n *is* 2*-good.*

Proof. The group S_λ^{ab} has exponent 1 or 2 for any λ. □

11.4. **Proof of Theorem 11.2.2.** The idea is to first use the good family of subgroups to reduce to the case where each G_i is abelian of invertible order. For such G, we identify representations of $\mathbf{FS}_G^{\mathrm{op}}$ with representations of $\mathbf{FWS}_\Lambda^{\mathrm{op}}$, where Λ is the group of characters of G. The theorem then follows from our results for Hilbert series of representations of $\mathbf{FWS}_\Lambda^{\mathrm{op}}$. We now go through the details.

Lemma 11.4.1. *Let $\underline{G} = (G_i)_{i \in I}$ be finite groups. For each i, let H_i be a subgroup of G_i whose order is invertible in $\mathbf{k}$. Define a functor*

$$\Phi \colon \operatorname{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}}_{\underline{G}}) \to \operatorname{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}}_{\underline{H}^{\mathrm{ab}}})$$

by letting $\Phi(M)(\underline{S})$ be the $[\underline{H}, \underline{H}]^{\underline{S}}$-coinvariants of $M(\underline{S})$. Then we have the following:

(a) *$\Phi(M)$ is a well-defined object of $\operatorname{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}}_{\underline{H}^{\mathrm{ab}}})$.*

(b) *If M is finitely generated then so is $\Phi(M)$.*

(c) *Let $\varphi_i \colon \mathcal{R}_{\mathbf{k}}(G_i) \to \mathcal{R}_{\mathbf{k}}(H_i^{\mathrm{ab}})$ be the map induced by restricting to H_i followed by taking $[H_i, H_i]$-coinvariants, and let $\varphi \colon \mathcal{R}_{\mathbf{k}}(\underline{G}) \to \mathcal{R}_{\mathbf{k}}(\underline{H}^{\mathrm{ab}})$ be the sum of the φ_i. Then $\mathrm{H}_{\Phi(M)}$ is the image of H_M under the ring homomorphism $\widehat{\operatorname{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{G})_{\mathbf{Q}}) \to \widehat{\operatorname{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{H}^{\mathrm{ab}})_{\mathbf{Q}})$ induced by φ.*

Proof. (a) For a tuple $\underline{S} = (S_i)_{i \in I}$ of sets, let $K(\underline{S})$ be the $\mathbf{k}$-subspace of $M(\underline{S})$ spanned by elements of the form $gm - m$ with $g \in [\underline{H}, \underline{H}]^{\underline{S}}$ and $m \in M(\underline{S})$. If $f \colon \underline{S} \to \underline{T}$ is a morphism in $\mathbf{FS}^I$ then the induced map $f^* \colon M(\underline{T}) \to M(\underline{S})$ carries $gm - m$ to $f^*(g)f^*(m) - f^*(m)$. Thus K is a $(\mathbf{FS}^{\mathrm{op}})^I$-submodule of M, and so M/K is a well-defined $(\mathbf{FS}^{\mathrm{op}})^I$-module. The group actions clearly carry through, and so $\Phi(M)$ is well-defined.

(b) Suppose $M \in \operatorname{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}}_{\underline{G}})$ is finitely generated. Then the restriction of M to $(\mathbf{FS}^{\mathrm{op}})^I$ is finitely generated by Propositions 11.1.8 and 4.2.4. The restriction of $\Phi(M)$ to $(\mathbf{FS}^{\mathrm{op}})^I$ is a quotient of the restriction of M, and is therefore finitely generated. Thus $\Phi(M)$ is finitely generated, by Proposition 4.2.5.

(c) This is clear. ∎

Lemma 11.4.2. *Let $\underline{G} = (G_i)_{i \in I}$ be groups, let $f \colon J \to I$ be a surjection, and let $f^*(\underline{G})$ be the resulting family of groups indexed by J. Let $\Phi \colon \mathbf{FS}^{\mathrm{op}}_{f^*(\underline{G})} \to \mathbf{FS}^{\mathrm{op}}_{\underline{G}}$ be the functor induced by disjoint union, i.e., $\Phi(\{S_j\}_{j \in J}) = \{T_i\}_{i \in I}$ where $T_i = \coprod_{j \in f^{-1}(i)} S_j$.*

(a) *Φ satisfies property (F); in particular, if M is finitely generated then so is $\Phi^*(M)$.*

(b) *Let $\varphi_i \colon \mathcal{R}_{\mathbf{k}}(G_i) \to \bigoplus_{j \in f^{-1}(i)} \mathcal{R}_{\mathbf{k}}(G_j)$ be the diagonal map, and let $\varphi \colon \mathcal{R}_{\mathbf{k}}(\underline{G}) \to \mathcal{R}_{\mathbf{k}}(f^*(\underline{G}))$ be the sum of the φ_i. Then $\mathrm{H}_{\Phi^*(M)}$ is the image of H_M under the ring homomorphism $\widehat{\operatorname{Sym}}(\mathcal{R}_{\mathbf{k}}(\underline{G})_{\mathbf{Q}}) \to \widehat{\operatorname{Sym}}(\mathcal{R}_{\mathbf{k}}(f^*(\underline{G}))_{\mathbf{Q}})$ induced by φ.*

Proof. (a) Consider the commutative diagram of categories

$$\begin{array}{ccc} \mathbf{FS}^{\mathrm{op}}_{f^*(\underline{G})} & \xrightarrow{\Phi} & \mathbf{FS}^{\mathrm{op}}_{\underline{G}} \\ \uparrow & & \uparrow \\ (\mathbf{FS}^{\mathrm{op}})^J & \xrightarrow{\Phi'} & (\mathbf{FS}^{\mathrm{op}})^I \end{array}$$

The functor Φ' is defined just like Φ; it satisfies property (F) by Proposition 8.4.9. The vertical maps satisfy property (F) by Proposition 11.1.8. Thus Φ satisfies property (F) by Propositions 4.2.2 and 4.2.3.

(b) This is clear. ∎

Lemma 11.4.3. *Suppose that $\underline{G} = (G_i)_{i \in I}$ is a family of commutative groups of exponents dividing N. Suppose that N is invertible in $\mathbf{k}$ and that $\mathbf{k}$ contains the Nth roots of unity. Let $\Lambda_i = \operatorname{Hom}(G_i, \mathbf{k}^\times)$ be the group of characters of G_i, and let $\underline{\Lambda} = (\Lambda_i)_{i \in I}$. Then there is an equivalence $\Phi \colon \operatorname{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}}_{\underline{G}}) \to \operatorname{Rep}_{\mathbf{k}}(\mathbf{FWS}^{\mathrm{op}}_{\underline{\Lambda}})$ respecting Hilbert series, i.e., $\mathrm{H}_M = \mathrm{H}_{\Phi(M)}$ for $M \in \operatorname{Rep}(\mathbf{FS}^{\mathrm{op}}_{\underline{G}})$.*

Before giving the proof, we offer two clarifications. First, $\mathbf{FWS}_{\underline{\Lambda}}$ denotes the category $\prod_{i \in I} \mathbf{FWS}_{\Lambda_i}$. An object of this category is a tuple of sets $\underline{S} = (S_i)_{i \in I}$ equipped with a weight function $\varphi_i \colon S_i \to \Lambda_i$ for each i. Second, H_M and $\mathrm{H}_{\Phi(M)}$ are both series in variables indexed by the characters of the G_i. This is why they are comparable.

Proof. Let M be a representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$. Let $\underline{S} = (S_i)_{i \in I}$ be a tuple of sets. Then we have a decomposition

$$M_{\underline{S}} = \bigoplus M_{\underline{S}, \underline{\varphi}},$$

where the sum is over weightings $\underline{\varphi}$ of $\underline{S}$, and $M_{\underline{S},\underline{\varphi}}$ is the subspace of $M_{\underline{S}}$ on which $\underline{G}^{\underline{S}}$ acts through $\underline{\varphi}$. If $\underline{f} \colon \underline{S} \to \underline{T}$ is a morphism in $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ then the map $\underline{f}_* \colon M_{\underline{S}} \to M_{\underline{T}}$ carries $M_{\underline{S},\underline{\varphi}}$ into $M_{\underline{T}, \underline{f}_*(\underline{\varphi})}$. We define $\Phi(M)$ to be the functor on $\mathbf{FWS}_{\underline{\Lambda}}^{\mathrm{op}}$ which assigns to a weighted set $(\underline{S}, \underline{\varphi})$ the space $M_{\underline{S},\underline{\varphi}}$. This construction can be reversed: given a representation M of $\mathbf{FWS}_{\underline{\Lambda}}^{\mathrm{op}}$, we can build a representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$ by defining $M_{\underline{S}}$ to be the sum of the $M_{\underline{S},\underline{\varphi}}$. We leave to the reader the verification that these constructions are quasi-inverse to each other. This shows that Φ is an equivalence. It is clear that it preserves Hilbert series: we note that the multinomial coefficients in the definition of $\mathrm{H}_{\Phi(M)}$ count, for each $M_{\underline{S},\underline{\varphi}}$, the number of $M_{\underline{S},\underline{\varphi}'}$ where $\underline{\varphi}'$ is a permutation of $\underline{\varphi}$. □

Proof of Theorem 11.2.2. Let M be a finitely generated representation of $\mathbf{FS}_{\underline{G}}^{\mathrm{op}}$, where $\underline{G} = (G_i)_{i \in I}$. For each $i \in I$, let $\{H_j\}_{j \in J_i}$ be a good collection of subgroups of $\underline{G}$ such that the exponent of each H_j^{ab} divides N. Let $J = \coprod_{i \in I} J_i$ and let $f \colon J \to I$ be the projection map. Then we have functors

$$\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{G}}^{\mathrm{op}}) \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{f^*(\underline{G})}^{\mathrm{op}}) \to \mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}_{\underline{H}^{\mathrm{ab}}}^{\mathrm{op}}).$$

Let M' be the image of M under the composition. By Lemmas 11.4.1 and 11.4.2, M' is finitely generated and $\mathrm{H}_{M'}$ is the image of H_M under the ring homomorphism corresponding to the natural additive map $\mathcal{R}_{\mathbf{k}}(\underline{G}) \to \mathcal{R}_{\mathbf{k}}(\underline{H}^{\mathrm{ab}})$. By Lemma 11.4.3 and Theorem 10.1.4, $\mathrm{H}_{M'}$ is $\mathcal{K}_N$. Thus by Lemma 3.4.5, H_M is $\mathcal{K}_N$. This completes the proof. □

12. An example with non-regular languages

Let $\mathbf{OI}_d^{=}$ be the subcategory of $\mathbf{OI}_d$ containing all objects but only those morphisms (f, g) for which all fibers of g have the same size (in other words, each of the d "colors" are used the same number of times). One easily sees that the inclusion $\mathbf{OI}_d^{=} \subset \mathbf{OI}_d$ satisfies property (S) (Definition 5.2.7), and so $\mathbf{OI}_d^{=}$ is Gröbner by Proposition 5.2.8 and Theorem 7.1.1. Endow $\mathbf{OI}_d^{=}$ with the restricted norm from $\mathbf{OI}_d$. Obviously, $\mathbf{OI}_1^{=} = \mathbf{OI}_1$, which is O-lingual (Theorem 7.1.1). We now examine what happens for larger d.

Proposition 12.1. *The normed category $\mathbf{OI}_2^{=}$ is* UCF*-lingual.*

Proof. Let $\mathcal{C} = \mathbf{OI}_2^{=}$ and let $\mathcal{C}' = \mathbf{OI}_2$. Let $x = [n]$ be an object of $\mathcal{C}$ and let $\Sigma = \{0, 1, 2\}$. We regard $|\mathcal{C}_x|$ as a subset of $|\mathcal{C}'_x|$ with the induced order, which is admissible by Proposition 5.2.8. Define the map $i \colon |\mathcal{C}'_x| \to \Sigma^\star$ as in the proof of Theorem 7.1.1. As shown there, if T is a poset ideal of $|\mathcal{C}'_x|$, then $i(T)$ is a regular language. Now, if T is a poset ideal of $|\mathcal{C}_x|$ and S is the poset ideal of $|\mathcal{C}'_x|$ it generates, then $T = S \cap |\mathcal{C}_x|$. Thus $i(T)$ is the intersection of the regular language $i(S)$ with the language $\mathcal{L} = i(|\mathcal{C}_x|)$. The language $\mathcal{L}$ consists of those words in Σ that contain exactly n 0's and use each of the symbols 1 and 2 the same number of times. This is a deterministic context-free (DCF) language (the proof is similar to [HU, Exercise 5.1]). As the intersection of a DCF language and a regular language is DCF [HU, Thm. 10.4], we see that $i(T)$ is DCF. Finally, DCF implies UCF (this is well-known, and follows from the proof of [HU, Thm. 5.4]), and so $i(T)$ is UCF. □

Corollary 12.2. *If M is a finitely-generated representation of $\mathbf{OI}_2^=$ then $\mathrm{H}_M(t)$ is an algebraic function of t.*

Example 12.3. Let $d \geq 1$ be arbitrary. Let $M_d \in \mathrm{Rep}_{\mathbf{k}}(\mathbf{OI}_d^=)$ be the principal projective at $[0]$, and let H_d be its Hilbert series. The space $M_d([n])$ has for a basis the set of all strings in $\{1, \ldots, d\}$ of length n in which the numbers $1, \ldots, d$ occur equally. The number of strings in which i occurs exactly n_i times is the multinomial coefficient

$$\frac{(n_1 + \cdots + n_d)!}{n_1! \cdots n_d!}.$$

It follows that

$$\mathrm{H}_d(t) = \sum_{n=0}^{\infty} \frac{(dn)!}{n!^d} t^{dn}.$$

In particular,

$$\mathrm{H}_1(t) = \frac{1}{1-t} \quad \text{and} \quad \mathrm{H}_2(t) = \frac{1}{\sqrt{1-4t^2}},$$

but $\mathrm{H}_d(t)$ is not algebraic for $d > 2$ [WS, Thm. 3.8]. □

Corollary 12.4. *If $d > 2$ then the normed category $\mathbf{OI}_d^=$ is not UCF-lingual.*

Let $\mathbf{FI}_d^=$ be the subcategory of $\mathbf{FI}_d$ defined in a way similar to $\mathbf{OI}_d^=$ with the induced norm. Then $\mathbf{FI}_d^=$ is quasi-Gröbner and the (non-exponential) Hilbert series of a finitely generated $\mathbf{FI}_2^=$-module is algebraic. We can interpret $\mathbf{FI}_d^=$ in terms of twisted commutative algebras. We just explain over the complex numbers. Given vector spaces $V_1, \ldots, V_d$, there is a natural map $V_1 \otimes \cdots \otimes V_d \to \mathrm{Sym}^d(V_1 \oplus \cdots \oplus V_d)$. Specializing to $V_i = \mathbf{C}^\infty$, we obtain a map $(\mathbf{C}^\infty)^{\otimes d} \to \mathrm{Sym}^d((\mathbf{C}^\infty)^{\oplus d})$. We thus have a map of algebras

$$\mathrm{Sym}((\mathbf{C}^\infty)^{\otimes d}) \to \mathrm{Sym}((\mathbf{C}^\infty)^{\oplus d}).$$

Let A be the image of this map. This is a bounded tca (see [SS2, §9]) generated in degree d. Then $\mathrm{Rep}_{\mathbf{C}}(\mathbf{FI}_d^=)$ is equivalent to the category of A-modules. It follows that the (non-exponential) Hilbert series of a finitely generated A-module is algebraic when $d = 2$.

Remark 12.5. We have shown, by very different means, that the Hilbert series of a finitely generated $\mathbf{FI}_d^=$-module over $\mathbf{C}$ is a D-finite function, for any $d \geq 1$. □

13. Examples of categories with infinite hom sets

We end with some examples of categories which have noetherian representation categories, and have infinite hom sets, in contrast to all of the examples previously studied. In §13.1 we study a linear-algebraic category built out of upper-unitriangular integer matrices. In §13.2 we introduce a generalization of **FI**-modules and prove a noetherianity result. As a special case, we improve Corollary 11.1.3 by allowing the group G to now be any polycyclic-by-finite group.

13.1. **Integral Borel categories.** If $n \leq m$, then a **Borel matrix** is an $m \times n$ integer matrix which is "upper unitriangular" in the sense that there exists $1 \leq s_1 < \cdots < s_n \leq m$ such that the (s_i, i) entries are all 1 (call these **pivots**), and the (j, i) entry is 0 if $j > s_i$. A **non-negative Borel matrix** is a Borel matrix whose entries are all non-negative.

Let $\mathscr{B}_{\mathbf{Z}}$ be the category whose objects are the non-negative integers $0, 1, 2, \ldots$ and where $\mathrm{Hom}_{\mathscr{B}_{\mathbf{Z}}}(n, m)$ is the set of $m \times n$ Borel matrices. Composition is defined by matrix multiplication. Let $\mathscr{B}_{\mathbf{Z}}^+$ be the subcategory of $\mathscr{B}_{\mathbf{Z}}$ where we only use non-negative Borel matrices. Note that $\mathrm{End}_{\mathscr{B}_{\mathbf{Z}}}(n)$ is the group of upper unitriangular integer matrices, while $\mathrm{End}_{\mathscr{B}_{\mathbf{Z}}^+}(n)$ is its "positive part."

Proposition 13.1.1. *The category $\mathscr{B}_{\mathbf{Z}}^+$ is Gröbner.*

Proof. Let $x = n$ be a non-negative integer, thought of as an object of $\mathscr{B}_{\mathbf{Z}}^+$. We claim that the poset $|S_x|$ (Definition 5.2.1) is noetherian. A given non-negative Borel matrix $m \times n$ generates all other non-negative Borel matrices which are obtained by inserting rows and also by increasing the values of any non-pivot entry which is allowed to be nonzero. The second point follows from the fact that the action of $\mathrm{End}(m)$ allows us to apply non-negative upwards row operations to the Borel matrix. For the first point, we explain how to insert a single row in position i: this follows from constructing a special $(m+1) \times m$ Borel matrix which is obtained from the identity $m \times m$ matrix by inserting the same row in position i with some extra 0's (we leave the details to the reader). So $|S_x|$ is a subposet of $(\mathbf{Z}^n_{\geq 0})^\star$ where $\mathbf{Z}^n_{\geq 0}$ is a poset with componentwise comparison. So $|S_x|$ is noetherian by Higman's lemma (Theorem 2.7).

Finally, we construct an ordering $\preceq$ on S_x. First, pick a total ordering on $\mathbf{Z}^n_{\geq 0}$ which extends the componentwise ordering. Let ψ and ψ' be non-negative Borel matrices of sizes $m \times n$ and $m' \times n$, respectively. Then we compare them as follows:

- If $m < m'$, then $\psi \preceq \psi'$.
- Otherwise $m = m'$. Let $s_1 < \cdots < s_m$ be the pivots of ψ and let $s'_1 < \cdots < s'_m$ be the pivots of ψ'. If $\{s_1, \ldots, s_m\} < \{s'_1, \ldots, s'_m\}$ in lexicographic order, then $\psi \preceq \psi'$.
- If $m = m'$ and the pivots of ψ and ψ' are the same, then we compare their rows lexicographically using the chosen ordering on $\mathbf{Z}^n_{\geq 0}$.

Then $\preceq$ extends the poset structure on $|S_x|$, so $\mathscr{B}_{\mathbf{Z}}^+$ is Gröbner. □

Corollary 13.1.2. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathscr{B}_{\mathbf{Z}}^+)$ is noetherian.*

Theorem 13.1.3. *If $\mathbf{k}$ is left-noetherian then $\mathrm{Rep}_{\mathbf{k}}(\mathscr{B}_{\mathbf{Z}})$ is noetherian.*

Proof. Let n be a non-negative integer, thought of as an object of $\mathscr{B}_{\mathbf{Z}}$, and let P_n be the principal projective module at n. We claim that given $x = \sum_{i=1}^r a_i f_i \in P_n(m)$, there exists $g \in \mathrm{End}_{\mathscr{B}_{\mathbf{Z}}^+}(m)$ such that $g \cdot f_1, \ldots, g \cdot f_r$ are non-negative Borel matrices. We prove this by induction on r. So let $h \in \mathrm{End}_{\mathscr{B}_{\mathbf{Z}}^+}(m)$ be such that $h \cdot f_1, \ldots, h \cdot f_{r-1}$ are non-negative. It is easy to find $g \in \mathrm{End}_{\mathscr{B}_{\mathbf{Z}}^+}(m)$ so that $g \cdot (h \cdot f_r)$ is non-negative: to construct g, we can perform arbitrary non-negative upwards row operations, so we can use the pivots to increase all negative values to non-negative ones. Then $g \cdot (h \cdot f_i)$ is non-negative for $i < r$ since $h \cdot f_i$ is non-negative. This proves the claim.

Finally, pick a submodule $M \subseteq P_n$ and pick generators $x_1, x_2, \ldots$. By the claim, we can replace each x_i by y_i, which is a linear combination of non-negative Borel matrices so that $y_1, y_2, \ldots$ also generates M (x_i and y_i differ by an invertible operator). Then the

$\mathscr{B}_{\mathbf{Z}}^+$-submodule generated by $y_1, y_2, \ldots$ can be identified with a submodule for the principal projective of $\mathscr{B}_{\mathbf{Z}}^+$ and hence is noetherian by Corollary 13.1.2. We conclude that it is generated by finitely many y_i, say $y_1, \ldots, y_N$. But then $x_1, \ldots, x_N$ generates M. $\square$

Remark 13.1.4. The group ring $\mathbf{k}[\mathbf{GL}_n(\mathbf{Z})]$ is not noetherian for $n \geq 2$, so this already implies that $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VI}_{\mathbf{Z}})$ is not noetherian. We give a quick proof of this fact now.

Given a subgroup $H \subseteq \mathbf{GL}_n(\mathbf{Z})$, we get a left ideal $I_H \subset \mathbf{k}[\mathbf{GL}_n(\mathbf{Z})]$ which is the kernel of the surjection $\mathbf{k}[\mathbf{GL}_n(\mathbf{Z})] \to \mathbf{k}[\mathbf{GL}_n(\mathbf{Z})/H]$. Then $H \subsetneqq H'$ implies $I_H \subsetneqq I_{H'}$, so it suffices to show that there is an infinite ascending chain of subgroups in $\mathbf{GL}_n(\mathbf{Z})$. This follows from the existence of free subgroups of rank 2 in $\mathbf{GL}_n(\mathbf{Z})$ (for example, when $n = 2$, take the subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$). $\square$

13.2. **$\mathbf{FI}_R$-modules.** Let $\mathbf{k}$ be a ring. Let (R, ϵ) be an augmented $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebra, that is, a $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebra R equipped with a surjection of $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebras $\epsilon \colon R \to \mathbf{k}$. Note that if $\mathbf{k}$ is commutative, then a $\mathbf{k}$-algebra structure on R is the same as a $(\mathbf{k} \otimes \mathbf{k}^{\mathrm{op}})$-algebra structure. We let $\mathfrak{a}$ be the kernel of ϵ, the **augmentation ideal**. For an R-module M we let $\Gamma(M)$ be the $\mathbf{k}$-submodule of M annihilated by $\mathfrak{a}$. For a finite set x, we write $R^{\otimes x}$ for the x-fold tensor product of R (over $\mathbf{k}$). If M is an $R^{\otimes y}$-module and x is a subset of y, we write $\Gamma_x(M)$ for the subspace of M annihilated by $\mathfrak{a}^{\otimes x}$.

An **$\mathbf{FI}_R$-module** is a rule M that attaches to every finite set x an $R^{\otimes x}$-module $M(x)$ and to every injection $f \colon x \to y$ of finite sets a map of $R^{\otimes x}$-modules $f_* \colon M(x) \to \Gamma_{y \setminus f(x)}(M(y))$ such that $(gf)_* = g_* f_*$ in the obvious sense. Here we regard $\Gamma_{y \setminus f(x)}(M(y))$ as an $R^{\otimes x}$-module via the homomorphism $f_* \colon R^{\otimes x} \to R^{\otimes y}$. Note that we have not actually defined a category $\mathbf{FI}_R$, but we still speak of $\mathbf{FI}_R$-modules. We write $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ for the category of $\mathbf{FI}_R$-modules. There are the usual notions of finite generation and noetherianity.

Let x be a finite set. We define the **principal projective $\mathbf{FI}_R$**-module at x by:

$$P_x(y) = \bigoplus_{f \colon x \to y} (R^{\otimes y} / \mathfrak{a}^{\otimes y \setminus f(x)}) \cdot e_f,$$

where the sum is over injections f. Thus $P_x(y)$ is spanned by the e_f, the annihilator of e_f is exactly $\mathfrak{a}^{\otimes y \setminus f(x)}$, and there are no other relations between the e_f. Note that $P_x(x)$ has a canonical element e_x, corresponding to the identity map $x \to x$. The following result justifies calling P_x the principal projective at x:

Lemma 13.2.1. *Let M be an $\mathbf{FI}_R$-module. Then the natural map $\mathrm{Hom}_{\mathbf{FI}_R}(P_x, M) \to M(x)$ given by evaluating on e_x is an isomorphism.*

Proof. It is clear that e_x generates P_x, and so the map is injective. Conversely, suppose that m is an element of $M(x)$. Given an injection of finite sets $f \colon x \to y$, the element $f_*(m)$ of $M(y)$ is annihilated by $\mathfrak{a}^{\otimes y \setminus f(x)}$, by definition. We therefore have a well-defined map of $R^{\otimes y}$ modules $P_x(y) \to M(y)$ given by $e_f \mapsto f_*(m)$. One readily verifies that this defines a map of $\mathbf{FI}_R$-modules taking e_x to m, which establishes surjectivity of the map in question. $\square$

The above lemma shows that the P_x are projective, and that every finitely generated $\mathbf{FI}_R$-module is a quotient of a finite direct sum of the P_x's. We now come to our main result:

Theorem 13.2.2. *Suppose that $R^{\otimes n}$ is left-noetherian for all $n \geq 0$. Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ is noetherian.*

Proof. It suffices to show that the principal projective P_x is noetherian. Let $P'_x \colon \mathbf{FI} \to \mathrm{Mod}_{R^{\otimes x}}$ be the principal projective representation of $\mathbf{FI}$ at x over the ring $R^{\otimes x}$. Thus $P'_x(y)$ is the free $R^{\otimes x}$-module with basis $\mathrm{Hom}_{\mathbf{FI}}(x, y)$. Suppose that $f \colon x \to y$ is an injection of finite sets. Then f_* induces an isomorphism $f_* \colon R^{\otimes x} \to R^{\otimes y}/\mathfrak{a}^{\otimes y \setminus f(x)}$. It follows that there is a natural isomorphism $\varphi_y \colon P'_x(y) \to P_x(y)$ given by $\lambda e_f \mapsto f_*(\lambda) e_f$. One readily verifies that if M is an $\mathbf{FI}_R$-submodule of P_x then $y \mapsto \varphi_y^{-1}(M(y))$ is a subobject of P'_x in the category $\mathrm{Rep}_{R^{\otimes x}}(\mathbf{FI})$. The noetherianity of this category (Corollary 7.1.3) now implies that P_x is noetherian, which completes the proof. □

We now relate $\mathbf{FI}_R$-modules to $\mathbf{FI}_G$ modules:

Proposition 13.2.3. *Let G be a group and let $R = \mathbf{k}[G]$ be its group algebra (augmented in the usual manner). Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_R)$ is canonically equivalent to $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$.*

Proof. Let M be an $\mathbf{FI}_G$-module. Then for each finite set x, we have a representation $M(x)$ of G^x, which can be thought of as an $\mathbf{k}[G^x] = R^{\otimes x}$-module. Given an injection $f \colon x \to y$ of finite sets, we have a map $f_* \colon M(x) \to M(y)$ of $\mathbf{k}$-modules. This map lands in the $G^{y \setminus f(x)}$ invariants of $M(y)$, and is G^x equivariant when G^x acts on the target via the homomorphism $G^x \to G^y$ induced by f. We therefore have a map $M(x) \to \Gamma_{y \setminus f(x)}(M(y))$ of $R^{\otimes x}$-modules. This shows that giving an $\mathbf{FI}_G$-module is exactly the same as giving an $\mathbf{FI}_R$-module. □

Corollary 13.2.4. *Let G be a group such that the group algebra $\mathbf{k}[G^n]$ is noetherian for all $n \geq 0$. Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ is noetherian.*

Recall that a group G is **polycyclic** if it has a finite composition series $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$ such that G_i/G_{i-1} is cyclic for $i = 1, \ldots, r$, and it is **polycyclic-by-finite** if it contains a polycyclic subgroup of finite index. It is known [Hal, §2.2, Lemma 3] that the group ring of a polycyclic-by-finite group over a left-noetherian ring is left-noetherian (there it is stated for the integral group ring, but the proof works for any left-noetherian coefficient ring). In fact, there are no other known examples of noetherian group algebras, but see [Iv] for related results. As the product of two polycyclic-by-finite groups is again polycyclic-by-finite the above corollary gives:

Corollary 13.2.5. *Let G be a polycyclic-by-finite group and let $\mathbf{k}$ be a left-noetherian ring. Then $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI}_G)$ is noetherian.*

Remark 13.2.6. The proof of this corollary uses the fact that FI-modules over the non-commutative ring $\mathbf{k}[G^n]$ are noetherian. This is the first real use of FI-modules over non-commutative rings that we are aware of. □

Remark 13.2.7. Everything in this section also applies to $\mathbf{FI}_d$-versions of the categories. □

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