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GROMOV-HAUSDORFF LIMITS OF KÄHLER MANIFOLDS WITH RICCI CURVATURE BOUNDED BELOW

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Abstract. We show that non-collapsed Gromov-Hausdorff limits of polarized Kähler manifolds, with Ricci curvature bounded below, are normal projective varieties, and the metric singularities of the limit space are precisely given by a countable union of analytic subvarieties. This extends a fundamental result of Donaldson-Sun, in which 2-sided Ricci curvature bounds were assumed. As a basic ingredient we show that, under lower Ricci curvature bounds, almost Euclidean balls in Kähler manifolds admit good holomorphic coordinates. Further applications are integral bounds for the scalar curvature on balls, and a rigidity theorem for Kähler manifolds with almost Euclidean volume growth.

1. Introduction

The structure of Gromov-Hausdorff limits of Riemannian manifolds with Ricci curvature bounded below has been studied extensively since the seminal work of Cheeger-Colding [3, 4, 5, 6], with a great deal of more recent important progress (see e.g. [10][12][13][9]). In the Kähler setting, the recent breakthrough work of Donaldson-Sun [21] has led to many important advances. They proved in particular that the Gromov-Hausdorff limit of a sequence of non-collapsed, polarized Kähler manifolds, with 2-sided Ricci curvature bounds, is a normal projective variety. Our first result is a generalization of this statement, removing the assumption of an upper bound for the Ricci curvature.

Theorem 1.1. *Given $n, d, v > 0$, there are constants $k_1, N > 0$ with the following property. Let (M_i^n, L_i, ω_i) be a sequence of polarized Kähler manifolds such that*

- *L_i is a Hermitian holomorphic line bundle with curvature $-\sqrt{-1}\omega_i$;*
- *$\mathrm{Ric}(\omega_i) > -\omega_i$, $\mathrm{vol}(M_i) > v$, and $\mathrm{diam}(M_i, \omega_i) < d$;*
- *The sequence (M_i^n, ω_i) converges in the Gromov-Hausdorff sense to a limit metric space X.*

Then each M_i^n can be embedded in a subspace of $\mathbb{CP}^N$ using sections of $L_i^{k_1}$, and the limit X is homeomorphic to a normal projective variety in $\mathbb{CP}^N$. Taking a subsequence and applying suitable projective transformations, the $M_i \subset \mathbb{CP}^N$ converge to X as algebraic varieties.

This result implies that, given bounds on n, d, v, only finitely many Hilbert polynomials appear. An immediate corollary is the following.

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Corollary 1.1. *Given $n, d, v > 0$, there are finitely many diffeomorphism types of polarized Kähler manifolds (M^n, L, ω), of dimension n, such that $\mathrm{Ric}(\omega) > -\omega$, the curvature of L is $-\sqrt{-1}\omega$, and $\mathrm{vol}(M) > v, \mathrm{diam}(M) < d$.*

The strategy of proof of Theorem 1.1 follows Donaldson-Sun [21], and a key step is the proof of the following partial C^0-estimate, conjectured originally by Tian [49] for Fano manifolds.

Theorem 1.2. *Given $n, d, v > 0$, there are $k_2, b > 0$ with the following property. Suppose that (M, L, ω) is a polarized Kähler manifold with $\mathrm{Ric}(\omega) > -\omega$, $\mathrm{vol}(M) > v$ and $\mathrm{diam}(M, \omega) < d$. Then for all $p \in M$, the line bundle L^{k_2} admits a holomorphic section s over M satisfying $|s\|_{L^2} = 1$, and $|s(p)| > b$.*

Tian [49] conjectured this result under a positive lower bound for the Ricci curvature, with $L = K_M^{-1}$, and proved it in the two-dimensional case [50]. Donaldson-Sun [21] showed the result with two-sided Ricci curvature bounds, but arbitrary polarizations, and later several extensions of their result were obtained (see e.g. [15, 16, 30, 48, 43]). The result assuming a lower bound for the Ricci curvature, with $L = K_M^{-1}$ was finally shown by Chen-Wang [17]. The improvement in our result is that we allow for general polarizations.

Our next result addresses the structure of the singular set of the limit space X. In the setting of Theorem 1.1, if the metrics along the sequence are Kähler-Einstein, Donaldson-Sun [21] showed that the metric singular set of X is the same as the complex analytic singular set of the corresponding projective variety (see also Corollary 4.1). In our setting this is not necessarily the case, however we have the following.

Theorem 1.3. *Let (X, d) be a Gromov-Hausdorff limit as in Theorem 1.1. Then for any $\epsilon > 0$, $X \setminus \mathcal{R}_\epsilon$ is contained in a finite union of analytic subvarieties of X. Furthermore, the singular set $X \setminus \mathcal{R}$ is equal to a countable union of subvarieties.*

The "almost regular" set $\mathcal{R}_\epsilon \subset X$ is defined to be the set of points p satisfying $\lim_{r\to 0} r^{-2n}\mathrm{Vol}(B(p,r)) > \omega_{2n} - \epsilon$ in terms of the volume ω_{2n} of the unit ball in $\mathbb{C}^n$. The regular set is then $\mathcal{R} = \cap_{\epsilon>0}\mathcal{R}_\epsilon$. Note that $\mathcal{R}_\epsilon$ is an open set, while in general $\mathcal{R}$ may not be open.

Cheeger-Colding [4] showed that even in the Riemannian setting the Hausdorff codimension of $X \setminus \mathcal{R}$ is at least 2, with more quantitative estimates obtained by Cheeger-Naber [10]. Moreover, in a recent deep work of Cheeger-Jiang-Naber [9], it was shown that for small ϵ the set $X \setminus \mathcal{R}_\epsilon$ has bounded $(2n-2)$-dimensional Minkowski content and is $2n-2$ rectifiable. These results show that the singular set behaves well from the perspective of geometric measure theory. On the other hand, the topology of the singular set could be rather complicated. In a recent paper of Li and Naber [34] (see also example 3.2 of [9]), it was shown that even assuming non-negative sectional curvature, non-collapsed limit spaces can have singular sets that are Cantor sets. Our Theorem 1.3 shows that in sharp contrast with this, in the polarized Kähler setting the singular set has strong rigidity properties. For example if we perturb the Kähler metrics along our sequence locally inside a holomorphic chart and assume that the geometric assumptions are preserved, then the metric singular set can change by at most a countable set of points.

A basic technical ingredient in this work is the following result, which is of independent interest.

Theorem 1.4. *There exists $\epsilon > 0$, depending on the dimension n, with the following property. Suppose that $B(p, \epsilon^{-1})$ is a relatively compact ball in a (not necessarily complete) Kähler manifold (M^n, p, ω), satisfying* $\mathrm{Ric}(\omega) > -\epsilon\omega$, *and*

$$d_{GH}\Big(B(p, \epsilon^{-1}), B_{\mathbb{C}^n}(0, \epsilon^{-1})\Big) < \epsilon,$$

where d_{GH} is the Gromov-Hausdorff distance. Then there is a holomorphic chart $F : B(p, 1) \to \mathbb{C}^n$ which is a $\Psi(\epsilon|n)$-Gromov-Hausdorff approximation to its image. In addition on $B(p, 1)$ we can write $\omega = i\partial\bar{\partial}\phi$ with $|\phi - r^2| < \Psi(\epsilon|n)$, where r is the distance from p.

Here, and throughout the paper, $\Psi(\epsilon_1, \dots, \epsilon_k | a_1, \dots, a_l)$ denotes a function such that for fixed a_i we have $\lim_{\epsilon_1,\dots,\epsilon_k \to 0} \Psi = 0$. This result is an extension of Proposition 1.3 of [36], where the bisectional curvature lower bound was assumed. See also [16, Proposition 1], where a similar conclusion is found under additional assumptions. A simple consequence of the result is that for any non-collapsed Gromov-Hausdorff limit of Kähler manifolds with Ricci curvature bounded below, the set $\mathcal{R}_\epsilon$ defined above has the structure of a complex manifold, for sufficiently small ϵ.

We give some further applications of this result. The first, Proposition 2.5, shows that under Gromov-Hausdorff convergence to a smooth Kähler manifold, the scalar curvature functions converge as measures. Here we state a simple corollary of this.

Corollary 1.2. *Given any $\epsilon > 0$, there is a $\delta > 0$ depending on ϵ, n, satisfying the following. Let $B(p, 1)$ be a relatively compact unit ball in a Kähler manifold (M, ω) satisfying* $\mathrm{Ric} > -1$, *and $d_{GH}(B(p, 1), B_{\mathbb{C}^n}(0, 1)) < \delta$. Then $\int_{B(p,1/2)} S < \epsilon$, where S is the scalar curvature of ω.*

When the manifold is polarized, non-collapsed, with Ricci curvature bounded below, then we obtain the following integral bound for the scalar curvature on any unit ball.

Proposition 1.1. *Let $B(p, 1)$ be a unit ball in a polarized Kähler manifold (M^n, L, ω), satisfying* $\mathrm{Ric} > -1$, *and* $\mathrm{vol}(B(p, 1)) > v > 0$. *There is a constant $C(n, v)$ depending on n, v such that $\int_{B(p,1)} S < C(n, v)$.*

This is closely related to a question posed by Yau [54, Problem 9. p. 278], on bounding the integral of scalar curvature on Riemannian manifolds with non-negative Ricci curvature. An argument similar to [38, Proposition 2.7] shows that under a bisectional curvature lower bound the same result holds even without the polarization condition. See also Petrunin [42] for an analogous result, where the sectional curvature is assumed to be bounded below, but non-collapsing is not required.

The final application is the following, which was proved previously by the first author [36] under the assumption of non-negative bisectional curvature.

Proposition 1.2. *There exists $\epsilon > 0$ depending on n, so that if M^n is a complete noncompact Kähler manifold with* $\mathrm{Ric} \geq 0$ *and* $\lim_{r\to\infty} r^{-2n}\mathrm{vol}(B(p, r)) \geq \omega_{2n} - \epsilon$, *then M is biholomophic to $\mathbb{C}^n$. Here ω_{2n} is the volume of the Euclidean unit ball.*

We conclude this introduction with a brief description of the contents of the paper. In Section 2 we prove Theorem 1.4 and the two applications mentioned above. We then use the charts provided by Theorem 1.4 in Section 3 to construct global holomorphic sections of high powers of our line bundles, following the approach of Donaldson-Sun [21]. This leads to the partial C^0 estimate, and the proof of Theorem 1.1. In Section 4 we study the relation between the complex analytic and metric singularities of X, proving Theorem 1.3. The argument in Section 3 uses the recent estimates of Cheeger-Jiang-Naber [9], but we show in the Appendix that our results can be obtained independently of [9] by following the approach of Chen-Donaldson-Sun [15]. In addition we prove a splitting result in the Appendix which is well known to experts but does not seem to be written up in the setting that we use.

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2. Holomorphic charts near regular points

Our main goal in this section is to prove Theorem 1.4. We will first construct a holomorphic chart by regularizing the metric using Perelman's pseudolocality [41] along the Ricci flow. The following is the basic input about the Ricci flow that we need.

Proposition 2.1. *There is a constant $D > 0$ such that given $a > 0$, for sufficiently small $\epsilon > 0$ the following holds. Let $B(p, \epsilon^{-1})$ be a relatively compact ball in a Kähler manifold (M^n, g) satisfying $\mathrm{Ric}(g) > -\epsilon g$, such that*

$$d_{GH}(B(p, \epsilon^{-1}), B_{\mathbb{C}^n}(0, \epsilon^{-1})) < \epsilon.$$

Then there is a Ricci flow solution $g(t)$ on $B(p, \epsilon^{-1/2})$ for $t \in [0, 1]$ with $g(0) = g$, such that

- *On the ball $B_{g(1)}(p, 10D)$, in suitable coordinates, we have*
$$|g(1) - g_{Euc}|_{C^5(g_{Euc})} < \Psi(\epsilon|n);$$
- *The curvature along the flow satisfies $|\mathrm{Rm}| < a/t$;*
- *We have the following estimates for the distance function along the flow:*
$$d_t(x, y) > d_0(x, y) - D\sqrt{t},$$
$$d_t(x, y) < D(d_0(x, y) + \sqrt{t}),$$
for $x, y \in B(p, \epsilon^{-1/2}/2)$ and $t \in [0, 1]$.

Proof. According to Cavalletti-Mondino [2], our assumptions imply that the $\Psi(\epsilon|n)$-almost Euclidean isoperimetric inequality holds in balls of radius $\epsilon^{-1/2}$ inside the ball $B(p, \epsilon^{-1})$. We can then apply He [25, Lemma 2.4] (see also Hochard [26] and Topping [53]) to conformally scale the metric g on a domain $U \subset B(p, \epsilon^{-1})$ to a complete Riemannian metric (U, h), such that $g = h$ on $B(p, \epsilon^{-1/2})$, the $\Psi(\epsilon|n)$-almost Euclidean isoperimetric inequality holds on balls of radius $\epsilon^{-1/2}/8$, and we have a lower bound $S_h \geq -\Psi(\epsilon|n)$ for the scalar curvature of h. As in [25] there exists a Ricci flow solution $h(t)$ for a definite time $t \in [0, T]$, satisfying $|\mathrm{Rm}| \leq A/t$ for $t \in (0, T]$. We can choose A arbitrarily small, and T as large as we like if ϵ is sufficiently small. The distance estimates follow from Lemma 3.5 and Lemma 3.7 in [25].

To see the claim about comparing $g(t)$ with the Euclidean metric, suppose we have a sequence of such flows, with $\epsilon_i \to 0$. The curvature estimates, and the non-collapsing estimate (see Lemma 3.1 in [25]) applied for large times, imply that in the limit we end up with a (stationary) flow of flat metrics on $\mathbb{R}^{2n}$, and the claim follows from this. □

We need the following estimate along a Ricci flow as above, similar to Lemma 2.3 in Huang-Tam [29].

Lemma 2.1. *Suppose that we have a Ricci flow on $B(p, \epsilon^{-1/2})$ as given by Proposition 2.1, for sufficiently small $\epsilon > 0$. Given constants $A, A_k, l > 0$ there are $C_k > 0$ satisfying the following. Let $f \geq 0$ be a smooth function on $B(p, \epsilon^{-1/2}) \times [0, 1]$, such that*

- *for all $k > 0$ f satisfies*
$$(\partial_t - \Delta) f(x, t) \leq \frac{A}{t} \max_{0 \leq s \leq t} f(x, s) + A_k t^k,$$
on $B(p, \epsilon^{-1/2}) \times (0, 1)$.
- *$\partial_t^k f|_{t=0} = 0$ for all $k \geq 0$ on $B(p, \epsilon^{-1/2})$.*
- *$\sup_{x \in B(p, \epsilon^{-1/2})} f(x, t) \leq A t^{-l}$ for $t \in (0, 1]$.*

Then we have $\sup_{x \in B(p,1)} f(x, t) \leq C_k t^k$ for all $k \geq 0$, and $t \in [0, 1]$.

Proof. The proof is similar to the first part of the proof of Huang-Tam [29], Lemma 2.3. We set $\phi(s)$ to be a cutoff function such that $\phi(s) = 0$ for $s \geq 3/4$, and $\phi(s) = 1$ for $s \leq 1/4$. Let $\Phi = \phi^m$ for suitable m to be chosen later, and we set $q = 1 - 2/m$. This satisfies

$$0 \leq \Phi \leq 1, \quad -C_m \Phi^q \leq \Phi' \leq 0, \quad |\Phi''| \leq C_m \Phi^q,$$

for a constant C_m depending on m. Let us define $\rho : B_{g(1)}(p, 9D) \to \mathbb{R}$ to be given by $\rho(x) = d_{g(1)}(p, x)^2/(9D)^2$. Then $|\nabla_{g(1)} \rho|, |\Delta_{g(1)} \rho| < C$, and so as in Lemma 2.2 in Huang-Tam [29], we have $|\nabla_{g(t)} \rho| < C t^{-ca}, |\Delta_{g(t)} \rho| < C t^{-1/2 - ca}$ for a dimensional constant c (the constant a is the constant in the estimate $|\mathrm{Rm}| < a/t$). We choose ϵ sufficiently small, so that a satisfies $ca < 1/4$. Then

$$|\nabla_{g(t)} \rho| < C t^{-1/4}, \quad |\Delta_{g(t)} \rho| < C t^{-3/4}.$$

We then set $\Psi(x) = \Phi(\rho(x))$, so that by definition Ψ vanishes outside of $B_{g(1)}(p, 9D)$ for all t. We also let $\theta(t) = \exp(-\alpha t^{1-\beta})$ for $\alpha > 0, \beta \in (0, 1)$. For any $K > A$, let $F = f t^{-K}$. Then we have a constant C (which may change from line to line) such that

$$\begin{aligned}
(\partial_t - \Delta) F(x, t) &\leq -K t^{-1-K} f(x, t) + \frac{A}{t} \max_{s \leq t} t^{-K} f(x, s) + A_k t^{k-K} \\
&\leq -\frac{A}{t} F(x, t) + \frac{A}{t} \max_{s \leq t} F(x, s) + C,
\end{aligned}$$

and in addition $F \leq A t^{-l-K}$. We will show that the smooth function

$$H(x, t) = \theta(t) \Psi(x) F(x, t),$$

is a priori bounded on $B(p, \epsilon^{-1/2}) \times [0, 1]$. Suppose that H achieves its maximum at a point (x_0, t_0).

At the maximum we have $\nabla H = 0$, therefore $\Psi\nabla F + F\nabla\Psi = 0$, and so

$$\nabla\Psi\cdot\nabla F = -\frac{F|\nabla\Psi|^2}{\Psi}.$$

Note that by the estimates above we have

$$|\Delta\Psi| = \big|\Phi''(\rho)|\nabla\rho|^2 + \Phi'(\rho)\Delta\rho\big| \leq C_m\Psi^q t_0^{-3/4},$$

and also

$$\frac{|\nabla\Psi|^2}{\Psi} \leq C_m\Psi^{2q-1}t_0^{-1/2}.$$

In addition, note that by the maximality of H at (x_0,t_0), for any $s\leq t_0$ we have

$$\theta(s)F(x_0,s)\leq\theta(t_0)F(x_0,t_0),$$

and so since θ is decreasing, we have $\max_{s\leq t_0}F(x_0,s)\leq F(x_0,t_0)$. It follows that at (x_0,t_0)

$$(\partial_t-\Delta)F\leq C.$$

At the maximum we can then compute

$$\begin{aligned}\Delta H &= \theta F\Delta\Psi + \theta\Psi\Delta F + 2\theta\nabla\Psi\cdot\nabla F\\ &\geq -C_m\theta F\Psi^q t_0^{-3/4} - C_m\theta F\Psi^{2q-1}t_0^{-1/2} + \theta\Psi\Delta F,\end{aligned}$$

and

$$\partial_t H = \theta\big(-\alpha(1-\beta)t_0^{-\beta}\Psi F + \Psi\partial_t F\big).$$

It follows that

$$\begin{aligned}0 &\leq (\partial_t-\Delta)H\\ &\leq \theta\Psi(\partial_t-\Delta)F + \theta\big[-\alpha(1-\beta)t_0^{-\beta}\Psi F + C_m\Psi^q F t_0^{-3/4} + C_m\Psi^{2q-1}Ft_0^{-1/2}\big]\\ &\leq \theta\big[C\Psi - \alpha(1-\beta)t_0^{-\beta}\Psi F\\ &\qquad + C_m(\Psi F)^q t_0^{-3/4-(1-q)(l+K)} + C_m(\Psi F)^{2q-1}t_0^{-1/2-(2-2q)(l+K)}\big],\end{aligned}$$

using that $F\leq At^{-l-K}$.

If we choose q very close to 1 (i.e. m very large), then we can choose $\beta\in(0,1)$ so that $\beta-3/4-(1-q)(l+K)>0$ and $\beta-1/2-(2-2q)(l+K)>0$. Then our previous inequality implies (multiplying through by t_0^β), that

$$0\leq -\alpha(1-\beta)\Psi F + C\Psi + C_m\big[(\Psi F)^q + (\Psi F)^{2q-1}\big].$$

We can now choose α so large that $\alpha(1-\beta)>2C_m+1$. Then

$$(\Psi F)\leq C\Psi + C_m\big[(\Psi F)^q + (\Psi F)^{2q-1} - 2(\Psi F)\big].$$

It follows that if ΨF is sufficiently large, then this leads to a contradiction, and so $\Psi F\leq C$. This implies that $H\leq C$ at the maximum, as required. This implies the estimate $F\leq C$ on $B_{g(1)}(p,8D)$, and the bounds for the distance along our Ricci flow imply that $B_{g(0)}(p,1)\subset B_{g(1)}(p,8D)$. □

Proposition 2.2. *Suppose that we are in the situation of Proposition 2.1, with sufficiently small ϵ. Then on a smaller ball $B(p,r)$ we have a holomorphic chart $F: B(p,r)\to\mathbb{C}^n$, such that for suitable $r_1,r_2>0$ the image of F satisfies*

$$B(0,r_1)\subset F(B(p,r))\subset B(0,r_2).$$

Proof. We consider the Ricci flow $g(t)$ on $B = B(p, \epsilon^{-1/2})$ provided by Proposition 2.1, and show that the metric $g(1)$ is "approximately Kähler" in the sense that on a smaller ball, $B(p, r_0)$, we have $|\nabla^i J_0| < C$ for $i \leq 5$. Here we denote by J_0 the fixed complex structure on $B(p, \epsilon^{-1/2})$ to distinguish it from another, time dependent family of almost complex structures $J(t)$ below. We follow the argument given by Kotschwar [32] for preserving the Kähler condition along a Ricci flow, using Lemma 2.1 (see also Huang-Tam [29], Shi [45]).

Following Kotschwar [32] we first define a family $J(t)$ of almost complex structures by $J(0) = J_0$, and

$$\frac{\partial}{\partial t} J^a_b = R^c_b J^a_c - R^a_c J^c_b.$$

It is convenient to introduce a differential operator D_t, in terms of which $D_t J = 0$ (see [32] for details). We also have $D_t g = 0$, and g remains Hermitian for the almost complex structure J.

Define $F, G \in \mathrm{End}(\wedge^2 T^* B)$ by

$$(F\eta)(X,Y) = \frac{1}{2}(\eta(X, JY) + \eta(JX, Y)), \quad (G\eta)(X,Y) = \eta(JX, JY).$$

We then let

$$\tilde{P} = \frac{1}{2}(\mathrm{Id} + G), \quad \hat{P} = \frac{1}{2}(\mathrm{Id} - G).$$

On the complexification $\wedge^2_{\mathbb{C}} T^* B$ we have

$$P^{(2,0)} = \frac{1}{2}\left(\hat{P} - \sqrt{-1} F\right), \quad P^{(1,1)} = \tilde{P}, \quad P^{(0,2)} = \overline{P^{(2,0)}},$$

for the orthogonal projection maps onto $\wedge^{2,0} B, \wedge^{1,1} B, \wedge^{0,2} B$. We will control the complex structure J along the flow in terms of the piece $\hat{R} = R \circ \hat{P}$, where $R : \wedge^2 T^* B \to \wedge^2 T^* B$ is the curvature operator of $g(t)$. The derivatives of J will then be controlled by derivatives $\nabla^i \hat{R}$. The evolution equations of these in turn depend on derivatives $\nabla^i \hat{P}$. Using that the initial metric is Kähler, at $t = 0$ we have $\partial_t^k \nabla^i \hat{R} = 0$ and $\partial_t^{k+1} \nabla^i \hat{P} = 0$ for all $k \geq 0$. For the calculations we also have the commutation formulas (see Kotschwar [33, Lemma 4.3]) for a tensor X

$$\begin{aligned} [D_t, \nabla] X &= \nabla R * X + R * \nabla X \\ [D_t - \Delta, \nabla_a] &= 2R_{abdc} \Lambda^c_d \nabla_b + 2R_{ab} \nabla_b, \end{aligned}$$

where Λ is a certain algebraic operation on tensors.

Let us write $\hat{S} = (\nabla R) \circ \hat{P}$ and $\hat{T} = (\nabla^2 R) \circ \hat{P}$. Here the action of $\hat{P}$ is such that for instance $\hat{S}(X, \eta) = (\nabla_X R) \hat{P}(\eta)$ for a vector X and 2-form η. Note that $\hat{S} = \nabla \hat{R} + R * \nabla \hat{P}$, and $\hat{T} = \nabla \hat{S} + \nabla R * \nabla \hat{P}$. The advantage of $\hat{S}$ and $\hat{T}$ over $\nabla \hat{R}, \nabla^2 \hat{R}$ is that their evolution equations only involve up to two derivatives of $\hat{P}$, and their norms are controlled by $|\nabla R|, |\nabla^2 R|$ since $\hat{P}$ is a projection. From [33, Proposition 4.5] we have

$$D_t \nabla \hat{P} = R * \nabla \hat{P} + \hat{P} * \hat{S},$$

and so we also have

$$\begin{aligned} D_t \nabla^2 \hat{P} &= \nabla D_t \nabla \hat{P} + \nabla R * \nabla \hat{P} + R * \nabla^2 \hat{P} \\ &= R * \nabla^2 \hat{P} + \nabla R * \nabla \hat{P} + \nabla \hat{P} * \hat{S} + \hat{P} * \hat{T} + \hat{P} * R * \nabla \hat{P}. \end{aligned}$$

Using the estimates for $\nabla^i R$ along the flow, it follows that

$$\begin{aligned}\partial_t|\nabla\hat{P}| &\leq \frac{A}{t}|\nabla\hat{P}|+c|\hat{S}|\\ \partial_t|\nabla^2\hat{P}| &\leq \frac{A}{t}|\nabla^2\hat{P}|+\frac{A}{t^{3/2}}|\nabla\hat{P}|+c|\hat{T}|,\end{aligned}\tag{2.1}$$

for a dimensional constant c.

For the evolution of the curvature we have (see [33, Proposition 4.7, Lemma 4.9])

$$\begin{aligned}(D_t-\Delta)R &= R*R,\\ [(D_t-\Delta)R]\circ\hat{P} &= R*\hat{R}+\hat{P}*R*\hat{R}.\end{aligned}$$

It follows that

$$\begin{aligned}(D_t-\Delta)\hat{R} &= [(D_t-\Delta)R]\circ\hat{P}+R*\nabla^2\hat{P}+\nabla R*\nabla\hat{P}\\ &= R*\hat{R}+\hat{P}*R*\hat{R}+R*\nabla^2\hat{P}+\nabla R*\nabla\hat{P},\end{aligned}$$

and so

$$(\partial_t-\Delta)|\hat{R}|\leq\frac{A}{t}|\hat{R}|+\frac{A}{t}|\nabla^2\hat{P}|+\frac{A}{t^{3/2}}|\nabla\hat{P}|.\tag{2.2}$$

Similarly we have

$$(D_t-\Delta)\hat{S}=[(D_t-\Delta)\nabla R]\circ\hat{P}+\nabla R*\nabla^2\hat{P}+\nabla^2R*\nabla\hat{P}.$$

Using the commutation relations,

$$[(D_t-\Delta)\nabla_aR]\circ\hat{P}=[\nabla(D_t-\Delta)R]\circ\hat{P}+[2R_{abdc}\Lambda^c_d\nabla_bR+2R_{ab}\nabla_bR]\circ\hat{P}$$

For the term involving Λ^c_d we have (see the calculation in [33, Proposition 4.13])

$$\hat{P}_{ijkl}R_{abdc}\Lambda^c_d\nabla_bR_{klmn}=R*\hat{S}+\nabla R*\hat{R}*\hat{P}.$$

It follows that

$$\begin{aligned}[(D_t-\Delta)\nabla_aR]\circ\hat{P} &= \nabla\big([(D_t-\Delta)R]\circ\hat{P}\big)+[(D_t-\Delta)R]*\nabla\hat{P}\\ &\quad +R*\hat{S}+\nabla R*\hat{R}*\hat{P}\\ &= \nabla R*\hat{R}+R*\nabla\hat{R}+\nabla\hat{P}*R*\hat{R}+\hat{P}*\nabla R*\hat{R}+\hat{P}*R*\nabla\hat{R}\\ &\quad +R*R*\nabla\hat{P}+R*\hat{S}\\ &= R*\hat{S}+\nabla R*\hat{R}+R*R*\nabla\hat{P}+R*\hat{R}*\nabla\hat{P}+\hat{P}*\nabla R*\hat{R}\\ &\quad +\hat{P}*R*\hat{S}+\hat{P}*R*R*\nabla\hat{P}.\end{aligned}$$

This implies

$$(\partial_t-\Delta)|\hat{S}|\leq\frac{A}{t}|\hat{S}|+\frac{A}{t^{3/2}}|\nabla^2\hat{P}|+\frac{A}{t^2}|\nabla\hat{P}|+\frac{A}{t^{3/2}}|\hat{R}|.\tag{2.3}$$

For $\hat{T}$, we have

$$(D_t-\Delta)\hat{T}=[(D_t-\Delta)\nabla^2R]\circ\hat{P}+\nabla^2R*\nabla^2\hat{P}+\nabla^3R*\nabla\hat{P}.$$

By the commutation relations again

$$\begin{aligned}[(D_t-\Delta)\nabla_a\nabla R]\circ\hat{P} &= [\nabla(D_t-\Delta)\nabla R]\circ\hat{P}\\ &\quad +[2R_{abdc}\Lambda^c_d\nabla_b\nabla R+2R_{ab}\nabla_b\nabla R]\circ\hat{P},\end{aligned}$$

By the same argument as [33, Proposition 4.13], the term involving Λ satisfies

$$R_{abdc}\Lambda^c_d\nabla_b\nabla R\circ\hat{P} = R*\hat{T}+\nabla^2R*\hat{R}*\hat{P}.$$

At the same time, we have

$$[\nabla(D_t-\Delta)\nabla R]\circ\hat{P} = \nabla\big([(D_t-\Delta)\nabla R]\circ\hat{P}\big)+[(D_t-\Delta)\nabla R]*\nabla\hat{P}.$$

Using the calculations above, and collecting various terms, we obtain

$$(\partial_t-\Delta)|\hat{T}| \le \frac{A}{t}|\hat{T}|+\frac{A}{t^2}|\nabla^2\hat{P}|+\frac{A}{t^{5/2}}|\nabla\hat{P}|+\frac{A}{t^2}|\hat{R}|+\frac{A}{t^{3/2}}|\hat{S}|. \tag{2.4}$$

Given a constant $K>0$, using Equations (2.1), (2.2), (2.3), (2.4), we have

$$\begin{aligned}
\partial_t t^{-K-1/2}|\nabla\hat{P}| &\le \frac{A-K-1/2}{t}t^{-K-1/2}|\nabla\hat{P}|+ct^{-K-1/2}|\hat{S}|\\
\partial_t t^{-K}|\nabla^2\hat{P}| &\le \frac{A-K}{t}t^{-K}|\nabla^2\hat{P}|+\frac{A}{t}t^{-K-1/2}|\nabla\hat{P}|+ct^{-K}|\hat{T}|\\
(\partial_t-\Delta)t^{-K-1}|\hat{R}| &\le \frac{A-K-1}{t}t^{-K-1}|\hat{R}|+\frac{A}{t^2}t^{-K}|\nabla^2\hat{P}|+\frac{A}{t^2}t^{-K-1/2}|\nabla\hat{P}|\\
(\partial_t-\Delta)t^{-K-1/2}|\hat{S}| &\le \frac{A-K-1/2}{t}t^{-K-1/2}|\hat{S}|+\frac{A}{t^2}t^{-K}|\nabla^2\hat{P}|\\
&\quad+\frac{A}{t^2}t^{-K-1/2}|\nabla\hat{P}|+\frac{A}{t}t^{-K-1}|\hat{R}|\\
(\partial_t-\Delta)t^{-K}|\hat{T}| &\le \frac{A-K}{t}t^{-K}|\hat{T}|+\frac{A}{t^2}t^{-K}|\nabla^2\hat{P}|+\frac{A}{t^2}t^{-K-1/2}|\nabla\hat{P}|\\
&\quad+\frac{A}{t}t^{-K-1}|\hat{R}|+\frac{A}{t}t^{-K-1/2}|\hat{S}|
\end{aligned}$$

Let $K>3A+2$ and define

$$\begin{aligned}
Y &= t^{-K}|\nabla^2\hat{P}|+t^{-K-1/2}|\nabla\hat{P}|\\
Z &= t^{-K}|\hat{T}|+t^{-K-1/2}|\hat{S}|+t^{-K-1}|\hat{R}|.
\end{aligned}$$

From the inequalities above we then obtain

$$\begin{aligned}
\partial_t Y &\le cZ\\
(\partial_t-\Delta)Z &\le \frac{A}{t^2}Y.
\end{aligned}$$

Note that Y is still smooth up to $t=0$, and $Y(x,0)=0$ for all x. It follows that

$$Y(x,t)\le tc\max_{s\le t}Z(x,s),$$

and therefore Z satisfies the inequality (for a larger choice of A)

$$(\partial_t-\Delta)Z(x,t)\le\frac{A}{t}\max_{s\le t}Z(x,s).$$

All t-derivatives of Z vanish at $t=0$ because the initial metric is Kähler. In addition since $\hat{P}$ is a projection map, the norms $|\hat{R}|,|\hat{S}|,|\hat{T}|$ are controlled by $|R|,|\nabla R|,|\nabla^2 R|$. By the curvature estimates along the flow we have $Z\le A/t^l$ for some A,l. We can therefore apply Lemma 2.1 to obtain that $Z\le C_kt^k$ on $B(p,1)$. In turn this also implies that $Y\le C_kt^k$.

We can apply the same argument to obtain estimates for further derivatives of the curvature composed with $\hat{P}$, inductively. Analogously to the above, we have inequalities

$$(\partial_t - \Delta)|\nabla^i R \circ \hat{P}| \leq \frac{A}{t}|\nabla^i R \circ \hat{P}| + C_k t^k,$$

where we are using the inductive assumption to control $|\nabla^j R \circ \hat{P}|$ for $j < i$. It follows that on a smaller ball $B(p, r_0)$ we have $|\nabla^i R \circ \hat{P}| < C_k t^k$, for $i < 5$, say. This in turn implies estimates $|\nabla^i \hat{P}| \leq C_k t^k$ for $0 < i < 5$.

We can now use this to control $\nabla^i J$ for $0 < i < 5$. Indeed, the evolution of ∇J has the form (see [32, Lemma 7])

$$D_t \nabla J = \hat{S} * J + R * \nabla \hat{P} * J + R * \nabla J,$$

and so

$$\partial_t |\nabla J| \leq \frac{A}{t}|\nabla J| + C_k t^k.$$

it follows that $|\nabla J| \leq C_k t^k$, since all t-derivatives of ∇J vanish at $t = 0$. For the higher derivatives of J we can use the commutation relation to get

$$D_t \nabla^i J = \nabla D_t \nabla^{i-1} J + R * \nabla^{i-1} J + \nabla R * \nabla^i J$$

and so we can inductively find inequalities of the form

$$\partial_t |\nabla^i J| \leq \frac{A}{t}|\nabla^i J| + C_k t^k,$$

for $0 < i < 5$, say. We therefore have $|\nabla^i J| \leq C_k t^k$ for $0 < i < 5$ on $B(p, r_0)$. From these bounds we find that the curvature endomorphism satisfies, for a local orthonormal frame e_i, that

$$|R(e_i, e_j)Je_k - JR(e_i, e_j)e_k| \leq C_k t^k,$$

and from this it follows that the Ricci endomorphism Rc also satisfies $|Rc(Je_i) - JRc(e_i)| \leq C_k t^k$. The evolution of J is given by $\partial_t J = J \circ Rc - Rc \circ J$, and so we find that $|J - J_0| \leq C_k t^k$. Similarly we can also obtain $|\nabla^i \partial_t J| \leq C_k t^k$, and so we can inductively obtain estimates $|\nabla^i (J - J_0)| \leq C_k t^k$ for $i < 5$, using the equations

$$D_t \nabla^i (J - J_0) = R * \nabla^i (J - J_0) + \nabla R * \nabla^{i-1}(J - J_0) + \nabla D_t \nabla^{i-1}(J - J_0).$$

Finally we can conclude that we have estimates $|\nabla^i J_0| \leq C_k t^k$ on $B(p, r_0)$ for $t \leq 1$.

By rescaling (and choosing ϵ smaller), we can assume that we have estimates $|\nabla^i J_0| \leq C$ with respect to the metric $g(1)$, on $B_g(p, 6D)$. By the properties of the flow $g(t)$ in Proposition 2.1 we can view $g(1)$ as a metric on the Euclidean ball $B_{\mathbb{C}^n}(0, 5D)$, close in C^5 to the Euclidean metric, and so J_0 has bounded derivatives in terms of the Euclidean metric. After a linear change of coordinates with bounded eigenvalues, we can assume that J_0 is standard at the origin, and it still has bounded derivatives. We can now find holomorphic functions on a small ball $B_{g(1)}(p, r_0)$ which are perturbations of the complex linear functions, for instance by the approach of Hörmander [28] using L^2-estimates to the Newlander-Nirenberg theorem. If we choose ϵ sufficiently small, then by the distance estimates along the flow (applied for small t) we can obtain

$$B_{g(1)}(p, r_0/3D) \subset B_{g(0)}(p, r_0/2D) \subset B_{g(1)}(p, r_0),$$

since by the curvature estimates, for any small $t_0 > 0$ we can assume that $g(t)$ is very close to $g(1)$ for $t \in [t_0, 1]$. Setting $r = r_0/2D$, we thus obtain a holomorphic chart as required. □

We next construct a bounded Kähler potential locally.

Proposition 2.3. *There exist $\epsilon, C > 0$ with the following property. Suppose that $B(p, \epsilon^{-1})$ is relatively compact in a Kähler manifold (M^n, ω) satisfying* $\mathrm{Ric}(\omega) > -\epsilon\omega$*, and in addition $d_{GH}(B(p, \epsilon^{-1}), B_{\mathbb{C}^n}(0, \epsilon^{-1}))$. Then on $B(p, C^{-1})$ we can write $\omega = i\partial\bar{\partial}\phi$ with $|\phi| < C$.*

Proof. In Proposition 2.2 we have constructed a holomorphic chart on a small ball around p, and by choosing ϵ smaller and scaling, we can assume that the chart

$$F : B(p, 1) \to \mathbb{C}^n$$

is defined on $B(p, 1)$, and $B(0, r_1) \subset \mathbb{C}^n$ is contained in its image. In terms of this chart, we can view our metric ω as defining a metric on $B(0, r_1)$. We first "glue" this metric onto $\mathbb{CP}^n$ in order to run the Ricci flow.

On $B(0, r_1)$ we can write $\omega = i\partial\bar{\partial}\psi$, where we can assume that $\psi \geq 0$. Consider the function

$$f(z) = \log(1 + |z|^2) - \log(1 + r_1^2/4),$$

on $\mathbb{C}^n$, that is negative in $B(0, r_1/2)$ and positive elsewhere. It follows that for sufficiently large K, if we take a regularized maximum

$$h(z) = \widetilde{\max}\{\psi, Kf\}$$

then $\eta = i\partial\bar{\partial}h$ is a well defined metric on $\mathbb{C}^n$ extending to a metric on $\mathbb{CP}^n$, such that $\eta = \omega$ in $B(0, r_1/4)$. The Kähler class $[\eta] = K[\omega_{FS}]$, where ω_{FS} is the Fubini-Study metric. Note that since ∇F is bounded, we have $F(B(p, \delta_1)) \subset B(0, r_1/4)$ for suitable $\delta_1 > 0$.

It follows from Tian-Zhang [52], that if K is large (not necessarily bounded a priori), then we have a well defined Kähler-Ricci flow solution η_t for $t \in [0, 1]$. Since for the initial metric $B(p, \delta_1)$ is Gromov-Hausdorff close to the Euclidean ball, we can again apply the result of Cavalletti-Mondino [2] and the pseudolocality theorem (alternatively we could apply the version of the pseudolocality theorem proved by Tian-Wang [51]). We find that if ϵ is sufficiently small, then for small $\delta_2, T > 0$ the metric η_T on the ball $B_{\eta_T}(p, \delta_2)$ is $\Psi(\epsilon|n, T)$-close to the Euclidean metric in C^5 (and it is Kähler), therefore $\eta_T = i\partial\bar{\partial}\phi_T$ for a potential ϕ_T satisfying $|\phi_T| < C$. We will fix T below, depending on the distance estimates following from the pseudolocality theorem.

Letting Ω be a fixed holomorphic volume form on $B_{\eta_T}(p, \delta_2)$, we can find a family of potentials ϕ_t for η_t on $B_{\eta_T}(p, \delta_2)$ by solving the equation

$$\frac{d}{dt}\phi_t = \log\frac{\eta_t^n}{\Omega}. \tag{2.5}$$

By pseudolocality we have the estimate $|\mathrm{Ric}(\eta_t)|_{\eta_t} < C/t$, and so the Ricci flow equation $\partial_t\eta_t = -\mathrm{Ric}(\eta_t)$ implies that the eigenvalues λ_t of η_t relative to η_T satisfy $C^{-1}t < |\lambda_t| < C/t$. Therefore

$$\left|\log\frac{\eta_t^n}{\Omega}\right| < C|\log t|,$$

and so from Equation (2.5), together with the bound for ϕ_T, we obtain $|\phi_0| < C$. Therefore the metric η has a bounded Kähler potential on the ball $B_{\eta_T}(p, \delta_2)$. As long as T is chosen sufficiently small, depending on the distance distortion estimates as in Proposition 2.1, this ball contains the ball $B_\eta(p, \delta_3)$ for suitable δ_3. □

We now prove Theorem 1.4

Proof of Theorem 1.4. Suppose that we have a sequence $\epsilon_i \to 0$, and corresponding balls $B(p_i, \epsilon_i^{-1})$ with metrics g_i as in the statement of the proposition. From Proposition 2.3, we can assume that for large i, we have holomorphic charts $F_i : B(p_i, 2) \to \mathbb{C}^n$, and Kähler potentials ϕ_i for g_i on $B(p_i, 2)$, with $|\phi_i| < C$. In addition, by assumption, the $B(p_i, 2)$ converge to $B_{\mathbb{C}^n}(0, 2)$ in the Gromov-Hausdorff sense.

We can now argue along the lines of the proof of Proposition 3.1 in [36] to show that for sufficiently large i, on smaller balls $B(p_i, \delta)$ with $\delta = \delta(n) > 0$ we can find holomorphic charts z_j^i, which give a $\Psi(i^{-1})$-Gromov-Hausdorff approximation to the Euclidean ball $B(0, \delta)$. For this, note first that under the holomorphic charts F_i above, we have $B(0, r_1) \subset F_i(B(p_i, 2))$ for some $r_1 > 0$. We can assume that $F_i(p_i) = 0$ for all i. Let $U_i \subset B(p_i, 2)$ denote the connected component of $F_i^{-1}(B(p_i, r_1/2))$ containing p_i. Then U_i is a Stein domain, which implies that we can apply the Hörmander L^2 existence theorem (see Demailly [14] Theorem 5.1) on U_i with the trivial line bundle equipped with metric $e^{-\phi_i}$. Note that the ϕ_i are uniformly bounded, and $B(p_i, r_2) \subset U_i$ for a fixed $r_2 > 0$ by the Cheng-Yau gradient estimate for F_i. As in [36], using Cheeger-Colding [3] and Cheeger-Colding-Tian [8, Section 9], we have harmonic functions w_j^i on $B(p_i, 2)$ which give a $\Psi(i^{-1})$-Gromov-Hausdorff approximation to $B(0, 1)$, and in addition satisfy

$$\fint_{B(p_i,1)} |\bar{\partial} w_j^i|^2 < \Psi(i^{-1}).$$

Using the L^2-estimate we can perturb the w_j^i on U_i to holomorphic functions z_j^i, which still give a $\Psi(i^{-1})$-Gromov-Hausdorff approximation from $B(p_i, r_2)$ to their image in Euclidean space. Just as in [36, Claim 3.2], on a smaller ball $B(p_i, \delta)$ the z_j^i define holomorphic charts for sufficiently large i.

Using our charts, we can now view the metrics ω_i as defining metrics on the Euclidean ball $B(0, \delta/2)$, with uniformly bounded potentials ϕ_i (i.e. we identify the functions z_j^i with z_j). The identity map on $B(0, \delta/2)$ is then a $\Psi(i^{-1})$-Gromov-Hausdorff approximation from ω_i to the Euclidean metric ω_{Euc}, while the gradient bound for the holomorphic functions z_i with respect to ω_i implies that we have a lower bound $\omega_i > C^{-1}\omega_{Euc}$. The ϕ_i satisfy $\Delta_{\omega_i}\phi_i = 2n$, and so using the gradient estimate, we can take a limit ϕ_∞ on $B(0, \delta/2)$. By the same argument as the proof of Claim 3.1 below, the function $\phi_\infty - r^2$ is pluriharmonic. It follows that $\widetilde{\phi}_i = \phi_i + (r^2 - \phi_\infty)$ are also Kähler potentials for ω_i, and by construction $|\widetilde{\phi}_i - r_i^2| < \Psi(i^{-1})$, where r_i is the ω_i-distance from p_i . □

Using such holomorphic charts, the following proposition allows us to define a complex structure on the almost regular set $\mathcal{R}_\epsilon$ in the limit space of Kähler manifolds with Ricci curvature bounded below, for sufficiently small ϵ.

Proposition 2.4. *There exists $\epsilon = \epsilon(n) > 0$ so that the following holds. Let (M_i^n, p_i, ω_i) be a sequence of Kähler manifolds (not necessarily complete) so that*

Ric $\geq -\epsilon$ *and* $B(p_i, \frac{2}{\epsilon}) \subset\subset M_i$, *with* $d_{GH}(B(p_i, 2\epsilon^{-1}), B_{\mathbb{C}^n}(0, 2\epsilon^{-1})) < \epsilon$. *Assume that* $(M_i^n, p_i) \to (X, p)$ *in the pointed Gromov-Hausdorff sense. Let* $(z_1^i, .., z_n^i)$ *be holomorphic charts on* $B(p_i, 10)$ *obtained using Theorem 1.4, and let us assume* $z_j^i \to z_j$ *on* $B(p, 5)$. *Then* $(z_1, .., z_n)$ *is a homeomorphism from* $B(p, 3)$ *to the image.*

Proof. The argument is similar to Proposition 6.1 of [35], and the main point is to prove the injectivity. Suppose that $x_1, x_2 \in B(p, 5)$, with $d(x_1, x_2) = 10d \neq 0$. We will show that the coordinates $(z_1, ..., z_n)$ separate them. Consider sequences $M_i \ni x_1^i \to x_1, M_i \ni x_2^i \to x_2$. Then for large i, $B(x_1^i, d) \cap B(x_2^i, d) = \emptyset$. Using volume comparison, we see that $B(x_1^i, \epsilon_0^{-1}d)$ is $\epsilon_0 d$-Gromov-Hausdorff close to a Euclidean ball if ϵ is sufficiently small, where ϵ_0 is the parameter from Theorem 1.4. It follows that we have holomorphic charts $(z_{k1}^i, ..., z_{kn}^i)$ around x_k $(k = 1, 2)$, which give Gromov-Hausdorff approximations from each $B(x_k^i, d)$ to the Euclidean ball. As in [36, Lemma 5.1], we can define plurisubharmonic weight functions

$$\psi_i = C(d)\phi_i + \sum_{k=1,2} \lambda\Big(d^{-2}\sum_{j=1}^n |z_{kj}^i|^2\Big) \log\left(\sum_{j=1}^n |z_{kj}^i|^2\right)$$

for a suitable constant $C(d)$, and cutoff function λ supported in $[0, 1/2)$, equal to 1 in $[0, 1/4]$. These have the property that $e^{-\psi_i}$ is not locally integrable at x_1^i, x_2^i.

Let us define the functions

$$f_i = \lambda\Big(d^{-2}\sum_{j=1}^n |z_{1j}^i|^2\Big),$$

which equal 1 in $B(x_1^i, d/2)$, and zero outside of $B(x_1^i, d)$. We have a uniform bound $\|\bar\partial f_i\|_{L^2(e^{-\psi_i})} < C$ using the weight functions $e^{-\psi_i}$, and so applying the Hörmander estimate (as in the proof of Theorem1.4 the ball $B(p_i, 6)$ is contained in a Stein domain), we can solve the equations $\bar\partial h_i = \bar\partial f_i$ on $B(p_i, 6)$, with estimates $\|h_i\|_{L^2(e^{-\psi_i})} < C$. Note that near x_1^i, x_2^i this implies that $\bar\partial h_i = 0$, and since $e^{-\psi_i}$ is not locally integrable near these points, we have $h_i(x_k^i) = 0$ for $k = 1, 2$. This shows that the holomorphic functions $f_i - h_i$ separate the points x_1^i, x_2^i. Since the construction is uniform in i, we obtain holomorphic functions of the z_i which separate x_1, x_2, and so x_1, x_2 must be separated by the z_i. If we now let $F = (z_1, \ldots, z_n)$, and $\Omega = F^{-1}(B(0, 4))$, we find that F is proper and injective, and therefore it is a homeomorphism. □

Note that these local charts define a holomorphic atlas on $\mathcal{R}_\epsilon$. For this it is enough to note that if $f_i = f_i(z_1^i, \ldots, z_n^i)$ are uniformly bounded holomorphic functions, and $f_i \to f$ under the Gromov-Hausdorff convergence, then f is a holomorphic function of $z_1, \ldots, z_n$.

To conclude this section, let us present two applications of Theorem 1.4.

Proposition 2.5. *Let* (M_i^n, ω_i, p_i) *be a sequence of complete Kähler manifolds with* Ric > -1 *and* $\mathrm{vol}(B(p_i, 1)) > v > 0$. *Assume that* $(M_i^n, p_i) \to (M^n, p)$ *in the pointed Gromov-Hausdorff sense, where* M^n *is a smooth Riemannian manifold. Then the scalar curvatures* S_i *of* M_i *converge to the scalar curvature* S *of* M *in the measure sense. That is to say, for any points* $M_i \ni q_i \to q \in M$, *and any* $r > 0$, *we have* $\int_{B(q_i,r)} S_i\omega_i^n \to \int_{B(q,r)} S\omega^n$ *as* $i \to \infty$.

Remark 2.1. *It is clear from the proof that this proposition is local in nature. That is to say, the completeness of the Kähler metrics is not necessary, as long as $B(q_i, r)$ is relatively compact in M_i.*

Proof. It suffices to prove that there exists a subsequence of M_i so that the proposition is true, and we only need to prove the result locally near p. By suitable scaling, we may assume that $d_{GH}(B(p, \frac{1}{\epsilon^2}), B_{\mathbb{C}^n}(0, \frac{1}{\epsilon^2})) < \epsilon^2$ and $\mathrm{Ric}(M_i) \geq -\epsilon^2$. Here $\epsilon = \epsilon(n)$ is the small constant in Theorem 1.4. Thus we have holomorphic charts $(z_1^i, ..., z_n^i)$ on $B(p_i, 10)$ so that the coordinate maps $(z_1^i, .., z_n^i) : B(p_i, 10) \to \mathbb{C}^n$ give $\Psi(\epsilon|n)$-Gromov-Hausdorff approximations to their images. Let us assume that the holomorphic charts $(z_1^i, ..., z_n^i)$ converge to a chart $(z_1, ..., z_n)$ on $B(p, 8)$ as in Proposition 2.4. This defines a complex structure J on the ball $B(p, 8)$. Note that a priori g is just a Riemannian metric on M, however we have the following.

Claim 2.1. *The metric g on M is compatible with the complex structure J on $B(p, 8)$. That is to say, g is a Kähler metric with respect to J.*

Proof. Note that the functions z_j^i are all holomorphic, hence harmonic, and so $z_1, .., z_n$ are all complex harmonic on $B(p, 8)$. Therefore, they are all smooth with respect to the Riemannian metric g. In particular, this shows that the complex structure J is smooth with respect to g. In addition, it follows that the composition of chart maps $(z_j)^{-1} \circ (z_j^i)$ gives a holomorphic $\Psi(i^{-1})$-Gromov-Hausdorff approximation from $B(p_i, 7)$ to its image in $B(p, 8)$.

Let u_i be such that $\sqrt{-1}\partial\overline{\partial} u_i = \omega_i$ on $B(p_i, 10)$, given by Theorem 1.4. As $\Delta u_i = 2n$, ∇u_i is uniformly bounded on $B(p_i, 9.5)$, and we can assume that u_i converges uniformly under the Gromov-Hausdorff convergence to a smooth function u on $B(p, 9)$, satisfying $\Delta u = 2n$. Using our charts, the u_i can be viewed as plurisubharmonic functions on $B(p, 9)$ converging uniformly to u, and so $\omega = \sqrt{-1}\partial\overline{\partial} u$ is a closed positive $(1, 1)$ current with smooth coefficients. To prove the claim, it suffices to prove that the metric g is the same as the Kähler metric ω. Note that ω is a well defined form on M, independent of the choice of bounded Kähler potentials u_i for ω_i. This follows since if above we choose different potentials u_i' converging to u', then $v_i = u_i - u_i'$ are pluriharmonic, and so is their uniform limit. So $\sqrt{-1}\partial\bar{\partial} u' = \sqrt{-1}\partial\bar{\partial} u$.

To compare these two smooth metrics, we can blow up a point p on M and compare the metrics on the tangent space. Let $\tau > 0$ be a small number. Let us rescale the distance on M_i and M by $\frac{1}{\tau}$. Let the rescaled manifolds be $(M_i^\tau, p_{\tau,i})$ and (M^τ, p_τ), and the metrics be $\omega_{\tau,i}$ and ω_τ, g_τ. Then by applying Theorem 1.4 again, on $B(p_{\tau,i}, 1)$ we have $\omega_{\tau,i} = \sqrt{-1}\partial\overline{\partial} u_{\tau,i}$. Let us say $u_{\tau,i} \to u_\tau$ on $B(p_\tau, 1)$. Notice that the limit of g_τ as $\tau \to 0$ is the Euclidean metric, and by Theorem 1.4 the limit potentials u_τ approach the distance squared $|z|^2$ from the origin as $\tau \to 0$. It follows that the limit of $\sqrt{-1}\partial\overline{\partial} u_\tau$ as $\tau \to 0$ is the Kähler form associated to the Euclidean metric. This implies our claim. □

From now on, for a function u on $B(p, 7)$, we may also think it is defined on $B(p_i, 6)$ by lifting via the coordinate map.

Claim 2.2. $\int_{B(p_i,5)} |\langle dz_k^i, d\overline{z_j^i}\rangle - \langle dz_k, d\overline{z_j}\rangle|^2 < \Psi(\frac{1}{i})$.

Proof. Pick $q \in B(p,6)$ and $M_i \ni q_i \to q$. From a standard covering argument, it suffices to prove that

$$\lim_{\rho\to 0}\lim_{i\to\infty} \fint_{B(q_i,\rho)} |\langle dz_k^i, \overline{dz_j^i}\rangle - \langle dz_k, \overline{dz_j}\rangle|^2 = 0. \tag{2.6}$$

By subtracting constants, we may assume $z_j^i(q_i) = z_j(q) = 0$ for all i, j, and in addition, applying a linear transformation we can assume that $\langle dz_k, d\bar{z}_j\rangle(q) = \delta_{kj}$. Let us rescale the distance on (M_i, q_i) and (M, q) by $\frac{1}{\rho}$. We also rescale z_j^i, z_j by $\frac{1}{\rho}$. As M is a smooth manifold, $\langle dz_k, \overline{dz_j}\rangle$ is a smooth function and so after scaling we have $\langle dz_k, \overline{dz_j}\rangle = \delta_{jk} + \Psi(\rho)$ on $B(q,1)$. It follows that for sufficiently large i, the z_j^i define a $\Psi(\rho)$-Gromov-Hausdorff approximation to the Euclidean ball, and so by [3], we have

$$\fint_{B(q_i,1)} |\langle dz_k^i, d\bar{z}_j^i\rangle - \delta_{jk}|^2 < \Psi(\rho). \tag{2.7}$$

The required equality (2.6) follows from this. ∎

Set $s_i = dz_1^i \wedge dz_2^i \wedge \ldots \wedge dz_n^i$ and $s = dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n$. Then Claim 2.2 implies that

$$\int_{B(p_i,5)} \big||s|^2 - |s_i|^2\big| < \Psi(\frac{1}{i}). \tag{2.8}$$

Lemma 2.2. $\lim\limits_{i\to\infty} \int_{B(p_i,4)} \big|\log|s_i|^2 - \log|s|^2\big|\,\omega_i^n = 0.$

Proof. The Poincaré-Lelong equation says

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|s_i|^2 = \mathrm{Ric}(M_i) \geq -\epsilon\omega_i. \tag{2.9}$$

According to Theorem 1.4, we may assume that on $B(p_i,10)$, $\omega_i = \sqrt{-1}\partial\overline{\partial}u_i$ and $u_i \to u$ uniformly. Let us say $|u_i| \leq 1000$ for all i. By (2.9), $\log|s_i|^2 + 2\pi\epsilon u_i$ is plurisubharmonic on $B(p_i,10)$. Set

$$\begin{aligned} v_i &= \log|s_i|^2 + 2\pi\epsilon u_i,\\ v &= \log|s|^2 + 2\pi\epsilon u. \end{aligned} \tag{2.10}$$

We need to show that v_i converges to v in L^1. If we were working on the same space, then the lemma would follow from the standard theory of plurisubharmonic functions since by (2.8), $\log|s_i|^2$ cannot go uniformly to $-\infty$.

Claim 2.3. *There exists a constant C, independent of i, so that $\int_{B(p_i,9)} |v_i| \leq C$.*

Proof. The argument will be similar to Proposition 2.7 of [38]. In the proof C will be a large constant independent of i. The value might change from line to line. Since v_i has an upper bound independent of i, it suffices to prove $\int_{B(p_i,7)} v_i$ is uniformly bounded from below. Let $G_i(x,y)$ be the Green's functions on $B(p_i,10)$. Set

$$F_i(x) = v_i(x) + \int_{B(p_i,10)} G_i(x,y)\Delta v_i(y)dy. \tag{2.11}$$

Then F_i is harmonic, with the same boundary values as v_i. By the maximum principle, $F_i \leq \sup\limits_{B(p_i,10)} v_i$. Let us say $F_i \leq C$. From (2.8), we can find a point

$x_i \in B(p_i, 1)$ so that $v(x_i) \geq -C$ for all i. Since v_i is subharmonic, $F_i(x_i) \geq -C$. Then by the Cheng-Yau gradient estimate [18], we have $|F_i| \leq C$ on $B(p_i, 9)$. Inserting $x = x_i$ in (2.11), we also obtain that $\int_{B(p_i,9)} \Delta v_i(y)dy \leq C$, using the lower bound for the Green's function. By changing the radius 10 to 11 in (2.11), we may assume that $\int_{B(p_i,10)} \Delta v_i(y)dy \leq C$.

By integrating (2.11), we find

(2.12)
$$\int_{B(p_i,9)} v_i(x) = \int_{B(p_i,9)} F_i(x) - \int_{B(p_i,10)} \left(\int_{B(p_i,9)} G_i(x,y)dx \right) \Delta v_i(y)dy \geq -C,$$

using the upper bound for the Green's function. □

To complete the proof of Lemma 2.2 we will use the argument in Hörmander [27], Theorem 4.1.9 on page 94. Because of Claim 2.3, by passing to a subsequence, we may assume that v_i converges weakly as a measure to w on $B(p, 7)$. By this we mean that for any smooth function u with compact support on $B(p, 9)$, $\int_{M_i} uv_i \to \int_M uw$. We emphasize that a priori, w is merely a measure. In Hörmander's proof, the convolution is used to mollify the functions. Since M_i is not necessarily Euclidean, we consider the heat flow. More precisely, let ϕ be a smooth cut-off function on M so that $\phi = 1$ on $B(p, 8)$ and $\phi = 0$ outside $B(p, 9)$. Recall that we use the charts to identify the different balls $B(p_i, 9)$, and by the gradient estimate we have a uniform lower bound for ω_i in the charts. It follows that we can assume $|\phi|, |\nabla\phi|, |\Delta\phi| < C$ with respect to the metric ω on M as well as with respect to the metrics ω_i. Define

$$\begin{aligned} v_{it}(x) &= \int_{M_i} H_i(x,y,t)\phi(y)v_i(y)dy, \\ w_t(x) &= \int_M H(x,y,t)\phi(y)w(y)dy, \end{aligned} \tag{2.13}$$

where H_i, H are heat kernels on M_i and M respectively. Note that for any $t > 0$, w_t is a function.

$$\begin{aligned} \frac{dv_{it}}{dt} &= \int_{M_i} H_i(x,y,t)v_i(y)\Delta\phi(y)dy + \int_{M_i} H_i(x,y,t)\phi(y)\Delta v_i(y)dy \\ &\quad + 2\int_{M_i} H_i(x,y,t)\langle\nabla v_i(y), \nabla\phi(y)\rangle dy. \end{aligned} \tag{2.14}$$

Notice that $\Delta\phi$ has support outside of $B(p_i, 7)$. As $v_i(y)$ has a uniform L^1 bound, according to Li-Yau's heat kernel estimate [39], on $B(p_i, 6)$, the first term will be bounded by $\Psi(t)$. The second term is nonnegative, since v_i is subharmonic. For the last term, we can do integration by parts to transform derivatives to the heat kernel and ϕ. By the estimate for the derivative of the heat kernel [31], we find that on $B(p_i, 6)$, the last term is bounded by $\Psi(t)$. Therefore, we find that on $B(p_i, 6)$,

$$\frac{dv_{it}}{dt} \geq -\Psi(t). \tag{2.15}$$

Let η be a nonnegative smooth function so that $\eta = 1$ on $B(p, 4)$, $\eta = 0$ outside $B(p, 5)$. According to the assumption,

$$\int_{M_i} v_i\eta \to \int_M w\eta. \tag{2.16}$$

It is clear from (2.13) that

$$\left|\int_M (w\eta - w_t\eta)\right| \le \Psi(t). \tag{2.17}$$

For each fixed $t > 0$, we have $H_i(x, y, t) \to H(x, y, t)$. Therefore, $v_{it} \to w_t$ uniformly on each compact set. By the almost monotonicity (2.15), given any $\delta > 0$, there exists $a > 0$ sufficiently small so that on $B(p_i, 6)$,

$$w_a - v_i + \delta > 0 \tag{2.18}$$

for all sufficiently large i. Note that by the volume convergence theorem of Colding [19], we have

$$\int_{M_i} w_a\eta \to \int_M w_a\eta. \tag{2.19}$$

Putting (2.16)-(2.19) together, we find that for sufficiently large i,

$$\int_{M_i} |w_a - v_i + \delta|\eta \le \Psi(a, \delta). \tag{2.20}$$

Then we obtain that

$$\lim_{\lambda\to+\infty} \liminf_{i\to\infty} \int_{E_i^\lambda} v_i = 0, \tag{2.21}$$

where $E_i^\lambda = \{x \in B(p_i, 4) | v_i(x) \le -\lambda\}$. Notice that on $B(p, 4)$, $\log|s|^2$ is bounded from below. Recall that v_i, v are defined in (2.10). Lemma 2.2 follows by putting (2.8) and (2.21) together. $\square$

Let h be a smooth function of compact support on $B(p, 4)$. Note that by the Poincaré-Lelong equation, the scalar curvature is given by

$$S_i = \frac{1}{2\pi}\Delta \log|s_i|^2. \tag{2.22}$$

Therefore

(2.23)

$$\begin{aligned} 2\pi\left(\int_{M_i} hS_i - \int_M hS\right) &= \int_{M_i} \log|s_i|^2\Delta h - \int_M \log|s|^2\Delta h \\ &= \int_{M_i} (\log|s_i|^2 - \log|s|^2)\Delta h + \int_{M_i} \log|s|^2(\Delta_{\omega_i} h - \Delta_\omega h) \\ &+ (\int_{M_i} \log|s|^2\Delta_\omega h - \int_M \log|s|^2\Delta h). \end{aligned}$$

As we noticed before, Δh with respect to ω_i is uniformly bounded. Therefore, according to Lemma 2.2, the first term approaches zero as $i \to \infty$. Note that by Claim 2.2, $\int_{M_i} |\Delta_\omega h - \Delta_{\omega_i} h| \to 0$. Therefore, the second term converges to zero. Finally, the last term converges to zero by the volume convergence theorem of Colding [19]. We obtained the following.

Lemma 2.3. *Let h be a smooth function of compact support on $B(p, 4)$. Then* $\lim_{i\to\infty} \int_{M_i} hS_i = \int_M hS$.

Note that $S_i + 2n \geq 0$ for all i. For any $r_1 < r < r_2$, we can find smooth functions f and g so that $0 \leq f, g \leq 1$; $f = 1$ on $B(p, r_1)$, f has compact support on $B(p, r)$; $g = 1$ on $B(p, r)$, g has compact support on $B(p, r_2)$. For sufficiently large i we have

$$\int_{M_i} f(S_i + 2n) \leq \int_{B(p_i, r)} (S_i + 2n) \leq \int_{M_i} g(S_i + 2n). \tag{2.24}$$

We can apply Lemma 2.3. Letting $i \to \infty$ and $r_1, r_2 \to r$, we obtain the proof of Proposition 2.5. □

Corollary 1.2 is immediate from Proposition 2.5. We will prove Proposition 1.1 after Proposition 3.3.

Finally we prove Proposition 1.2. We state it again here for convenience.

Proposition 2.6. *There exists $\epsilon(n) > 0$ so that if M^n is a complete noncompact Kähler manifold with* Ric ≥ 0 *and $\lim_{r \to \infty} r^{-2n} \mathrm{vol}(B(p, r)) \geq \omega_{2n} - \epsilon$, then M is biholomophic to $\mathbb{C}^n$.*

The argument is very similar to [36, Theorem 4.1], with the three circle theorem replaced by the three annulus type result from Donaldson-Sun [22, Proposition 3.7]. In our setting, the relevant statement is the following.

Lemma 2.4. *Given $\delta \in (0, 1)$ there exists $\epsilon > 0$ with the following property. Suppose that $\lim_{r \to \infty} r^{-2n} \mathrm{vol}(B(p, r)) \geq \omega_{2n} - \epsilon$. Then for any $r > 0$, and any holomorphic function f on $B(p, 2r)$ we have that*

$$⨍_{B(p,r)} |f|^2 \geq 2^{2(1-\delta)} ⨍_{B(p,r/2)} |f|^2 \ \textit{ implies } ⨍_{B(p,2r)} |f|^2 \geq 2^{2(1-\delta)} ⨍_{B(p,r)} |f|^2,$$

and

$$⨍_{B(p,r)} |f|^2 \geq 2^{2(1+\delta)} ⨍_{B(p,r/2)} |f|^2 \ \textit{ implies } ⨍_{B(p,2r)} |f|^2 \geq 2^{2(1+\delta)} ⨍_{B(p,r)} |f|^2.$$

Proof. By the volume monotonicity we have $r^{-2n} \mathrm{vol}(B(p, r)) \geq \omega_{2n} - \epsilon$ for all r. After rescaling, we can assume that in the statement of the Lemma we have $r = 1$. We can then argue by contradiction, just as in [22, Proposition 3.7] (see also Ding [20, Theorem 0.7]). If the first conclusion were to fail, then we could extract a limiting harmonic function f on the Euclidean ball $B_{\mathbb{C}^n}(0, 2)$, such that

$$⨍_{B(0,1)} |f|^2 \geq 2^{2(1-\delta)} ⨍_{B(0,1/2)} |f|^2, \ \text{ but } ⨍_{B(0,2)} |f|^2 \leq 2^{2(1-\delta)} ⨍_{B(0,1)} |f|^2.$$

Such an f would have to be homogeneous of degree $1 - \delta$, but there is no such harmonic function on Euclidean space. The other conclusion follows similarly. □

Proof of Proposition 2.6. First note that once ϵ is sufficiently small, we can apply Theorem 1.4 to arbitrary balls in M. In particular for any R we obtain holomorphic functions $z_1^R, \ldots, z_n^R$ which provide a $\Psi(\epsilon)$-Gromov-Hausdorff approximation from $B(p, 2R)$ to $B_{\mathbb{C}^n}(0, 2R)$. We can assume $z_i^R(p) = 0$. Once ϵ is sufficiently small, we have

$$⨍_{B(p,2R)} |z_i^R|^2 \leq 2^3 ⨍_{B(p,R)} |z_i^R|^2,$$

and so iterating Lemma 2.4 we have

$$\fint_{B(p,2^k)} |z_i^R|^2 \leq C2^{3k} \fint_{B(p,1/2)} |z_i^R|^2,$$

whenever $2^k < R$. Let us define u_i^R in the span of the z_i^R, so that

$$\fint_{B(p,1)} u_i^R \bar{u}_j^R = \delta_{ij}.$$

The estimate above implies that we can extract limit holomorphic functions $u_1, \ldots, u_n$ on M as $R \to \infty$, that are L^2-orthonormal on $B(p,1)$, and

$$\fint_{B(p,R)} |u_i|^2 \leq CR^{3/2}$$

for all $R > 1$. We will show that $(u_1, \ldots, u_n)$ provides a biholomorphism from M to $\mathbb{C}^n$ if ϵ is sufficiently small.

We next prove the properness of the map given by the u_i. For $R > 1$ let us choose a new basis v_i^R for the span of the u_i, so that

$$\fint_{B(p,R)} v_i^R \overline{v_j^R} = c_i^R \delta_{ij}, \text{ and} \fint_{B(p,1)} v_i^R \overline{v_j^R} = \delta_{ij},$$

for some constants c_i^R. We then have $\sum |v_i^R|^2 = \sum |u_i|^2$. Let $\lambda_i^R = \sup_{B(p,R)} |v_i^R|$, and define $w_i^R = v_i^R / \lambda_i^R$. Then just as in [36, Claim 4.2], an argument by contradiction shows that if ϵ is sufficiently small, then the Rw_i^R give an $\frac{R}{100n}$-Gromov-Hausdorff approximation from $B(p,R)$ to the Euclidean ball $B_{\mathbb{C}^n}(0,R)$. In particular $\sum |w_i^R|^2 > 1/2$ on $\partial B(p,R)$. At the same time, using Lemma 2.4 with $\delta = 1/2$, and the fact that $v_i^R(p) = 0$ (so that v_i^R has at least linear growth on small scales), we have

$$\fint_{B(p,R)} |v_i^R|^2 \geq C^{-1} R \fint_{B(p,1)} |v_i^R|^2 = C^{-1} R.$$

It follows that $\lambda_i^R \geq C^{-1/2} R^{1/2}$. Therefore on $\partial B(p,R)$ we have

$$\sum |u_i|^2 = \sum |v_i^R|^2 \geq C^{-1/2} R^{1/2} \sum |w_i^R|^2 \geq \frac{1}{2} C^{-1/2} R^{1/2}.$$

This implies that the map given by $(u_1, \ldots, u_n) : M \to \mathbb{C}^n$ is proper.

We assert that the holomorphic n-form $du_1 \wedge \ldots \wedge du_n$ cannot vanish at any point. Note that on each ball $B(p,R)$ the functions u_i are obtained as a limit of u_i^R which satisfy that $du_1^R \wedge \ldots \wedge du_n^R$ is nowhere vanishing. Therefore if $du_1 \wedge \ldots \wedge du_n$ were to vanish at a point, then it would have to be identically zero. In this case the image of $(u_1, \ldots, u_n)$ would have dimension at most $n-1$ in $\mathbb{C}^n$, so the preimage of a point would be a compact subvariety in M of dimension at least 1. This contradicts that by Theorem 1.4 we can find holomorphic charts on arbitrarily large balls in M. Since $\mathbb{C}^n$ is simply connected, it follows that $(u_1, \ldots, u_n)$ must be a biholomorphism. □

3. Constructing projective embeddings

Suppose that (M_i^n, ω_i, L_i) is a sequence of compact polarized Kähler manifolds with $\mathrm{Ric}(M_i) > -1, \mathrm{diam}(M_i) < d, \mathrm{vol}(M_i) > v$ and such that the curvature of L_i is the Kähler metric ω_i. Let us assume that the sequence M_i converges in the Gromov-Hausdorff sense to the limit X. Our goal is to show that X is homeomorphic to a normal projective variety, following Donaldson-Sun [21]. As in [21], the main step is the construction of holomorphic sections of suitable powers of L_i, uniformly in i. We can follow the argument in [21] fairly closely, the main new difficulty being that in our setting we do not have smooth convergence of the metrics on the regular set. To overcome this we will use the existence of good holomorphic charts on the regular set (or rather the set $\mathcal{R}_\epsilon$ for small ϵ) provided by Theorem 1.4. The argument is simplified by using the recent estimate of Cheeger-Jiang-Naber [9] on the codimension 2 Minkowski content of the singular set, but see the Appendix for a proof which avoids this.

We prove the following.

Proposition 3.1. *Given $\nu, \zeta > 0$ there are $K, \epsilon, C > 0$ with the following property. Let (M^n, L, ω) be a polarized Kähler manifold such that* $\mathrm{Ric}(\omega) > -\epsilon\omega$*, and* $\mathrm{vol}(B(q,1)) > \nu$ *for all $q \in M$. Suppose that $d_{GH}(B(p,\epsilon^{-1}), B(o,\epsilon^{-1})) < \epsilon$ for a metric cone (V,o). Then L^m admits a holomorphic section s over M for some $m < K$, such that $\|s\|_{L^2} \leq C$, and $||s(x)| - e^{-md(x,p)^2/2}| < \zeta$ for $x \in B(p,1)$.*

Note that if (M^n, ω) satisfies $\mathrm{Ric} > -1, \mathrm{diam}(M) < d, \mathrm{vol}(M) > v$, then by volume comparison, there is a $\nu > 0$, depending on d, v such that $\mathrm{vol}(B(p,r)) > \nu r^{2n}$ for all $r \leq 1$. Thus scaled up copies of M will satisfy the local non-collapsing assumption in the statement of the proposition.

Proof. We will argue by contradiction, so suppose that the sequence $(M_i^n, p_i, L_i, \omega_i)$ satisfying the assumptions converges in the pointed Gromov-Hausdorff sense to (V, o). We will show that there is an $m > 0$ such that for sufficiently large i, L_i^m admits a suitable section for the points p_i.

From Theorem 1.4, volume comparison, and Cheeger-Colding theory, we have an $\epsilon = \epsilon(n) > 0$ such that if $q \in V$ satisfies $r^{-2n}\mathrm{vol}(B(q,r)) > \omega_{2n} - \epsilon$ for some $r < \epsilon$, then we have a holomorphic chart on $B(q, \epsilon r)$. In terms of this we define the singular set Σ of V as points q which satisfy $\lim_{r\to 0} r^{-2n}\mathrm{vol}(B(q,r)) \leq \omega_{2n} - \epsilon$. Note that Σ is a closed set. Let Σ_ρ be the ρ-neighborhood of Σ and set $U = B(o,R)\backslash\Sigma_\rho$. Here R is sufficiently large and ρ if sufficiently small, to be chosen later, depending on the parameters n, ν.

We can cover U by a finite number of small geodesic balls U'^j ($j = 1, .., N$, N depends on R, ρ) with center q_j, such that the balls $U^j = \epsilon U'^j$ with the same centers, but radius scaled by ϵ, still cover U. The radii of the U'^j can be chosen to be a fixed small constant, depending on R, ρ.

Let U_i be the lift of U back to M_i, under the Gromov-Hausdorff approximation. By Theorem 1.4 and our choice of ϵ, for sufficiently large i, we have uniform holomorphic charts $(U_i^j, (z_{i1}^j, ..., z_{in}^j))$ covering U_i. Moreover, by Proposition 2.4 the holomorphic charts on U_i^j converge to charts on U^j, with holomorphic transition functions. Thus U admits a holomorphic structure.

From Theorem 1.4, on each U_i^j, there is a function ρ_{ij} so that $\sqrt{-1}\partial\overline{\partial}\rho_{ij} = \omega_i$ and ρ_{ij} is very close to the distance squared from the center of U_i^j. In particular these ρ_{ij} are uniformly bounded, independent of i. Observe $\Delta\rho_{ij} = 2n$, so Yau's estimate gives us uniform gradient bound on a smaller interior domain. Therefore, by passing to a subsequence, we may assume $\rho_{ij} \to \rho_j$ uniformly.

Claim 3.1. *On each U^j, $\rho_j - r^2$ is a pluriharmonic function with respect to the holomorphic structure constructed above, where r is the distance function to the vertex $o \in V$.*

Proof. First, $\rho_j - r^2$ is a bounded Lipschitz function. To prove the result, we show that if α is any smooth $(n-1, n-1)$ form with respect to the holomorphic structure, with compact support in U^j, then

$$\int_{U^j} (\rho_j - r^2)\partial\overline{\partial}\alpha = \int_{U^j} \partial\bar{\partial}(\rho_j - r^2) \wedge \alpha = 0.$$

According to Cheeger-Colding [3], there is a sequence of functions h_i on the U_i^j converging to r^2 as $i \to \infty$, and such that

$$\lim_{i\to\infty} \int_{U_i^j} |\omega_i - \sqrt{-1}\partial\bar{\partial}h_i|^2_{\omega_i}\, \omega_i^n = 0.$$

The claim then follows from the fact that under our charts

$$\lim_{i\to\infty} \left| \int_{U_i^j} \partial\overline{\partial}(\rho_{ij} - h_i) \wedge \alpha \right| \le \lim_{i\to\infty} \int_{U_i^j} |\partial\overline{\partial}(\rho_{ij} - h_i)|_{\omega_i} |\alpha|_{\omega_i} \omega_i^n = 0.$$

Here we used that $|\alpha|_{\omega_i}$ is uniformly bounded since our holomorphic charts have uniformly bounded gradients, and so the ω_i have uniform lower bounds in terms of the charts. □

Since U_i^j is contained in a holomorphic chart, the line bundle L_i over U_i^j is isometric to a trivial holomorphic line bundle with weight $e^{-\rho_{ij}}$. Let s_{ij} be a holomorphic section over U_i^j so that $|s_{ij}| = e^{-\rho_{ij}} \neq 0$. Note that if $U_i^j \cap U_i^k \neq \emptyset$, the (holomorphic) transition functions $f_{ijk} = \frac{s_{ik}}{s_{ij}}$ are uniformly bounded. Therefore, after taking a subsequence, we may assume that the line bundles L_i converge to a Hermitian holomorphic line bundle (L, h) over U.

The line bundle L is trivial over each U_j with weight $e^{-\rho_j}$. By Claim 3.1 the metric $e^{r^2}h$ on L is flat over U, but since U is not necessarily simply connected, $(L, e^{r^2}h)$ need not be a trivial holomorphic line bundle with the flat metric. To deal with the possible presence of holonomy we follow Donaldson-Sun's argument. For the reader's convenience, we include some details. Pick a point $q \in U \cap \partial B_V(o, 1)$. As U is connected, we can join q with q_j (recall q_j is the center of geodesic balls U_j) by smooth curves $l_j \subset U$ (in terms of holomorphic structure of U). Now let s' be a vector in the fibre of L over q so that $|s'| = 1$. We can parallel transport s' along l_j, to get vectors s'_j in the fibre of L over q_j. Let us parallel transport s'_j in the geodesic ball U_j. Since L is flat and U_j is contained in a holomorphic chart, s'_j is well-defined.

Claim 3.2. *Given any $\delta > 0$, if we replace L by L^m, where m is some number that is bounded by a constant $K(\delta, n, \nu, R, \rho)$, then in the overlap $U_j \cap U_k$ we can ensure that $|s'_j - s'_k|_{e^{r^2}h} < \delta$.*

Proof. Notice that the norms under $e^{r^2}h$ of the s'_j are all equal to one. So $s'_j = s'_k e^{\sqrt{-1}\theta_{jk}}$. As $U_j \cap U_k$ is connected, θ_{jk} is constant. $(s'_j)^m = (s'_k)^m e^{\sqrt{-1}m\theta_{jk}}$. To prove the claim, we just need to find m so that $m\theta_{jk}$ is close to 2π times an integer for all $j, k \le N$. It follows from elementary number theory that such m exists. The bound follows from the fact that N depends only on n, ν, R, ρ. □

Let us fix a very small $\delta = \delta(n, \nu)$, to be determined later. From now on, we replace L by L^m, where $m < K(\delta, n, \nu, R, \rho)$. Let us still call the new line bundle L. Let us also rescale the metric $(M_i, p_i, \omega_i, L_i)$ by $(M_i, p_i, m\omega_i, L_i^m)$. Since m is a fixed number, the new sequence which we still call (M_i, p_i, ω_i) will converge to (V, o). We can consider the same $U \subset V$. A priori, the charts U_i^j might be different, but we shall make the centers q_j be the same. Note that $|s'_j - s'_k|_{e^{r^2}h}$ reflects the holonomy and homotopy preserves the holonomy since L is flat. Thus the claim implies that $|s'_j - s'_k|_{e^{r^2}h} < \delta$. Note that s'_j is a holomorphic section of L over U^j. Under the convergence $L_i \to L$, we can find holomorphic sections s_j^i on L_i over U_i^j so that $s_j^i \to s'_j$. Then, for sufficiently large i we have $|s_j^i - s_k^i| \leq 2\delta e^{-\frac{1}{2}r^2} \leq 2\delta$. Moreover, $|s_j^i| \leq 10e^{-\frac{1}{2}r^2}$ for large i.

By a standard partition of unity, we can glue the sections s_j^i together to a smooth section $\hat{s}_i$ of L_i over U_i so that $|\overline{\partial}\hat{s}_i|^2 < \gamma$, $|\hat{s}_i - s_j^i| < \min(10\delta, 20e^{-\frac{1}{2}r^2})$. Here γ is a small number depending only on n, ν, δ, R, ρ, and we are using the smooth structure given by our holomorphic charts.

Similarly to [21], we introduce the first standard cut-off function ψ_i^1, supported in $B(p_i, R)$, and the second cut-off function ψ_i^2, supported outside $B(p_i, \rho)$.

To define the third cut-off function ψ_i^3, first recall the cut-off function in [40], page 871. More precisely, if $\epsilon' \ll \epsilon \ll 1$ (constants independent of i) we define a cut-off function $\psi(t) = 1$, if $t \geq \epsilon$; $\psi(t) = (\frac{t}{\epsilon})^\epsilon$ if $2\epsilon' \leq t \leq \epsilon$; $\psi(t) = (2\frac{\epsilon'}{\epsilon})^\epsilon(\frac{t}{\epsilon'} - 1)$, if $\epsilon' \leq t \leq 2\epsilon'$; $\psi(t) = 0$ otherwise.

Let $\Sigma^i \subset M_i$ so that Σ^i converges to the singular set Σ under the Gromov-Hausdorff approximation. Let Σ_r^i be the r-tubular neighborhood of Σ^i. For $x \in B(p_i, R)$, let $d_i(x) = dist(x, \Sigma^i)$. According to Cheeger-Jiang-Naber's theorem [9] and the volume convergence theorem [19], for $\epsilon' < r < 2\epsilon$, if i is sufficiently large, $\mathrm{vol}(\Sigma_r^i \cap B(p_i, R)) \leq C(n, \nu, R)r^2$. Let the third cut-off function be $\psi_i^3(x) = \psi(d_i(x))$. Note $|\psi'|$ is decreasing from $2\epsilon'$ to ϵ and $10|\psi'(2t)| \geq |\psi'(t)|$ for any $2\epsilon' \leq t \leq \frac{1}{2}\epsilon$. By the calculation in [40], for i large enough, we can make $\int_{B(p_i,R)} |\nabla\psi_i^3|^2 e^{-r^2}$ as small as we want, provided ϵ' and ϵ are small enough.

Recall that U is the complement of Σ_ρ in $B(o, R)$. Let us assume $\rho < \frac{1}{10}\epsilon'$. As in [21], the smooth section $\tilde{s}_i = \psi_i^1\psi_i^2\psi_i^3\hat{s}_i$ satisfies

- $\tilde{s}_i$ supported in $B(p_i, R)\backslash(B(p_i, \rho) \cup \Sigma_{\epsilon'}^i)$.
- $\int |\overline{\partial}\tilde{s}_i|^2 < \gamma_2$ (can be as small as we want, if we set the parameters $R, \rho, \epsilon, \epsilon'$ properly).
- $|\nabla\tilde{s}_i| \leq C(n, \nu)$ on $U_i\backslash\Sigma_{10\epsilon}^i$
- $|\tilde{s}_i - s_j^i| \leq C(n, \nu)\delta$ on a slightly smaller subdomain of U_i.

By Hörmander's L^2 estimate, we can find a holomorphic section s_i on M_i so that if we set $s''_i = s_i - \tilde{s}_i$, then $\int_{M_i} |s''_i|^2 \leq 10\gamma_2$. As s_i has uniform L^2 bound, $|\nabla s_i|$ is uniformly bounded. Thus, on $U_i\backslash\Sigma_{10\epsilon}^i$, $\nabla s''_i$ is uniformly bounded. Therefore, by the integral estimate of s''_i, s''_i is very small in $V_i = U_i\backslash\Sigma_{\epsilon_0}^i$. Here ϵ_0 is a small number depending only on n, ν, γ_2. This means that on V_i, s_i is close to $\tilde{s}_i$, hence $|s_i|^2$ is close to $e^{-r_i^2}$ on V_i (here r_i is the distance to p_i). But as a set, V_i is Hausdorff close to $B(p_i, R)$. Then by the gradient estimate of s_i, we find that $|s_i|^2$ is very close to $e^{-r_i^2}$ on $B(p_i, R)$.

Since $\tilde{s}_i$ vanishes outside of $B(p_i, R)$, s_i'' is holomorphic on $M_i \backslash B(p_i, R)$. As $\int |s_i''|^2 \leq 10\gamma_2$, we can make sure that $|s_i''|^2 < c(n, \nu, d)\gamma_2$ on $M_i \backslash B(p_i, 2R)$. Now we can choose appropriate parameters so that Proposition 3.1 holds. □

Now let us take a look at the special case when the cone V in Proposition 3.1 splits off $\mathbb{R}^{2n-2}$. Note that when $|\mathrm{Ric}|$ is bounded, we actually have $V = \mathbb{R}^{2n}$ in this case, by Cheeger-Colding-Tian [8]. In general (V, o) is isometric $\mathbb{R}^{2n-2} \times W$, where (W, o') is a two dimensional metric cone. Let us write the metric on W as $dr^2 + r^2 d\theta^2$, where $0 \leq \theta \leq \alpha$ and α is the cone angle of W. By [8], the factor $\mathbb{R}^{2n-2}$ has a natural linear complex structure. The conical metric on W also determines a natural complex structure. Thus (V, o) can be identified with $\mathbb{C}^n$. Let the standard holomorphic coordinates be given by $(z_1, .., z_{n-1}, z_n)$, where $z_1, .., z_{n-1}$ are the standard linear coordinates on the first factor $\mathbb{R}^{2n-2}$ and $z_n(r, \theta) = r^{\frac{2\pi}{\alpha}} e^{\frac{2\pi\sqrt{-1}\theta}{\alpha}}$.

Fix ζ small, and let s be the holomorphic section of L^m constructed in Proposition 3.1. For simplicity of notation let us replace L by L^m (i.e. replace ω by $m\omega$). Set $h = -\log |s|^2$, so that $\sqrt{-1}\partial\overline{\partial}h = \omega$. In addition h is close to the distance squared from p. In particular, if we define Ω' as the sublevel set $h < 1000$, then if ζ is chosen small, we have $B(p, 10) \subset \Omega' \subset\subset B(p, 100)$. Let Ω be the connected component of Ω' containing $B(p, 10)$. Then Ω is a Stein manifold.

Using the same argument as Lemma 4.6 of [35] (the proof there only requires the Ricci curvature lower bound), we have the following.

Lemma 3.1. *We can find n complex harmonic functions $w_1', ..., w_n'$ on $B(p, 100)$ so that w_k' is $\Psi(\epsilon|n, \nu)$-close to z_k under the Gromov-Hausdorff approximation. Furthermore, $\int_{B(p,100)} |\overline{\partial} w_k'|^2 \leq \Psi(\epsilon|n, \nu)$.*

We can solve the $\overline{\partial}$ problem on Ω by using the weight e^{-h}. By a similar argument to before, we find holomorphic functions $w_1, \ldots, w_n$ on $B(p, 10)$ which are $\Psi(\epsilon|n, \nu)$-close to the $z_1, \ldots, z_n$. By using the same argument as on page 18 of [35], we find that if ϵ is sufficiently small, $(w_1, .., w_n)$ gives a holomorphic chart on $B(p, 5)$. We have therefore obtained the following result (see also Proposition 12 in [15]).

Proposition 3.2. *Let (M, L, ω) be a polarized Kähler manifold satisfying $\mathrm{Ric} > -1$ and $\mathrm{vol}(B(p, 1)) > \nu$ for all $p \in M$. There exists $\epsilon = \epsilon(n, \nu)$ so that the following holds. Assume that $d_{GH}(B(p, \epsilon^{-1}), B_V(o, \epsilon^{-1})) < \epsilon$, where (V, o) is a metric cone splitting off $\mathbb{R}^{2n-2}$. Then there exists a holomorphic chart $(w_1, ..., w_n)$ on $B(p, 5)$ such that $(w_1, ..., w_n)$ is $\Psi(\epsilon|n, \nu)$-close to a standard holomorphic coordinate chart on (V, o).*

As a consequence of this we have the following result analogous to Proposition 2.5.

Proposition 3.3. *Suppose that (M_i^n, L_i, ω_i) is a sequence of polarized Kähler manifolds with $\mathrm{Ric} > -1$, $\mathrm{vol}(B(q_i, 1)) > \nu > 0$ for all $q_i \in M_i$. For $p_i \in M_i$, assume that (M_i, p_i) converges to a metric cone (V, o) in the pointed Gromov-Hausdorff sense, where $V = \mathbb{R}^{2n-2} \times W$ with (W, o') a two-dimensional cone. Then*

$$\lim_{i \to \infty} \int_{B(p_i, 1)} S_i \omega_i^n = \omega_{2n-2}(2\pi - \alpha),$$

where ω_{2n-2} is the area of the unit ball in $\mathbb{R}^{2n-2}$ and $\alpha \in (0, 2\pi)$ is the cone angle of W. Note that the distributional scalar curvature of W is $(2\pi - \alpha)\delta_{o'}$.

Proof. The proof of Proposition 2.5 can be used essentially verbatim, using that under our assumptions Proposition 3.2 gives suitable holomorphic charts on $B(p_i, 10)$. The main difference is that now on the limit space the function $|s| = |dz_1 \wedge \ldots \wedge dz_n|$ vanishes along the singular set, and so $\log|s|$ is unbounded. Instead of the statement of Lemma 2.2, we have that for any neighborhood U of the singular set (identified with a subset of $B(p_i, 4)$ using the chart),

$$\lim_{i\to\infty} \int_{B(p_i,4)\setminus U} \left|\log|s_i|^2 - \log|s|^2\right| \omega_i^n = 0.$$

Then just as in (2.23) we will have

$$\lim_{i\to\infty} \int_{M_i\setminus U} \log|s_i|^2 \Delta_{\omega_i} h = \int_{V\setminus U} \log|s|^2 \Delta_\omega h. \tag{3.1}$$

Notice that on sufficiently small neighborhoods U of the singular set, the integrals of both $\log|s|$ and $\log|s_i|$ can be made arbitrarily small. The former by direct calculation, and the latter by the estimate (2.20). It then follows from (3.1) that

$$\lim_{i\to\infty} \int_{M_i} \log|s_i|^2 \Delta_{\omega_i} h = \int_V \log|s|^2 \Delta_\omega h,$$

which implies the required result. □

Using this result, together with Cheeger-Jiang-Naber's [9] bounds we now prove Proposition 1.1, which we state again for the reader's convenience.

Proposition 3.4. *Let $B(p,1)$ be a unit ball in a polarized Kähler manifold (M^n, L, ω) satisfying* $\mathrm{Ric} > -1$*, such that* $\mathrm{vol}(B(p,1)) > v > 0$*. Then* $\int_{B(p,1)} S < C(n,v)$.

Proof. For any $l > 0$, let A_l denote the supremum of $\int_{B(p,1)} |S|$ over all unit balls as in the statement, with the additional condition that $\mathrm{Ric} < l$. Our goal is to show that A_l is bounded independently of l. Note that any A_l is finite by volume comparison. Also it is convenient to replace the condition $\mathrm{vol}(B(p,1)) > v > 0$ by

$$r^{-2n}\mathrm{vol}(B(q,r)) > v' > 0, \text{ for all } q \in B(p,1) \text{ and } r < 1,$$

since this condition is preserved when passing to smaller balls. The two conditions imply each other for suitable v, v' by volume comparison.

Let us recall the following notion from Cheeger-Jiang-Naber [9, Definition 1.3]. A ball $B(x,r)$ in a metric space is (k,ϵ)-symmetric if there is a metric cone $X' = \mathbf{R}^k \times C(Z)$ with vertex x' splitting an isometric factor of $\mathbf{R}^k$, such that $d_{GH}(B(x,r), B(x',r)) < \epsilon r$. From Corollary 1.2 and Proposition 3.3 we find that there are constants $\epsilon, C_1 > 0$ with the following property. If $B(q,\epsilon^{-1})$ is a ball in a polarized Kähler manifold with $\mathrm{Ric} > -1, \mathrm{vol}(B(q,1)) > v$, and $B(q,\epsilon^{-1})$ is $(2n-2,\epsilon^2)$-symmetric, then $\int_{B(q,1)} |S| < C_1$.

Let $r > 0$ be small, to be chosen later, and set $k = 2n-3$. As in [9], let $S^{2n-3}_{\epsilon^2,r}$ denote the points $x \in B(p,1)$ such that $B(x,s)$ is not $(2n-2,\epsilon^2)$-symmetric for any $s \in [r,1)$. Let us choose $x_1, \ldots, x_{N_r} \in S^{2n-3}_{\epsilon^2,r}$ such that $B(x_i,r)$ cover $S^{2n-3}_{\epsilon^2,r}$, while $B(x_i, r/3)$ are disjoint. By [9, Remark 1.10] we have that $N_r r^{2n-3} \le C_\epsilon$. In addition, if $\mathrm{Ric} < l$, then after scaling the ball $B(x_i,r)$ to unit size, it will still have Ricci curvature bounded by l. We can assume that r^{-2} is an integer so the scaled up manifold is still polarized. It follows by scaling that

$$\int_{B(x_i,r)} |S| < r^{2n-2} A_l.$$

If $y \notin \bigcup B(x_i, r)$, then by definition there is an $s \in [r, 1)$ such that $B(y, s)$ is $(2n-2, \epsilon^2)$-symmetric. It follows, after rescaling the result above that

$$\int_{B(y,\epsilon s)} |S| < (\epsilon s)^{2n-2} C_1.$$

We can now cover $B(p, 1) \setminus \bigcup B(x_i, r)$ by such balls $B(y_j, \epsilon s_j)$, such that the $B(y_j, \epsilon s_j/5)$ are disjoint. We then have

$$\sum_j (\epsilon s_j)^{2n} < C_2,$$

and

$$\int_{B(p,1)\setminus\bigcup_i B(x_i,r)} |S| \leq \sum_j \int_{B(y_j,\epsilon s_j)} |S| \leq C_1 \sum_j (\epsilon s_j)^{2n-2} < C_1 C_2 (\epsilon r)^{-2},$$

using $s_j \geq r$. In sum we get

$$\int_{B(p,1)} |S| < N_r r^{2n-2} A_l + C_1 C_2 (\epsilon r)^{-2} \leq r C_\epsilon A_l + C_1 C_2 (\epsilon r)^{-2}.$$

We now choose r so that $r C_\epsilon < 1/2$. It follows that

$$A_l \leq \frac{1}{2} A_l + C',$$

where C' is independent of l. This implies our result. □

Let us now return to the setting of the beginning of the section, i.e. (M_i^n, L_i, ω_i) are polarized Kähler manifolds with $\mathrm{Ric}(\omega_i) > -\omega_i, \mathrm{diam}(M_i) < d, \mathrm{vol}(M_i) < v$, converging to X in the Gromov-Hausdorff sense. Given Proposition 3.1, an argument by contradiction implies the partial C^0-estimate and separation of points:

Proposition 3.5. *Given any point $p \in X$, take a sequence $M_i \ni p_i \to p$. There exist $\delta = \delta(n, v, d) > 0$, $K(n, v, d) \in \mathbb{N}$ and holomorphic sections s_i over $L_i^m (m < K(n, v, d))$ so that $\int |s_i|^2 = 1$, $|s_i(p_i)| \geq \delta$.*

Furthermore, given any two points $p, q \in X$ with $d(p, q) > a > 0$ and sequences $M_i \ni p_i \to p, M_i \ni q_i \to q$, we can find holomorphic sections s_i^1, s_i^2 of $L^m (m < K(n, v, a, d))$ and $\delta = \delta(n, v, a, d)$ so that

- *$\int |s_i^1|^2 + |s_i^2|^2 < 1$;*
- *$|s_i^1(p_i)| = \delta, s_i^1(q_i) = 0$;*
- *$s_i^2(p_i) = 0, |s_i^2(q_i)| = \delta$.*

By following the same arguments as in [21, Section 4.3.1] we can prove that X is homeomorphic to a projective variety and after suitable projective embeddings a subsequence of the M_i converge to X as algebraic varieties. In addition, Proposition 3.2 implies that X is complex analytically regular near the points $p \in X$ which admit tangent cones splitting off $\mathbb{R}^{2n-2}$. The remainder of X has Hausdorff dimension at most $2n - 4$ by [4], which implies as in [21] that X is normal. This completes the proof of Theorem 1.1.

4. Complex analytic and metric singularities

As in the previous section, given n, d, v, let (M_i^n, ω_i, L_i) be polarized Kähler manifolds with $\mathrm{Ric}(M_i) > -1, \mathrm{diam}(M_i) < d, \mathrm{vol}(M_i) > v$, such that M_i converge in the Gromov-Hausdorff sense to X. Then X has the structure of a normal projective variety, and it is also a metric space. In this section we will study the relation between the singular sets of X in the metric sense and in the complex analytic sense.

Pick a point $p \in X$, and take a sequence $M_i \ni p_i \to p$. As a consequence of Proposition 3.1, for sufficiently large i, there exists $r_0 > 0$ independent of i and a smooth u_i on $B(p_i, r_0)$ so that $\sqrt{-1}\partial\overline{\partial}u_i = \omega_i$ and $u_i(x)$ is close to $d^2(x, p_i)$. Since $\mathrm{Ric}(\omega_i) > -\omega_i$, $\Theta_i = \mathrm{Ric}(\omega_i) + \sqrt{-1}\partial\overline{\partial}u_i$ is a closed positive current on $B(p_i, r_0)$. Now assume that p is a complex analytically regular point on X(i.e., a non-singular point on the variety X). By shrinking the value of r_0 if necessary, by solving the $\bar{\partial}$-problem with weights e^{-u_i}, we can find uniform holomorphic charts $(z_1^i, ..., z_n^i)$ on $B(p_i, r_0)$ so that the charts converge to a holomorphic chart $(U, (z_1, .., z_n))$ near p. As before, we can use these charts to identify the balls $B(p_i, r_0/2)$ with corresponding subsets of U. Let us say $z_j(p) = 0$ for all $j = 1, ..., n$, and define

$$v_i = \frac{1}{2\pi} \log |dz_1^i \wedge dz_2^i \wedge \ldots \wedge dz_n^i|^2 + u_i.$$

Note that $\Theta_i = \sqrt{-1}\partial\overline{\partial}v_i \geq 0$. Since U is an open set, $|dz_1^i \wedge dz_2^i \wedge \ldots \wedge dz_n^i|$ cannot go to zero uniformly, as $i \to \infty$. Thus v_i cannot go uniformly to $-\infty$. By taking a subsequence, we may assume that v_i converges, in L^1_{loc} sense (with respect to the Lebesgue measure given by the charts), to a plurisubharmonic function v on U. As u_i has uniformly bounded gradient (note that $\Delta u_i = 2n$), we can also assume that $u_i \to u$ uniformly. Define

$$\mathrm{Ric} = \sqrt{-1}\partial\overline{\partial}(v - u). \tag{4.1}$$

Then Ric, as a closed $(1, 1)$ current, is well defined on the complex analytically regular part of X. Note that Ric is a closed positive current, up to $\sqrt{-1}\partial\overline{\partial}u$ for a bounded function u. Thus the Lelong number for Ric, given by

$$\frac{1}{2\pi} \liminf_{x \to p} \frac{\log |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n|^2}{\log |z(x)|}, \tag{4.2}$$

is well defined. Here the numerator can be defined as the limit of $\log |dz_1^i \wedge dz_2^i \wedge \ldots \wedge dz_n^i|^2$.

The main result in this section is the following

Proposition 4.1. *A point $p \in X$ is regular in the metric sense if and only if it is complex analytically regular and the Lelong number for* Ric *vanishes at p.*

Proof. It is clear from Theorem 1.4 that if p is regular in the metric sense, then p must be complex analytically regular. Now let us prove that if X is complex analytically regular at p and the Lelong number for Ric is zero at p, then p is a regular point in X in the metric sense. We first need some preliminary results.

Claim 4.1. *There exists $a > 0$, $b = b(n, v, d) > 0$ so that for all $r < a$, and any point $q \in \partial B(p, r)$, there exists a holomorphic function f on $B(p, 4r)$ so that $f(p) = 0$, $\sup_{B(p,2r)} |f| = 1$ and $|f(q)| > b$.*

Proof. Assume that the claim is false. Then there exist sequences $r_i, b_i \to 0$ and $q_i \in \partial B(p, r_i)$ so that for all holomorphic functions f_i on $B(p, 4r_i)$ with $f_i(p) = 0$ and $\sup_{B(p,2r_i)} |f_i| = 1$, $|f_i(q_i)| < b_i$. By passing to a subsequence, we may assume that $(X_i, p_i, d_i) = (X, p, \frac{d}{r_i})$ converges to a metric cone (V, o) in the pointed Gromov-Hausdorff sense. Assume $q_i \to q \in \partial B(o, 1)$.

To get a contradiction, we can prove results similar to Theorem 1.4 and Proposition 2.9 of [22] (alternatively, Proposition 6.1 in [36]). The proof follows by a very minor modification, so we skip the details. Then on (V, o) we can find a holomorphic function vanishing at o but nonzero at q, which we can lift to (X_i, p_i) for sufficiently large i. The lifted holomorphic functions will have a uniform lower bound at q_i, giving a contradiction. It is clear from the argument that b depends only on n, v, d. □

Claim 4.2. *Let p be a complex analytically regular point on X. Let $(z_1, ..., z_n)$ be a holomorphic chart near p, such that $z_j(p) = 0$ for all j. Then there exists $\alpha = \alpha(n, v, d) > 0$, $C > 0, c > 0$ so that $cr(q)^\alpha \leq |z(q)| \leq Cr(q)$ for all q sufficiently close to p. Here r is the distance function to p.*

Proof. The inequality $|z(q)| \leq Cr(q)$ follows directly from the gradient estimate. Now we prove the first inequality. Let $a > 0, b = b(n, v, d) > 0$ be the constants in Claim 4.1. Let us fix a small $r_0 < a$. We may assume that $B(p, 2r_0)$ is contained in the holomorphic chart $(z_1, ..., z_n)$. For any $\rho > 0$, let U_ρ be the open set so that $|z| < \rho$. Since $(z_1, .., z_n)$ is a holomorphic chart, for δ sufficiently small, we may assume that $U_\delta \subset B(p, 2r_0)$. Pick an arbitrary $q \in \partial B(p, r_0)$. According to Claim 4.1, there exists a holomorphic function f on $B(p, 4r_0)$ so that $f(p) = 0$, $\sup_{B(p,2r_0)} |f| = 1$, $|f(q)| > b$. If $q \in U_\delta$, we restrict f to U_δ. As $f(p) = 0$, by the standard Hadamard three circle theorem on $B_{\mathbb{C}^n}(0, \delta)$ we find $\frac{|f(q)|}{|z(q)|} \leq \frac{\sup_{U_\delta} |f|}{\delta}$, thus $|z(q)| \geq b\delta$. If $q \notin U_\delta$, then by definition $|z(q)| \geq b\delta$. Since q is arbitrary on $\partial B(p, r_0)$, we find that $U_{b\delta} \subset B(p, r_0)$. Iterating this result, we obtain that $U_{b^k\delta} \subset B(p, 2^{1-k}r_0)$, which implies the first inequality $cr(q)^\alpha \leq |z(q)|$. □

Claim 4.3. *Assume that p is not a regular point in the metric sense. Then there exist $\epsilon > 0$ and $r_0 > 0$ satisfying the following. For all $r < r_0$, if nonzero holomorphic functions $f_1, .., f_n$ on $B(p, 4r)$ satisfy $f_j(p) = 0$ and $\int_{B(p,r)} f_j \overline{f}_k = 0$ for $j \neq k$, then there exists $1 \leq l \leq n$ so that*

$$\frac{⨍_{B(p,2r)} |f_l|^2}{⨍_{B(p,r)} |f_l|^2} \geq 2^{2+10n\epsilon}.$$

Remark 4.1. *From the proof it follows that ϵ depends only on $\omega_{2n} - \lim_{r\to 0} \frac{\mathrm{vol}(B(p,r))}{r^{2n}}$, where ω_{2n} is the volume of the unit ball of $\mathbb{C}^n$.*

Proof. If this is not the case, then we can find sequences $\epsilon_i \to 0$, $r_i \to 0$, and nonzero holomorphic functions $f_{i1}, ..., f_{in}$ on $B(p, 4r_i)$ so that $f_{ij}(p) = 0$ and $\int_{B(p,r_i)} f_{ij} \overline{f}_{ik} = 0$ for $j \neq k$. Also for all j,

$$\frac{⨍_{B(p,2r_i)} |f_{ij}|^2}{⨍_{B(p,r_i)} |f_{ij}|^2} < 2^{2+10n\epsilon_i}.$$

Define $(X_i, p_i, d_i) = (X, p, \frac{d}{r_i})$. By passing to a subsequence, we may assume that (X_i, p_i) converges in the pointed Gromov-Hausdorff sense to a tangent cone (V, o) at p. We trivially lift f_{ij} to $B(p_i, 4)$ on X_i. By normalization, we may assume that $⨍_{B(p_i,1)} |f_{ij}|^2 = 1$ for all j. Then after taking a further subsequence, f_{ij} converges uniformly on each compact set of $B(o, 2)$ to linearly independent complex harmonic functions h_j. These satisfy $⨍_{B(o,1)} |h_j|^2 = 1$, $h_j(o) = 0$ for all j, and $⨍_{B(o,2)} |h_j|^2 \leq 4$. By the spectral decomposition for the Laplacian on the cross section, the h_j can be extended as degree one homogeneous complex harmonic functions on V. Therefore we have $2n$ linearly independent real harmonic functions of linear growth which all vanish at o. Then it is well known that (V, o) is isometric to $\mathbb{R}^{2n}$ (see Proposition 5.2 in the Appendix). This contradicts the assumption that p is not a regular point. □

Now let $p \in X$ be a regular point in the complex analytic sense, but not in the metric sense. We claim that the Lelong number of Ric at p is positive. Let $(z_1, .., z_n)$ be a holomorphic chart around p. For r_0 small, we may assume $B(p, 2r_0)$ is contained in the chart. By suitable scaling and orthogonalization of z_j, we may assume

$$z_j(p) = 0, \quad ⨍_{B(p,2r_0)} |z_j|^2 = 1, \quad (j = 1, ..., n), \tag{4.3}$$

$$\int_{B(p,2r_0)} z_j \overline{z}_k = 0, \text{ for } j \neq k. \tag{4.4}$$

By scaling the metric, let us assume without loss of generality that $r_0 = 1$. Then by the gradient estimate

$$|dz_1 \wedge \ldots \wedge dz_n| \leq C = C(n, v, d) \text{ on } B(p, 1). \tag{4.5}$$

Let E be the linear space spanned by $z_1, ..., z_n$. On E, we have two norms, given by L^2 integration over $B(p, 2)$ and $B(p, 1)$. After a simultaneous diagonalization with respect to these two norms, we may assume that z_j are also L^2 orthogonal on $B(p, 1)$. Let ϵ be the positive number appearing in Claim 4.3. We may assume that $B(p, 10)$ is sufficiently close to a metric cone, such that

$$\frac{⨍_{B(p,2)} |z_j|^2}{⨍_{B(p,1)} |z_j|^2} \geq 2^{2-\epsilon} \tag{4.6}$$

for all j, since on the cone there are no non-constant sublinear harmonic functions. According to Claim 4.3, we can find l so that

$$\frac{⨍_{B(p,2)} |z_l|^2}{⨍_{B(p,1)} |z_l|^2} \geq 2^{2+10n\epsilon}.$$

Define $z'_j = z_j 2^{-\epsilon}$ for $j \neq l$; $z'_l = z_l 2^{(n-1)\epsilon}$, so that

$$dz'_1 \wedge dz'_2 \wedge \ldots \wedge dz'_n = dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n$$

at any point. Moreover, from (4.6) we have

$$⨍_{B(p,1)} |2z'_k|^2 \leq 2^{-\epsilon}$$

for all k. Using the gradient estimate we obtain that on $B(p, \frac{1}{2})$,

$$|dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n| = |dz'_1 \wedge dz'_2 \wedge \ldots \wedge dz'_n| \leq C2^{-0.5n\epsilon} \leq C2^{-\epsilon}.$$

Here C is the same constant as in (4.5). By iteration, we obtain that for all $0 < r < 1$, on $B(p, r)$,

$$|dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n| \leq 2Cr^{\epsilon}.$$

According to Claim 4.2, Claim 4.3, the Lelong number for $2\pi\text{Ric}$ at p is given by

$$\liminf_{x \to p} \frac{\log |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n|^2(x)}{\log |z(x)|} \geq \liminf_{x \to p} \frac{2\log(2Cr(x)^{\epsilon})}{\log(cr(x)^{\alpha})} = \frac{2\epsilon}{\alpha} > 0.$$

This means that if p is complex analytically regular and the Lelong number for Ric vanishes at p, then p is regular in the metric sense.

We are left to prove that if p is regular in the metric sense, then the Lelong number for Ric vanishes at p. Since p is metric regular, for any fixed small $\epsilon > 0$, by scaling, we may assume that $B(p, \frac{1}{\epsilon})$ is ϵ-Gromov Hausdorff close to a ball in $\mathbb{C}^n$. Then we can find a holomorphic map $F = (f_1, ..., f_n)$ on $B(p, 100)$ which gives a $\Psi(\epsilon|n)$-Gromov Hausdorff approximation to its image in $\mathbb{C}^n$. As in the proof of Theorem 1.4, F is a holomorphic chart on $B(p, 1)$. Without loss of generality, we may assume $f_j(p) = 0, \int_{B(p,1)} f_j\overline{f_k} = 0$ for $j \neq k$. We have

$$\frac{⨍_{B(p,2)} |f_j|^2}{⨍_{B(p,1)} |f_j|^2} \leq 4 + \Psi(\epsilon|n), \tag{4.7}$$

and since $B(p, r)$ is $\Phi(\epsilon|n)r$-Gromov-Hausdorff close to a Euclidean ball for all $r > 0$, just as in Lemma 2.4 we conclude that

$$4 - \Psi(\epsilon|n) \leq \frac{⨍_{B(p,2r)} |f_j|^2}{⨍_{B(p,r)} |f_j|^2} \leq 4 + \Psi(\epsilon|n) \tag{4.8}$$

for all $r < 1$. For any $r > 0$, we may assume that the $f_1, ..., f_n$ are orthogonal simultaneously with respect to the L^2 inner products on $B(p, 1)$ and $B(p, r)$. Now let c_j be constants (depending on r) so that $\sup_{B(p,r)} |c_j f_j| = r$. Define $f_j' = c_j f_j$. As in the proof of Proposition 2.6, arguing as in Claim 4.2 of [36], we see that $F' = (f_1', .., f_n')$ is a $\Psi(\epsilon|n)r$-Gromov-Hausdorff approximation to a ball in $\mathbb{C}^n$. The Cheeger-Colding [3] estimate (see Equation (2.7)) implies that

$$\sup_{B(p,r)} |df_1' \wedge df_2' \wedge \ldots \wedge df_n'| \geq c(n) > 0,$$

and note that by (4.8),

$$⨍_{B(p,2r)} |f_j|^2 \geq c(n)r^{2+\Psi(\epsilon|n)}.$$

Thus

$$|c_j| \leq C(n)r^{-\Psi(\epsilon|n)},$$

and so

$$\sup_{B(p,r)} |df_1 \wedge df_2 \wedge \ldots \wedge df_n| \geq c(n)r^{\Psi(\epsilon|n)}.$$

It follows that the Lelong number at p satisfies

$$\liminf_{x \to p} \frac{\log |df_1 \wedge df_2 \wedge \ldots \wedge df_n|^2(x)}{2\pi \log |F(x)|} \leq \liminf_{x \to p} \frac{2\log(c(n)r(x)^{\Psi(\epsilon|n)})}{2\pi \log(Cr(x))} = \Psi(\epsilon|n).$$

As ϵ is arbitrary, we find that the Lelong number at p is zero. The proof of Proposition 4.1 is complete. □

Let A be the complex analytically singular set of X. Let c be a positive constant, and let H_c be set of points whose Lelong number for Ric is at least c on $X\backslash A$. By Siu's theorem [47], H_c is a complex analytic set of $X\backslash A$. We thank Professor Siu for providing the proof of the following.

Lemma 4.1. *The topological closure of H_c in X is a complex analytic set.*

Proof. The problem is local on X. For any $p \in X$, take a sequence $M_i \ni p_i \to p$. For sufficiently large i, there exists $r_0 > 0$ independent of i and smooth functions u_i on $B(p_i, 2r_0)$ so that $\sqrt{-1}\partial\overline{\partial}u_i = \omega_i$ and $u_i(x)$ is close to $d^2(x, p_i)$. Write $U = B(p, r_0)$ and assume $u_i \to u$ uniformly on U. Now assume that p is a complex analytically singular point on X. Then $\Theta = \mathrm{Ric} + \sqrt{-1}\partial\overline{\partial}u$ is a closed positive $(1, 1)$ current on $U\backslash A$. It is clear that the Lelong number for Θ is the same as Lelong number for Ric. By shrinking U if necessary, we may assume that (U, p) is a normal subvariety of $(\Omega, 0) \subset (\mathbb{C}^N, 0)$. We can trivially extend Θ as a positive closed $(N - n + 1, N - n + 1)$-current $\hat{\Theta}$ on $\Omega\backslash A$. Since U is a normal variety, A has complex dimension at most $n - 2$. Thus the codimension of A in Ω is at least $N - n + 2$. According to [46], $\hat{\Theta}$ extends to a closed positive current on Ω. By applying Siu's theorem again, we proved the lemma. □

The following is a generalization of Donaldson-Sun's Proposition 4.14 in [21], where the Einstein case was treated (although their proof applies in the case of bounded Ricci curvature too).

Corollary 4.1. *Suppose that the M_i above have uniformly bounded Ricci curvature. Then the metric singular set coincides with the complex analytic singular set on X.*

Proof. Let p be a complex analytically regular point. It suffices to prove that p is metric regular. According to Proposition 4.1, it suffices to show that the Lelong number for Ric vanishes at p. Recall the bounded function u in the last lemma. Let us assume $|\mathrm{Ric}(M_i)| \leq C$. Then as a closed positive $(1, 1)$ current, $\Theta = \mathrm{Ric} + C\sqrt{-1}\partial\overline{\partial}u \leq 2C\sqrt{-1}\partial\overline{\partial}u$. Note that the Lelong number for Θ is the same as for Ric at p. By monotonicity, the Lelong number of Θ is no greater than the Lelong number of the positive $(1, 1)$-current $2C\sqrt{-1}\partial\overline{\partial}u$. But u is bounded, therefore the Lelong number for Ric vanishes at p.

□

Theorem 1.3 is a direct consequence of Proposition 4.1. For the reader's convenience, we rewrite it here.

Theorem 4.1. *Let (X, d) be a Gromov-Hausdorff limit as in Theorem 1.1. Then for any $\epsilon > 0$, $X \setminus \mathcal{R}_\epsilon$ is contained in a finite union of analytic subvarieties of X. Furthermore, $X \setminus \mathcal{R}$ is equal to a countable union of subvarieties.*

Here $\mathcal{R}$ consists of points $x \in X$ with tangent cone $\mathbb{C}^n$, while $\mathcal{R}_\epsilon$ is the set of points p so that $\omega_{2n} - \lim\limits_{r\to 0} \frac{\mathrm{Vol}(B(p,r))}{r^{2n}} < \epsilon$.. In view of the main result of [35], we can use the same argument to obtain the following.

Theorem 4.2. *Let (X, p) be the pointed Gromov-Hausdorff limit of complete Kähler manifolds (M_i^n, p_i) with bisectional curvature lower bound -1 and* $\mathrm{vol}(B(p_i, 1)) \geq$

$v > 0$. Then X is homeomorphic to a normal complex analytic space. The metric singular set $X \setminus \mathcal{R}$ is exactly given by a countable union of complex analytic sets, and for any $\epsilon > 0$, each compact subset of $X \setminus \mathcal{R}_\epsilon$ is contained in a finite union of subvarieties.

5. Appendix

In the proof of Proposition 3.1 we used the estimate of Cheeger-Jiang-Naber [9] for the volumes of tubular neighborhoods of the singular set, in order to control the cutoff functions ψ_i^3. In this appendix we first give an alternative argument, following the approach of Chen-Donaldson-Sun [15], which is independent of the results in [9]. First note that the cone $V = \mathbb{R}^{2n-2} \times W$ for a two-dimensional cone (W, o') has singular set $\mathbb{R}^{2n-2} \times \{o'\}$, and so we can directly see the required estimate for the volumes of its tubular neighborhoods. Therefore Propositions 3.2 and 3.3 hold without appealing to [9].

We can now argue similarly to [15] to show that the singular sets in any cone (V, o) arising in Proposition 3.1 have good cutoff functions (even closer to what we do are Propositions 12, 13 and 14 of the arXiv version of [48]). More precisely, suppose that (M_i, L_i, ω_i) is a sequence as in Proposition 3.1 such that (M_i, p_i) converges to a cone (V, o) for some $p_i \in M_i$. Recall that the singular set is $\Sigma \subset V$, consisting of points $q \in V$ such that $\lim_{r\to 0} r^{-2n}\mathrm{vol}(B(q, r)) \leq \omega_{2n} - \epsilon$, where ϵ is obtained from Theorem 1.4. We then have the following.

Proposition 5.1. *For any compact set $K \subset V$ and $\kappa > 0$ there is a function χ on V, equal to 1 on a neighborhood of $K \cap \Sigma$, supported in the κ-neighborhood of $K \cap \Sigma$, and such that $\int_K |\nabla\chi|^2 < \kappa$.*

Proof. Let us fix K, κ. Suppose that we have distance functions d_i on $B(p_i, 2R) \sqcup B(o, 2R)$, realizing the Gromov-Hausdorff convergence, where R is large so that $K \subset B(o, R)$. For $q \in B(o, R)$, and $\rho \in (0, 1)$, define

$$V(i, q, \rho) = \rho^{2-2n} \int_{U_i(q,\rho)} (S_i + 2n)\omega_i^n,$$

where $U_i(q, \rho) = \{x \in B(p_i, 2) \,:\, d_i(x, q) < \rho\}$. Note that $S_i + 2n \geq 0$.

Let us denote by $\mathcal{D} \subset K\cap\Sigma$ the set of points which have a tangent cone splitting off $\mathbb{R}^{2n-2}$, and let $S_2 = \Sigma \setminus \mathcal{D}$. Proposition 3.3 implies that for any $x \in \mathcal{D}$ there exists a $\rho_x > 0$ such that $V(i, x, \rho_x) < A$ for a fixed constant A, for sufficiently large i. At the same time by Proposition 3.3 we also have a constant $c_0 > 0$ (depending on n, v, d, ϵ) such that for any $x \in \mathcal{D}$ and $\delta > 0$ there is an $r_x < \delta$ such that $V(i, x, r_x) > c_0$ for sufficiently large i (here note that the two dimensional cones appearing in tangent cones of points in $\mathcal{D}$ have cone angles bounded strictly away from 2π).

By Cheeger-Colding's [4] the Hausdorff dimension of S_2 is at most $2n - 4$, so for any small $\epsilon > 0$ we can cover $S_2 \cap K$ with balls B_μ such that

$$\sum_\mu r_\mu^{2n-3} < \epsilon.$$

The set

$$J = (K \cap \Sigma) \setminus \cup_\mu B_\mu$$

is compact, $J \subset \mathcal{D}$, and so it is covered by the balls $B(x, \rho_x)$ with $x \in \mathcal{D}$. We choose a finite subcover corresponding to $x_1, \ldots, x_N$, and set

$$W = \bigcup_{j=1}^{N} B(x_j, \rho_{x_j}) \subset V,$$

$$W_i = \bigcup_{j=1}^{N} U_i(x_j, \rho_{x_j}) \subset M_i.$$

For sufficiently large i we then get an estimate

$$\int_{W_i} (S_i + 2n)\omega_i^n < C, \tag{5.1}$$

where C depends on ϵ, N, but not on i.

We claim that the compact set $J \subset \mathcal{D}$ has finite $(2n-2)$-dimensional Hausdorff measure. To prove this, recall that for any small $\delta > 0$ and all $x \in \mathcal{D} \cap J$ we have $r_x < \delta$ such that $V(i, x, r_x) > c_0$ for large i. By a Vitali type covering argument we can find a disjoint, finite sequence of balls $B(y_k, r_{y_k})$ in W, for $k = 1, \ldots, N'$ such that $B(y_k, 5r_{y_k})$ cover all of J. It follows that

$$\mathcal{H}_\delta^{2n-2}(J) \leq \sum_{k=1}^{N'} 5^{2n-2} r_{y_k}^{2n-2}.$$

At the same time for each y_k, we have the estimate

$$r_{y_k}^{2-2n} \int_{U_i(y_k, r_{y_k})} (S_i + 2n)\omega_i^n > c_0,$$

for sufficiently large i. Taking i even larger we can assume that the $U_i(y_k, r_{y_k})$ are disjoint, since they converge in the Gromov-Hausdorff sense to the disjoint balls $B(y_k, r_{y_k})$. Using (5.1) we therefore have

$$\sum_{k=1}^{N'} c_0 r_{y_k}^{2n-2} < C.$$

Since δ was arbitrary (and C is independent of δ), this implies that $\mathcal{H}^{2n-2}(J) \leq C'$.

It follows that J has capacity zero, in the sense that for any $\kappa > 0$ we can find a cutoff function η_1 supported in the κ-neighborhood of J, such that $\|\nabla \eta_1\|_{L^2} \leq \kappa/2$, and $\eta_1 = 1$ on a neighborhood U of J (see for instance [1, Lemma 2.2] or [23, Theorem 3, p. 154]). The set $(K \cap \Sigma) \setminus U$ is compact, and so it is covered by finitely many of our balls B_μ from before. Because of this, as in [21], we can find a good cutoff function η_2, with $\|\nabla \eta_2\|_{L^2} \leq \kappa/2$ (if ϵ at the beginning was sufficiently small), such that η_2 is supported in the κ-neighborhood of $(K \cap \Sigma) \setminus U$ and with $\eta_2 = 1$ on a neighborhood of $(K \cap \Sigma) \setminus U$. Then $\eta = 1 - (1 - \eta_1)(1 - \eta_2)$ gives the required cutoff function. □

We next prove a result essentially contained in Cheeger-Colding-Minicozzi [7], that we used in the proof of Proposition 4.1.

Proposition 5.2. *Let (V, o) denote a tangent cone of a non-collapsed limit space of manifolds with Ricci curvature bounded from below. Suppose that there are k linearly independent harmonic functions $u^1, \ldots, u^k$ on V that are homogeneous of degree one. Then we have a splitting $V = \mathbb{R}^k \times Y$.*

Proof. By assumption we have a sequence $B(p_i, 2)$ of balls in Riemannian manifolds with $\mathrm{Ric} > -i^{-1}$, such that $B(p_i, 2) \to B(o, 2)$ in the Gromov-Hausdorff sense. We will prove the following: for any $\delta > 0$, we can find an $r > 0$ and δ-splitting maps $u_i : B(p_i, r) \to \mathbb{R}^k$ for sufficiently large i. Since V is a cone, after scaling up by r^{-1} and taking a diagonal sequence, we find a sequence $B(p_i', 1) \to B(o, 1)$ such that each $B(p_i', 1)$ admits an i^{-1}-splitting map. From this it follows that $B(o, 1/2)$ splits an isometric factor of $\mathbb{R}^k$. For this, see Cheeger-Colding [3], or Cheeger-Naber [11, Definition 1.20, Lemma 1.21].

Before we begin let us recall the notion of a δ-splitting map. A map $u = (u^1, \ldots, u^k) : B(p, r) \to \mathbb{R}^k$ is a δ-splitting map, if it is harmonic, and satisfies

(1) $|\nabla u| < 1 + \delta$,
(2) $⨍_{B_r(p)} |\langle \nabla u^\alpha, \nabla u^\beta\rangle - \delta^{\alpha\beta}|^2 < \delta^2$,
(3) $r^2 ⨍_{B_r(p)} |\nabla^2 u^\alpha|^2 < \delta^2$.

Consider again our sequence $B(p_i, 2) \to B(o, 2)$. We can assume that

$$⨍_{B(o,2)} \langle \nabla u^\alpha, \nabla u^\beta \rangle = \delta^{\alpha\beta},$$

and since the u^α are homogeneous, this implies that for all r we have

$$⨍_{B(o,r)} \langle \nabla u^\alpha, \nabla u^\beta \rangle = \delta^{\alpha\beta}.$$

We can find a sequence of harmonic functions u_i^α on $B(p_i, 2)$ such that under the Gromov-Hausdorff convergence we have $u_i^\alpha \to u^\alpha$ uniformly on each compact set, and moreover for any $0 < r < 2$,

$$\lim_{i\to\infty} ⨍_{B(p_i,r)} \langle \nabla u_i^\alpha, \nabla u_i^\beta \rangle = \delta^{\alpha\beta}. \tag{5.2}$$

Let f_i denote a harmonic function of the form u_i^α or $\frac{1}{\sqrt{2}}(u_i^\alpha \pm u_i^\beta)$ for $\alpha \neq \beta$. By the Bochner formula $\Delta|\nabla f_i|^2 \geq -\Psi(i^{-1})|\nabla f_i|^2$, and so by the mean value inequality, for sufficiently large i we have

$$\sup_{B(p_i,1.5)} |\nabla f_i|^2 \leq C.$$

It follows that for any $0 < r < 1.5$ we have

$$\lim_{i\to\infty} ⨍_{B(p_i,r)} |\nabla f_i|^2 = 1. \tag{5.3}$$

As the gradient of f_i is uniformly bounded, we find the above convergence is uniform on the interval $a < r < 1$, where $a > 0$ is any constant.

Note that $\sup_{B(p_i,r)} |\nabla f_i|^2 \geq 1/2$, and so for large i

$$\sup_{B(p_i,1)} |\nabla f_i|^2 \leq 2C \sup_{B(p_i,r)} |\nabla f_i|^2.$$

Given $\epsilon > 0$, we can then choose $r_0 > 0$ depending on ϵ, C, such that for all sufficiently large i there is some $r \in (r_0, 1/10)$, perhaps depending on i, satisfying

$$\sup_{B(p_i,3r)} |\nabla f_i|^2 \leq (1-\epsilon)^{-1} \sup_{B(p_i,r)} |\nabla f_i|^2.$$

Consider now the functions $v_i = \sup_{B(p_i,3r)} |\nabla f_i|^2 - |\nabla f_i|^2$. Then on $B(p_i, 3r)$, $v_i \geq 0$,

$$\Delta v_i \leq \Psi(i^{-1}) \sup_{B(p_i,3r)} |\nabla f_i|^2 = \Psi(i^{-1}),$$

and

$$\inf_{B(p_i,r)} v_i = \sup_{B(p_i,3r)} |\nabla f_i|^2 - \sup_{B(p_i,r)} |\nabla f_i|^2 \leq \epsilon \sup_{B(p_i,3r)} |\nabla f_i|^2.$$

From the weak Harnack inequality, once i is sufficiently large,

$$\fint_{B(p_i,2r)} v_i \leq C(\Psi(i^{-1}) + \epsilon \sup_{B(p_i,3r)} |\nabla f_i|^2) \leq 2\epsilon C \sup_{B(p_i,3r)} |\nabla f_i|^2.$$

This implies

$$(1 - 2C\epsilon) \sup_{B(p_i,3r)} |\nabla f_i|^2 \leq \fint_{B(p_i,2r)} |\nabla f_i|^2,$$

where C depends only on the non-collapsing constant, through the Sobolev inequality. Recall r (depending on i) has a lower bound r_0, and so by (5.3),

$$\lim_{i\to\infty} \fint_{B(p_i,2r)} |\nabla f_i|^2 = 1.$$

It follows that for sufficiently large i we have

$$\sup_{B(p_i,2r_0)} |\nabla f_i|^2 \leq \sup_{B(p_i,3r)} |\nabla f_i|^2 \leq 1 + C\epsilon.$$

Therefore,

$$\fint_{B(p_i,2r_0)} ||\nabla f_i|^2 - 1| < \Psi(\epsilon).$$

We now apply this to $f_i = u_i^\alpha$, and to $f_i = \frac{1}{\sqrt{2}}(u_i^\alpha \pm u_i^\beta)$, and use the polarization identity

$$\frac{1}{2}|\nabla(u_i^\alpha + u_i^\beta)|^2 - \frac{1}{2}|\nabla(u_i^\alpha - u_i^\beta)|^2 = 2\langle \nabla u_i^\alpha, \nabla u_i^\beta \rangle.$$

Using also (5.2) we find that for sufficiently large i (depending on ϵ), we have

$$\fint_{B(p_i,2r_0)} \left| \langle \nabla u_i^\alpha, \nabla u_i^\beta \rangle - \delta^{\alpha\beta} \right| < \Psi(\epsilon).$$

Since at the same time $|\nabla u_i^\alpha|^2 \leq 1 + \Psi(\epsilon)$, we can use the Bochner formula, using a cutoff function ϕ as in Cheeger-Colding [3] supported in $B(p_i, 2r_0)$, equal to 1 in $B(p_i, r_0)$. We find that for sufficiently large i,

$$\begin{aligned}\fint_{B(p_i,r_0)} |\nabla^2 u_i^\alpha|^2 &\leq \fint_{B(p_i,2r_0)} \frac{1}{2}\phi\Delta(|\nabla u_i^\alpha|^2 - 1) - \Psi(1/i) \fint_{B(p_i,2r_0)} \phi |\nabla u_i^\alpha|^2 \\ &\leq C r_0^{-2} \fint_{B(p_i,2r_0)} \left| |\nabla u_i^\alpha|^2 - 1 \right| - \Psi(1/i) \\ &\leq r_0^{-2}\Psi(\epsilon).\end{aligned}$$

If ϵ is chosen sufficiently small, depending on $\delta > 0$, then this shows that $u_i = (u_i^1, \dots, u_i^k)$ is a δ-splitting map on $B(p_i, r_0)$, for sufficiently large i. □

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