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GROSS SUBSTITUTES CONDITION AND DISCRETE CONCAVITY FOR MULTI-UNIT VALUATIONS: A SURVEY

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Abstract Efficient allocation of indivisible goods is an important problem in mathematical economics and operations research, where the concept of Walrasian equilibrium plays a fundamental role. As a sufficient condition for the existence of a Walrasian equilibrium, the concept of gross substitutes condition for valuation functions is introduced by Kelso and Crawford (1982). Since then, several variants of gross substitutes condition as well as a discrete concavity concept, called M$^\natural$-concavity, have been introduced to show the existence of an equilibrium in various models. In this paper, we survey the relationship among Kelso and Crawford's gross substitutes condition and its variants, and discuss the connection with M$^\natural$-concavity. We also review various characterizations and properties of these concepts.

Keywords: Discrete optimization, gross substitutes condition, discrete convexity, equilibrium, auction, allocation

1. Introduction

The problem of efficiently allocating indivisible (or discrete) goods is one of the main research topics in economics and operations research. Auction with multiple differentiated goods is a typical example of efficient allocation of goods. In recent years, there has been a growing use of auctions for goods such as spectrum licenses in telecommunication, electrical power, landing slots at airports, etc. A fundamental concept in the allocation problem is Walrasian equilibrium, which is a pair of an allocation of goods and a price vector satisfying a certain property.

Walrasian equilibrium

We explain the concept of Walrasian equilibrium using the auction market model. Let us consider an auction market with n types of goods, denoted by $N = \{1, 2, \ldots, n\}$, which are to be allocated to m buyers (or bidders, consumers, etc.). We have $u(i) \in \mathbb{Z}_+$ units available for each good $i \in N$. We denote the integer interval as $[\mathbf{0}, u]_\mathbb{Z} = \{x \in \mathbb{Z}^n \mid \mathbf{0} \le x \le u\}$, where $\mathbf{0} \le x \le u$ denotes the component-wise inequalities. Each vector $x \in [\mathbf{0}, u]_\mathbb{Z}$ is called a *bundle*; a bundle x corresponds to a (multi)-set of goods, where $x(i)$ represents the multiplicity of good $i \in N$. Each buyer b has his valuation function $f_b : [\mathbf{0}, u]_\mathbb{Z} \to \mathbb{R}$; the value $f_b(x)$ represents the degree of satisfaction for a bundle x in monetary terms.

Given a price vector $p \in \mathbb{R}^n$, each buyer b wants to have a bundle x which maximizes the value $f_b(x) - p^\top x$. For each buyer b and $p \in \mathbb{R}^n$, define

$$D(f_b, p) \quad = \quad \arg\max\{f_b(x) - p^\top x \mid x \in [\mathbf{0}, u]_\mathbb{Z}\},$$

which is called the *demand correspondence*. On the other hand, the auctioneer wants to find a price vector under which all items are sold. Hence, all of the auctioneer and buyers are happy if we can find a pair of a set of vectors $x_1^*, x_2^*, \ldots, x_m^*$ and a price vector p^* satisfying the conditions that $x_b^* \in D(f_b, p^*)$ for each buyer b and $\sum_{b=1}^{m} x_b^* = u$. Such a pair is called a *Walrasian equilibrium* (also called a *competitive equilibrium*).

Gross substitutes condition

Although a Walrasian equilibrium possesses a variety of desirable properties as an allocation of goods, it does not always exist. Kelso and Crawford [21] showed that in the case where each good is available by *single unit* (i.e., $u = \mathbf{1}$), a Walrasian equilibrium always exists under a natural assumption on buyers' valuation functions, called gross substitutes condition. We say that a valuation function $f = f_b$ satisfies *gross substitutes condition* if it satisfies the following:

(GS) $\forall p, q \in \mathbb{R}^n$ with $p \leq q$, $\forall x \in D(f, p)$, $\exists y \in D(f, q)$:

$$y(i) \geq x(i) \quad (\forall i \in N \text{ with } q(i) = p(i)).$$

Intuitively, the condition (GS) means that a buyer still wants to get items that do not change in price after the prices of other items increase. Since the concept of the gross substitutes condition is introduced by Kelso and Crawford [21], this condition has been widely used in various models such as matching, housing, and labor markets (see, e.g., [2, 5–7, 16, 17, 23]) and has become a benchmark condition in auction design. Kelso and Crawford [21] also proposed an auction procedure (i.e., an algorithm) to compute an equilibrium (see also [18]). Gul and Stacchetti [17] proposed two conditions equivalent to the condition (GS), and showed that (GS) is also "necessary" condition for the existence of an equilibrium in the sense that if one of buyers' valuation functions violates the condition (GS), then an equilibrium does not exist in general. The condition (GS), however, is not sufficient for the existence of an equilibrium in the *multi-unit* case where multiple units are available for each type of goods (see Section 8 for such an example).

In the multi-unit case, Danilov, Koshevoy, and Murota [9] made an attempt to obtain a general sufficient condition for the existence of an equilibrium corresponding to the concept of concavity for *divisible* goods, and showed the existence of an equilibrium by assuming a concept of discrete concavity for valuation functions, called *M$^\natural$-concavity**.

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be *M$^\natural$-concave* if it satisfies the following exchange property:

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j),$$

where $\mathrm{dom}\, f = \{x \in \mathbb{Z}^n \mid f(x) > -\infty\}$, $\mathrm{supp}^+(x) = \{i \in N \mid x(i) > 0\}$, $\mathrm{supp}^-(x) = \{i \in N \mid x(i) < 0\}$ for $x \in \mathbb{Z}^n$, χ_i is the characteristic vector of $i \in N$, and $\chi_0 = \mathbf{0}$. The concept of M$^\natural$-concavity is introduced by Murota and Shioura [29] as a variant of M-concavity by Murota [25], and plays a central role in the theory of discrete convex analysis, which is a theory of discrete convexity/concavity for functions on integer lattice points (see [27, 28]).

Since then the relationship between M$^\natural$-concavity and the gross substitutes conditions has been discussed in the literature. The first important result, given by Fujishige and Yang [16], is the equivalence between M$^\natural$-concavity and (GS) for valuation functions on $\{0, 1\}^n$ (i.e., single-unit case). Then, Murota and Tamura [35] and Danilov, Koshevoy, and Lang [8] independently investigated the connection between M$^\natural$-concavity and the gross substitutes condition; in particular, stronger variants of gross substitutes condition are introduced in

*An M$^\natural$-concave function is called $\mathcal{IGP}$-concave function in [9] and PM-function in [8], while they are equivalent concepts.

[35] and in [8] to characterize M$^\natural$-concavity (labeled as (PRJ-GS) and (SWGS), respectively, in this paper; see Section 4). Another stronger variant of gross substitutes condition is also considered by Milgrom and Strulovici [24] which is obtained by regarding every unit of goods as distinct goods; the condition is labeled as (SS) in Section 4.

Aim of this paper

In this paper, we survey the relationship among Kelso and Crawford's gross substitutes condition (GS) and other variants of gross substitutes condition by Murota and Tamura [35], Danilov, Koshevoy, and Lang [8], and Milgrom and Strulovici [24]. Especially, our focus is mainly on *multi-unit valuation functions* (i.e., functions on integer lattice points) which appear in several recent papers [8, 15, 24, 35], while *single-unit valuation functions* (i.e., functions on 0-1 vectors), which are often dealt with in the literature, are also considered as an important special case. We investigate the relationship among gross substitutes conditions from the viewpoint of discrete convex analysis. In particular, we reveal the connection of gross substitutes conditions with M$^\natural$-concavity, and clarify polyhedral structure of gross substitutes valuation functions. We also explain the connection with other conditions such as the single improvement condition and submodularity, and present characterizations in terms of indirect utility function. For most of theorems and propositions shown in this paper, we provide rigorous proofs from the viewpoint of discrete convex analysis for a better understanding of the connection between gross substitutes conditions and discrete concavity.

The organization of this paper is as follows. In Section 2, we present various examples of valuation functions which satisfy M$^\natural$-concavity, most of which often appear in mathematical economics. It turns out from the results shown in Section 4 that these functions are also examples of gross substitutes valuation functions.

In Section 3, we explain some definitions and notation necessary in this paper. We also review some fundamental results in (discrete) convex analysis which are relevant to this paper. Readers may skip this section and refer to it when necessary.

The main results in this paper are given in Section 4, where we present variants of gross substitutes condition and their connection with M$^\natural$-concavity. In addition to the gross substitutes conditions (GS), (PRJ-GS), (SWGS), and (SS) mentioned above, we also consider the following simple condition which is obtained by adding to (GS) an extra inequality:

(GS&LAD) $\forall p, q \in \mathbb{R}^n$ with $p \le q$, $\forall x \in D(f, p)$, $\exists y \in D(f, q)$:

$$y(i) \ge x(i) \quad (\forall i \in N \text{ with } q(i) = p(i)), \qquad \sum_{i=1}^{n} y(i) \le \sum_{i=1}^{n} x(i).$$

The extra inequality $\sum_{i \in N} y(i) \le \sum_{i \in N} x(i)$ means that if prices are increased, then a buyer wants less items than before. We also give characterizations of gross substitutes valuation functions in terms of the polyhedral structure of their concave closures. In this and following sections, we put all proofs at the end of each section so that readers not interested in proofs can skip them.

In Section 5, we consider the single improvement condition introduced by Gul and Stacchetti [17] and its stronger variant, and discuss the relationship with gross substitutes conditions.

Submodularity for valuation functions is recognized as an important property in mathematical economics. In Section 6, we show that gross substitutes conditions imply submodularity, and also provide a characterization of gross substitutes conditions using submodularity to clarify the difference between two properties.

Given a valuation function and a price vector for goods, the indirect utility function is defined as the maximum value of the valuation of a bundle of goods minus the bundle's total price. The indirect utility function represents the structure of the corresponding valuation function from the dual viewpoint (i.e., viewpoint of prices). In Section 7, we investigate the structure of gross substitutes valuation functions through indirect utility functions, and give some characterizations.

As an application of the results presented in this paper, we discuss the existence and computation of a Walrasian equilibrium in Section 8. We show that the assumption of gross substitutes conditions implies the existence of an equilibrium and makes it possible to compute an equilibrium by a simple algorithm.

In Appendix, we give rigorous proofs of the theorems on characterizations of supermodularity and submodularity of polyhedral convex functions given in Section 3.3 for readers' convenience.

2. Examples of Valuation Functions

We present various examples of valuation functions which satisfy M$^\natural$-concavity (hence they also satisfy gross substitutes conditions such as (GS) and (GS&LAD)). See [27, Chapter 6] and [28] for more examples of M$^\natural$-concave valuation functions.

The first three examples below show some simple classes of M$^\natural$-concave valuation functions which often appear in mathematical economics, while more complex classes of M$^\natural$-concave valuation functions are presented in the latter four examples.

Example 2.1 (additive (linear) valuations)**.** An *additive* valuation function is defined as a function

$$f(x) = a^\top x = \sum_{i \in N : x(i)=1} a(i) \qquad (x \in \{0,1\}^n),$$

where $a \in \mathbb{R}^n_+$ is a vector representing the value or price of each good in N. More generally, we consider a linear function $f(x) = a^\top x$ defined on an integer interval $[\mathbf{0}, u]_{\mathbb{Z}}$. Every linear function is an M$^\natural$-concave function. □

Remark 2.2. Budgeted version of additive valuations often appears in the literature. A *budgeted additive* (or *budget additive*) valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is given as

$$f(x) = \begin{cases} a^\top x & (\text{if } x \in \{0,1\}^n \text{ and } a^\top x \le B), \\ -\infty & (\text{otherwise}), \end{cases}$$

where $a \in \mathbb{R}^n_+$ and $B \in \mathbb{R}_+$ is a real number representing the budget of a buyer. A budgeted additive valuation is not M$^\natural$-concave in general, while it is M$^\natural$-concave if all values $a(i)$ are the same. □

Example 2.3 (unit-demand valuations)**.** Suppose that a buyer wants to buy at most one good. Then, it is natural to consider a valuation function defined as

$$f(x) = \max\{a(i) \mid i \in N,\ x(i) = 1\} \qquad (x \in \{0,1\}^n),$$

where $a \in \mathbb{R}^n_+$. Such a function is called a *unit-demand* valuation function. Instead, we may consider the following valuation function, where a function takes a finite value at the zero vector or any unit vector:

$$g(x) = \begin{cases} 0 & (\text{if } x = \mathbf{0}), \\ a(i) & (\text{if } x = \chi_i \text{ for some } i \in N), \\ -\infty & (\text{otherwise}). \end{cases}$$

Both of functions f and g are M$^\natural$-concave. □

Remark 2.4. Suppose that a buyer is "single-minded," i.e., the buyer wants to buy a specific set $S^* \subseteq N$ of goods, and finds no value for any other sets. In such a case, we often use a valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given as

$$\operatorname{dom} f = \{0,1\}^n, \qquad f(x) = \begin{cases} \alpha & (\text{if } x = \chi_S \text{ for some } S \supseteq S^*), \\ 0 & (\text{otherwise}), \end{cases}$$

where $\alpha > 0$ and χ_S is the characteristic vector of $S \subseteq N$. This function is called a *single-minded* valuation function; single-minded valuation functions are not M$^\natural$-concave in general.

Instead, we may consider a valuation function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\operatorname{dom} g \subseteq \{0,1\}^n, \qquad g(x) = \begin{cases} \alpha & (\text{if } x = \chi_S \text{ for some } S \supseteq S^*), \\ -\infty & (\text{otherwise}), \end{cases}$$

with $\alpha > 0$. This function satisfies M$^\natural$-concavity. □

Example 2.5 (symmetric concave valuations)**.** A valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with $\operatorname{dom} f = \{0,1\}^n$ is called *symmetric* if there exists a univariate function $\varphi : \mathbb{Z} \to \mathbb{R}$ such that $f(x) = \varphi(\sum_{i=1}^n x(i))$. Such f is called a *symmetric concave* valuation function if φ is a concave function, in addition. Every symmetric concave valuation function is M$^\natural$-concave. □

Example 2.6 (laminar concave functions)**.** Let $\mathcal{T} \subseteq 2^N$ be a laminar family, i.e., $X \cap Y = \emptyset$ or $X \subseteq Y$ or $X \supseteq Y$ holds for every $X, Y \in \mathcal{T}$. For $Y \in \mathcal{T}$, let $\varphi_Y : \mathbb{Z}_+ \to \mathbb{R}$ be a univariate concave function. Define a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ by

$$\operatorname{dom} f = [\mathbf{0}, u]_{\mathbb{Z}}, \qquad f(x) = \sum_{Y \in \mathcal{T}} \varphi_Y\left(\sum_{i \in Y} x(i)\right) \qquad (x \in [\mathbf{0}, u]_{\mathbb{Z}}),$$

which is called a *laminar concave function* [27] (also called *S-valuation* in the preprint version of [5]). Every laminar concave function is an M$^\natural$-concave function. □

Example 2.7 (Maximum-weight bipartite matching and its extension)**.** Consider a complete bipartite graph G with two vertex sets N and J, where N and J correspond to workers and jobs, respectively. We assume that for every $(i, v) \in N \times J$, profit $w(i, v) \in \mathbb{R}_+$ can be obtained by assigning worker i to job v. Consider a matching $M \subseteq N \times J$ between workers and jobs which maximizes the total profit. Define $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ by

$$\begin{aligned} &\operatorname{dom} f = \{0,1\}^n, \\ &f(\chi_X) = \max\{\sum_{(i,v) \in M} w(i,v) \mid M : \text{matching in } G \text{ s.t. } \partial_N M = X\} \quad (X \subseteq N), \end{aligned}$$

where $\partial_N M$ denotes the set of vertices in N covered by edges in M. Then, f is an M$^\natural$-concave function. Such f is called an *OXS valuation* in [23].

We consider a more general setting where each i corresponds to a *type* of workers and there are $u(i) \in \mathbb{Z}_+$ workers of type i. In a similar way as above, we can define a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ by

$$\begin{aligned} &\operatorname{dom} f = [\mathbf{0}, u]_{\mathbb{Z}}, \\ &f(x) = \max\{\sum_{i \in N} \sum_{v \in J} w(i,v) a(i,v) \mid a : N \times J \to \mathbb{Z}_+ \\ &\qquad\qquad \text{s.t. } \sum_{v \in J} a(i,v) = x(i) \ (\forall i \in N)\} \quad (x \in [\mathbf{0}, u]_{\mathbb{Z}}). \end{aligned}$$

This f is an M$^\natural$-concave function.

A much more general example of M$^\natural$-concave functions can be obtained from the maximum-weight network flow problem (see [27]). $\square$

Example 2.8 (quadratic functions)**.** Let $A = (a(i,k) \mid i,k \in N) \in \mathbb{R}^{n\times n}$ be a symmetric matrix, i.e., $a(i,k) = a(k,i)$ for $i,k \in N$. A quadratic function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\mathrm{dom}\, f = [\mathbf{0}, u]_{\mathbb{Z}}, \qquad f(x) = \sum_{i\in N}\sum_{k\in N} a(i,k)x(i)x(k) \quad (x \in [\mathbf{0}, u]_{\mathbb{Z}})$$

is M$^\natural$-concave if the matrix A satisfies the following condition:

$$a(i,k) \le 0 \ (\forall i,k \in N), \quad a(i,k) \le \max\{a(i,\ell), a(k,\ell)\} \text{ if } \{i,k\} \cap \{\ell\} = \emptyset.$$

In particular, a quadratic function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\mathrm{dom}\, f = [\mathbf{0}, u]_{\mathbb{Z}}, \qquad f(x) = \sum_{i\in N} a(i)x(i)^2 + b\sum_{i<k} x(i)x(k) \quad (x \in [\mathbf{0}, u]_{\mathbb{Z}})$$

with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ is M$^\natural$-concave if $0 \ge b \ge 2\max_{i\in N} a(i)$ [27]. $\square$

Example 2.9 (weighted rank functions)**.** Let $\mathcal{I} \subseteq 2^N$ be the family of independent sets of a matroid, and $w \in \mathbb{R}^n_+$. Define a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\mathrm{dom}\, f = \{0,1\}^n, \qquad f(\chi_X) = \max\{\sum_{i\in Y} w(i) \mid Y \subseteq X,\ Y \in \mathcal{I}\} \quad (X \subseteq N),$$

which is called the *weighted rank function.* If $w(i) = 1$ $(i \in N)$, then f is the ordinary rank function of matroid $(N, \mathcal{I})$. Every weighted rank function is M$^\natural$-concave [28]. $\square$

3. Preliminaries

In this section we explain the definitions of various concepts and notation used in this paper, and review some fundamental results in (discrete) convex analysis which are relevant to this paper. Readers may skip this section and refer to it when necessary.

3.1. Definitions and notation

We denote by $\mathbb{R}$ the set of reals and by $\mathbb{Z}$ the set of integers. Also, denote by $\mathbb{R}_+$ the set of nonnegative reals and by $\mathbb{Z}_+$ the set of nonnegative integers. Throughout this paper, let n be a positive integer and denote $N = \{1, 2, \ldots, n\}$. The characteristic vector of a subset $X \subseteq N$ is denoted by χ_X $(\in \{0,1\}^n)$, i.e.,

$$\chi_X(i) = \begin{cases} 1 & (\text{if } i \in X), \\ 0 & (\text{otherwise}). \end{cases}$$

In particular, we use the notation $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_N$, and $\chi_i = \chi_{\{i\}}$ for $i \in N$. We also denote $\chi_0 = \mathbf{0}$.

Let $x = (x(i) \mid i \in N) \in \mathbb{R}^n$ be a vector. We define

$$\mathrm{supp}^+(x) = \{i \in N \mid x(i) > 0\}, \quad \mathrm{supp}^-(x) = \{i \in N \mid x(i) < 0\},$$
$$\|x\|_1 = \sum_{i\in N} |x(i)|, \qquad x(Y) = \sum_{i\in Y} x(i) \quad (Y \subseteq N).$$

For vectors $x, y \in \mathbb{R}^n$, the inequality $x \le y$ denotes the component-wise inequality, i.e., it means that $x(i) \le y(i)$ for every $i \in N$.

For vectors $\ell, u \in \mathbb{R}^n$ with $\ell \leq u$, we define the interval $[\ell, u]$ $(\subseteq \mathbb{R}^n)$ by

$$[\ell, u] = \{x \in \mathbb{R}^n \mid \ell(i) \leq x(i) \leq u(i)\ (\forall i \in N)\},$$

and the integer interval $[\ell, u]_{\mathbb{Z}}$ $(\subseteq \mathbb{Z}^n)$ by $[\ell, u]_{\mathbb{Z}} = [\ell, u] \cap \mathbb{Z}^n$. For any set $S \subseteq \mathbb{R}^n$, the *convex closure* of S, denoted by $\mathrm{conv}(S)$, is the smallest closed convex set containing S.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function defined on $\mathbb{R}^n$. The *effective domain* of f, denoted by $\mathrm{dom}\, f$, is the set

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n \mid -\infty < f(x) < +\infty\}.$$

The sets of maximizers and minimizers, denoted by $\arg\max f$ and $\arg\min f$, respectively, are defined by

$$\arg\max f = \{x \in \mathbb{R}^n \mid f(x) \geq f(y)\ (\forall y \in \mathbb{R}^n)\},$$
$$\arg\min f = \{x \in \mathbb{R}^n \mid f(x) \leq f(y)\ (\forall y \in \mathbb{R}^n)\};$$

$\arg\max f$ and $\arg\min f$ can be empty sets. For a vector $p \in \mathbb{R}^n$, we define the function $f[p] : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ by

$$f[p](x) = f(x) + p^{\top} x \qquad (x \in \mathbb{R}^n).$$

For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined on the integer lattice points $\mathbb{Z}^n$, we also define $\mathrm{dom}\, f$, $\arg\max f$, $\arg\min f$, and $f[p]$ in a similar way.

For vectors $x, y \in \mathbb{R}^n$, denote by $x \wedge y, x \vee y \in \mathbb{R}^n$ the vectors such that

$$(x \wedge y)(i) = \min\{x(i), y(i)\}, \quad (x \vee y)(i) = \max\{x(i), y(i)\} \qquad (i \in N).$$

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (or $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$) is said to be *supermodular* if it satisfies the following *supermodular inequality*:

$$f(x) + f(x) \leq f(x \vee y) + f(x \wedge y) \qquad (\forall x, y \in \mathbb{R}^n),$$

where we admit the inequality $+\infty \leq +\infty$ (or $-\infty \leq -\infty$). Similarly, a function f is said to be *submodular* if it satisfies the following *submodular inequality*:

$$f(x) + f(x) \geq f(x \vee y) + f(x \wedge y) \qquad (\forall x, y \in \mathbb{R}^n).$$

That is, f is submodular if and only if $-f$ is supermodular. We also define supermodularity and submodularity for functions $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ (or $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$) defined on the integer lattice points in a similar way.

3.2. Basic concepts in convex analysis

We explain basic concepts in convex analysis and show some properties of polyhedral convex functions which are relevant to this paper; see [38] for more accounts. While we consider functions defined on $\mathbb{R}^n$ in this section, some of the concepts explained below naturally extend to functions defined on $\mathbb{Z}^n$ by regarding them as functions f with $\mathrm{dom}\, f \subseteq \mathbb{Z}^n$.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function. The *epigraph* of f is the set given as

$$\{(x, \alpha) \mid x \in \mathbb{R}^n,\ \alpha \in \mathbb{R},\ \alpha \geq f(x)\}\ (\subseteq \mathbb{R}^n \times \mathbb{R}).$$

We say that f is *convex* if its epigraph is a convex set in $\mathbb{R}^n \times \mathbb{R}$. A function f is called *concave* if $-f$ is convex. When $f > -\infty$, f is convex if and only if

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \quad (\forall x, y \in \mathbb{R}^n,\ \forall \alpha \in [0, 1]).$$

A convex function is said to be *polyhedral convex* if its epigraph is a polyhedron in $\mathbb{R}^n \times \mathbb{R}$. A concave function f is called *polyhedral concave* if $-f$ is polyhedral convex.

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, the *convex conjugate* $f^\bullet : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and the *concave conjugate* $f^\circ : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ of f are defined by

$$f^\bullet(p) = \sup_{x \in \mathbb{R}^n} \{p^\top x - f(x)\} \qquad (p \in \mathbb{R}^n),$$
$$f^\circ(p) = \inf_{x \in \mathbb{R}^n} \{p^\top x - f(x)\} \qquad (p \in \mathbb{R}^n).$$

Note that

$$f^\circ(p) = -\sup_{x \in \mathbb{R}^n} \{(-p)^\top x - (-f(x))\} = -(-f)^\bullet(-p) \qquad (\forall p \in \mathbb{R}^n).$$

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, the *convex closure* $\check{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and the *concave closure* $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ of f are defined by

$$\begin{aligned} \check{f}(x) &= \sup\{p^\top x + \alpha \mid p \in \mathbb{R}^n,\ \alpha \in \mathbb{R},\ p^\top y + \alpha \le f(y)\ (\forall y \in \mathbb{R}^n)\} \quad (x \in \mathbb{R}^n), \\ \bar{f}(x) &= \inf\{p^\top x + \alpha \mid p \in \mathbb{R}^n,\ \alpha \in \mathbb{R},\ p^\top y + \alpha \ge f(y)\ (\forall y \in \mathbb{R}^n)\} \quad (x \in \mathbb{R}^n). \end{aligned} \tag{3.1}$$

Theorem 3.1. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a function with* $\mathrm{dom}\, f \ne \emptyset$.
(i) *If f is a polyhedral convex function, then the convex conjugate function $f^\bullet$ is also a polyhedral convex function with $f^\bullet > -\infty$.*
(ii) *The function $(f^\bullet)^\bullet$ is equal to the convex closure of f, i.e., $(f^\bullet)^\bullet = \check{f}$. In particular, if f is a polyhedral convex function, then $(f^\bullet)^\bullet = f$.*

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. For any $x \in \mathrm{dom}\, f$ and $d \in \mathbb{R}^n$, we define the *directional derivative* of f at x with respect to d by

$$f'(x; d) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha}$$

if the limit exists. Note that if f is polyhedral convex or polyhedral concave, then $f'(x; d)$ is well defined for every $x \in \mathrm{dom}\, f$ and $d \in \mathbb{R}^n$ by $f'(x; d) = \{f(x + \alpha d) - f(x)\}/\alpha$ with a sufficiently small $\alpha > 0$. The *subdifferential* $\partial f(x)$ of f at x is defined by

$$\partial f(x) = \{p \in \mathbb{R}^n \mid f(y) \ge f(x) + p^\top (y - x)\ (\forall y \in \mathbb{R}^n)\}.$$

Theorem 3.2. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, f \ne \emptyset$. *Then, it holds that*

$$f'(x; d) = \max\{p^\top d \mid p \in \partial f(x)\} \qquad (\forall x \in \mathrm{dom}\, f,\ \forall d \in \mathbb{R}^n).$$

Theorem 3.3. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a polyhedral convex function with* $\mathrm{dom}\, f \ne \emptyset$. *For every $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$, we have*

$$p \in \partial f(x) \iff x \in \arg\min f[-p] \iff x \in \partial f^\bullet(p) \iff p \in \arg\min f^\bullet[-x].$$

For a polyhedral convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, a *linearity domain* of f is a nonempty subset of $\mathrm{dom}\, f$ given as $\arg\min f[-p]$ for some $p \in \mathbb{R}^n$. Similarly, a *linearity domain* of a polyhedral concave function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is a nonempty subset of $\mathrm{dom}\, f$ given as $\arg\max f[-p]$ for some $p \in \mathbb{R}^n$.

3.3. Fundamental results in discrete convex analysis

Discrete convex analysis is a theory of discrete convexity/concavity for functions on integer lattice points (see [25–28]). We review some fundamental results in discrete convex analysis. See [27] for more accounts.

The key concepts in discrete convex analysis are the two kinds of discrete concavity/convexity, called M$^\natural$-concavity and L$^\natural$-concavity. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be *M$^\natural$-concave* if it satisfies the following exchange property:

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \leq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *M$^\natural$-convex* if $-f$ is M$^\natural$-concave.

A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be *L$^\natural$-concave* if it satisfies the following condition:

$$g(p) + g(q) \leq g((p - \lambda \mathbf{1}) \vee q) + g(p \wedge (q + \lambda \mathbf{1})) \qquad (\forall p, q \in \mathbb{Z}^n,\ \forall \lambda \in \mathbb{Z}_+).$$

A function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *L$^\natural$-convex* if $-g$ is L$^\natural$-concave.

It is easy to see from its definition that an L$^\natural$-concave function is a supermodular function, while M$^\natural$-concavity implies submodularity.

Theorem 3.4. *An M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is a submodular function.*

The concepts of M$^\natural$-concavity and L$^\natural$-concavity, originally defined for function on $\mathbb{Z}^n$, are generalized to polyhedral concave function [27, 30]. A polyhedral M$^\natural$-concave function is defined by a condition similar to (M$^\natural$-EXC); see [27, 30] for a precise definition of polyhedral M$^\natural$-concavity. For a polyhedral concave function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, we say that g is a *polyhedral L$^\natural$-concave* if it satisfies the following condition:

$$g(p) + g(q) \leq g((p - \lambda \mathbf{1}) \vee q) + g(p \wedge (q + \lambda \mathbf{1})) \qquad (\forall p, q \in \mathbb{R}^n,\ \forall \lambda \in \mathbb{R}_+).$$

A polyhedral L$^\natural$-concave function g is said to be *integral* if each linearity domain g is an integral polyhedron, i.e., for every $x \in \mathbb{R}^n$, the set $\arg\max g[-x]$ is an integral polyhedron if it is not empty. A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *(integral) polyhedral L$^\natural$-convex* if $-g$ is (integral) polyhedral L$^\natural$-concave.

By the definitions of (polyhedral) L$^\natural$-concavity, the restriction of a polyhedral L$^\natural$-concave function on the integer lattice point $\mathbb{Z}^n$ is an L$^\natural$-concave function on $\mathbb{Z}^n$. Moreover, it is known (see, e.g., [27, Theorem 7.26]) that a function g with bounded $\mathrm{dom}\, g$ is integral polyhedral L$^\natural$-concave if and only if it can be represented as the concave closure of some L$^\natural$-concave function on $\mathbb{Z}^n$.

Maximizers of M$^\natural$-concave and L$^\natural$-concave functions can be characterized by local properties.

Theorem 3.5. *For an M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ and a vector $x^* \in \mathrm{dom}\, f$, we have*

$$f(x^*) \geq f(y)\ (\forall y \in \mathbb{Z}^n) \quad \Longleftrightarrow \quad f(x^*) \geq f(x^* - \chi_i + \chi_j)\ (\forall i, j \in N \cup \{0\}).$$

Theorem 3.6. *For an L$^\natural$-concave function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ and a vector $p^* \in \mathrm{dom}\, g$, we have*

$$g(p^*) \geq g(q)\ (\forall q \in \mathbb{Z}^n) \quad \Longleftrightarrow \quad g(p^*) \geq \max\{g(p^* + \chi_X), g(p^* - \chi_X)\}\ (\forall X \subseteq N).$$

Based on Theorems 3.5 and 3.6, maximization of M$^\natural$-concave and L$^\natural$-concave functions can be solved by the following steepest ascent algorithms.

Algorithm M$^\natural$_STEEPEST_ASCENT_UP
Step 0: Let $x := x_0$ be a vector in $\mathrm{dom}\, f$ such that $x^* \geq x_0$ for some maximizer x^* of f.
Step 1: Find $j \in N \cup \{0\}$ that maximizes $f(x + \chi_j)$.
Step 2: If $f(x + \chi_j) \leq f(x)$, then output x and stop [x is a maximizer of f].
Step 3: Set $x := x + \chi_j$ and go to Step 1.

Algorithm L$^\natural$_STEEPEST_ASCENT_UP
Step 0: Let $p := p_0$ be a vector in $\mathrm{dom}\, g$ such that $p^* \geq p_0$ for some maximizer p^* of g.
Step 1: Find $X \subseteq N$ that maximizes $g(p + \chi_X)$.
Step 2: If $g(p + \chi_X) \leq g(p)$, then output p and stop [p is a maximizer of g].
Step 3: Set $p := p + \chi_X$ and go to Step 1.

Theorem 3.7. *For an M$^\natural$-concave function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with bounded* $\mathrm{dom}\, f$*, the algorithm* M$^\natural$_STEEPEST_ASCENT_UP *finds a maximizer of f in a finite number of iterations.*
Theorem 3.8. *For an L$^\natural$-concave function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with bounded* $\mathrm{dom}\, g$*, the algorithm* L$^\natural$_STEEPEST_ASCENT_UP *finds a maximizer of g in a finite number of iterations.*
Detailed analysis of the number of iterations required by the two algorithms is given in [27, Chapter 10] and [22, 32].

M$^\natural$-concave and L$^\natural$-concave functions can be characterized by the sets of maximizers. A polyhedron $S \subseteq \mathbb{R}^n$ is called a *generalized polymatroid* (*g-polymatroid*, for short)

$$S = \{x \in \mathbb{R}^n \mid \mu(X) \leq x(X) \leq \rho(X) \ (\forall X \subseteq N)\} \tag{3.2}$$

given by a pair of submodular/supermodular functions $\rho : 2^N \to \mathbb{R} \cup \{+\infty\}$, $\mu : 2^N \to \mathbb{R} \cup \{-\infty\}$ satisfying the inequality

$$\rho(X) - \mu(Y) \geq \rho(X \setminus Y) - \mu(Y \setminus X) \qquad (\forall X, Y \subseteq N).$$

It is known that g-polymatroids can be characterized by their edges. Recall that an edge is a 1-dimensional face of a polyhedron.
Theorem 3.9. *A pointed polyhedron is a g-polymatroid if and only if every edge of the polyhedron is parallel to a vector $+\chi_i - \chi_j$ for some distinct $i, j \in N \cup \{0\}$.*

If ρ and μ are integer-valued, then S is an integral polyhedron; in such a case, we say that S is an *integral g-polymatroid.* An *M$^\natural$-convex set* is the set of integral vectors in an integral g-polymatroid.

We say that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is *concave-extensible* if there exists a concave function $\hat{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ such that $\hat{f}(x) = f(x)$ for all integer lattice points $x \in \mathbb{Z}^n$. Hence, a function f is concave-extensible if and only if the concave closure $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ of f given by (3.1) satisfies $f(x) = \bar{f}(x)$ for all $x \in \mathbb{Z}^n$.
Theorem 3.10. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function with bounded* $\mathrm{dom}\, f$*.*
(i) *f is M$^\natural$-concave if and only if* $\arg\max f[-p]$ *is an M$^\natural$-convex set for every $p \in \mathbb{R}^n$.*
(ii) *Suppose that f is concave-extensible. Then, f is M$^\natural$-concave if and only if* $\arg\max \bar{f}[-p]$ *is an integral g-polymatroid for every $p \in \mathbb{R}^n$.*

A polyhedron $S \subseteq \mathbb{R}^n$ is called an *L$^\natural$-convex polyhedron* if it can be represented as

$$S = \{p \in \mathbb{R}^n \mid p^\top(\chi_i - \chi_j) \leq \gamma(i, j) \ (\forall i, j \in N \cup \{0\},\ i \neq j)\}$$

with some values $\gamma(i, j) \in \mathbb{R} \cup \{+\infty\}$ $(i, j \in N \cup \{0\},\ i \neq j)$ (see, e.g., [27, Chapter 5] for the original definition of L$^\natural$-convex polyhedron).

Theorem 3.11. *A polyhedral concave function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is polyhedral $L^\natural$-concave if and only if for every $x \in \mathbb{R}^n$ with $\arg\max g[-x] \neq \emptyset$, the set $\arg\max g[-x]$ is an $L^\natural$-convex polyhedron.*

We consider the maximization of the sum of two M$^\natural$-concave functions and give an optimality criterion. Note that the sum of two M$^\natural$-concave functions is not M$^\natural$-concave in general.

Theorem 3.12. *For $M^\natural$-concave functions $f, h : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ and a vector $x^* \in \mathrm{dom}\, f \cap \mathrm{dom}\, h$, we have*

$$f(x^*) + h(x^*) = \max\{f(x) + h(x) \mid x \in \mathbb{Z}^n\}$$

if and only if there exists some $p^* \in \mathbb{R}^n$ such that

$$f[-p^*](x^*) = \max\{f[-p^*](x) \mid x \in \mathbb{Z}^n\}, \qquad h[p^*](x^*) = \max\{h[p^*](x) \mid x \in \mathbb{Z}^n\}.$$

We then explain the relationship between M$^\natural$-concavity and L$^\natural$-concavity. The concave conjugate of an M$^\natural$-concave function has L$^\natural$-concavity, and this property characterizes M$^\natural$-concavity of a function.

Theorem 3.13. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function with nonempty bounded* $\mathrm{dom}\, f$.
(i) *f is $M^\natural$-concave if and only if its conjugate function $f^\circ : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is polyhedral $L^\natural$-concave.*
(ii) *If f is an integer-valued $M^\natural$-concave function, then f° is an integral polyhedral $L^\natural$-concave function (i.e., a polyhedral $L^\natural$-concave function such that each linearity domain is an integral polyhedron).*

We finally present some theorems on characterizations of supermodularity and submodularity of polyhedral convex functions in terms of linearity domains. It should be noted that $\mathrm{dom}\, f$ is assumed to be full-dimensional in Theorem 3.14, while such an assumption is not required in Theorems 3.15 and 3.16.

Theorem 3.14. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function such that* $\mathrm{dom}\, f$ *is an n-dimensional polyhedron. Then, the following three conditions are equivalent:*
(i) *f is a supermodular function, i.e., $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$ $(\forall x, y \in \mathbb{R}^n)$.*
(ii) *for every linearity domain $S \subseteq \mathbb{R}^n$ of f and $x, y \in S$ with $x \leq y$, we have $[x, y] \subseteq S$.*
(iii) *every linearity domain of f can be represented by a system of inequalities of the form $a^\top x \leq b$ with $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$ satisfying $a \geq \mathbf{0}$ or $a \leq \mathbf{0}$.*

Theorem 3.15. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function such that* $\mathrm{dom}\, f$ *is a pointed polyhedron. Suppose that every 1-dimensional linearity domain of f is parallel to a nonzero vector $v \in \mathbb{R}^n$ with $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$. Then, f is a supermodular function.*

Theorem 3.16. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral convex function. Then, the following three conditions are equivalent:*
(i) *f is a submodular function, i.e., $f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$ $(\forall x, y \in \mathbb{R}^n)$.*
(ii) *every linearity domain $S \subseteq \mathbb{R}^n$ of f satisfies $x \wedge y, x \vee y \in S$ for every $x, y \in S$.*
(iii) *every linearity domain of f can be represented by a system of inequalities of the form $a^\top x \leq b$ with $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$ satisfying $|\mathrm{supp}^+(a)| \leq 1$ and $|\mathrm{supp}^-(a)| \leq 1$.*

These theorems were proven in the paper by Danilov and Lang [10] in Russian, and referred to in the paper by Danilov, Koshevoy, and Lang [8] without proofs. For readers' convenience, we provide rigorous proofs of the theorems in Appendix.

4. Gross Substitutes Conditions and Their Connection with Discrete Concavity

In this section we present several variants of gross substitutes condition and discuss their connection with the concept of M$^\natural$-concavity in discrete convex analysis. We also give characterizations of gross substitutes valuation functions in terms of the polyhedral structure of their concave closures. Proofs of theorems are given at the end of this section.

Throughout this paper, a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is often called a *multi-unit valuation* function, while a function f with $\mathrm{dom}\, f \subseteq \{0, 1\}^n$ is called a *single-unit valuation* function to put emphasis on the fact that only one unit is available for each type of goods.

4.1. Gross substitutes condition and its stronger variants

For a valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ and $p \in \mathbb{R}^n$, we define a set $D(f, p) \subseteq \mathrm{dom}\, f$ by

$$D(f, p) = \arg\max f[-p] = \{x \in \mathbb{Z}^n \mid f[-p](x) \geq f[-p](y)\ (\forall y \in \mathbb{Z}^n)\};$$

recall that $f[-p](x) = f(x) - p^\top x$ $(x \in \mathbb{Z}^n)$. A set $D(f, p)$ is called *demand correspondence* (or a *demand set*).

The original definition of gross substitutes condition by Kelso and Crawford [21] is as follows, where $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is a function:

(GS) $\forall p, q \in \mathbb{R}^n$ with $p \leq q$, $\forall x \in D(f, p)$, $\exists y \in D(f, q)$:

$$y(i) \geq x(i) \quad (\forall i \in N \text{ with } q(i) = p(i)).$$

In addition to (GS), we consider four variants of gross substitutes condition, each of which is stronger than the original condition. The first two conditions (PRJ-GS) and (SWGS), called *projected gross substitutes condition* and *step-wise gross substitutes condition*, are introduced by Murota and Tamura [35] and Danilov, Koshevoy, and Lang [8], respectively.

(PRJ-GS) $\forall p, q \in \mathbb{R}^n$ with $p \leq q$, $\forall p_0, q_0 \in \mathbb{R}$ with $p_0 \leq q_0$,
$\forall x \in D(f, p - p_0\mathbf{1})$, $\exists y \in D(f, q - q_0\mathbf{1})$:

$$y(i) \geq x(i) \quad (\forall i \in N \text{ with } q(i) = p(i)), \qquad y(N) \leq x(N) \text{ if } p_0 = q_0,$$

(SWGS) $\forall p \in \mathbb{R}^n$, $\forall k \in N$, $\forall x \in D(f, p)$, at least one of (i) and (ii) holds true:
(i) $\forall \lambda \in \mathbb{R}_+$: $x \in D(f, p + \lambda\chi_k)$.
(ii) $\exists \lambda \in \mathbb{R}_+$, $\exists y \in D(f, p + \lambda\chi_k)$: $y(k) = x(k) - 1$ and $y(i) \geq x(i)$ $(\forall i \in N \setminus \{k\})$.

It is easy to see that (PRJ-GS) with $p_0 = q_0 = 0$ implies (GS).

The third variant is the following condition introduced by Murota, Shioura, and Yang [33]:

(GS&LAD) $\forall p, q \in \mathbb{R}^n$ with $p \leq q$, $\forall x \in D(f, p)$, $\exists y \in D(f, q)$:

$$y(i) \geq x(i) \quad (\forall i \in N \text{ with } q(i) = p(i)), \qquad y(N) \leq x(N).$$

Note that the condition (GS&LAD) coincides with (PRJ-GS) if $p_0 = q_0 = 0$. The condition (GS&LAD) can also be seen as a combination of (GS) and the condition called the *law of the aggregate demand* (see [24]; see also [19]):

(LAD) $\forall p, q \in \mathbb{R}^n$ with $p \leq q$, $\forall x \in D(f, p)$, $\exists y \in D(f, q)$: $y(N) \leq x(N)$.

The forth variant of gross substitutes condition, called the *strong substitute condition* [24], is defined as follows, where the effective domain of a valuation function is assumed to be contained in the set $\mathbb{Z}^n_+$ of non-negative integral vectors:

a multi-unit valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with $\mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$ satisfies the *strong substitute condition* if, when we regard every unit of every good as a different good, the corresponding (single-unit) valuation function satisfies the original gross substitutes condition (GS).

We give a more rigorous formulation of the strong substitute condition. Let $u \in \mathbb{Z}^n_+$ be a vector such that $\mathrm{dom}\, f \subseteq [\mathbf{0}, u]_{\mathbb{Z}}$. We define a set N^{B} and a single-unit valuation function $f^{\mathrm{B}} : \{0,1\}^{N^{\mathrm{B}}} \to \mathbb{R} \cup \{-\infty\}$ as follows:

$$\begin{aligned} N^{\mathrm{B}} &= \{(i,\beta) \mid i \in N,\ \beta \in \mathbb{Z},\ 1 \le \beta \le u(i)\}, \\ f^{\mathrm{B}}(x^{\mathrm{B}}) &= f(x) \text{ for } x^{\mathrm{B}} \in \{0,1\}^{N^{\mathrm{B}}},\ \text{where } x(i) = \sum_{\beta=1}^{u(i)} x^{\mathrm{B}}(i,\beta) \quad (i \in N). \end{aligned} \tag{4.1}$$

Using the function f^{B}, the strong substitute condition is described as follows:

(SS) The function f^{B} given by (4.1) satisfies the condition (GS).

Finally, recall that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is called M$^\natural$-concave if it satisfies the following condition, where $\chi_0 = \mathbf{0}$:

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \le f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

Note that the condition (M$^\natural$-EXC) is "price-free," i.e., it does not use price vectors p and q.

We show that each of the conditions (PRJ-GS), (SWGS), (GS&LAD), and (SS) is equivalent to M$^\natural$-concavity under some natural assumption. While the condition (GS) is weaker than the other conditions in general, it is also equivalent to the others for single-unit valuation functions.

Recall that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is concave-extensible if and only if the concave closure $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ of f given by (3.1) satisfies $f(x) = \bar{f}(x)$ for all $x \in \mathbb{Z}^n$. We note that every single-unit valuation function f (i.e., $\mathrm{dom}\, f \subseteq \{0,1\}^n$) is concave-extensible.

Theorem 4.1. *Let* $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$.

(i) *Suppose that f is concave-extensible and* $\mathrm{dom}\, f$ *is bounded. Then, the following equivalences hold:*

$$\text{(M}^\natural\text{-EXC)} \iff \text{(PRJ-GS)} \iff \text{(SWGS)} \iff \text{(GS\&LAD)}.$$

Moreover, if $\mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$ *holds in addition, then each of the four conditions is equivalent to the condition* (SS).

(ii) *Suppose that* $\mathrm{dom}\, f \subseteq \{0,1\}^n$. *Then, the following equivalences hold:*

$$\text{(M}^\natural\text{-EXC)} \iff \text{(PRJ-GS)} \iff \text{(SWGS)} \iff \text{(GS\&LAD)} \iff \text{(SS)} \iff \text{(GS)}.$$

We see that each of the five conditions in Theorem 4.1 (i) implies (GS), while the converse does not hold in general, as shown in the following example.

Example 4.2. There exists a multi-unit valuation function which is concave-extensible and satisfies (GS), but satisfies none of (M$^\natural$-EXC), (PRJ-GS), (SWGS), (GS&LAD), and (SS).

Let $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ be a function such that

$$\begin{aligned} &\mathrm{dom}\, f = \{x \in \mathbb{Z}^2 \mid 0 \le x(1) \le 2,\ 0 \le x(2) \le 1\}, \\ &f(0,0) = 0, \quad f(1,0) = 3, \quad f(2,0) = 6, \\ &f(0,1) = 4, \quad f(1,1) = 5, \quad f(2,1) = 6. \end{aligned}$$

It is not difficult to see that f is concave-extensible and satisfies (GS). We also have

$$f(0,1) + f(2,0) = 4 + 6, \qquad f(0,0) + f(2,1) = 0 + 6, \qquad f(1,1) + f(1,0) = 5 + 3,$$

from which $f(0,1)+f(2,0) > \max\{f(0,0)+f(2,1), f(1,1)+f(1,0)\}$ follows. Hence, f does not satisfy the condition (M$^\natural$-EXC) with $x = (0,1)$ and $y = (2,0)$, i.e., f is not M$^\natural$-concave. By this fact and Theorem 4.1 (i), f satisfies none of (PRJ-GS), (SWGS), (GS&LAD), and (SS). □

Based on the theorem and the example above, in the following sections we refer to each of the conditions (PRJ-GS), (SWGS), (GS&LAD), and (SS) as a *strong gross substitutes condition* for multi-unit valuation functions, and (GS) as the *weak gross substitutes condition.*

While M$^\natural$-concavity is a stronger condition than (GS), they are equivalent for some classes of valuation functions. One such class is single-unit valuation functions, as shown in Theorem 4.1 (ii). Another such class is valuation functions $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ satisfying the following property:

$$\text{dom}\, f \text{ is bounded and contained in a hyperplane of the form } x(N) = \lambda \text{ with } \lambda \in \mathbb{Z}. \quad (4.2)$$

Theorem 4.3. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function satisfying the condition* (4.2)*. Then, f is M$^\natural$-concave if and only if it is concave-extensible and satisfies* (GS)*.*

Remark 4.4. The class of M$^\natural$-concave functions satisfying the condition (4.2) coincides with the class of functions called *M-concave functions* (see [26, 27] for the original definition of M-concave functions). Hence, Theorem 4.3 can be regarded as the equivalence between M-concavity and (GS).

The following relationship holds for M-concave and M$^\natural$-concave functions (see, e.g., [27, Section 6.1]):

a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is M$^\natural$-concave if and only if the function $\tilde{f} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & (\text{if } x_0 + x(N) = 0), \\ -\infty & (\text{otherwise}) \end{cases} \qquad ((x_0, x) \in \mathbb{Z} \times \mathbb{Z}^n) \qquad (4.3)$$

is M-concave.

Based on this relationship and the equivalence between M-concavity and (GS) in Theorem 4.3, Murota and Tamura [35] introduced the condition (PRJ-GS) by reformulating the condition (GS) for $\tilde{f}$ in terms of the original function f. □

Remark 4.5. The assumption of concave-extensibility used in Theorem 4.1 (i) is a natural assumption for multi-unit valuation functions since the gross substitutes condition for valuations of indivisible items is a counterpart of concavity condition for valuations of divisible items, and every single-unit valuation function is concave-extensible.

Indeed, either of the conditions (GS), (PRJ-GS), and (GS&LAD) is not enough to obtain a good property of valuation functions for multi-unit valuation functions, without the assumption of concave-extensibility. For example, every univariate function with bounded domain satisfies the conditions (GS), (PRJ-GS), and (GS&LAD); this means that none of (GS), (PRJ-GS), and (GS&LAD) implies a good property for univariate functions. □

4.2. More variants of gross substitutes condition

We also consider the following variant of the condition (GS), where the price vector q in (GS) is restricted to a vector of the form $p + \lambda\chi_k$ with $k \in N$ and $\lambda > 0$:

(GS′) $\forall p \in \mathbb{R}^n$, $\forall k \in N$, $\forall \lambda \in \mathbb{R}_+$, $\forall x \in D(f, p)$, $\exists y \in D(f, p + \lambda\chi_k)$:

$$y(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}).$$

Similarly, we consider the following variant of (GS&LAD):

(GS&LAD′) $\forall p \in \mathbb{R}^n$, $\forall k \in N$, $\forall \lambda \in \mathbb{R}_+$, $\forall x \in D(f,p)$, $\exists y \in D(f, p+\lambda\chi_k)$:

$$y(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}), \qquad y(N) \leq x(N).$$

The following property can be shown easily by a simple inductive argument.

Proposition 4.6. *For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$, we have* (GS) $\iff$ (GS′) *and* (GS&LAD) $\iff$ (GS&LAD′).

We consider another variants of (GS) and (GS&LAD), where the roles of p (resp., x) and q (resp., y) are interchanged:

(GS$_-$) $\forall q, p \in \mathbb{R}^n$ with $q \geq p$, $\forall y \in D(f,q)$, $\exists x \in D(f,p)$:

$$x(i) \leq y(i) \quad (\forall i \in N \text{ with } p(i) = q(i)).$$

(GS&LAD$_-$) $\forall q, p \in \mathbb{R}^n$ with $q \geq p$, $\forall y \in D(f,q)$, $\exists x \in D(f,p)$:

$$x(i) \leq y(i) \quad (\forall i \in N \text{ with } p(i) = q(i)), \qquad x(N) \geq y(N).$$

Theorem 4.7. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function.*
(i) *Suppose that f is concave-extensible and* $\mathrm{dom}\, f$ *is bounded. Then, f satisfies the condition* (GS&LAD$_-$) *if and only if it satisfies* (GS&LAD).
(ii) *Suppose that* $\mathrm{dom}\, f \subseteq \{0,1\}^n$. *Then, f satisfies* (GS$_-$) *if and only if it satisfies* (GS).

For multi-unit valuation functions which are not concave-extensible, the condition (GS$_-$) (resp., (GS&LAD$_-$)) is independent of (GS) (resp., (GS&LAD)) in general, as shown in the following examples.

Example 4.8. We show an example of a function which satisfies (GS$_-$) but not (GS). Let $f : \mathbb{Z}^3 \to \mathbb{R} \cup \{-\infty\}$ be a function given by

$$\begin{aligned} \mathrm{dom}\, f &= \{x \in \mathbb{Z}^3 \mid 0 \leq x(1) \leq 2,\ 0 \leq x(2) \leq 3,\ 0 \leq x(3) \leq 6,\ 3x(1) + 2x(2) + x(3) = 6\} \\ &= \{(2,0,0), (0,3,0), (0,0,6), (1,1,1), (1,0,3), (0,2,2), (0,1,4)\}, \\ f(x) &= 0 \qquad (x \in \mathrm{dom}\, f). \end{aligned}$$

In a brute-force manner, we can check that f satisfies the condition (GS$_-$).

We show that f does not satisfy the condition (GS). For $p = (0,0,0)$ and $q = p + \varepsilon\chi_3$ with $\varepsilon > 0$, we have

$$D(f,p) = \mathrm{dom}\, f, \qquad D(f,q) = \{(2,0,0), (0,3,0)\}.$$

For $x = (1,1,1) \in D(f,p)$, each of the vectors in $D(f,q)$ violates $y(1) \geq x(1)$ or $y(2) \geq x(2)$. That is, (GS) does not hold for f.

On the other hand, the function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by $g(x) = f(-x)$ is an example of a function which satisfies (GS) but not (GS$_-$). □

Example 4.9. We show an example of a function which satisfies (GS&LAD) but not (GS&LAD$_-$). Let $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ be a function given by

$$\mathrm{dom}\, f = \{(0,0), (1,0), (2,0), (3,0), (0,1), (3,1)\}, \qquad f(x) = 0 \ (x \in \mathrm{dom}\, f).$$

Note that $\mathrm{dom}\, f$ is bounded and f is not concave-extensible. It is not difficult to see that f satisfies (GS&LAD).

We show that f does not satisfy the condition (GS&LAD$_-$). For $q = (0,0)$ and $p = q - \varepsilon\chi_2$ with $\varepsilon > 0$, we have

$$D(f,q) = \mathrm{dom}\, f, \qquad D(f,p) = \{(0,1),(3,1)\}.$$

For $y = (2,0) \in D(f,q)$, each of the vectors x in $D(f,p)$ violates $x(1)+x(2) \geq y(1)+y(2)$ or $x(1) \leq y(1)$. That is, (GS&LAD$_-$) does not hold for f.

On the other hand, the function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ given by $g(x) = f(-x)$ is an example of a function which satisfies (GS&LAD$_-$) but not (GS&LAD). $\square$

4.3. Polyhedral structure of gross substitutes valuations

We investigate the polyhedral structure for the concave closure of a gross substitutes valuation function. Note that the concave closure $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ of a valuation function f given by (3.1) is a polyhedral concave function if the effective domain of f is bounded.

Recall that a *linearity domain* $S \subseteq \mathbb{R}^n$ of a polyhedral concave function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is a set given as

$$S = \arg\max g[-p] = \{x \in \mathbb{R}^n \mid g[-p](x) \geq g[-p](y)\ (\forall y \in \mathbb{R}^n)\}$$

for some vector $p \in \mathbb{R}^n$. Consider the following conditions in terms of linearity domains of the concave closure $\bar{f}$:

(1DLD) Every 1-dimensional linearity domain of $\bar{f}$ is parallel to a nonzero vector $+\chi_i - \chi_j$ with $i, j \in N \cup \{0\}$.

(1DLD′) Every 1-dimensional linearity domain of $\bar{f}$ is parallel to a nonzero vector $v \in \mathbb{R}^n$ with $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$.

(LD) Every linearity domain of $\bar{f}$ is a g-polymatroid.

Theorem 4.10. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function, and $\bar{f} : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be its concave closure.*

(i) *Suppose that* $\mathrm{dom}\, f$ *is bounded. Then, the following relations hold:*

$$\begin{array}{ccccc} \text{(GS\&LAD)} & \Longrightarrow & \text{(1DLD)} & \Longleftrightarrow & \text{(LD)} \\ \Downarrow & & \Downarrow & & \\ \text{(GS)} & \Longrightarrow & \text{(1DLD}'\text{)}. & & \end{array}$$

(ii) *Suppose that* $\mathrm{dom}\, f$ *is bounded and f is concave-extensible. Then, the following relations hold:*

$$\begin{array}{ccccccc} \text{(M}^\natural\text{-EXC)} & \Longleftrightarrow & \text{(GS\&LAD)} & \Longleftrightarrow & \text{(1DLD)} & \Longleftrightarrow & \text{(LD)} \\ & & \Downarrow & & \Downarrow & & \\ & & \text{(GS)} & \Longrightarrow & \text{(1DLD}'\text{)}. & & \end{array}$$

(iii) *Suppose that* $\mathrm{dom}\, f \subseteq \{0,1\}^n$. *Then, all the six conditions in* (ii) *are equivalent.*

For multi-unit valuation functions which are not concave-extensible, neither (LD) nor (1DLD) implies (GS&LAD) in general, as shown in the following example.

Example 4.11. We show an example of a multi-unit valuation function which satisfies both of (LD) and (1DLD) but not (GS&LAD).

Let $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ be a function given by

$$\mathrm{dom}\, f = \{(0,0),(3,0),(0,1),(1,1),(2,1),(3,1)\}, \qquad f(x) = 0\ (x \in \mathrm{dom}\, f).$$

It is not difficult to see that f satisfies both of (LD) and (1DLD) since the concave closure $\bar{f}$ of f is given as

$$\bar{f}(x) = \begin{cases} 0 & \text{(if } 0 \leq x(1) \leq 3 \text{ and } 0 \leq x(2) \leq 1\text{)}, \\ -\infty & \text{(otherwise)}. \end{cases}$$

This also shows that f is not concave-extensible since $\bar{f}(1,0) = 0 \neq -\infty = f(1,0)$.

We show that f does not satisfy the condition (GS&LAD). For $p = (0,0)$ and $q = p+\varepsilon\chi_2$ with $\varepsilon > 0$, we have $D(f,p) = \mathrm{dom}\, f$ and $D(f,q) = \{(0,0),(3,0)\}$. For $x = (1,1) \in D(f,p)$, each of $y = (0,0)$ and $y = (3,0)$ violates the condition $y(1) \geq x(1)$ or $y(1) + y(2) \leq x(1) + x(2)$. That is, (GS&LAD) does not hold for f. □

We also show that (1DLD′) implies none of (GS) and (1DLD) for multi-unit valuation functions.

Example 4.12 (Danilov, Koshevoy, and Lang [8, Example 6])**.** We show an example of a multi-unit valuation function which satisfies (1DLD′) but none of (GS) and (1DLD). Let $f : \mathbb{Z}^3 \to \mathbb{R} \cup \{-\infty\}$ be a function given by

$$\mathrm{dom}\, f = [\mathbf{0}, u]_{\mathbb{Z}} \text{ with } u = (2,3,6), \qquad f(x) = \min\{3x(1) + 2x(2) + x(3), 6\} \ (x \in \mathrm{dom}\, f).$$

Note that this function f is nondecreasing and concave-extensible, and the concave closure $\bar{f} : \mathbb{R}^3 \to \mathbb{R} \cup \{-\infty\}$ of f is a function with $\mathrm{dom}\, \bar{f} = [\mathbf{0}, u]$ such that

$$\bar{f}(x) = \min\{3x(1) + 2x(2) + x(3), 6\} \qquad (x \in \mathrm{dom}\, \bar{f}).$$

We see that every 1-dimensional linearity domain of $\bar{f}$ is parallel to one of the vectors in

$$\{\chi_1,\ \chi_2,\ \chi_3,\ 2\chi_1 - 3\chi_2,\ \chi_1 - 3\chi_3,\ \chi_2 - 2\chi_3\}.$$

Note that every vector v in this set satisfies $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$. Therefore, f satisfies (1DLD′) but not (1DLD).

We show that f does not satisfy the condition (GS). For $p = (3/2, 1, 1/2)$ and $q = p+\varepsilon\chi_3$ with $0 < \varepsilon < 1/2$, we have

$$\begin{aligned} D(f,p) &= \{x \in [\mathbf{0}, u]_{\mathbb{Z}} \mid 3x(1) + 2x(2) + x(3) = 6\} \\ &= \{(2,0,0),(0,3,0),(0,0,6),(1,1,1),(1,0,3),(0,2,2),(0,1,4)\}, \\ D(f,q) &= \{(2,0,0),(0,3,0)\}. \end{aligned}$$

For $x = (1,1,1) \in D(f,p)$, none of the vectors in $D(f,q)$ satisfies both of $y(1) \geq x(1)$ and $y(2) \geq x(2)$. That is, (GS) does not hold for f. □

4.4. Proofs

We prove Theorem 4.10 first, and then Theorems 4.3, 4.1, and 4.7 in turn.

4.4.1. Proof of Theorem 4.10 (i)

The implications "(GS&LAD) $\Longrightarrow$ (GS)" and "(1DLD) $\Longrightarrow$ (1DLD′)" are easy to see. The equivalence of (LD) and (1DLD) follows from Theorem 3.9 and the fact that each edge of a linearity domain of $\bar{f}$ is a 1-dimensional linearity domain of $\bar{f}$.

Below we prove the implications "(GS) $\Longrightarrow$ (1DLD′)" and "(GS&LAD) $\Longrightarrow$ (1DLD)."

Proof of "(GS) $\Longrightarrow$ (1DLD′)". Let $S \subseteq \mathbb{R}^n$ be a 1-dimensional linearity domain of $\bar{f}$. Since S is a 1-dimensional bounded polyhedron, there exist two distinct vectors $x, y \in S$ which are extreme points of S. To show (1DLD′), we prove that $v = x - y$ satisfies $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$.

Since $x \neq y$, we may assume that $x(k) > y(k)$ for some $k \in N$. Since $\mathrm{dom}\, f$ is bounded, there exists some $p \in \mathbb{R}^n$ and some sufficiently small $\delta > 0$ such that $D(f,p) = \{x\}$ and $D(f, p + \delta\chi_k) = \{y\}$. The condition (GS) applied to the price vectors p, $p + \delta\chi_k$, and the vector $x \in D(f,p)$ implies that

$$y(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}). \tag{4.4}$$

If $\mathrm{supp}^-(x-y) = \emptyset$, then we are done since $\mathrm{supp}^+(x-y) = \{k\}$ holds. Hence, we assume $h \in \mathrm{supp}^-(x-y)$ holds for some $h \in N \setminus \{k\}$. Then, in a similar way as in the discussion above with the roles of x and y changed, we can show that $x(i) \geq y(i)$ for all $i \in N \setminus \{h\}$, which, together with (4.4), implies $x(i) = y(i)$ for $i \in N \setminus \{h, k\}$. From this follows that $\mathrm{supp}^+(x-y) = \{k\}$ and $\mathrm{supp}^-(x-y) = \{h\}$. □

Proof of "(GS&LAD) $\Longrightarrow$ (1DLD)". The proof below is similar to that for the implication "(GS) $\Longrightarrow$ (1DLD′)" shown above. Let $S \subseteq \mathbb{R}^n$, $x, y \in S$, $k \in N$, $p \in \mathbb{R}^n$, and $\delta > 0$ be as in the proof of "(GS) $\Longrightarrow$ (1DLD′)." Then, the condition (GS&LAD) implies that

$$y(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}), \qquad y(N) \leq x(N). \tag{4.5}$$

If $y(i) = x(i)$ holds for all $i \in N \setminus \{k\}$, then we are done since $x - y$ is parallel to χ_k. Hence, we assume $x(h) < y(h)$ for some $h \in N \setminus \{k\}$. Then, in a similar way we can show that

$$x(i) \geq y(i) \quad (\forall i \in N \setminus \{h\}), \qquad x(N) \leq y(N),$$

which, together with (4.5), implies

$$x(i) = y(i) \quad (\forall i \in N \setminus \{h, k\}), \qquad x(N) = y(N).$$

From this follows that $x - y$ is parallel to $+\chi_k - \chi_h$. Hence, (1DLD) holds. □

4.4.2. Proof of Theorem 4.10 (ii)

The equivalence between (M$^\natural$-EXC) and (LD) is immediate from Theorem 3.10 (ii). Hence, it suffices to show "(M$^\natural$-EXC) $\Longrightarrow$ (GS&LAD)" by Theorem 4.10 (i), where we use the following property of M$^\natural$-concave functions.

Theorem 4.13 ([29, Theorem 4.2]). *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be an M$^\natural$-concave function. For $x, y \in \mathrm{dom}\, f$ with $x(N) < y(N)$, there exists some $j \in \mathrm{supp}^-(x-y)$ such that*

$$f(x) + f(y) \leq f(x + \chi_j) + f(y - \chi_j).$$

Proof of "(M$^\natural$-EXC) $\Longrightarrow$ (GS&LAD)". Let $p, q \in \mathbb{R}^n$ be vectors with $p \leq q$, and let $x \in D(f, p)$. Let $y^* \in D(f, q)$ be a vector such that $\|y^* - x\|_1$ is the minimum among all vectors in $D(f, q)$. To show the condition (GS&LAD), we prove that y^* satisfies the following:

$$y^*(i) \geq x(i) \quad (\forall i \in N \text{ with } q(i) = p(i)), \qquad y^*(N) \leq x(N). \tag{4.6}$$

Assume, to the contrary, that there exists some $k \in N$ such that $q(k) = p(k)$ and $y^*(k) < x(k)$. Then, (M$^\natural$-EXC) implies that there exists some $h \in \mathrm{supp}^-(x - y^*) \cup \{0\}$ such that

$$f(x) + f(y^*) \leq f(x - \chi_k + \chi_h) + f(y^* + \chi_k - \chi_h).$$

Hence, we have

$$\begin{aligned} f[-p](x) + f[-q](y^*) &\leq f[-p](x - \chi_k + \chi_h) + f[-q](y^* + \chi_k - \chi_h) + p(h) - q(h) \\ &\leq f[-p](x - \chi_k + \chi_h) + f[-q](y^* + \chi_k - \chi_h) \end{aligned}$$

since $p(k) = q(k)$ and $p(h) \leq q(h)$. This, together with $x \in D(f, p)$ and $y^* \in D(f, q)$, implies that $y^* + \chi_k - \chi_h \in D(f, q)$ (and $x - \chi_k + \chi_h \in D(f, p)$), a contradiction to the choice of y^* since $\|(y^* + \chi_k - \chi_h) - x\|_1 < \|y^* - x\|_1$. Hence, the former condition in (4.6) holds.

To prove $y^*(N) \leq x(N)$, we then assume, to the contrary, that $y^*(N) > x(N)$. Theorem 4.13 implies that there exists some $h \in \mathrm{supp}^-(x - y^*)$ such that

$$f(x) + f(y^*) \leq f(x + \chi_h) + f(y^* - \chi_h).$$

Hence, we have

$$\begin{aligned} f[-p](x) + f[-q](y^*) &\leq f[-p](x + \chi_h) + f[-q](y^* - \chi_h) + p(h) - q(h) \\ &\leq f[-p](x + \chi_h) + f[-q](y^* - \chi_h) \end{aligned}$$

since $p(h) \leq q(h)$. This implies that $y^* - \chi_h \in D(f, q)$ (and $x + \chi_h \in D(f, p)$), a contradiction to the choice of y^* since $\|(y^* - \chi_h) - x\|_1 < \|y^* - x\|_1$. Therefore, we have the latter condition in (4.6). □

4.4.3. Proof of Theorem 4.10 (iii)

Since single-unit valuation functions have bounded effective domains and are concave-extensible, we can apply Theorem 4.10 (ii). Hence, it suffices to show that the implication "(1DLD′) $\Longrightarrow$ (1DLD)" holds for single-unit valuation functions.

Let $S \subseteq \mathbb{R}^n$ be a 1-dimensional linearity domain of $\bar{f}$. Since S is a 1-dimensional bounded polyhedron, there exist two distinct vectors $x, y \in S$ which are extreme points of S. By (1DLD′), the vector $v = x - y$ satisfies $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$. Since $\mathrm{dom}\, f \subseteq \{0, 1\}^n$, we have $x, y \in \{0, 1\}^n$ and $v \in \{0, +1, -1\}^n$. Hence, v is a vector of the form $+\chi_i - \chi_j$ with $i, j \in N \cup \{0\}$. This shows that (1DLD) holds.

4.4.4. Proof of Theorem 4.3

Suppose that f satisfies the condition (4.2). For such a function f, the condition (GS&LAD) is equivalent to (GS) since the inequality $y(N) \leq x(N)$ always holds for $x, y \in \mathrm{dom}\, f$. Hence, the statement of Theorem 4.3 follows immediately from the equivalence between (GS&LAD) and (M$^\natural$-EXC) in Theorem 4.10 (ii) (see Section 4.4.2).

4.4.5. Proof of Theorem 4.1

By Theorem 4.10 (ii) and (iii) shown in Sections 4.4.2 and 4.4.3, it suffices to show that "(PRJ-GS) $\Longleftrightarrow$ (M$^\natural$-EXC)," "(SS) $\Longleftrightarrow$ (M$^\natural$-EXC)" (in the case with $\mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$), "(SWGS) $\Longrightarrow$ (1DLD)," and "(M$^\natural$-EXC) $\Longrightarrow$ (SWGS)" for concave-extensible, multi-unit valuation functions with bounded effective domains.

Proof of "(PRJ-GS) $\Longleftrightarrow$ (M$^\natural$-EXC)". Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible valuation function with bounded $\mathrm{dom}\, f$, and $\tilde{f} : \mathbb{Z} \times \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function given by (4.3). Then, we have the following sequence of equivalences (see Remark 4.4):

$$\text{(PRJ-GS) for } f \iff \text{(GS) for } \tilde{f} \iff \text{M-concavity for } \tilde{f} \iff \text{(M}^\natural\text{-EXC) for } f,$$

where the first equivalence can be obtained by a simple calculation based on the definition (4.3) of $\tilde{f}$, the second is by Theorem 4.3 proved in Section 4.4.4, and the third is by the relation between M$^\natural$-concavity and M-concavity. □

Proof of "(SS) $\Longleftrightarrow$ (M$^\natural$-EXC)". Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a valuation function with $\mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$, and define a single-unit valuation function $f^{\mathrm{B}} : \{0, 1\}^{N^{\mathrm{B}}} \to \mathbb{R} \cup \{-\infty\}$ by (4.1). It is not difficult to check that f satisfies (M$^\natural$-EXC) if and only if f^{B} satisfies (M$^\natural$-EXC). Hence, we have the following sequence of equivalences:

$$\text{(M}^\natural\text{-EXC) for } f \iff \text{(M}^\natural\text{-EXC) for } f^{\mathrm{B}} \iff \text{(GS) for } f^{\mathrm{B}} \iff \text{(SS) for } f,$$

where the second equivalence follows from Theorem 4.10 (iii) proved in Section 4.4.3 and the fact that f^{B} is a single-unit valuation function, and the third equivalence follows immediately from the definition of the condition (SS). $\square$

Proof of "(SWGS) $\Longrightarrow$ (1DLD)". The proof is similar to that for "(GS&LAD) $\Longrightarrow$ (1DLD)" in Section 4.4.1. Let $S \subseteq \mathbb{R}^n$ be a 1-dimensional linearity domain of $\bar{f}$. Since $\operatorname{dom} f$ is bounded and f is concave-extensible, there exists some $p \in \mathbb{R}^n$ such that S is equal to the convex closure $\operatorname{conv}(D(f,p))$ of $D(f,p)$. Let $x, y \in S$ be the extreme points of S. Then, we have $x, y \in D(f,p)$, and S is parallel to $x - y$. We show that the vector $x - y$ is parallel to a nonzero vector $+\chi_i - \chi_j$ with $i, j \in N \cup \{0\}$.

Since $x \neq y$, we may assume that $x(k) > y(k)$ for some $k \in N$. The condition (SWGS) applied to $p \in \mathbb{R}^n$, $k \in N$, and $x \in D(f,p)$ implies that there exist some $\delta \geq 0$ and $z \in D(f, p + \delta\chi_k)$ such that

$$z(k) = x(k) - 1, \qquad z(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}); \tag{4.7}$$

this holds since $D(f, p + \delta'\chi_k) = \{y\}$ for some $\delta' > 0$. We have

$$\begin{aligned} f[-p](z) - \delta(x(k) - 1) &= f[-(p + \delta\chi_k)](z) \\ &\geq f[-(p + \delta\chi_k)](y) \\ &= f[-p](y) - \delta y(k) \geq f[-p](y) - \delta(x(k) - 1), \end{aligned}$$

where the first inequality is by $z \in D(f, p + \delta\chi_k)$ and the second by $x(k) > y(k)$. Hence, it holds that $f[-p](z) \geq f[-p](y)$, which, combined with $y \in D(f,p)$, implies $z \in D(f,p)$. Since $\operatorname{conv}(D(f,p)) = S$ is a 1-dimensional polyhedron with extreme points x and y, the vector z can be expressed as a convex combination of x and y. This, together with (4.7), implies that

$$y(k) < x(k), \qquad y(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}). \tag{4.8}$$

If $y(i) = x(i)$ holds for all $i \in N \setminus \{k\}$, then we are done since (4.8) implies that $x - y$ is parallel to $+\chi_k$. Hence, we assume $x(h) < y(h)$ for some $h \in N \setminus \{k\}$. Then, in a similar way as in the discussion above with the roles of x and y changed, we can show that

$$y(h) > x(h), \qquad y(i) \leq x(i) \quad (\forall i \in N \setminus \{h\}),$$

which, together with (4.8), implies

$$x(i) = y(i) \quad (\forall i \in N \setminus \{h, k\}), \qquad x(N) = y(N).$$

From this follows that $x - y$ is parallel to $+\chi_k - \chi_h$. $\square$

Proof of "(M$^\natural$-EXC) $\Longrightarrow$ (SWGS)". Let $p \in \mathbb{R}^n$, $k \in N$, and $x \in D(f,p)$. Also, let

$$\lambda^* = \sup\{\lambda \mid \lambda \in \mathbb{R}_+,\ x \in D(f, p + \lambda\chi_k)\}.$$

If $\lambda^* = +\infty$, then (i) in (SWGS) holds. Hence, we assume $\lambda^* < +\infty$. Note that $x \in D(f, p + \lambda^*\chi_k)$ holds.

For $\lambda \in \mathbb{R}_+$, $f[-(p + \lambda\chi_k)]$ is an M$^\natural$-concave function. Hence, Theorem 3.5 implies that

$$x \in D(f, p + \lambda\chi_k) = \arg\max f[-(p + \lambda\chi_k)]$$

if and only if

$$f[-(p + \lambda \chi_k)](x) \geq f[-(p + \lambda \chi_k)](x - \chi_i + \chi_j) \qquad (\forall i, j \in N \cup \{0\}),$$

which can be rewritten as

$$\lambda(-\chi_i(k) + \chi_j(k)) \geq f(x - \chi_i + \chi_j) - f(x) - (-p(i) + p(j)) \qquad (\forall i, j \in N \cup \{0\}). \quad (4.9)$$

Since $x \in D(f, p)$, the inequality (4.9) holds with $\lambda = 0$, i.e., we have

$$f(x - \chi_i + \chi_j) - f(x) - (-p(i) + p(j)) \leq 0 \qquad (\forall i, j \in N \cup \{0\}).$$

Hence, the value λ^* is given as

$$\lambda^* = \min\{-f(x - \chi_k + \chi_j) + f(x) + (-p(k) + p(j)) \mid j \in (N \cup \{0\}) \setminus \{k\}\}.$$

Let $h \in (N \cup \{0\}) \setminus \{k\}$ be such that $\lambda^* = -f(x - \chi_k + \chi_h) + f(x) + (-p(k) + p(h))$. Then, we have

$$f[-(p + \lambda^* \chi_k)](x) = f[-(p + \lambda^* \chi_k)](x - \chi_k + \chi_h),$$

implying that $x - \chi_k + \chi_h \in D(f, p + \lambda^* \chi_k)$. Therefore, (ii) in (SWGS) holds. □

4.4.6. Proof of Theorem 4.7

We can prove the equivalence between (GS&LAD$_-$) and M$^\natural$-concavity in the same way as in the proof of the equivalence between (GS&LAD) and M$^\natural$-concavity in Section 4.4.2. Hence, we obtain the equivalence between (GS&LAD$_-$) and (GS&LAD). In a similar way, we can prove the equivalence between (GS$_-$) and (GS) for single-unit valuation functions.

Bibliographical notes

The equivalence between (M$^\natural$-EXC) and (LD) for concave-extensible multi-unit valuation functions (Theorem 4.10 (ii)) is due to Murota [25]. The equivalence among (M$^\natural$-EXC), (GS), and (1DLD) for single-unit valuation functions (Theorem 4.1 (ii) and Theorem 4.10 (iii)) was proved in Fujishige and Yang [16] (see also Danilov, Koshevoy, and Lang [8]). Danilov, Koshevoy, and Lang [8] also showed the equivalence between (1DLD) and (1DLD′) for single-unit valuation functions (Theorem 4.10 (iii)).

The relationship between the gross substitutes condition and M$^\natural$-concavity was generalized to multi-unit valuation functions by Murota and Tamura [35] and Danilov, Koshevoy, and Lang [8]. In particular, the two variants of gross substitutes condition, (PRJ-GS) and (SWGS), were introduced in [35] and [8], respectively, to give characterizations of M$^\natural$-concave valuation functions in terms of gross substitutes conditions. The equivalence among (M$^\natural$-EXC), (PRJ-GS), and (1DLD) (Theorem 4.1 (i) and Theorem 4.10 (ii)) is due to Murota and Tamura [35], while the equivalence among (M$^\natural$-EXC), (SWGS), and (1DLD), and the implication "(GS) $\Longrightarrow$ (1DLD′)" (Theorem 4.1 (i) and Theorem 4.10 (i), (ii)) was shown by Danilov, Koshevoy, and Lang [8].

While the condition (SS) was implicitly considered in several earlier papers (see, e.g., [1, p. 617] and [4, p. 391]), it was explicitly stated for the first time in [24]. The condition (GS&LAD) was introduced by Murota, Shioura, and Yang [33], where the equivalence with M$^\natural$-concavity was shown (Theorem 4.1 (i)).

5. Single Improvement Condition and Gross Substitutes Condition

In this section, we discuss the relationship between gross substitutes conditions and some variants of the single improvement condition.

5.1. Theorems

The following condition, called the *single improvement condition* in Gul and Stacchetti [17], states that if a vector x is not a maximizer of the function $f[-p]$, then we can find a vector with a larger function value of $f[-p]$ in a "neighborhood" of x.

(SI) $\forall p \in \mathbb{R}^n$, $\forall x \in \mathrm{dom}\, f$ with $x \notin D(f,p)$, $\exists i, j \in N \cup \{0\}$ with $i \neq j$:
$f[-p](x - \chi_i + \chi_j) > f[-p](x)$.

A stronger variant of the single improvement condition, which we call the *strong single improvement condition*, is considered in Murota and Tamura [35].

(SSI) $\forall p \in \mathbb{R}^n$, $\forall x, y \in \mathrm{dom}\, f$ with $f[-p](y) > f[-p](x)$,
$\exists i \in \mathrm{supp}^+(x-y) \cup \{0\}$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$: $f[-p](x - \chi_i + \chi_j) > f[-p](x)$.

It turns out that under some mild assumptions, the condition (SI) is equivalent to (SSI) and strong gross substitutes conditions (i.e., (PRJ-GS), (SWGS), (GS&LAD), and (SS)).

Theorem 5.1. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function.*
(i) *Suppose that* $\mathrm{dom}\, f$ *is bounded. Then, the following relations hold:*

$$\text{(SSI)} \Longrightarrow \text{(SI)} \Longrightarrow \text{(GS\&LAD)}.$$

(ii) *Suppose that* $\mathrm{dom}\, f$ *is bounded and f is concave-extensible. Then, the following relations hold:*

$$\text{(SSI)} \iff \text{(SI)} \iff \text{(GS\&LAD)}.$$

(iii) *Suppose that* $\mathrm{dom}\, f \subseteq \{0,1\}^n$. *Then, the following relations hold:*

$$\text{(SSI)} \iff \text{(SI)} \iff \text{(GS\&LAD)} \iff \text{(GS)}.$$

For valuation functions which are not concave-extensible, the converse of the two implications in Theorem 5.1 (i) does not hold in general.

Example 5.2 (Murota and Tamura [35, Remark 15])**.** We show an example of a valuation function which satisfies (SI) but not (SSI). Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that

$$\begin{aligned}
&\mathrm{dom}\, f = \{(-1,1), (-1,0), (0,1), (0,0), (0,-1), (1,0), (1,-1)\}, \\
&f(-1,1) = -3,\ f(-1,0) = -4,\ f(0,1) = -1,\ f(0,0) = -10, \\
&f(0,-1) = -3,\ f(1,0) = 0,\ f(1,-1) = -1.
\end{aligned}$$

We can check that f satisfies (SI). On the other hand, if we take $p = (0,0)$, $x = (-1,0)$, and $y = (1,0)$, there exists no $i \in \mathrm{supp}^+(x-y) \cup \{0\}$ and $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ with $f[-p](x - \chi_i + \chi_j) > f[-p](x)$, i.e., (SSI) does not hold. □

Example 5.3. The condition (GS&LAD) does not imply (SI) in general. Let $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ be the function given in Example 4.9. Note that f satisfies (GS&LAD), and is not concave-extensible. To see that f does not satisfy (SI), let $p = (-1,-10)$ and $x = (0,1)$. Then, we have $D(f,p) = \{(3,1)\}$ and $y = (3,1)$ is the unique vector which satisfies $f[-p](y) > f[-p](x)$. Hence, (SI) does not hold. □

5.2. Proofs

5.2.1. Proof of Theorem 5.1 (i)

It is not difficult to see that (SSI) implies (SI). It is shown that (SI) implies (GS&LAD′) which is equivalent to (GS&LAD) by Proposition 4.6. To prove (GS&LAD′) for f, we first show that (SI) implies the following property which is slightly stronger than (SWGS):

(SWGS+) $\forall p \in \mathbb{R}^n$, $\forall k \in N$, $\forall x \in D(f,p)$, at least one of (i) and (ii) holds true:
(i) $\forall \lambda \in \mathbb{R}_+$: $x \in D(f, p + \lambda \chi_k)$.
(ii) $\exists \lambda \in \mathbb{R}_+$, $\exists y \in D(f, p + \lambda \chi_k)$: $x \in D(f, p + \lambda' \chi_k)$ $(0 \leq \forall \lambda' \leq \lambda)$,
$y(k) = x(k) - 1$, $y(i) \geq x(i)$ $(\forall i \in N \setminus \{k\})$, and $y(N) \leq x(N)$.

We then prove (GS&LAD′) for f by using (SWGS+).

[Proof of "(SI) $\Longrightarrow$ (SWGS+)"] Let $p \in \mathbb{R}^n$, $k \in N$, and $x \in D(f, p)$. Put

$$\lambda = \sup\{\lambda' \mid x \in D(f, p + \lambda'\chi_k)\}.$$

If $\lambda = +\infty$, then we are done; the condition (i) in (SWGS+) holds. Hence, we assume $\lambda < +\infty$, and show that the condition (ii) in (SWGS+) holds.

By the definition of λ, it holds that

$$x \in D(f, p + \lambda'\chi_k) \quad (0 \leq \forall\lambda' \leq \lambda), \qquad x \notin D(f, p + \lambda'\chi_k) \quad (\forall\lambda' > \lambda). \tag{5.1}$$

Since $x \in D(f, p + \lambda\chi_k)$ and $\mathrm{dom}\, f$ is bounded, there exists a sufficiently small $\varepsilon > 0$ such that

$$\varepsilon < f[-(p + \lambda\chi_k)](x) - \max\{f[-(p + \lambda\chi_k)](z) \mid z \in \mathbb{Z}^n \setminus D(f, p + \lambda\chi_k)\}, \tag{5.2}$$
$$D(f, p + (\lambda + \varepsilon)\chi_k) = \{z \in D(f, p + \lambda\chi_k) \mid z(k) = \min\{z'(k) \mid z' \in D(f, p + \lambda\chi_k)\}\}.$$

Put $q = p + (\lambda + \varepsilon)\chi_k$. Since $x \notin D(f, q)$ by (5.1), the condition (SI) implies that there exist some distinct $i_*, j_* \in N \cup \{0\}$ such that

$$f[-q](x - \chi_{i_*} + \chi_{j_*}) > f[-q](x). \tag{5.3}$$

We show that the vector $y = x - \chi_{i_*} + \chi_{j_*}$ satisfies the desired conditions. By (5.3), it holds that

$$\begin{aligned} 0 &< f[-q](y) - f[-q](x) \\ &= f[-(p + \lambda\chi_k)](y) - f[-(p + \lambda\chi_k)](x) + \varepsilon(\chi_k(i_*) - \chi_k(j_*)) \\ &\leq \varepsilon(\chi_k(i_*) - \chi_k(j_*)), \end{aligned} \tag{5.4}$$

where the last inequality is by $x \in D(f, p + \lambda\chi_k)$. From this inequality follows $i_* = k$ and $y(k) = x(k) - 1$. Since $j_* \neq i_*$, we also have $y(i) \geq x(i)$ $(\forall i \in N \setminus \{k\})$ and $y(N) \leq x(N)$. By (5.2) and (5.4), it holds that

$$\begin{aligned} f[-(p + \lambda\chi_k)](y) &> f[-(p + \lambda\chi_k)](x) - \varepsilon \\ &> \max\{f[-(p + \lambda\chi_k)](z) \mid z \in \mathbb{Z}^n \setminus D(f, p + \lambda\chi_k)\}, \end{aligned}$$

from which follows that $y \in D(f, p + \lambda\chi_k)$.

[Proof of "(SWGS+) $\Longrightarrow$ (GS&LAD′)"] Let $p \in \mathbb{R}^n$, $k \in N$, $\lambda \in \mathbb{R}_+$, and $x \in D(f, p)$. We show the existence of a vector $y \in \mathbb{Z}^n$ with

$$y \in D(f, p + \lambda\chi_k), \qquad y(i) \geq x(i) \quad (\forall i \in N \setminus \{k\}), \qquad y(N) \leq x(N). \tag{5.5}$$

Such y can be found by the following algorithm which is based on (SWGS+):

Step 0: Let $t = 1$, $x_1 = x$ and $\lambda_1 = 0$.
Step 1: If $x_t \in D(f, p + \lambda\chi_k)$ holds, then output $y = x_t$ and stop.
Otherwise (i.e., $x_t \notin D(f, p + \lambda\chi_k)$), go to Step 2.
Step 2: Find $\lambda_{t+1} \in \mathbb{R}_+$ with $\lambda_{t+1} \geq \lambda_t$ and $x_{t+1} \in D(f, p + \lambda_{t+1}\chi_k)$ such that

$$x_t \in D(f, p + \lambda'\chi_k) \qquad (\lambda_t \leq \forall\lambda' \leq \lambda_{t+1}), \tag{5.6}$$
$$x_{t+1}(k) = x_t(k) - 1, \quad x_{t+1}(i) \geq x_t(i) \ (\forall i \in N \setminus \{k\}), \quad x_{t+1}(N) \leq x_t(N). \tag{5.7}$$

Step 3: Set $t := t + 1$ and go to Step 1.

The algorithm terminates in a finite number of iterations since the value $x_t(k)$ decreases in each iteration and $\mathrm{dom}\, f$ is bounded. Suppose that the algorithm terminates at the r-th iteration.

We first show by induction that $\lambda_t < \lambda$ holds for $t = 1, 2, \ldots, r$. Suppose that $\lambda_t < \lambda$ holds for some $t \leq r - 1$. Since the algorithm does not stop in the t-th iteration, $x_t \notin D(f, p + \lambda\chi_k)$ holds. Moreover, we have $x_t \in D(f, p + \lambda'\chi_k)$ $(\lambda_t \leq \forall\lambda' \leq \lambda_{t+1})$ by (5.6), implying that $\lambda_{t+1} < \lambda$.

We then show the correctness of the algorithm. The existence of λ_{t+1} and x_{t+1} in Step 2 follows from (SWGS+) since $\lambda_t < \lambda$, $x_t \in D(f, p + \lambda_t\chi_k)$, and $x_t \notin D(f, p + \lambda\chi_k)$ holds. If $x_t \in D(f, p + \lambda\chi_k)$ holds in Step 1, then the condition (5.7) for $t = 1, 2, \ldots, r - 1$ implies that $y = x_r$ satisfies (5.5).

5.2.2. Proof of Theorem 5.1 (ii) and (iii)

By Theorem 5.1 (i), we need to prove that (GS&LAD) implies (SSI) in the case where $\mathrm{dom}\, f$ is bounded and f is concave-extensible.

Suppose that f satisfies (GS&LAD). By Theorem 4.1 (i), f is M$^\natural$-concave. By using (M$^\natural$-EXC), we show that for $p \in \mathbb{R}^n$ and $x, y \in \mathrm{dom}\, f$ with $f[-p](y) > f[-p](x)$, the following condition holds:

$$\exists i \in \mathrm{supp}^+(x-y) \cup \{0\},\ \exists j \in \mathrm{supp}^-(x-y) \cup \{0\} : f[-p](x - \chi_i + \chi_j) > f[-p](x). \tag{5.8}$$

The proof is given by induction on the value $\|x - y\|_1$. If $y = x - \chi_i + \chi_j$ holds for some $i \in \mathrm{supp}^+(x-y) \cup \{0\}$ and $j \in \mathrm{supp}^-(x-y) \cup \{0\}$, then we have (5.8) immediately. Hence, we may assume the following condition:

$$\|x - y\|_1 \geq 3 \qquad \text{or} \qquad |x(k) - y(k)| \geq 2 \text{ for some } k \in N. \tag{5.9}$$

Since function $f[-p]$ also satisfies (M$^\natural$-EXC), we have

$$f[-p](x) + f[-p](y) \leq f[-p](x - \chi_i + \chi_j) + f[-p](y + \chi_i - \chi_j) \tag{5.10}$$

for some $i \in \mathrm{supp}^+(x-y) \cup \{0\}$ and $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ with $i \neq j$. By (5.10), we have at least one of $f[-p](x) < f[-p](x - \chi_i + \chi_j)$ and $f[-p](y) \leq f[-p](y + \chi_i - \chi_j)$. If the former holds, then we have the claim (5.8). Hence, we assume the latter holds.

Put $y' = y + \chi_i - \chi_j$, where $y' \neq x$ by (5.9). We have

$$f[-p](y') \geq f[-p](y) > f[-p](x), \qquad \|x - y'\|_1 < \|x - y\|_1.$$

Hence, the induction hypothesis implies that there exists $i' \in \mathrm{supp}^+(x - y') \cup \{0\}$ and $j' \in \mathrm{supp}^-(x - y') \cup \{0\}$ such that $f[-p](x - \chi_{i'} + \chi_{j'}) > f[-p](x)$. Since

$$\mathrm{supp}^+(x - y') \cup \{0\} \subseteq \mathrm{supp}^+(x - y) \cup \{0\}, \qquad \mathrm{supp}^-(x - y') \cup \{0\} \subseteq \mathrm{supp}^-(x - y) \cup \{0\},$$

we have the claim (5.8).

Bibliographical notes

For single-unit valuation functions, equivalence of the conditions (GS) and (SI) (Theorem 5.1 (iii)) is shown in Gul and Stacchetti [17, Theorem 1], while the equivalence of (SI) and (SSI) (Theorem 5.1 (iii)) is due to Murota and Tamura [35, Theorem 12 (c)].

For multi-unit valuation functions, Milgrom and Strulovici [24, Theorem 7] showed that (SI) implies (GS), while we proved in Theorem 5.1 (i) that (SI) implies (GS&LAD), a stronger condition than (GS). For concave-extensible multi-unit valuation functions, the equivalence between (SI) and (SS) (strong substitute condition, see Section 4) was shown by Milgrom and Strulovici [24, Theorem 13]. The equivalence between (SSI) and M$^\natural$-concavity (cf. Theorem 5.1 (ii)) was shown in Murota and Tamura [35, Theorem 7 (b)].

6. Submodularity and Gross Substitutes Condition

In this section, we discuss the relationship between gross substitutes conditions and submodularity for valuation functions. Recall that a valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be *submodular* if it satisfies

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y) \qquad (\forall x, y \in \mathbb{Z}^n), \tag{6.1}$$

where we admit the inequality of the form $-\infty \geq -\infty$. This inequality is equivalent to the following local submodular inequality under some mild assumption on $\mathrm{dom}\, f$:

$$f(x + \chi_i) + f(x + \chi_j) \geq f(x) + f(x + \chi_i + \chi_j) \qquad (\forall x \in \mathbb{Z}^n,\ \forall i, j \in N,\ i \neq j). \tag{6.2}$$

Recall that an M$^\natural$-convex set is the set of integral vectors in an integral g-polymatroid; see Section 3.3.

Proposition 6.1. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is an M$^\natural$-convex set. Then, f is a submodular function if and only if it satisfies* (6.2)*.*

Remark 6.2. Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that $\mathrm{dom}\, f$ is contained in a hyperplane of the form $x(N) = \lambda$ for some $\lambda \in \mathbb{Z}$. Then, f satisfies the submodular inequality (6.1) since for every distinct $x, y \in \mathbb{Z}^n$ at least one of $x \wedge y$ and $x \vee y$ is not in $\mathrm{dom}\, f$. This observation shows that we need some additional assumption on $\mathrm{dom}\, f$ to derive some meaningful (or useful) property from the submodular inequality (6.1). □

6.1. Theorems

The gross substitutes condition (GS) implies submodularity of valuation functions.

Theorem 6.3. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$.*

(i) *Suppose that f is concave-extensible and* $\mathrm{dom}\, f$ *is bounded. If f satisfies the condition* (GS)*, then f is a submodular function.*

(ii) *Suppose that* $\mathrm{dom}\, f \subseteq \{0, 1\}^n$*. If f satisfies the condition* (GS)*, then f is a submodular function.*

Note that the converse of the statements above does not hold in general, i.e., the class of gross substitutes valuation functions is a proper subclass of submodular valuation functions.

Example 6.4. A budgeted additive valuation function considered in Remark 2.2 is a submodular function, while it does not satisfy the condition (GS) in general.

For example, by setting $n = 3$, $a = (1, 1, 2)$, and $B = 2$, we obtain the following budgeted additive valuation function:

$$f(x) = \begin{cases} x(1) + x(2) + 2x(3) & (\text{if } x \in \{0,1\}^3 \text{ and } x(1) + x(2) + 2x(3) \leq 2), \\ 2 & (\text{if } x \in \{0,1\}^3 \text{ and } x(1) + x(2) + 2x(3) > 2), \\ -\infty & (\text{otherwise}). \end{cases}$$

It can be easily checked that f is a submodular function.

For $p = (0.5, 0.5, 1)$ and $q = (0.5 + \delta, 0.5, 1)$ with a sufficiently small $\delta > 0$, we have

$$D(f, p) = \{(0, 0, 1), (1, 1, 0)\}, \qquad D(f, q) = \{(0, 0, 1)\}.$$

Hence, $y = (0, 0, 1)$ is the unique vector in $D(f, q)$. If we take $x = (1, 1, 0) \in D(f, p)$, then for $i = 2$ we have $q(i) = p(i)$ and $y(i) = 0 < 1 = x(i)$, i.e., function f does not satisfy (GS). □

It is natural to ask which property distinguishes gross substitutes valuation functions from submodular valuation functions. The following theorem gives a characterization of the gross substitutes condition by the combination of submodularity and an additional property.

Theorem 6.5. *Let* $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ *be a function such that* $\mathrm{dom}\, f$ *is an* $\mathrm{M}^\natural$*-convex set with* $\mathbf{0} \in \mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$.

(i) *Suppose that* $\mathrm{dom}\, f$ *is bounded and* f *is concave-extensible. Then,* f *satisfies the condition* (GS&LAD) *if and only if it is a submodular function satisfying the following condition:*

$$\begin{aligned} &f(x + \chi_i + \chi_j) + f(x + \chi_k) \\ &\le \max\{f(x + \chi_i + \chi_k) + f(x + \chi_j), f(x + \chi_j + \chi_k) + f(x + \chi_i)\} \\ &\qquad\qquad\qquad\qquad (\forall x \in \mathbb{Z}^n,\ \forall i, j, k \in N \text{ with } k \notin \{i, j\}), \end{aligned} \tag{6.3}$$

where i *and* j *can be the same.*

(ii) *Suppose that* $\mathrm{dom}\, f \subseteq \{0,1\}^n$*. Then,* f *satisfies the condition* (GS) *if and only if it is a submodular function satisfying* (6.3).

6.2. Proofs

6.2.1. Proof of Proposition 6.1

In the proof we use the following property of an $\mathrm{M}^\natural$-convex set.

Proposition 6.6. *Let* $S \subseteq \mathbb{Z}^n$ *be an* $\mathrm{M}^\natural$*-convex set. For* $\ell, u \in S$ *with* $\ell \le u$*, we have* $[\ell, u]_{\mathbb{Z}} \subseteq S$.

Proof. Since S is an $\mathrm{M}^\natural$-convex set, it can be described as

$$S = \{x \in \mathbb{Z}^n \mid \mu(Y) \le x(Y) \le \rho(Y)\ (\forall Y \subseteq N)\}$$

with some functions $\mu : 2^N \to \mathbb{Z} \cup \{-\infty\}$ and $\rho : 2^N \to \mathbb{Z} \cup \{+\infty\}$ (cf. (3.2)). For every $z \in [\ell, u]_{\mathbb{Z}}$, it holds that

$$\mu(Y) \le \ell(Y) \le z(Y) \le u(Y) \le \rho(Y) \qquad (\forall Y \subseteq N),$$

implying that $z \in S$. Hence, the claim follows. □

To prove Proposition 6.1, it suffices to show that local submodular inequality (6.2) implies submodularity. We prove the submodular inequality for $x, y \in \mathbb{Z}^n$ by induction on the value $\|x - y\|_1$.

If $x \wedge y \notin \mathrm{dom}\, f$ or $x \vee y \notin \mathrm{dom}\, f$, then we have

$$f(x) + f(y) \ge -\infty = f(x \wedge y) + f(x \vee y).$$

Hence, we assume $x \wedge y \in \mathrm{dom}\, f$ and $x \vee y \in \mathrm{dom}\, f$. Since $\mathrm{dom}\, f$ is an $\mathrm{M}^\natural$-convex set, this assumption, together with Proposition 6.6, implies that $[x \wedge y, x \vee y]_{\mathbb{Z}} \subseteq \mathrm{dom}\, f$.

We may also assume that

$$\sum_{i \in \mathrm{supp}^+(x-y)} (x(i) - y(i)) \ge 1 \quad \text{and} \quad \sum_{i \in \mathrm{supp}^-(x-y)} (y(i) - x(i)) \ge 2, \qquad \text{or}$$
$$\sum_{i \in \mathrm{supp}^+(x-y)} (x(i) - y(i)) \ge 2 \quad \text{and} \quad \sum_{i \in \mathrm{supp}^-(x-y)} (y(i) - x(i)) \ge 1$$

since otherwise the submodular inequality follows immediately from the local submodular inequality. We here consider the former case only; the latter case can be dealt with similarly.

Let $k \in \mathrm{supp}^-(x-y)$. Then, it holds that

$$x + \chi_k,\ (x \wedge y) + \chi_k \in [x \wedge y, x \vee y]_{\mathbb{Z}} \subseteq \mathrm{dom}\, f,$$
$$\|(x+\chi_k) - y\|_1 < \|x-y\|_1, \quad \|x - ((x \wedge y) + \chi_k)\|_1 < \|x-y\|_1.$$

Hence, the induction hypothesis implies that

$$\begin{aligned} f(x+\chi_k) + f(y) &\geq f((x+\chi_k) \wedge y) + f((x+\chi_k) \vee y) \\ &= f((x \wedge y) + \chi_k) + f(x \vee y), \qquad (6.4) \\ f(x) + f((x \wedge y) + \chi_k) &\geq f(x \wedge ((x \wedge y) + \chi_k)) + f(x \vee ((x \wedge y) + \chi_k)) \\ &= f(x \wedge y) + f(x + \chi_k). \qquad (6.5) \end{aligned}$$

From (6.4) and (6.5) we obtain $f(x) + f(y) \geq f(x \wedge y) + f(x \vee y)$.

6.2.2. Proof of Theorem 6.3

We here give a proof based on a polyhedral property of valuation functions with (GS). By Theorem 4.10 (i), function f satisfies the condition (1DLD′), i.e., every 1-dimensional linearity domain of the concave closure $\bar{f}$ of f is parallel to a nonzero vector $v \in \mathbb{R}^n$ with $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$. Since the concave closure $\bar{f}$ is a polyhedral concave function, Theorem 3.15 implies that $\bar{f}$ is a submodular function, from which follows that f is also a submodular function since it is concave-extensible.

6.2.3. Proof of Theorem 6.5

Our proof of Theorem 6.5 is based on the following characterization of an M$^\natural$-concave function by a local exchange property:

Theorem 6.7 (cf. [25, Theorem 3.1], [27, Theorem 6.4])**.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is an M$^\natural$-convex set. Then, f is M$^\natural$-concave if and only if it satisfies the following local exchange property:*

$\forall x, y \in \mathrm{dom}\, f$ *with* $\|x-y\|_1 + |x(N) - y(N)| \leq 4$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$*:*

$$f(x) + f(y) \leq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

We first prove a refined version of this characterization for M$^\natural$-concave functions f with $\mathbf{0} \in \mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$ by using Theorem 6.7.

Theorem 6.8. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that* $\mathrm{dom}\, f$ *is an M$^\natural$-convex set with $\mathbf{0} \in \mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$. Then, f is M$^\natural$-concave if and only if it satisfies* (6.2), (6.3), *and the following inequality*

$$2f(x+\chi_i) \geq f(x) + f(x+2\chi_i) \qquad (\forall x \in \mathbb{Z}^n,\ \forall i \in N). \qquad (6.6)$$

Proof. We see that the local exchange property in Theorem 6.7 is equivalent to the combination of (6.2), (6.3), (6.6), and the following inequality:

$$\begin{aligned} &f(x+\chi_i+\chi_j) + f(x+\chi_k+\chi_h) \\ &\leq \max\{f(x+\chi_i+\chi_k) + f(x+\chi_j+\chi_h), f(x+\chi_j+\chi_k) + f(x+\chi_i+\chi_h)\} \\ &\qquad\qquad (\forall x \in \mathbb{Z}^n,\ \forall i,j,k,h \in N \text{ with } \{i,j\} \cap \{k,h\} = \emptyset). \qquad (6.7) \end{aligned}$$

By Theorem 6.7, it suffices to show that (6.3) implies the inequality (6.7).

In the following, we denote $a(s,t) = f(x+\chi_s+\chi_t) \in \mathbb{R} \cup \{-\infty\}$ and $b(s) = f(x+\chi_s) \in \mathbb{R} \cup \{-\infty\}$ for every $s, t \in N$. Then, the inequality (6.7) can be rewritten as

$$a(i,j) + a(k,h) \leq a(i,k) + a(j,h) \text{ or } a(i,j) + a(k,h) \leq a(j,k) + a(i,h) \text{ (or both)}, \qquad (6.8)$$

which is proved below by a case-by-case analysis. We may assume

$$a(i,j) = f(x + \chi_i + \chi_j) > -\infty, \qquad a(k,h) = f(x + \chi_k + \chi_h) > -\infty$$

since otherwise the inequality (6.8) holds immediately. By Proposition 6.6, this assumption, combined with $\mathbf{0} \in \mathrm{dom}\, f \subseteq \mathbb{Z}^n_+$, implies that $b(s) > -\infty$ for $s \in \{i, j, k, h\}$.

By the condition (6.3), we may assume, without loss of generality, that

$$a(i,j) + b(k) \le a(i,k) + b(j). \tag{6.9}$$

We consider six inequalities

$$\begin{aligned} a(k,h) + b(j) &\le a(j,h) + b(k), & (6.10)\\ a(k,h) + b(j) &\le a(j,k) + b(h), & (6.11)\\ a(i,j) + b(h) &\le a(i,h) + b(j), & (6.12)\\ a(i,j) + b(h) &\le a(j,h) + b(i), & (6.13)\\ a(k,h) + b(i) &\le a(i,k) + b(h), & (6.14)\\ a(k,h) + b(i) &\le a(i,h) + b(k). & (6.15) \end{aligned}$$

The condition (6.3) implies that
- either (6.10) or (6.11) (or both) holds,
- either (6.12) or (6.13) (or both) holds,
- either (6.14) or (6.15) (or both) holds.

Hence, it suffices to consider the following four cases:
Case 1: (6.10) holds,
Case 2: (6.11) and (6.12) hold,
Case 3: (6.13) and (6.14) hold,
Case 4: (6.11), (6.13), and (6.15) hold.

In Case 1, the inequalities (6.9) and (6.10) imply

$$a(i,j) + a(k,h) \le [a(i,k) + b(j) - b(k)] + [a(j,h) + b(k) - b(j)] = a(i,k) + a(j,h),$$

i.e., the first inequality in (6.8) holds. In Case 2, the inequalities (6.11) and (6.12) imply

$$a(k,h) + a(i,j) \le [a(j,k) + b(h) - b(j)] + [a(i,h) + b(j) - b(h)] = a(j,k) + a(i,h),$$

i.e., the second inequality in (6.8) holds. We can also obtain the first inequality in (6.8) from (6.13) and (6.14) in a similar way in Case 3.

We finally consider Case 4. From (6.9), (6.11), (6.13), and (6.15) follows that

$$\begin{aligned} a(i,j) + a(k,h) &\le \frac{1}{2}\{a(i,k) + a(j,h) + a(i,h) + a(j,k)\}\\ &\le \max\{a(i,k) + a(j,h), a(i,h) + a(j,k)\}, \end{aligned}$$

i.e., (6.8) holds. This concludes the proof. □

We now prove Theorem 6.5, where only the claim (i) is proven since the claim (ii) follows from (i) and the equivalence between (GS) and (GS&LAD) for single-unit valuation functions shown in Theorem 4.1 (ii).

Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible function such that $\mathrm{dom}\, f$ is a bounded M$^\natural$-convex set. By Theorem 4.1 (i) and Theorem 6.8, f satisfies (GS&LAD) if and only if it satisfies (6.2), (6.3), and (6.6). This implies the claim (i) of Theorem 6.5, since (6.2) is equivalent to submodularity by Proposition 6.1 and concave-extensibility implies (6.6).

Bibliographical notes

The relationship between the condition (GS) and submodularity (Theorem 6.3) was shown by Gul and Stacchetti [17, Lemma 5] in the case with $\mathrm{dom}\, f = \{0,1\}^n$ and by Danilov, Koshevoy, and Lang [8, Corollary 4] for concave-extensible multi-unit valuation functions. Submodularity of an M$^\natural$-concave function was shown in Murota and Shioura [31, Theorem 3.8] (see Theorem 3.4).

Reijnierse, van Gellekom, and Potters [37] showed a characterization of the condition (GS) using submodularity (Theorem 6.5 (ii)) in the case with $\mathrm{dom}\, f = \{0,1\}^n$ (see [37, Theorem 10]; see also [7, Theorem 13.5]), which motivates us the generalization to multi-unit valuation functions in Theorem 6.5 (i).

7. Indirect Utility Functions of Gross Substitutes Valuations

For a valuation function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ with nonempty bounded $\mathrm{dom}\, f$, the *indirect utility function* (also referred to as the *dual profit function* [24]) is a function $f^{\mathrm{IU}} : \mathbb{R}^n \to \mathbb{R}$ given by

$$f^{\mathrm{IU}}(p) = \max\{f(x) - p^\top x \mid x \in \mathrm{dom}\, f\} \qquad (p \in \mathbb{R}^n).$$

The value $f^{\mathrm{IU}}(p)$ of the indirect utility function represents the maximum utility of a buyer with a valuation function f at a given price vector p. The indirect utility function f^{IU} is essentially equivalent to the concave conjugate f° of a valuation function in the following sense:

$$f^{\mathrm{IU}}(p) = -\min\{p^\top x - f(x) \mid x \in \mathrm{dom}\, f\} = -f^\circ(p) \qquad (p \in \mathbb{R}^n). \tag{7.1}$$

In this section, we show various properties and characterizations of indirect utility functions associated with gross substitutes valuation functions.

7.1. Theorems

Any strong gross substitutes condition for a valuation function (i.e., (PRJ-GS), (SWGS), (GS&LAD), or (SS)) can be characterized by another discrete convexity concept for the indirect utility function, called *L$^\natural$-convexity*. Recall that a polyhedral convex function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *polyhedral L$^\natural$-convex* if it satisfies the following condition:

$$g(p) + g(q) \geq g((p - \lambda \mathbf{1}) \vee q) + g(p \wedge (q + \lambda \mathbf{1})) \qquad (\forall p, q \in \mathbb{R}^n,\ \forall \lambda \in \mathbb{R}_+).$$

This inequality with $\lambda = 0$ implies the submodularity of g, in particular. Hence, L$^\natural$-convexity is a stronger condition than submodularity.

Theorem 7.1. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a concave-extensible function with bounded* $\mathrm{dom}\, f$*, and $f^{\mathrm{IU}} : \mathbb{R}^n \to \mathbb{R}$ be its indirect utility function. Then, f satisfies* (GS&LAD) *if and only if f^{IU} is polyhedral L$^\natural$-convex.*

As shown in Theorem 7.1, any strong gross substitutes condition is equivalent to L$^\natural$-convexity of the indirect utility function. On the other hand, for a single-unit valuation function, any strong gross substitutes condition and the weak gross substitutes condition (i.e., the condition (GS)) are equivalent to submodularity, a weaker condition than L$^\natural$-convexity.

Theorem 7.2. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function with* $\mathrm{dom}\, f \subseteq \{0,1\}^n$*, and $f^{\mathrm{IU}} : \mathbb{R}^n \to \mathbb{R}$ be its indirect utility function. Then,*

$$\begin{aligned} f \text{ satisfies (GS)} &\iff f \text{ satisfies (GS\&LAD)} \\ &\iff f^{\mathrm{IU}} \text{ is polyhedral L}^\natural\text{-convex} \iff f^{\mathrm{IU}} \text{ is submodular.} \end{aligned}$$

Theorem 7.2 shows the equivalence between (GS) and submodularity of the indirect utility function for a single-unit valuation function. It is natural to ask whether this equivalence extends to the case of multi-unit valuation functions. If a multi-unit valuation function satisfies a strong gross substitutes condition such as (GS&LAD), then its indirect utility function is polyhedral L$^\natural$-convex by Theorem 7.1, and therefore it is submodular. In fact, submodularity of the indirect utility function can be obtained from the weak gross substitutes condition (GS).

Theorem 7.3. *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a valuation function with bounded* $\mathrm{dom}\, f$*. If f satisfies the condition* (GS)*, then its indirect utility function $f^{\mathrm{IU}} : \mathbb{R}^n \to \mathbb{R}$ is submodular.*

On the other hand, submodularity of the indirect utility function does not imply the condition (GS) for a multi-unit valuation function, as shown in the following example.

Example 7.4 (Danilov, Koshevoy, and Lang [8, Example 6])**.** We show an example of a function which does not satisfy the condition (GS), while the function is concave-extensible and its indirect utility function is submodular. Consider the function f in Example 4.12, which does not satisfy (GS). Then, the indirect utility function $f^{\mathrm{IU}} : \mathbb{R}^3 \to \mathbb{R}$ is given as

$$f^{\mathrm{IU}}(p) = \max_{1 \leq j \leq 8} \{f(x_j) - p^\top x_j\} \qquad (p \in \mathbb{R}^3), \tag{7.2}$$

where vectors x_j are given as follows:

$$x_1 = (0,0,0),\ x_2 = (2,0,0),\ x_3 = (0,3,0),\ x_4 = (0,0,6),$$
$$x_5 = (2,3,0),\ x_6 = (2,0,6),\ x_7 = (0,3,6),\ x_8 = (2,3,6) = u.$$

The formula (7.2) follows from the fact that every extreme point of linearity domains of $\bar{f}$ is in $\{x_1, x_2, \ldots, x_8\}$. Then, we can show by using (7.2) that every linearity domain of f^{IU} is represented by a system of inequalities of the form $a^\top p \leq b$ with $b \in \mathbb{R}$ and a vector $a \in \mathbb{R}^n$ satisfying $|\mathrm{supp}^+(a)| \leq 1$ and $|\mathrm{supp}^-(a)| \leq 1$. Hence, f^{IU} is a submodular function by Theorem 3.16. □

Remark 7.5. Let us consider the following statement given in [24, Theorem 2].

> **Statement A.** Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a function such that $\mathrm{dom}\, f$ is a finite integer interval. Then, f satisfies the condition (GS) if and only if its indirect utility function $g : \mathbb{R}^n \to \mathbb{R}$ given by (7) is submodular.

As pointed out in Example 7.4, the "if" part of Statement A does not hold in general for multi-unit valuation functions, while the "only if" part holds true. □

7.2. Proofs

7.2.1. Proof of Theorem 7.1

The relation (7.1) and Theorem 3.13 (i) imply that f is M$^\natural$-concave if and only if its indirect utility function is polyhedral L$^\natural$-convex. Since M$^\natural$-concavity is equivalent to (GS&LAD) by Theorem 4.1 (i), we obtain the claim of the theorem.

7.2.2. Proof of Theorem 7.2

By Theorem 4.1 (ii) and Theorem 7.1, it suffices to show that submodularity of f^{IU} implies L$^\natural$-convexity of f^{IU} for a single-unit valuation function f.

Suppose that f^{IU} is a submodular function. Since $\mathrm{dom}\, f$ is bounded, we have $\mathrm{dom}\, f^{\mathrm{IU}} = \mathbb{R}^n$ and it is full-dimensional, in particular. We prove the L$^\natural$-convexity of f^{IU} by using Theorem 3.11, which can be restated as follows: a polyhedral concave function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is polyhedral L$^\natural$-concave if and only if each linearity domain $S \subseteq \mathbb{R}^n$ of g satisfies the following condition:

(∗) S can be represented as $S = \{p \in \mathbb{R}^n \mid a_j^\top p \leq b_j \ (j = 1, 2, \ldots, m)\}$ with vectors $a_1, a_2, \ldots, a_m \in \{0, \pm 1\}^n$ satisfying $|\mathrm{supp}^+(a_j)| \leq 1$ and $|\mathrm{supp}^-(a_j)| \leq 1$ and $b_1, b_2, \ldots, b_m \in \mathbb{R}$.

Note that the condition (∗) for some linearity domain S of f^{IU} implies the same condition (∗) for each face of S, which is also a linearity domain of f^{IU}. Hence, it suffices to prove the condition (∗) for all full-dimensional linearity domains of f^{IU}.

Since f^{IU} is the indirect utility function of function f with $\mathrm{dom}\, f \subseteq \{0, 1\}^n$, each full-dimensional linearity domain S of f^{IU} is associated with some $\chi_X \in \mathrm{dom}\, f$ and given as

$$\begin{aligned} S &= \{p \in \mathbb{R}^n \mid p^\top \chi_X - p^\top \chi_Y \leq f(\chi_X) - f(\chi_Y) \ (\forall \chi_Y \in \mathrm{dom}\, f)\} \\ &= \{p \in \mathbb{R}^n \mid p^\top (\chi_{X \setminus Y} - \chi_{Y \setminus X}) \leq f(\chi_X) - f(\chi_Y) \ (\forall \chi_Y \in \mathrm{dom}\, f)\}. \end{aligned}$$

Since f^{IU} is submodular, Theorem 3.16 implies that each facet of S is orthogonal to a nonzero vector v with $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$. Hence, we have $|\mathrm{supp}^+(\chi_{X \setminus Y} - \chi_{Y \setminus X})| \leq 1$ and $|\mathrm{supp}^-(\chi_{X \setminus Y} - \chi_{Y \setminus X})| \leq 1$ if $p^\top (\chi_{X \setminus Y} - \chi_{Y \setminus X}) \leq f(\chi_X) - f(\chi_Y)$ is a facet-defining inequality for S. Hence, the polyhedron S can be rewritten as

$$\begin{aligned} S = \{p \in \mathbb{R}^n \mid\ & p^\top (\chi_{X \setminus Y} - \chi_{Y \setminus X}) \leq f(\chi_X) - f(\chi_Y) \\ & (\forall \chi_Y \in \mathrm{dom}\, f,\ |X \setminus Y| \leq 1,\ |Y \setminus X| \leq 1)\}. \end{aligned}$$

This shows that S satisfies the condition (∗).

7.2.3. Proof of Theorem 7.3

We give two proofs of Theorem 7.3. The first proof is based on Theorem 3.16, a characterization of submodularity for polyhedral convex functions, while the second proof shows the submodularity of f^{IU} directly. In the proofs, the following properties of f^{IU} are used.

Proposition 7.6. *For $p \in \mathbb{R}^n$, it holds that*

$$f^{\mathrm{IU}}(p) = \max\{\bar{f}(x) - p^\top x \mid x \in \mathrm{dom}\, \bar{f}\} = -(\bar{f})^\circ(p), \tag{7.3}$$

$$\partial f^{\mathrm{IU}}(p) = -\arg\max \bar{f}[-p]. \tag{7.4}$$

Proof. We see from the definition of the concave closure (3.1) and the linear programming duality that the value $\bar{f}(x)$ can be represented as a convex combination of values $f(y)$ ($y \in \mathrm{dom}\, f$). Hence, we have

$$f^{\mathrm{IU}}(p) = \max\{f(x) - p^\top x \mid x \in \mathrm{dom}\, f\} = \max\{\bar{f}(x) - p^\top x \mid x \in \mathrm{dom}\, \bar{f}\}.$$

The equation (7.4) follows from (7.3) and Theorem 3.3. □

First Proof

Since f satisfies (GS), it follows from Theorem 4.10 (i) that f satisfies the condition (1DLD′), i.e., every 1-dimensional linearity domain of the concave closure $\bar{f}$ is parallel to a nonzero vector $v \in \mathbb{R}^n$ with $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$. By (7.3), the condition (1DLD′) is equivalent to the following condition for f^{IU}:

every $(n-1)$-dimensional linearity domain $S \subseteq \mathbb{R}^n$ is orthogonal to a nonzero vector $v \in \mathbb{R}^n$ with $|\mathrm{supp}^+(v)| \leq 1$ and $|\mathrm{supp}^-(v)| \leq 1$,

which, combined with Theorem 3.16, implies that f^{IU} is submodular since f^{IU} is a polyhedral convex function.

Second Proof

Since f^{IU} is a polyhedral convex function with $\mathrm{dom}\, f = \mathbb{R}^n$, it is not difficult to see that the following three conditions are equivalent:

(i) f^{IU} is submodular, i.e., $f^{\mathrm{IU}}(p) + f^{\mathrm{IU}}(q) \geq f^{\mathrm{IU}}(p \vee q) + f^{\mathrm{IU}}(p \wedge q)$ $(\forall p, q \in \mathbb{R}^n)$.

(ii) $f^{\mathrm{IU}}(p+\lambda\chi_i)+f^{\mathrm{IU}}(p+\mu\chi_j) \geq f^{\mathrm{IU}}(p)+f^{\mathrm{IU}}(p+\lambda\chi_i+\mu\chi_j)$ $(\forall p \in \mathbb{R}^n,\ \forall i, j \in N, i \neq j,\ \forall \lambda, \mu \in \mathbb{R}_+)$.

(iii) $(f^{\mathrm{IU}})'(p; +\chi_j) \geq (f^{\mathrm{IU}})'(p + \lambda\chi_i; +\chi_j)$ $(\forall p \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}_+, \forall i, j \in N,\ i \neq j)$.

Based on this equivalence, we will prove below that the condition (iii) holds.

By Theorem 3.2 and (7.4), we have

$$\begin{aligned}(f^{\mathrm{IU}})'(p; +\chi_j) &= \max\{x(j) \mid x \in \partial f^{\mathrm{IU}}(p)\} \\ &= -\min\{x(j) \mid x \in \arg\max \bar{f}[-p]\} = -\min\{x(j) \mid x \in D(f, p)\}, \end{aligned} \tag{7.5}$$

where the last equality is by the fact that $\arg\max \bar{f}[-p]$ is equal to the convex closure $\mathrm{conv}(D(f, p))$ of $D(f, p)$. Similarly, we have

$$(f^{\mathrm{IU}})'(p + \lambda\chi_i; +\chi_j) = -\min\{y(j) \mid y \in D(f, p + \lambda\chi_i)\}. \tag{7.6}$$

The condition (GS) implies that for every $x \in D(f, p)$, there exists some $y \in D(f, p + \lambda\chi_i)$ such that $x(k) \leq y(k)$ for all $k \in N \setminus \{i\}$, from which follows that

$$\min\{x(j) \mid x \in D(f, p)\} \leq \min\{y(j) \mid y \in D(f, p + \lambda\chi_i)\}. \tag{7.7}$$

By (7.5), (7.6), and (7.7), we have $(f^{\mathrm{IU}})'(p; +\chi_j) \geq (f^{\mathrm{IU}})'(p + \lambda\chi_i; +\chi_j)$, i.e., the condition (iii) holds.

Bibliographical notes

The statement of Theorem 7.3 is proven in several papers in different setting. Ausubel and Milgrom [2] proved the statement in the case where $\mathrm{dom}\, f = \{0, 1\}^n$ (see [2, Theorem 10]); they also proved the converse in this case (see Theorem 7.2). Milgrom and Strulovici [24] proved the statement in the case where $\mathrm{dom}\, f$ is a finite integer interval (see [24, Theorem 2]). While we assumed the boundedness of $\mathrm{dom}\, f$ in Theorem 7.3, more general setting is considered in Danilov, Koshevoy, and Lang [8], where $\mathrm{dom}\, f$ can be unbounded but is contained in a translation of $\mathbb{Z}^n_+$ (see [8, Proposition 4 and Theorem 1]).

8. Application to Walrasian Equilibrium

As an application of the results presented in this paper, we discuss the existence and computation of a Walrasian equilibrium.

8.1. Walrasian equilibrium

We consider an auction market model with m buyers who want to buy goods in N. Assume that for good of type $i \in N$, the number of units available is given by a positive integer $u(i)$. We denote the set of buyers by $B = \{1, 2, \ldots, m\}$. For $b \in B$, let $f_b : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ be a valuation function of buyer b. We assume that $\mathrm{dom}\, f_b$ is bounded and satisfies

$$\{\mathbf{0}, u\} \subseteq \mathrm{dom}\, f_b \subseteq \mathbb{Z}^n_+,$$

and f_b is monotone nondecreasing in the effective domain, i.e., $f_b(x) \leq f_b(y)$ whenever $x, y \in \mathrm{dom}\, f_b$ and $x \leq y$.

Table 1: Functions f_1 and f_2

$f_1(x)$	$x(1)=0$	$x(1)=1$	$x(1)=2$
$x(2)=0$	0	3	6
$x(2)=1$	4	5	6

$f_2(x)$	$x(1)=0$	$x(1)=1$	$x(1)=2$
$x(2)=0$	0	4	4
$x(2)=1$	1	5	5

We say that a pair $(\{x_b^* \mid b \in B\}, p^*)$ of a set of vectors $x_b^* \in \operatorname{dom} f_b$ $(b \in B)$ and a (non-negative) price vector $p^* \in \mathbb{R}^n_+$ is a *Walrasian equilibrium* if it satisfies the following conditions:

$$x_b^* \in D(f_b, p^*) \quad (\forall b \in B), \qquad \sum_{b=1}^{m} x_b^* = u. \tag{8.1}$$

The vector $p^* \in \mathbb{R}^n_+$ is called a *Walrasian equilibrium price vector.*

In the case of the model with multi-unit valuation functions, the condition (GS) is not sufficient to show the existence of a Walrasian equilibrium.

Example 8.1. We show an example of multi-unit valuation functions satisfying (GS) for which a Walrasian equilibrium does not exist. This example is essentially the same as the one in Milgrom and Strulovici [24, Section 1].

Let $u = (2,1)$ and $f_1, f_2 : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$ be valuation functions with

$$\operatorname{dom} f_b = \{x \in \mathbb{Z}^2 \mid 0 \le x(1) \le 2,\ 0 \le x(2) \le 1\}$$

given as in Table 1. Note that f_1 is the same as the function f in Example 4.2. We see that f_1 and f_2 are concave-extensible functions and satisfy the condition (GS); moreover, f_2 is an M$^\natural$-concave function, while f_1 is not.

Suppose, to the contrary, that there exists a Walrasian equilibrium $(\{x_1^*, x_2^*\}, p^*)$. Then, we have $x_1^* + x_2^* = u$ and

$$\begin{aligned}
&f_1(x_1^*) + f_2(x_2^*)\\
&= f_1[-p^*](x_1^*) + f_2[-p^*](x_2^*) + (p^*)^\top u\\
&= \max\{f_1[-p^*](x_1) + f_2[-p^*](x_2) + (p^*)^\top u \mid x_i \in \operatorname{dom} f_i\ (i=1,2),\ x_1 + x_2 = u\}\\
&= \max\{f_1(x_1) + f_2(x_2) \mid x_i \in \operatorname{dom} f_i\ (i=1,2),\ x_1 + x_2 = u\},
\end{aligned}$$

where the second equality is by $x_1^* \in D(f_1, p^*)$ and $x_2^* \in D(f_2, p^*)$. We see that $x_1^* = (1,1)$ and $x_2^* = (1,0)$ are the unique vectors maximizing the value $f_1(x_1^*) + f_2(x_2^*)$ under the condition $x_1^* + x_2^* = u$. For these vectors, we have

$$\begin{aligned}
x_1^* \in D(f_1, p^*) &\iff p^*(1) = 1 \text{ and } p^*(2) \le 0,\\
x_2^* \in D(f_2, p^*) &\iff 0 \le p^*(1) \le 4 \text{ and } p^*(2) \ge 1.
\end{aligned}$$

Hence, there exists no $p^* \in \mathbb{R}^2$ satisfying both of $x_1^* \in D(f_1, p^*)$ and $x_2^* \in D(f_2, p^*)$, a contradiction. □

This example shows that some stronger condition than (GS) is required to guarantee the existence of a Walrasian equilibrium. Below we show that if valuation functions satisfy a strong gross substitutes condition such as (GS&LAD), then a Walrasian equilibrium exists. Proof is given at the end of this section.

Theorem 8.2. *Consider the auction market model mentioned above.*
(i) *Suppose that valuation functions $f_b : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ $(b \in B)$ are concave-extensible*

and satisfy the condition (GS&LAD)*. Then, there exists a Walrasian equilibrium.*
(ii) *Suppose that* $u = \mathbf{1}$ *and valuation functions* $f_b : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ $(b \in B)$ *satisfy* $\mathrm{dom}\, f_b = \{0,1\}^n$ *and the condition* (GS)*. Then, there exists a Walrasian equilibrium.*

Based on the theorem above, we assume to the end of this section that valuation functions $f_b : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ $(b \in B)$ are concave-extensible and satisfy the condition (GS&LAD). This implies that each f_b is M$^\natural$-concave by Theorem 4.1 (i).

We next discuss how to compute a Walrasian equilibrium. For this purpose, we define a new function $L : \mathbb{R}^n_+ \to \mathbb{R}$ by

$$L(p) = p^\top u + \sum_{b=1}^{m} f_b^{\mathrm{IU}}(p) \qquad (p \in \mathbb{R}^n_+),$$

where f_b^{IU} is the indirect utility function of f_b for $b \in B$. The function L is called the *Lyapunov function* [1]. Note that L is a polyhedral convex function since each function f_b^{IU} is polyhedral convex.

The Lyapunov function is useful in finding a Walrasian equilibrium, due to the following properties. Proof is given at the end of this section.

Theorem 8.3. *Suppose that valuation functions* $f_b : \mathbb{Z}^n \to \mathbb{R} \cup \{-\infty\}$ $(b \in B)$ *are concave-extensible and satisfy the condition* (GS&LAD)*.*
(i) *A vector* $p^* \in \mathbb{R}^n_+$ *is a Walrasian equilibrium price vector if and only if it is a minimizer of Lyapunov function* L*.*
(ii) *Lyapunov function* L *is a polyhedral* L$^\natural$*-convex function.*
(iii) *Suppose that each valuation function* f_b *is integer-valued. Then, Lyapunov function* L *is an integral polyhedral* L$^\natural$*-convex function and has an integral minimizer.*

In the following, we assume that each f_b is an integer-valued function. Then, Theorem 8.3 implies that the problem of finding a Walrasian equilibrium price vector can be reduced to the problem of finding an integral minimizer of the Lyapunov function. Since the Lyapunov function is integral polyhedral L$^\natural$-convex by Theorem 8.3 (iii), we can obtain an integral minimizer by the following algorithm, which is an application of the algorithm L$^\natural$_Steepest_Ascent_Up in Section 3.3 to the function $-L$.

Algorithm Find_Equilibrium
Step0: Set $p := \mathbf{0}$.
Step1: Find $X \subseteq N$ that minimizes $L(p + \chi_X)$.
Step2: If $L(p + \chi_X) \le L(p)$, then output p and stop.
Step3: Set $p := p + \chi_X$ and go to Step 1.

By Theorem 3.8, the algorithm outputs in a finite number of iterations an integral minimizer of L, which is a Walrasian equilibrium price vector.

We finally discuss some implementation issues of the algorithm. Let us first consider how to find X in Step 1. Since the Lyapunov function is polyhedral L$^\natural$-convex, the function $\rho(X) = L(p + \chi_X)$ is a submodular function. Hence, some of the existing submodular minimization algorithms (e.g., [20, 39]) can be applied to find such X in Step 1.

We then consider the computation of the values of the Lyapunov function. If we can access the values of valuation functions f_b, then the values of the indirect utility functions f_b^{IU} can be computed by using the algorithm M$^\natural$_Steepest_Ascent_Up since $f_b^{\mathrm{IU}}(p)$ for $p \in \mathbb{R}^n$ is given as the maximization of the M$^\natural$-concave function $f[-p]$:

$$f^{\mathrm{IU}}(p) = \max\{f[-p](x) \mid x \in \mathrm{dom}\, f\} \qquad (p \in \mathbb{R}^n).$$

In the literature of auctions, however, it is often assumed that the access to buyers' valuation functions is impossible since valuation functions contain buyers' private information. Instead, it is often assumed that the information about the demand correspondences $D(f_b, p)$ is available. Even in such a case, we can still apply the algorithm FIND_EQUILIBRIUM to find an equilibrium by evaluating the *difference* of function values $L(p+\chi_X) - L(p)$ instead of the function value $L(p + \chi_X)$. This is possible since the values $f_b^{\mathrm{IU}}(p + \chi_X) - f_b^{\mathrm{IU}}(p)$ for $b \in B$ can be computed by using the demand correspondences $D(f_b, p)$ as follows:

$$f_b^{\mathrm{IU}}(p + \chi_X) - f_b^{\mathrm{IU}}(p) = -\min\{y(X) \mid y \in D(f_b, p)\} \qquad (\forall p \in \mathbb{R}^n,\ \forall X \subseteq N,\ \forall b \in B);$$

this equation follows from some known results in discrete convex analysis. See [1] and the full-paper version of [33] for more details on the implementation issues.

8.2. Proofs

8.2.1. Proof of Theorem 8.2

We prove the claim (i) only since (ii) follows from (i) and Theorem 4.1 (ii).

The assumption of f_b implies that each f_b is an M$^\natural$-concave function by Theorem 4.1 (i). Let $x_b^* \in \mathrm{dom}\, f_b$ $(b = 1, 2, \ldots, m)$ be vectors satisfying $\sum_{b=1}^m x_b^* = u$ and

$$\sum_{b=1}^m f_b(x_b^*) = \max\left\{ \sum_{b=1}^m f_b(x_b) \;\middle|\; x_b \in \mathrm{dom}\, f_b\ (b \in B),\ \sum_{b=1}^m x_b = u \right\}.$$

To show the existence of a Walrasian equilibrium, we prove that there exists some nonnegative vector $p^* \in \mathbb{R}^n$ such that

$$x_b^* \in \arg\max f_b[-p^*] = D(f_b, p^*) \qquad (\forall b \in B). \tag{8.2}$$

Define functions $g, h : \mathbb{Z}^{n\times m} \to \mathbb{R} \cup \{-\infty\}$ by

$$g(\tilde{x}) = \sum_{b=1}^m f_b(x_b) \qquad (\tilde{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{Z}^{n\times m}),$$
$$h(\tilde{x}) = \begin{cases} 0 & (\text{if } \sum_{b=1}^m x_b = u), \\ -\infty & (\text{otherwise}) \end{cases} \qquad (\tilde{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{Z}^{n\times m}).$$

Then, it can be shown that both of g and h satisfy (M$^\natural$-EXC), i.e., they are M$^\natural$-concave functions. Moreover, the vector $\tilde{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ maximizes the function value of the sum of g and h, i.e.,

$$g(\tilde{x}^*) + h(\tilde{x}^*) = \max\{g(\tilde{x}) + h(\tilde{x}) \mid \tilde{x} \in \mathbb{Z}^{n\times m}\}.$$

Therefore, it follows from Theorem 3.12 that there exists $\tilde{p}^* = (p_1^*, p_2^*, \ldots, p_m^*) \in \mathbb{R}^{n\times m}$ satisfying

$$\tilde{x}^* \in \arg\max g[-\tilde{p}^*] \cap \arg\max h[\tilde{p}^*].$$

This implies that

$$x_b^* \in \arg\max f_b[-p_b^*] \qquad (\forall b \in B), \tag{8.3}$$
$$\arg\max h[\tilde{p}^*] \neq \emptyset. \tag{8.4}$$

Since

$$h[\tilde{p}^*](\tilde{x}) = \sum_{b=1}^m (p_b^*)^\top x_b$$

holds for every $\tilde{x} = (x_1, x_2, \ldots, x_m) \in \operatorname{dom} h$, we have $p_1^* = p_2^* = \cdots = p_m^*$ by the condition (8.4); otherwise, $\sup h[\tilde{p}^*] = +\infty$ and $\arg\max h[\tilde{p}^*] = \emptyset$, a contradiction. Putting $p^* = p_1^*$ $(= p_2^* = \cdots = p_m^*)$, the condition (8.3) is rewritten as (8.2).

To conclude the proof, we show that p^* defined above is non-negative. By Proposition 6.6, we have $[\mathbf{0}, u]_{\mathbb{Z}} \subseteq \operatorname{dom} f_b$ for each $b \in B$ since $\mathbf{0}, u \in \operatorname{dom} f_b$. We have $0 \leq x_b^*(i) \leq u(i)$ for $b \in B$ and $\sum_{b=1}^m x_b^*(i) = u(i)$. Hence, for each $i \in N$, there exists some $b \in B$ such that $x_b^*(i) < u(i)$. Therefore, we have $x_b^* + \chi_i \in \operatorname{dom} f_b$. Since $x_b^* \in \arg\max f_b[-p^*]$, it holds that $f_b[-p^*](x_b^*) \geq f_b[-p^*](x_b^* + \chi_i)$, from which follows that

$$p^*(i) \geq f_b(x_b^* + \chi_i) - f_b(x_b^*) \geq 0,$$

where the last inequality is by the monotonicity of f_b. Therefore, p^* is a non-negative vector.

8.2.2. Proof of Theorem 8.3

We first note that the assumption of f_b implies that each f_b is an M$^\natural$-concave function by Theorem 4.1 (i).

[Proof of (i)] Since Lyapunov function L is a polyhedral convex function, a vector p^* is a minimizer of L if and only if the subdifferential $\partial L(p^*)$ of L at p^* contains the zero vector $\mathbf{0}$. The subdifferential $\partial L(p^*)$ is given as

$$\partial L(p^*) = u + \sum_{b=1}^m \partial f_b^{\mathrm{IU}}(p^*) = u - \sum_{b=1}^m \arg\max \bar{f}_b[-p^*],$$

where the second equality is by (7.4). Note that $\sum_{b=1}^m \partial f_b^{\mathrm{IU}}(p^*)$ (resp., $\sum_{b=1}^m \arg\max \bar{f}_b[-p^*]$) denotes the Minkowski sum of the sets $\partial f_b^{\mathrm{IU}}(p^*)$ (resp., $\arg\max \bar{f}_b[-p^*]$). Hence, we have $\mathbf{0} \in \partial L(p^*)$ if and only if $u \in \sum_{b=1}^m \arg\max \bar{f}_b[-p^*]$.

We claim that

$$\left(\sum_{b=1}^m \arg\max \bar{f}_b[-p^*]\right) \cap \mathbb{Z}^n = \sum_{b=1}^m D(f_b, p^*).$$

Since f_b is an M$^\natural$-concave function, $D(f_b, p^*)$ is an M$^\natural$-convex set by Theorem 3.10. Moreover, $\arg\max \bar{f}_b[-p^*]$ is equal to the convex closure $\operatorname{conv}(D(f_b, p^*))$. Hence, it holds that

$$\begin{aligned}\left(\sum_{b=1}^m \arg\max \bar{f}_b[-p^*]\right) \cap \mathbb{Z}^n &= \left(\sum_{b=1}^m \operatorname{conv}(D(f_b, p^*))\right) \cap \mathbb{Z}^n \\ &= \sum_{b=1}^m \operatorname{conv}(D(f_b, p^*)) \cap \mathbb{Z}^n = \sum_{b=1}^m D(f_b, p^*),\end{aligned}$$

where the second equality is due to M$^\natural$-convexity of $D(f_b, p^*)$ (cf. [27, Theorem 4.23]).

From the claim above, we have $u \in \sum_{b=1}^m \arg\max \bar{f}_b[-p^*]$ if and only if $u \in \sum_{b=1}^m D(f_b, p^*)$, which holds if and only if there exists $x_b^* \in D(f_b, p^*)$ $(b \in B)$ such that $\sum_{b=1}^m x_b^* = u$, i.e., p^* is a Walrasian equilibrium price vector. Hence, claim (i) holds.

[Proof of (ii)] By Theorem 7.1, each f_b^{IU} is a polyhedral L$^\natural$-convex function. Since polyhedral L$^\natural$-convexity is closed under the addition of functions, the claim (ii) follows.

[Proof of (iii)] Since f_b is an integer-valued M$^\natural$-concave function, its indirect utility function f_b^{IU} is an integral polyhedral L$^\natural$-convex function by (7.1) and Theorem 3.13 (ii). Since the sum of integral polyhedral L$^\natural$-convex functions is also integral polyhedral L$^\natural$-convex (cf. [27, Chapter 7]), L is integral polyhedral L$^\natural$-convex. Hence, there exists an integral minimizer of L.

Bibliographical notes

Kelso and Crawford [21] showed the existence of a Walrasian equilibrium in the setting of single-unit valuation functions with the condition (GS) (Theorem 8.2 (ii)), and presented an algorithm (more precisely, an auction procedure) to compute a Walrasian equilibrium. Danilov, Koshevoy, and Murota [9] showed the existence of a Walrasian equilibrium in the case of multi-unit valuation functions by using $\mathrm{M}^\natural$-concavity (cf. Theorem 8.2 (i)); moreover they proved the existence in a more general model with *producers* in addition to buyers.

The algorithm for computing an equilibrium shown in this section is essentially the same as the auction procedure by Ausubel [1], which can be seen as a reformulation of an auction procedure by Gul and Stacchetti [18]. The algorithm works if we can obtain the information of buyers' demand correspondences $D(f_b, p)$, even if we do not know the function values of f_b (see Ausubel [1]). In the case where buyers' valuation functions are directly accessible, we can alternatively use an algorithm of Murota and Tamura [34] based on the reduction to the M-convex submodular flow problem.

9. Conclusion

In this paper, we surveyed the relationships among Kelso and Crawford's gross substitutes condition (GS), other variants of gross substitutes condition, and discrete concavity, where multi-unit valuation functions were mainly considered. We also reviewed various characterizations and properties of these concepts.

In Section 4, the condition (GS) and its variants such as (PRJ-GS), (SWGS), (GS&LAD), and (SS) were presented, and their connection with the concept of $\mathrm{M}^\natural$-concavity in discrete convex analysis was discussed. We also gave characterizations of gross substitutes valuation functions in terms of the polyhedral properties of their concave closures such as (1DLD), (1DLD$'$), and (LD).

The single improvement condition (SI) and its stronger variant (SSI) were considered in Section 5, where the relationships with gross substitutes conditions were discussed.

We discussed in Section 6 the relationships between gross substitutes conditions and submodularity, which is recognized as an important property in mathematical economics. It was shown that gross substitutes conditions imply submodularity, and a characterization of gross substitutes conditions using submodularity was provided.

In Section 7, we investigated the structure of gross substitutes valuation functions from the viewpoint of indirect utility functions, and gave some characterizations. In particular, we showed that gross substitutes conditions for a valuation function can be characterized by another kind of discrete convexity called $\mathrm{L}^\natural$-convexity for the associated indirect utility function.

As an application of the results presented in this paper, the existence and computation of a Walrasian equilibrium were discussed in Section 8. It was shown that the assumption of gross substitutes conditions implies the existence of an equilibrium and makes it possible to compute an equilibrium by applying a steepest descent algorithm to the Lyapunov function.

In this survey, we mainly put emphasis on mathematical properties and characterizations of gross substitutes conditions. It is known that gross substitutes conditions for valuation functions also provide various nice algorithmic properties; a typical example is the computation of a Walrasian equilibrium discussed in Section 8, where a strong gross substitutes condition plays a crucial role. For readers who are interested in algorithmic aspects of gross substitutes conditions as well as other related concepts, we suggest an (unpublished) paper by Paes Leme [36] which is available online.

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A. Appendix: Characterizations of Supermodularity and Submodularity of Polyhedral Convex Functions

In this section we give proofs of the theorems (Theorems 3.14, 3.15 and 3.16 in Section 3.3) on characterizations of supermodularity and submodularity of polyhedral convex functions.

A.1. Proof of Theorem 3.14

Recall that $\mathrm{dom}\, f$ is assumed to be full-dimensional. In the following, we prove the implications "(i) $\Longrightarrow$ (ii)," "(ii) $\Longrightarrow$ (iii)," and "(iii) $\Longrightarrow$ (i)" in turn.

[(i) $\Longrightarrow$ (ii)] Let $S \subseteq \mathbb{R}^n$ be a linearity domain of f and $x, y \in S$ with $x \leq y$. We show that every extreme point $z \in \mathbb{R}^n$ of the interval $[x, y]$ is contained in S, which implies $[x, y] \subseteq S$ since $[x, y]$ is a convex set.

Let $p \in \mathbb{R}^n$ be a vector with $S = \arg\min f[-p]$. Also, let $z \in \mathbb{R}^n$ be an extreme point of the interval $[x, y]$, i.e., $z(i) = x(i)$ or $z(i) = y(i)$ for each $i \in N$. Then, there exists another extreme point z' of $[x, y]$ such that $z \wedge z' = x$ and $z \vee z' = y$. Since f is supermodular, the function $f[-p]$ is also supermodular, and therefore it holds that

$$f[-p](z) + f[-p](z') \leq f[-p](x) + f[-p](y).$$

Since $x, y \in S = \arg\min f[-p]$, we have $z, z' \in S$.

[(ii) $\Longrightarrow$ (iii)] Since $\mathrm{dom}\, f$ is n-dimensional, the condition (iii) holds for f if and only if every $(n-1)$-dimensional linearity domain is contained in a hyperplane with a nonnegative orthonormal vector. We show the latter condition is implied by the condition (ii).

Assume, to the contrary, that there exists some $(n-1)$-dimensional linearity domain $S \subseteq \mathrm{dom}\, f$ such that it is contained in a hyperplane $a^\top y = b$ with $b \in \mathbb{R}$ and an orthonormal vector $a \in \mathbb{R}^n$ satisfying $\mathrm{supp}^+(a) \neq \emptyset$ and $\mathrm{supp}^-(a) \neq \emptyset$. We may assume, without loss of generality, that $a(1) > 0$ and $a(2) < 0$.

Let $x \in \mathbb{R}^n$ be a point in the relative interior of S. Then, there exist sufficiently small $\lambda, \mu > 0$ such that $x + \lambda\chi_1 + \mu\chi_2 \in S$, where $\lambda a(1) + \mu a(2) = 0$. Since x and $x + \lambda\chi_1 + \mu\chi_2$ are contained in S, we have $x + \lambda\chi_1 \in S$ and $x + \mu\chi_2 \in S$ by the condition (ii). On the other hand, it holds that

$$a^\top(x + \lambda\chi_1) > a^\top x = b > a^\top(x + \mu\chi_2),$$

which implies that none of $x + \lambda\chi_1$ and $x + \mu\chi_2$ is contained in the hyperplane $a^\top y = b$, and therefore $x + \lambda\chi_1$ and $x + \mu\chi_2$ are not in S, a contradiction.

[(iii) $\Longrightarrow$ (i)] For $x, y \in \mathbb{R}^n$, we show the inequality

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y). \tag{A.1}$$

We may assume that $\mathrm{supp}^+(x - y) \neq \emptyset$, $\mathrm{supp}^-(x - y) \neq \emptyset$, and $x \wedge y, x \vee y \in \mathrm{dom}\, f$ since otherwise the inequality (A.1) holds immediately.

Claim: For $\ell, u \in \mathrm{dom}\, f$ with $\ell \leq u$, we have $[\ell, u] \subseteq \mathrm{dom}\, f$.
[Proof of Claim] By the condition (iii), $\mathrm{dom}\, f$ can be described as

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n \mid a_j^\top x \leq b_j \ (j = 1, 2, \ldots, t'),\ a_j^\top x \geq b_j \ (j = t' + 1, t' + 2, \ldots, t)\}$$

with real numbers b_j $(j = 1, 2, \dots, t)$ and vectors a_j $(j = 1, 2, \dots, t)$ such that $a_j \geq \mathbf{0}$. For every $z \in [\ell, u]$, it holds that

$$a_j^\top z \leq a_j^\top u \leq b_j \ (j = 1, 2, \dots, t'), \ a_j^\top z \geq a_j^\top \ell \geq b_j \ (j = t'+1, t'+2, \dots, t),$$

implying that $z \in \mathrm{dom}\, f$. Hence, the claim follows. [End of Claim]

Since $x \wedge y, x \vee y \in \mathrm{dom}\, f$, the claim above implies that $[x \wedge y, x \vee y] \subseteq \mathrm{dom}\, f$. We define a function $\varphi : [0, 1]^2 \to \mathbb{R}$ by

$$\varphi(\lambda, \mu) = f((x \wedge y) + \lambda u + \mu v), \qquad \text{where } u = x - (x \wedge y), \ v = y - (x \wedge y).$$

It should be noted that $(x \wedge y) + \lambda u + \mu v \in [x \wedge y, x \vee y] \subseteq \mathrm{dom}\, f$ for every $\lambda, \mu \in [0, 1]$. Then, the inequality (A.1) can be rewritten in terms of φ as follows:

$$\varphi(1, 0) - \varphi(0, 0) \leq \varphi(1, 1) - \varphi(0, 1).$$

To prove this inequality, it suffices to show that

$$\varphi'((\lambda, 0); +\chi_1) \leq \varphi'((\lambda, 1); +\chi_1) \qquad (\forall \lambda \in [0, 1)) \tag{A.2}$$

since φ is a continuous function.

By the condition (iii) for f, we have the following property of φ:

every 1-dimensional linearity domain of φ is contained in a hyperplane of the form $a'\lambda + a''\mu = b$ with $a', a'', b \in \mathbb{R}$ such that $a' \geq 0, \ a'' \geq 0$.

Hence, there exists a sequence $D_1, D_2, \dots, D_k$ of 2-dimensional linearity domains of φ such that

(a) $(\lambda, 0) \in D_1$ and $(\lambda + \delta, 0) \in D_1$ for a sufficiently small $\delta > 0$,

(b) $(\lambda, 1) \in D_k$ and $(\lambda + \delta, 1) \in D_k$ for a sufficiently small $\delta > 0$,

(c) For $h = 1, 2, \dots, k-1$, the set $D_h \cap D_{h+1}$ is contained in a hyperplane $a'_h\lambda + a''_h\mu = b_h$ with $a'_h \geq 0, \ a''_h \geq 0$, and $b_h \in \mathbb{R}$ satisfying

$$D_h \subseteq \{(\lambda, \mu) \mid a'_h\lambda + a''_h\mu \leq b_h\}, \quad D_{h+1} \subseteq \{(\lambda, \mu) \mid a'_h\lambda + a''_h\mu \geq b_h\}.$$

Hence, the value $\varphi'((\lambda, \mu); +\chi_1)$ is nondecreasing with respect to $\mu \in [0, 1)$, which implies (A.2).

A.2. Proofs of Theorems 3.15 and 3.16

We first prove Theorem 3.16, and then Theorem 3.15.

A polyhedron $S \subseteq \mathbb{R}^n$ is called *distributive* [11] if for every $x, y \in S$ we have $x \wedge y, x \vee y \in S$. Hence, the condition (ii) in Theorem 3.16 can be restated as follows:

(ii) every linearity domain of f is a distributive polyhedron.

A polyhedral characterization of distributive polyhedra is shown by Felsner and Knauer [11].

Theorem A.1 (Felsner and Knauer [11, Theorem 4])**.** *A polyhedron $S \subseteq \mathbb{R}^n$ is distributive if and only if it can be represented by a system of inequalities of the form $a^\top x \leq b$ with a vector $a \in \mathbb{R}^n$ satisfying $|\mathrm{supp}^+(a)| \leq 1$ and $|\mathrm{supp}^-(a)| \leq 1$ and $b \in \mathbb{R}$.*

We give a proof of Theorem 3.16. The equivalence between (ii) and (iii) is immediate from Theorem A.1. In the following, we prove the implications "(i) $\Longrightarrow$ (ii)" and "(iii) $\Longrightarrow$ (i)," where the proofs are similar to those for the corresponding statements in Theorem 3.14.

[(i) $\Longrightarrow$ (ii)] Let $S \subseteq \mathbb{R}^n$ be a linearity domain of f. Then, there exists some $p \in \mathbb{R}^n$ such that $S = \arg\min f[-p]$. Since f is submodular, the function $f[-p]$ is also submodular, i.e., it satisfies

$$f[-p](x) + f[-p](y) \geq f[-p](x \wedge y) + f[-p](x \vee y) \qquad (\forall x, y \in \mathbb{R}^n).$$

If $x, y \in S = \arg\min f[-p]$, then this inequality implies $x \wedge y, x \vee y \in S$. Hence, S is a distributive polyhedron.

[(iii) $\Longrightarrow$ (i)] For $x, y \in \mathrm{dom}\, f$, we show the inequality

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y). \tag{A.3}$$

We may assume that $\mathrm{supp}^+(x-y) \neq \emptyset$ and $\mathrm{supp}^-(x-y) \neq \emptyset$ since otherwise the inequality (A.3) holds immediately.

Claim: $x \wedge y, x \vee y \in \mathrm{dom}\, f$.
[Proof of Claim] The condition (iii) implies that the set $\mathrm{dom}\, f$ can be also represented by a system of inequalities of the form $a^\top x \leq b$ with a vector $a \in \mathbb{R}^n$ satisfying $|\mathrm{supp}^+(a)| \leq 1$ and $|\mathrm{supp}^-(a)| \leq 1$ and $b \in \mathbb{R}$. Hence, $\mathrm{dom}\, f$ is a distributive polyhedron by Theorem A.1, from which the claim follows. [End of Claim]

We define a function $\varphi : [0,1]^2 \to \mathbb{R}$ by

$$\varphi(\lambda, \mu) = f((x \wedge y) + \lambda u + \mu v), \qquad \text{where } u = x - (x \wedge y),\ v = y - (x \wedge y).$$

It should be noted that $(x \wedge y) + \lambda u + \mu v \in \mathrm{dom}\, f$ $(\lambda, \mu \in [0,1])$ by Claim and the convexity of f. Then, the inequality (A.3) can be rewritten in terms of φ as follows:

$$\varphi(1,0) - \varphi(0,0) \geq \varphi(1,1) - \varphi(0,1).$$

To prove this inequality, it suffices to show that

$$\varphi'((\lambda, 0); +\chi_1) \geq \varphi'((\lambda, 1); +\chi_1) \qquad (\forall \lambda \in [0,1)) \tag{A.4}$$

since φ is a continuous function.

By the condition (iii) for f, we have the following property of φ:
every 1-dimensional linearity domain of φ is contained in a hyperplane of the form $a'\lambda + a''\mu = b$ with $a', a'', b \in \mathbb{R}$ such that $a' \geq 0,\ a'' \leq 0$.

Hence, there exists a sequence $D_1, D_2, \ldots, D_k$ of 2-dimensional linearity domains of φ such that

(a) $(\lambda, 0) \in D_1$ and $(\lambda + \delta, 0) \in D_1$ for a sufficiently small $\delta > 0$,
(b) $(\lambda, 1) \in D_k$ and $(\lambda + \delta, 1) \in D_k$ for a sufficiently small $\delta > 0$,
(c) For $h = 1, 2, \ldots, k-1$, the set $D_h \cap D_{h+1}$ is contained in a hyperplane $a'_h\lambda + a''_h\mu = b_h$ with $a'_h \geq 0$, $a''_h \leq 0$, and $b_h \in \mathbb{R}$ satisfying

$$D_h \subseteq \{(\lambda, \mu) \mid a'_h\lambda + a''_h\mu \geq b_h\}, \quad D_{h+1} \subseteq \{(\lambda, \mu) \mid a'_h\lambda + a''_h\mu \leq b_h\}.$$

Hence, the value $\varphi'((\lambda, \mu); +\chi_1)$ is nonincreasing with respect to $\mu \in [0,1)$, which implies (A.4). This concludes the proof of Theorem 3.16.

We finally prove Theorem 3.15 by using Theorem 3.16 and the following property on the convex conjugate of polyhedral convex submodular functions.

Theorem A.2 (Topkis [44, Corollary 2.7.3]; see also Murota [27, Theorem 8.1])**.** *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a polyhedral convex function. If* f *is a submodular function, then its convex conjugate function* $f^\bullet : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is a supermodular function.*

Proof of Theorem 3.15. Put $g = f^\bullet$. Since f is a polyhedral convex function with pointed $\mathrm{dom}\, f$, the function g is a polyhedral convex function such that its effective domain $\mathrm{dom}\, g$ is n-dimensional. As shown below, g is a submodular function. Since $g^\bullet = (f^\bullet)^\bullet = f$ holds by Theorem 3.1 (ii), f is a supermodular function by Theorem A.2.

We now prove the submodularity of g. Since $\mathrm{dom}\, g$ is n-dimensional, $(n-1)$-dimensional linearity domains of g have one-to-one correspondence to 1-dimensional linearity domains of f, and an $(n-1)$-dimensional linearity domain of g is orthogonal to a vector $a \in \mathbb{R}^n$ if and only if the corresponding 1-dimensional linearity domain of f is parallel to the vector a. Hence, the assumption for f implies that every $(n-1)$-dimensional linearity domain of g is orthogonal to a vector $a \in \mathbb{R}^n$ satisfying $|\mathrm{supp}^+(a)| \leq 1$ and $|\mathrm{supp}^-(a)| \leq 1$. This implies the condition (iii) in Theorem 3.16, and therefore function g is submodular by Theorem 3.16. $\square$

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