Ground Reducibility is EXPTIME-complete

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Abstract. We prove that ground reducibility is EXPTIME-complete in the general case. EXPTIME-hardness is proved by encoding the emptiness problem for the intersection of recognizable tree languages. It is more difficult to show that ground reducibility belongs to DEXPTIME. We associate first an automaton with disequality constraints $\mathcal{A}_{\mathcal{R},t}$ to a rewrite system $\mathcal{R}$ and a term t. This automaton is deterministic and accepts at least one term iff t is not ground reducible by $\mathcal{R}$. The number of states of $\mathcal{A}_{\mathcal{R},t}$ is $O(2^{\|\mathcal{R}\|\|t\|})$ and the size of its constraints is polynomial in the size of $\mathcal{R}$, t. Then we prove some new pumping lemmas, using a total ordering on the computations of the automaton. Thanks to these lemmas, we can show that emptiness for an automaton with disequality constraints can be decided in a time which is polynomial in the number of states and exponential in the size of the constraints. Altogether, we get a simply exponential time deterministic algorithm for ground reducibility decision.

1 Introduction

Ground reducibility of a term t w.r.t. a term rewriting system $\mathcal{R}$ expresses that all ground instances (instances without variables) of t are reducible by $\mathcal{R}$. This property is fundamental in automating inductive proofs in equational theories without constructors [9]. It is also related to *sufficient completeness* in algebraic specifications (see e.g. [11]). Roughly, it expresses that all cases have been covered by $\mathcal{R}$ and that t will be reducible for any inputs. Many papers have been devoted to decision of ground reducibility. Let us report a brief history of the milestones, starting only in 1985 with the general case.

Ground reducibility was first shown decidable by D. Plaisted [13]. The algorithm is however quite complex: a tower of 9 exponentials though there is no explicit complexity analysis in the paper. D. Kapur et al. [11] gave another decidability proof which is conceptually simpler, though still very complicated, and whose complexity is a tower of 7 exponentials in the size of $\mathcal{R}$, t. More precisely, they show that checking the reducibility of all ground instances of t can be reduced to checking the reducibility of all ground instances of t of depth smaller than $N(\mathcal{R})$ where $N(\mathcal{R})$ is a tower of 5 exponentials in the size of $\mathcal{R}$. A third proof was proposed by E. Kounalis [12]. The result is generalized to co-ground reducibility and the expected complexity is 5 exponentials, though there is no explicit complexity analysis in the paper. These three algorithms use combinatorial arguments and some "pumping property": if there is a deep enough irreducible instance of t, then there is also a smaller instance which is also irreducible. This yielded the idea of making explicit the pumping argument as a pumping lemma in some tree language. In support of this idea, when both t and the left members of $\mathcal{R}$ are *linear*, i.e. each variable appears only once, then the set of reducible instances of t is accepted by a finite tree automaton [8]. Hence the set of irreducible ground instances is also accepted by a tree automaton, by complement. This easily gives a simply exponential algorithm in the linear case. (As we will see this algorithm is optimal).

H. Comon expressed first the problem of ground reducibility as an emptiness problem for some tree language [3]. He also gave a decision proof whose complexity is even worse than the former ones. A.-C Caron, J.-L. Coquid and M. Dauchet [2, 5] proved a very beautiful result in 1993, enlighting the pumping properties and their difficulty. They actually show a more general result: the first-order theory of unary encompassment predicates is decidable. And it turns out that ground reducibility can be expressed as a simple formula in this logic. Their technique consists in associating an automaton with each formula, in the spirit of Buchi's and Rabin's method. The kind of automata which is appropriate here is what they call *reduction automata*, a particular case of *automata with constraints* introduced by M. Dauchet in 1981. Such tree automata have the ability to check for equality or disequality of some subtrees before applying a transition rule. In general, emptiness of languages recognized by such automata is undecidable. However, when we only allow a fixed number of equality tests on each computation branch, then emptiness becomes decidable. Unfortunately, their result does not give any information about possible efficient algorithms. The complexity which results from their proof is not better than Plaisted's bound. We tried to specialize the tree automata technique for ground reducibility and we got in this way a triple exponential bound [4]. This is better than previous methods, but still far from the lower bound.

The problem in all works about ground reducibility is that they give a bound on the depth of a minimal irreducible instance of t (or a minimal term accepted by the automaton). However, after establishing carefully such an upper bound, they use a brute-force algorithm, checking the reducibility of all terms of depth smaller than the bound, which increases the complexity by a double exponential.

We use here a different approach. We still rely on automata with disequality constraints. However, we do not try to give a bound on the depth of an accepted term. Rather, we show a stronger result: with an appropriate notion of minimality, a minimal term accepted by the automaton contains at most an exponential number of distinct subterms. To prove this, we use a generalization of pumping to arbitrary replacements for which the term is decreasing according to some well chosen well founded ordering. With a few more ingredients, this yields an algorithm for deciding the emptiness of an automaton with disequality constraints which runs in polynomial time w.r.t. the number of states and in exponential time w.r.t. the size of the constraints. On the other hand, we show that ground reducibility of t w.r.t. $\mathcal{R}$ can be reduced to the emptiness problem for an automaton $\mathcal{A}$ with disequality constraints whose number of states is an exponential in the size of $\mathcal{R}$ and t and whose constraints are polynomial in size. Altogether, we have a simply exponential algorithm for ground reducibility.

This result is optimal since ground reducibility is EXPTIME-hard, already for linear rewrite systems and linear t. A $O(2^{\frac{n}{\log n}})$ lower bound was proved by Kapur et al [10]. We give here a simple proof of EXPTIME-hardness. It is known that the emptiness problem for the intersection of n recognizable languages is EXPTIME-complete, see [7, 14]. We show here that this problem is reducible to ground reducibility in polynomial time.

In section 2, we recall the definition of automata with disequality constraints. In section 3, we show how to construct an automaton with disequality constraints whose emptiness is equivalent to the ground reducibility of t w.r.t. $\mathcal{R}$ and we analyze carefully the complexity of such a construction, and the size of the automaton. Section 4 is devoted to to pumping lemmas for automata with disequality constraints. These lemmas are applied in section 5 to derive an optimal algorithm which checks the emptiness of the (language recognized by) an automaton with disequality constraints. Finally, we study the lower bound of ground reducibility in section 6.

2 Automata with disequality constraints

$\mathcal{F}$ will always be a fixed finite set of function symbols (together with their arity), and $\mathcal{X}$ a set of variables. The set of terms built on $\mathcal{F}$ is written $\mathcal{T}(\mathcal{F}, \mathcal{X})$ and its subset of ground terms is written $\mathcal{T}(\mathcal{F})$. A *position* is a string of positive integers. The concatenation of two positions p and p' is denoted pp' and Λ is the empty string. The length of a string p is $|p|$. Positions are ordered according the prefix ordering: $p \prec p'$ iff there is a string p'' such that $pp'' = p'$. The position p, p' are called *parallel*, $p \parallel p'$, iff $p \not\preccurlyeq p'$ and $p' \not\preccurlyeq p$.

As usual, a finite term t can be viewed as a mapping from its set of positions $Pos(t)$ into $\mathcal{F}$. For instance, if $t = f(g(a), b)$, $Pos(t) = \{\Lambda, 1, 11, 2\}$ and e.g. $t(1) = g$. The subset of maximal position of t w.r.t. $\preccurlyeq$, also called subset of *leaves positions* is denoted $Posl(t)$. If $p \in Pos(t)$, we write $t|_p$ for the subterm of t at position p and $t[s]_p$ for the term obtained by replacing $t|_p$ by s (at position p) in t.

We assume the reader familiar with (constrained) term rewriting systems (see [6] for a survey). Let us only recall that a term t is *ground reducible* by a rewrite system $\mathcal{R}$ iff all the ground instances of t are reducible by $\mathcal{R}$. The rewriting relation associated to a rewrite system $\mathcal{R}$ is denoted $\xrightarrow[\mathcal{R}]{}$ and its reflexive transitive closure is denoted $\xrightarrow[\mathcal{R}]{*}$. A term t is *ground reducible* by a rewrite system $\mathcal{R}$ iff all the ground instances of t are reducible by $\mathcal{R}$.

We use the subsumption quasi-ordering on terms: $s \geqslant t$ if there is a substitution σ such that $s\sigma = t$. Two terms are *similar* if $s \geqslant t$ and $t \geqslant s$. The set of variables occurring in a term t is denoted $Var(t)$. Finally, the *size* of a term t, which is denoted by $\|t\|$, is the cardinal $|Pos(t)|$ of its positions set, and the size of a rewrite system $\mathcal{R}$, which is denoted $\|\mathcal{R}\|$, is the sum of the sizes of its left members[1].

Definition 1. *An* automaton with disequality constraints *(or ADC for short) is a tuple $(Q, Q^{\mathrm{f}}, \Delta)$ where Q is a finite set of states, Q^{f} is the subset of Q of* final states *and Δ is a finite set of transition rules of the form: $f(q_1, \ldots, q_n) \xrightarrow{c} q$ where $f \in \mathcal{F}$ has arity n, $q_1, \ldots, q_n, q \in Q$ and c is a boolean combination without negation of constraints $\pi \neq \pi'$ where π, π' are positions.*

The empty conjunction is written $\top$. The state q is called *target state* of the rule $f(q_1, \ldots, q_n) \xrightarrow{c} q$. A ground term $t \in \mathcal{T}(\mathcal{F})$ *satisfies* a constraint $\pi \neq \pi'$ (which we write $t \models \pi \neq \pi'$) if both π and π' are positions of t and $t|_\pi \neq t|_{\pi'}$. This notion of satisfaction is extended to conjunctions and disjunctions as expected. (In particular $t \models \top$ for every t).

Definition 2. *A* run *of the automaton $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ on a term t is a mapping ρ from $Pos(t)$ into Δ such that, for every $p \in Pos(t)$, if $t(p) = f$ with arity n then $\rho(p)$ is a rule $f(q_1, \ldots, q_n) \xrightarrow{c} q$ and 1. for every $1 \leq i \leq n$, $\rho(p \cdot i)$ is a rule whose target is q_i 2. $t|_p \models c$. If only the first condition is met by ρ, ρ will be called a* weak run.

Runs of $\mathcal{A}$ can also be seen as ground terms over the alphabet Δ (terms of $\mathcal{T}(\Delta)$), the arity of a "symbol" $f(q_1, \ldots, q_n) \xrightarrow{c} q$ in Δ being n, the arity of the symbol f in $\mathcal{F}$.

A term $t \in \mathcal{T}(\mathcal{F})$ is *accepted by* $\mathcal{A}$. there is a run ρ of $\mathcal{A}$ on t such that $\rho(\Lambda)$ is a rule whose target is a final state of Q^{f}. In this case, ρ is called an *accepting run*. The language $L(\mathcal{A})$

[1] This may be a non standard definition but the size of right hand sides of rules is not relevant for our purpose.

of $\mathcal{A}$ is the subset of $\mathcal{T}(\mathcal{F})$ of its accepted terms. Equivalently, $L(\mathcal{A})$ is the set of all terms $t \in \mathcal{T}(\mathcal{F})$ which can be reduced to a final state $q \in Q^{\mathrm{f}}$ by the constrained rewrite system Δ:

$$L(\mathcal{A}) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists q \in Q^{\mathrm{f}},\ t \xrightarrow[\Delta]{*} q\}$$

A *regular language* is a language of some standard tree automata, i.e. of an ADC all the constraints of which are $\top$.

Example 1. Let $\mathcal{F} = \{f, a, b\}$ and $Q = \{q\} = Q^{\mathrm{f}}$,

$$\Delta = \{r_1 :\ a \to q \quad r_2 :\ b \to q \quad r_3 :\ f(q, q) \xrightarrow{1 \neq 2} q\}.$$

This defines an automaton (which accepts the terms irreducible by the rule $f(x, x) \to a$). The term $f(a, b)$ is accepted since $\rho = r_3(r_1, r_2)$ is a run on t such that r_3 yields a final state. The term $f(a, a)$ is not accepted by $\mathcal{A}$: there is a weak run $r_3(r_1, r_1)$ on $f(a, a)$ but the disequality of r_3 is not satisfied.

Note that in general ADC can be non-deterministic (more than one run on a term) or not completely specified (no run on some term). However, given a run ρ, there is a unique term $[\rho] \in \mathcal{T}(\mathcal{F})$ *associated* to ρ.

Definition 3. *Let $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ be an ADC and ρ a weak run of $\mathcal{A}$ on t. An* equality *of ρ is a triple of positions (p, π, π') such that $p, p\pi, p\pi' \in Pos(t)$, $\pi \neq \pi'$ is in the constraint of $\rho(p)$ and $t|_{p\pi} = t|_{p\pi'}$.*

In particular, a weak run without any equality is a run. The equalities in a run are also classified according to a particular position $p_0 \in Pos(t)$:

- (p, π, π') is *close* to p_0 iff $p \preccurlyeq p_0 \prec p\pi$ or $p \preccurlyeq p_0 \prec p\pi'$
- (p, π, π') is *far* (or *remote*) from p_0 if $p\pi \preccurlyeq p_0$ or $p\pi' \preccurlyeq p_0$

These two possible situations are depicted in figure 1.

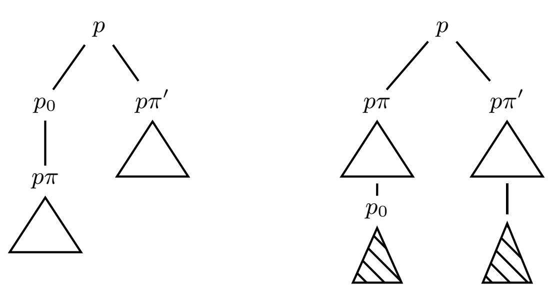

Fig. 1. An equality close to p_0 An equality far from p_0.

3 Reducing ground reducibility to an emptiness problem for ADC

In this section, we show how to construct an ADC whose emptiness is equivalent to the ground reducibility problem and we show precisely the size of such an automaton. We start with an ADC accepting the set of irreducible ground terms (*normal forms*).

3.1 Normal forms ADC

Let $\mathcal{L}$ be the set of left hand sides of a given rewrite system $\mathcal{R}$. Let $\mathcal{L}_1$ be the subset of the linear terms in $\mathcal{L}$, let $\mathcal{L}_2$ be its complement in $\mathcal{L}$ and let $\mathcal{L}_3$ be the set of linearized versions of terms in $\mathcal{L}_2$ (i.e. terms obtained by replacing in some $t \in \mathcal{L}_2$ each occurrence of a variable by a new variable, yielding a linear term).

The initial set of states Q_0 consists in all strict subterms of elements in $\mathcal{L}_1 \cup \mathcal{L}_3$ plus two special states: a single variable x which will accept all terms, and q_r which will accept only reducible terms of $\mathcal{R}$ (hence is a failure state). We assume that all terms are considered up to variable renaming (in particular any two terms are assumed to share no variables in what follows).

The set of states $Q_{\mathrm{NF}(\mathcal{R})}$ of the normal forms automaton consists in all unifiable subsets of $Q_0 \setminus \{q_\mathrm{r}\}$ plus the state q_r. Each element of $Q_{\mathrm{NF}(\mathcal{R})}$ is denoted q_u where u is the most general unifier (*mgu*) of the state – if it is not the special symbol "r".

The transition rules set, denoted $\Delta_{\mathrm{NF}(\mathcal{R})}$, is the set of rules of the form:

$$f(q_{u_1}, \ldots, q_{u_n}) \xrightarrow{c} q_u$$

with:

1. if one of the q_{u_i}'s is q_r or if $f(u_1, \ldots, u_n)$ is an instance of some $s \in \mathcal{L}_1$, then $q_u = q_\mathrm{r}$ and $c = \top$
2. if $f(u_1, \ldots, u_n)$ is not an instance of any term in $\mathcal{L}_1$, then u is the mgu of all terms $v \in Q_0 \setminus \{q_\mathrm{r}\}$ (including the variable x) such that $f(u_1, \ldots, u_n)$ is an instance of v
3. when $q_u \neq q_\mathrm{r}$, the constraint c is defined by:

$$\bigwedge_{\substack{\ell \in \mathcal{L}_2 \\ u,\ell \text{ are unifiable}}} \quad \bigvee_{\substack{x \text{ is a var. of } \ell \\ \ell|_\pi = \ell|_{\pi'} = x \\ \pi \not\equiv \pi'}} (\pi \neq \pi')$$

Note that the unifier in the second condition always exists because one of the states of Q_0 is q_x, $x \in \mathcal{X}$. The final states of the normal forms automaton are all states, except q_r.

$$Q^\mathrm{f}_{\mathrm{NF}(\mathcal{R})} := Q_{\mathrm{NF}(\mathcal{R})} \setminus \{q_\mathrm{r}\}$$

The normal forms automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ is defined by the above constructed sets:

$$\mathcal{A}_{\mathrm{NF}(\mathcal{R})} := (Q_{\mathrm{NF}(\mathcal{R})}, Q^\mathrm{f}_{\mathrm{NF}(\mathcal{R})}, \Delta_{\mathrm{NF}(\mathcal{R})})$$

This automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ is not necessary complete (the automaton may have no run on terms that are reducible by a non-left linear rule). It is however deterministic.

Example 2. The normal forms automaton $\mathcal{A}$ in Example 1 is the normal form automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ where $\mathcal{R} = \{f(x,x) \to a\}$. Note that $\mathcal{A}$ is indeed deterministic but that there exists no run of $\mathcal{A}$ on e.g. the reducible term $f(a,a)$.

Proposition 1. *The automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ accepts the set of terms of $\mathcal{T}(\mathcal{F})$ that are irreducible by $\mathcal{R}$. Its number of states is an exponential in the size of $\mathcal{R}$. Each constraint occurring in a rule of $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ has a size bounded by $O(\|\mathcal{R}\|^3)$.*

Proof. The constraints in the rules of $\Delta_{\mathrm{NF}(\mathcal{R})}$ are conjunctions of disjunctions of disequality atoms. The size of each of these constraints can be bounded according to the respective sizes of conjunctions, disjunctions and atoms.

$$\begin{aligned}
\text{size} &\leq \overbrace{\sum_{\ell \in \mathcal{L}_2}}^{\#\text{ of }\wedge} \overbrace{\sum_{\pi,\pi' \in Pos(\ell)}}^{\#\text{ of }\vee} (|\pi| + |\pi'|) \\
&\leq \sum_{\ell \to r \in \mathcal{R}} \sum_{\pi \in Pos(\ell)} \Big(|\pi| + \sum_{\pi' \in Pos(\ell)} |\pi'| \Big) \\
&\leq \sum_{\ell \to r \in \mathcal{R}} \sum_{\pi \in Pos(\ell)} (|\pi| + \|\ell\|^2) \\
&\leq \sum_{\ell \to r \in \mathcal{R}} (\|\ell\|^3 + \|\ell\|^2) \\
&\leq \|R\|^3
\end{aligned}$$

Concerning the number of states, if is sufficient to remark that it is of the same magnitude as the cardinal of the closure of $Q_0 \setminus \{q_{\mathrm{r}}\}$ by mgu.

We prove the first part of Proposition 1 in the two following paragraphs.

Correctness. Lemmas 1 and 2 show that $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ recognises normal forms of $\mathcal{R}$ only.

Lemma 1. *Let $s \in \mathcal{T}(\mathcal{F})$. If $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$ for some $q_u \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$, then s is an instance of u and $u = \sup\{v \mid q_v \in Q_{\mathrm{NF}(\mathcal{R})}$ and s is an instance of $v\}$ (sup is considered w.r.t. $\geqslant$).*

Proof. By induction on the length of the derivation $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$. □

Lemma 2. *Let $s \in \mathcal{T}(\mathcal{F})$. If $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$ for some $q_u \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$, then s is a normal form of $\mathcal{R}$.*

Proof. By induction on the length of the derivation $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$.
If the length is 1, s is a constant of $\mathcal{F}$. In that case there exists a rule $s \to q_u \in \Delta_{\mathrm{NF}(\mathcal{R})}$ and thus $s \notin \mathcal{L}_1$ by the first construction condition, which means that s is a normal form.
Assume $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} f(q_{u_1}, \ldots, q_{u_n}) \xrightarrow{c} q_u \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$, and let $s = f(s_1, \ldots, s_n)$. For each $1 \leq i \leq n$, we have $s_i \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_{u_i}$ and $q_{u_i} \neq q_{\mathrm{r}}$ by the first condition of the construction. Thus each s_i is a normal form by induction hypothesis. Assume now that s is reducible by $\mathcal{R}$. This means that it must be an instance of some term in $\mathcal{L}$, say $f(l_1, \ldots, l_n)$. We have two cases for $f(l_1, \ldots, l_n)$:

1. if $f(l_1, \ldots, l_n) \in \mathcal{L}_1$, for each $1 \leq i \leq n$, s_i is an instance of l_i. By Lemma 1, this implies that for each $1 \leq i \leq n$, u_i is also an instance of l_i, thus $f(u_1, \ldots, u_n)$ is an instance of $f(l_1, \ldots, l_n) \in \mathcal{L}_1$ which contradicts the existence of the rule $f(q_{u_1}, \ldots, q_{u_n}) \xrightarrow{c} q_u$ ($q_u \neq q_{\mathrm{r}}$) in $\Delta_{\mathrm{NF}(\mathcal{R})}$, by the first construction condition
2. if $f(l_1, \ldots, l_n) \in \mathcal{L}_2$, s is an instance of $f(l_1, \ldots, l_n) = l$ iff s is an instance of the linearised version of l, and, for every distinct positions π, π' of l such that $l|_\pi \equiv l|_{\pi'}$, we have $s|_\pi \equiv s|_{\pi'}$. This last condition implies $s \models \neg c$ by construction of c. Hence $s \not\models c$, which of course contradicts the application of the rule $f(q_{u_1}, \ldots, q_{u_n}) \xrightarrow{c} q_u$ in the last step of the derivation $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$.

(end of proof of Lemma 2) □

Completeness. $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ recognises every ground normal forms of $\mathcal{R}$.

Lemma 3. *Let s be a term of $\mathcal{T}(\mathcal{F})$ which is a normal form of $\mathcal{R}$. There exists $q_u \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$ such that $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$ and s is an instance of u.*

Proof. By induction on s.
If s is a constant, then it is not (an instance of) any term in $\mathcal{L}_1$ thus we have $s \to q_x \in \Delta_{\mathrm{NF}(\mathcal{R})}$ by construction.
For the induction step, let $s = f(s_1, \ldots, s_n)$. The subterms $s_1, \ldots, s_n$ are normal forms of $\mathcal{R}$. Thus by induction hypothesis, we have states $q_{u_1}, \ldots, q_{u_n} \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})} = Q_{\mathrm{NF}(\mathcal{R})} \setminus \{q_{\mathrm{r}}\}$ such that $s_i \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_{u_i}$ and s_i is an instance of u_i for all $i \leq n$. Thus s is an instance of $f(u_1, \ldots, u_n)$.
We proceed by contradiction. Assume that no rule in $\Delta_{\mathrm{NF}(\mathcal{R})}$ with a target in $Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$ is applicable to $f(q_{u_1}, \ldots, q_{u_n})$. Then we are in one of the following cases.

1. One of the q_{u_i}'s is q_{r} or $f(u_1, \ldots, u_n)$ is an instance of some term in $\mathcal{L}_1$ (first condition in the construction of $\Delta_{\mathrm{NF}(\mathcal{R})}$). This would contradict respectively the induction hypothesis and the irreducibility of s by Lemma 1.
2. There exists $f(q_{u_1}, \ldots, q_{u_n}) \xrightarrow{c} q_u$ in $\Delta_{\mathrm{NF}(\mathcal{R})}$ for some c and $t \neq \mathrm{r}$ but $s \not\models c$. This contradicts the irreducibility of s again, by construction of the constraint c.

Hence, there exists a term u, such that $s \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q_u$. Moreover, by construction, u is the most general unifier of the terms $v \in Q_0 \setminus \{q_{\mathrm{r}}\}$ such that $f(u_1, \ldots, u_n)$ is an instance of v. Thus s is an instance of u. □ (end of proof of Proposition 1) □

3.2 Ground reducibility and ADC

If t is a linear term, then its ground reducibility is equivalent to the emptiness of the intersection of $L(\mathcal{A}_{\mathrm{NF}(\mathcal{R})})$ with the (regular) set of instances of t. Since the class ADC is closed by intersection with a regular language (it can be computed in time the product of the sizes of both automata), deciding ground reducibility amounts to decide emptiness of an ADC whose number of states is $\mathrm{O}(2^{\|\mathcal{R}\|} \times \|t\|)$ and constraints have a size $\mathrm{O}(\|\mathcal{R}\|^3)$.

It is a bit more difficult when t is not linear since, in such a situation, the set of irreducible instances of t is not necessarily recognized by an ADC. For this reason, we have to compute directly an automaton whose language is empty iff t is ground reducible by $\mathcal{R}$. This ADC is denoted:

$$\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t} = (Q_{\mathrm{NF}(\mathcal{R}),t}, Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R}),t}, \Delta_{\mathrm{NF}(\mathcal{R}),t})$$

We start with the above normal forms ADC constructed in section 3.1: $\mathcal{A}_{\mathrm{NF}(\mathcal{R})} := (Q_{\mathrm{NF}(\mathcal{R})}, Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}, \Delta_{\mathrm{NF}(\mathcal{R})})$.

Let $\mathcal{S}_t = \{t\sigma|_p \mid p \in \mathit{Pos}(t)\}$ where σ ranges over substitutions whose domain is the set of of variables occurring at least twice in t into $Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$. The cardinal of $\mathcal{S}_t$ is thus exponential in the size $\|t\|$ of t.

1. $Q_{\mathrm{NF}(\mathcal{R}),t} := \mathcal{S}_t \times Q_{\mathrm{NF}(\mathcal{R})}$
2. the final states set $Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R}),t} := \{[u, q] \mid q \in Q_{\mathrm{NF}(\mathcal{R})},\ u \text{ is an instance of } t\}$
3. For all $f(q_1, \ldots, q_n) \xrightarrow{c} q_{n+1} \in \Delta_{\mathrm{NF}(\mathcal{R})}$ and all $u_1, \ldots, u_n \in \mathcal{S}_t$, $\Delta_{\mathrm{NF}(\mathcal{R}),t}$ contains the following rules:
   (a) $f([u_1, q_1], \ldots, [u_n, q_n]) \xrightarrow{c \wedge c'} [f(u_1, \ldots, u_n), q_{n+1}]$
   if $f(u_1, \ldots, u_n)$ is an instance of t and c' is defined below.

(b) $f([u_1, q_1], \ldots, [u_n, q_n]) \xrightarrow{c} [f(u_1, \ldots, u_n), q_{n+1}]$
if $[f(u_1, \ldots, u_n), q_{n+1}] \in Q_{\mathrm{NF}(\mathcal{R}),t}$ and we are not in the first case.
(c) $f([u_1, q_1], \ldots, [u_n, q_n]) \xrightarrow{c} [q_{n+1}, q_{n+1}]$ in all other cases.

Remark that $[q_{n+1}, q_{n+1}]$ is indeed one of the states of $Q_{\mathrm{NF}(\mathcal{R}),t}$
The constraint c' is constructed in three steps.

Lifting. First, all disequality constraints which are checked "in the t part" of $f(u_1, \ldots, u_n) = t\sigma$, are "lifted" to the root position; this is explained in the following construction. From $f(u_1, \ldots, u_n)$ we can retrieve the rules of $\mathcal{A}_{NF(\mathcal{R})}$ which are applied at any position $p \in Pos(t)$ in a run on $f(u_1, \ldots, u_n)$ ($\mathcal{A}_{NF(\mathcal{R})}$ is deterministic). Let c_p be the constraint of the rule applied at position p.
We write $p \cdot c$ the constraint defined by induction on c as below and let c'_1 be $\bigwedge_{p \in Pos(t)} p \cdot c_p$.

$$
\begin{array}{ll}
p \cdot \top := \top,\ p \cdot \bot := \bot & p \cdot (c_1 \wedge c_2) := p \cdot c_1 \wedge p \cdot c_2 \\
p \cdot (\pi \neq \pi') := p\pi \neq p\pi' & p \cdot (c_1 \vee c_2) := p \cdot c_1 \vee p \cdot c_2
\end{array}
$$

Extension. The second step consists in ensuring that all disequality constraints are "deep enough", i.e. below the positions of t: for each constraint $p \neq p'$ in c'_1, such that p or p' is a strict prefix of some position of t, we apply the following rule. We get then a constraint c'_2.

$$
p \neq p' \rightarrow \bigvee_{\substack{p\pi \in Posl(t) \wedge p'\pi \notin Pos(t) \\ \text{or } p'\pi \in Posl(t) \wedge p\pi \notin Pos(t)}} p\pi \neq p'\pi
$$

Variables. After this preparation, we take into account non-linearities of t: they imply equality constraints at the root, hence, by equational deduction, new disequality constraints can be inferred: We let c' be the constraints obtained by saturation of c'_2 using the following deduction rule for each distinct positions p_1 and p_2 in $Pos(t)$ such that $t|_{p_1} \equiv t|_{p_2}$ is a variable:

$$
p_1\pi \neq p' \vdash p_2\pi \neq p'
$$

Example 3. Let $\mathcal{F} = \{f, a, b\}$, $t = f(x, f(x, y))$ and $\mathcal{R} = f(x, x) \rightarrow a\}$. The automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ is (see Examples 1 and 2):

$$
\mathcal{A}_{\mathrm{NF}(\mathcal{R})} = \Big(\{q\}, \{q\}, \{a \rightarrow q; b \rightarrow q; f(q, q) \xrightarrow{1 \neq 2} q\}\Big)
$$

Then the automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ will contain additionally the rule $f([q_q, q], [q_{f(q,q)}, q]) \xrightarrow{1 \neq 2 \wedge 1 \neq 22} q$.

Proposition 2. *The term t is ground reducible by $\mathcal{R}$ iff the language accepted by $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ is empty. The number of states of this automaton is* $\mathrm{O}(2^{c \times \|t\| \times \|\mathcal{R}\|})$ *where c is a constant. The global size of the constraints of transition rules is* $\mathrm{O}(\|t\|^4 \times \|\mathcal{R}\|^3)$.

Moreover, the number of rules of the automaton is $\mathrm{O}(2^{c \times \|t\| \times \|\mathcal{R}\| \times \alpha} \times |\mathcal{F}|)$ where α is the maximal arity of a function symbol of $\mathcal{F}$ and $|\mathcal{F}|$ is the number of function symbols.

Proof. To bound $|Q_{\mathrm{NF}(\mathcal{R}),t}|$, let us recall that $Q_{\mathrm{NF}(\mathcal{R}),t} := \mathcal{S}_t \times Q_{\mathrm{NF}(\mathcal{R})}$, and that $|Q_{\mathrm{NF}(\mathcal{R})}| = 2^{d\times\|\mathcal{R}\|}$ where d is a constraint. Moreover by construction, $|\mathcal{S}_t| \leq |Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}|^{\|t\|} = \mathrm{O}(2^{d\times\|\mathcal{R}\|\times\|t\|})$ which give upper bound $\mathrm{O}(2^{c\times\|t\|\times\|\mathcal{R}\|})$ for the number of states of $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$.

The constraints in the rules of $\Delta_{\mathrm{NF}(\mathcal{R}),t}$ are constraints of $\Delta_{\mathrm{NF}(\mathcal{R})}$ or are conjunction of such constraints and a c' constructed as above. The c''s are conjunctions of disjunctions of conjunctions of disequality atoms and there size after each construction step is bounded below.

Lifting.
$$\underset{p\in Pos(t)}{\overset{\bigwedge}{\sum}} \; \underset{\ell\to r\in\mathcal{R}}{\overset{\bigwedge}{\sum}} \; \underset{\pi,\pi'\in Pos(\ell)}{\overset{\bigvee}{\sum}} (|\pi| + |\pi'| + 2|p|)$$

Extension.
$$\underset{p\in Pos(t)}{\overset{\bigwedge}{\sum}} \; \underset{\ell\to r\in\mathcal{R}}{\overset{\bigwedge}{\sum}} \; \underset{\pi,\pi'\in Pos(\ell)}{\overset{\bigvee}{\sum}} \; \underset{p'\in Pos(t)}{\overset{\bigvee}{\sum}} (|\pi| + |\pi'| + 2\|t\|)$$

Variables.
$$\begin{aligned}
&\overset{\bigwedge}{\|t\|\times} \; \underset{\ell\to r\in\mathcal{R}}{\overset{\bigwedge}{\sum}} \; \underset{\pi,\pi'\in Pos(\ell)}{\overset{\bigvee}{\sum}} \; \overset{\bigvee}{\|t\|\times} \; \overset{\bigwedge}{2\|t\|\times} (|\pi| + |\pi'| + 2\|t\|)\\
&\leq 2\|t\|^3 \times \sum_{\ell\to r\in\mathcal{R}} \sum_{\pi,\pi'\in Pos(\ell)} (|\pi| + |\pi'| + 2\|t\|)\\
&\leq 2\|t\|^3 \times \|\mathcal{R}\|^3 + 2\|t\|^3 \times 2\|t\| \times \|R\|^2\\
&\leq 2\|t\|^3 \times \|\mathcal{R}\|^3 + 4\|t\|^4 \times \|R\|^2
\end{aligned}$$

The if direction of the first part of Proposition 2 follows from the following lemma.

Lemma 4. *Every ground instance of t which is irreducible by $\mathcal{R}$ is accepted by $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$*

Proof. Let τ be a substitution from the set of variables of t to the ground terms of $\mathcal{T}(\mathcal{F})$ such that $t\tau$ is irreducible by $\mathcal{R}$. $t \in L(\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t})$ is a consequence of the following fact, proved by multiset induction.

Lemma 5. *Let τ be a substitution such that $t\tau$ is ground and irreducible by $\mathcal{R}$. For all multiset $\{\{u_1, \ldots, u_m\}\}$ of subterms of t, there exists: a substitution σ from the variables of t to $Q_{\mathrm{NF}(\mathcal{R})}$ and final states $q^{\mathrm{f}}_1, \ldots, q^{\mathrm{f}}_m \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$ such that for all $1 \leq i \leq m$, $u_i\tau$ is reduced by $\Delta_{\mathcal{R},t}$ into the state $[u_i\sigma, q^{\mathrm{f}}_i]$.*

Note that the substitution σ is the same for every u_i.

Proof. Let $\{\{x_1, \ldots, x_m\}\}$ be a multiset of variables of t. By hypothesis, for all $1 \leq i \leq m$, $x_i\tau$ is a normal form of $\mathcal{R}$. Thus, $x_i\tau \in L(\mathcal{A}_{\mathrm{NF}(\mathcal{R})})$ by Proposition 1. This means that there exists final states $q^{\mathrm{f}}_1, \ldots, q^{\mathrm{f}}_m \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$ such that for each $i \leq m$, $x_i\tau \xrightarrow[\Delta_{\mathrm{NF}(\mathcal{R})}]{*} q^{\mathrm{f}}_i$
Thus each $x_i\tau$ is reduced by $\Delta_{\mathrm{NF}(\mathcal{R}),t}$ into the state $[q^{\mathrm{f}}_i, q^{\mathrm{f}}_i]$. We can moreover assume that for all $1 \leq i_1, i_2 \leq m$ such that $x_{i_1} = x_{i_2}$, we have $q^{\mathrm{f}}_{i_1} = q^{\mathrm{f}}_{i_2}$. This give the substitution σ from $\{x_1, \ldots, x_m\}$ to $\{q^{\mathrm{f}}_1, \ldots, q^{\mathrm{f}}_m\}$.
Let $\{\{u_1, \ldots, u_m\}\}$ be a multiset of subterms of t, such that one u_j ($1 \leq j \leq m$) at least is not a variable. We let $u_j = f(v_1, \ldots, v_n)$. By induction hypothesis for the multiset:

$$\{\{u_1, \ldots, u_{j-1}, v_1, \ldots, v_n, u_{j+1}, \ldots, u_m\}\}$$

there exists final states $q^{\mathrm{f}}_1, \ldots, q^{\mathrm{f}}_{m+n-1} \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$ and a substitution τ from the set of variable of t to $Q_{\mathrm{NF}(\mathcal{R})}$ such that for all $1 \leq i < j$ and all $j < i \leq m$, $u_i\tau$ is reduced by $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ into the state $[u_i\sigma, q^{\mathrm{f}}_i]$.
Moreover, by hypothesis, $u_j\tau$ is irreducible by $\mathcal{R}$, thus accepted by $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ which is deterministic (see Proposition 1). Thus, $\Delta_{\mathrm{NF}(\mathcal{R})}$ contains a transition rule of the form: $f(q^{\mathrm{f}}_j, \ldots, q^{\mathrm{f}}_{j+n-1}) \xrightarrow{c} q^{\mathrm{f}}$ with $q^{\mathrm{f}} \in Q^{\mathrm{f}}_{\mathrm{NF}(\mathcal{R})}$ and $u_j\tau \models c$. And $\Delta_{\mathrm{NF}(\mathcal{R}),t}$ contains a transition rule:

$$f([v_1\sigma, q^{\mathrm{f}}_j], \ldots, [v_n\sigma, q^{\mathrm{f}}_{j+n-1}]) \xrightarrow{c''} [f(v_1, \ldots, v_n)\sigma, q^{\mathrm{f}}]$$

where $c'' = c \wedge c'$ if $f(v_1, \ldots, v_n)\sigma$ is an instance of t and $c'' = c$ otherwise.
This gives the final states: $q^{\mathrm{f}}_1, \ldots, q^{\mathrm{f}}_{i-1}, q^{\mathrm{f}}, q^{\mathrm{f}}_{i+1}, \ldots, q^{\mathrm{f}}_m$ and the substitution σ we wanted for the multiset $\{\{u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m\}\}$ □

The only if direction of the first part of Proposition 2 is now proved with the help of three intermediate lemmas. The automaton $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ does not recognise only irreducible ground instances of t. However, we are going to show that if u is accepted then we can construct a term u' which is an irreducible instance of t and which is still accepted (u' is thus a witness for non ground reducibility of t by $\mathcal{R}$).

Lemma 6. *Each term of $L(\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t})$ is a normal form of $\mathcal{R}$.*

Proof. By construction, if we transform $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ with a projection on the first component of the states of $Q_{\mathrm{NF}(\mathcal{R}),t}$, we obtain exactly the normal form ADC $\mathcal{A}_{\mathrm{NF}(\mathcal{R})}$ of Proposition 1. □

Lemma 7. *Each term of $L(\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t})$ is a ground instance of the linearised version of t.*

Proof. We may show by induction the more general fact that each term of $\mathcal{T}(\mathcal{F})$ recognized by $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ in the state $[u\sigma, q] \in Q_{\mathrm{NF}(\mathcal{R}),t}$ ($u\sigma \in \mathcal{S}_t$ and $u = t|_p$ for some $p \in \mathit{Pos}(t)$) is a ground instance of the linearised version of u, by induction on u. □

Lemma 8. *Let ρ be a run of $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ and a position $p \in \mathit{Pos}(\rho)$ such the target state of $\rho(p)$ is $[u, q]$. Then for all position $p' \in \mathit{Pos}(u)$, if $u(p') = q' \in Q_{\mathrm{NF}(\mathcal{R})}$, then the target state of $\rho(pp')$ is $[q', q']$.*

Proof. by induction on ρ. □

Now, we can terminate the proof of the only if direction of the first part of Proposition 2. Assume we have $s \in L(\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t})$. Let ρ be a run of $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$ on this s. By Lemma 7, s is a ground instance of the linearised version of t. It could happen though that s is a actually not an instance of t itself, because we have in s two distinct subterms at positions $p_1, p_2 \in \mathit{Pos}(t)$ corresponding to the same variable in t ($t(p_1) = t(p_2) \in \mathcal{X}$).
The idea is then to construct s', a ground instance of t by replacing $s|_{p_1}$ with $s|_{p_2}$ in s. Lets associate to each variable $x \in \mathit{Var}(t)$ the set $\mathrm{occ}_t(x)$ of positions of x in t,

$$\mathrm{occ}_t(x) := \{p \in \mathit{Pos}(t) \mid t(p) = x\}$$

And let $[t\sigma, q]$ be the (final) target state of $\rho(\Lambda)$.
For each variable x in t, we note $q_x := x\sigma$, which is by construction a state of $Q_{\mathrm{NF}(\mathcal{R})}$. From Lemma 8, for each $p \in \mathrm{occ}_t(x)$, the target state of $\rho(p)$ is $[q_x, q_x]$.

We then construct a weak run ρ' as follows: for each $x \in Var(t)$, if $|\mathrm{occ}_t(x)| \geq 1$, we choose $p \in \mathrm{occ}_t(x)$ and do the replacement $\rho[\rho|_p]_{p'}$ for all other $p' \in \mathrm{occ}_t(x) \setminus \{p\}$. To show that ρ' is indeed a run of $\mathcal{A}_{\mathrm{NF}(\mathcal{R}),t}$, we have to check that the constraints in ρ' are still valid after the replacements, in other words, that ρ' contains no equalities. This follows from the fact that the constraint of $\rho(\Lambda)$ is satisfied by s, in particular the subconstraint c' constructed as above.
We shall first remark that c' contains every constraints which are valid in ρ and may not be valid in ρ', because of the first part, *lifting*, of the construction of c'. There are only two kind of equalities which may occur in ρ' (see Lemma 9 in section 4):

1. (p, π, π') such that there is α such that $p\pi\alpha \in \mathrm{occ}_t(x)$ for some variable x, we performed the replacement $\rho[\rho|_{p\pi\alpha}]_\beta$ (for some other $\beta \in \mathrm{occ}_t(x)$), and $s|_{p\pi'\alpha} = s|_\beta$, which is the cause of the equality $s|_{p\pi} = s|_{p\pi'}$. This situation is not possible because of the transformations performed in the part *extension* in the construction of c'.
2. $(p, \alpha\pi, \pi')$ such that $p\alpha \in \mathrm{occ}_t(x)$ for some variable x, we performed the replacement $\rho[\rho|_{p\alpha}]_\beta$ (for some other $\beta \in \mathrm{occ}_t(x)$), and $s|_{p\pi'} = s|_{\beta\pi}$. This situation is made impossible by the completion in the part called *variables* in the construction of c'.

The term s' is the term of $\mathcal{T}(\mathcal{F})$ associated to ρ' and it is a ground instance of t. Moreover, by Lemma 6, s' is irreducible by $\mathcal{R}$. $\square$

(end of proof of Proposition 2) $\square$

4 Generalised pumping lemmas

This is the crux part of our proof. We assume here a given ADC $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ and a well founded ordering $\gg$, total on runs[2] of $\mathcal{A}$ containing the strict superterm relation (i.e. $\rho \gg \rho|_p$ for all position $p \in Pos(\rho) \setminus \{\Lambda\}$), and monotonic (i.e. $\rho \gg \rho'$ implies that for every ground term s and any position $p \in Pos(s)$, $s[\rho]_p \gg s[\rho']_p$).

Definition 4. *A* pumping *(w.r.t. $\gg$) is a replacement $\rho[\rho']_p$ where ρ, ρ' are runs such that the target state of $\rho'(\Lambda)$ is the same as the target state of $\rho(p)$ and $\rho \gg \rho[\rho']_p$*

This definition generalises the usual pumping definition: we get the usual pumping if we choose for $\gg$ the *embedding ordering*.

Lemma 9. *Every pumping $\rho[\rho']_p$ is a weak run and every equality in $\rho[\rho']_p$ is either far from p or close to p.*

Proof. $\rho[\rho']_p$ is a weak run because the target states of $\rho(p)$ and $\rho'(\Lambda)$ are the same. Let (p', π, π') be an equality of $\rho[\rho']_p$. By definition, ρ and ρ' are runs, thus they do not contain equalities, thus $p' \not\geq p$ and $p' \not\parallel p$. The only remaining possibilities are equalities close to p or far from p. $\square$

Hence, in the following, we may refer to close and far equalities in pumpings, forgetting the position p. Given a large enough run ρ, we will successively show how to construct a weak run by pumping which does not contain any close equality (this uses combinatorial arguments only) then we show how to remove far equalities by further successive pumpings.

[2] Let us recall that runs of $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ can be seen as terms of $\mathcal{T}(\Delta)$.

4.1 Pumping without creating close equalities

Given an ADC $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ and an integer k, we let:

$$g(\mathcal{A}, k) := (e \times k + 1) \times |Q| \times 2^{c(\mathcal{A})} \times c(\mathcal{A})!$$

where e is the exponential basis ($e := \sum_{n=0}^{+\infty} \frac{1}{n!}$) and $c(\mathcal{A})$ is the number of distinct suffixes of positions π, π' in an atom $\pi \neq \pi'$ occurring in a constraint of transition rules of $\mathcal{A}$. Then we have a pumping lemma which generalises those of [5, 4].

Lemma 10. *Let k be an integer. If ρ is a run of $\mathcal{A}$ and $p_1, \ldots, p_{g(\mathcal{A},k)}$ are positions of ρ such that $\rho|_{p_1} \gg \ldots \gg \rho|_{p_{g(\mathcal{A},k)}}$ then there are indices $i_0 < i_1 < \ldots < i_k$ such that the pumping $\rho[\rho|_{p_{i_j}}]_{p_{i_0}}$ does not contain any close equality.*

Example 4. This example illustrates the principle of the proof of Lemma 10. Let $\mathcal{F}$ contain a ternary symbol f and let the ADC $\mathcal{A}$ contain the following transition rule:

$$r: \quad f(q_1, q_2, q_3) \xrightarrow{1 \neq 31 \wedge 1 \neq 32} q$$

Consider moreover the following run (which is also depicted on figure 2):

$$\rho = r(u_0, v_0, r(u_1, v_1, r(u_2, v_2, r(u_3, v_3, r(u_4, v_4, r(u_5, v_5, r(u_6, v_6, v)))))))$$

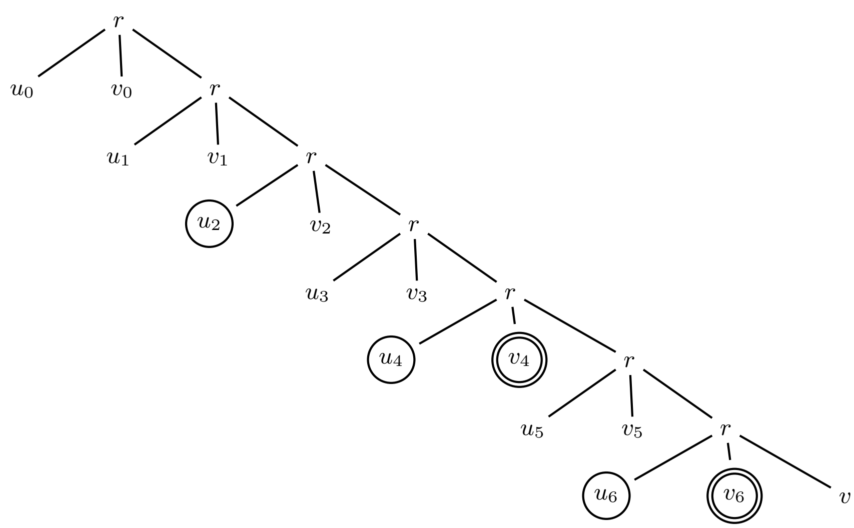

Fig. 2. A run with a possible pumping.

We show that ρ is large enough so as to be able to find a pumping which does not create any close equality. Assume first that the replacement of the subtree at position 3 in ρ by any other subtree rooted by r (except ρ itself) creates a close equality. This means that, for all $i = 2, \ldots, 6$, $u_i = u_0$ or $v_i = v_0$. Then it is possible to extract a subsequence of three indices i_1, i_2, i_3 such that $(u_0 = u_{i_1} = u_{i_2} = u_{i_3}) \vee (v_0 = v_{i_1} = v_{i_2} = v_{i_3})$. Assume we are in the first case of the disjunction and that, for instance $u_0 = u_2 = u_4 = u_6$. Now, we try to replace the subterm $r(u_2, v_2, \ldots)$ with $r(u_4, v_4, \ldots)$ and $r(u_6, v_6, v)$ respectively. Since $u_2 = u_4 = u_6 \neq u_1$, if each of these replacements creates a close equality, we must have $v_1 = v_4 = v_6$. Finally, replacing $r(u_4, v_4, \ldots)$ by $r(u_6, v_6, v)$, we cannot create a close equality since $u_6 = u_4 \neq u_3$ and $v_6 = v_4 \neq v_3$.

Proof. (Lemma 10) We can first extract from $p_1 \ldots p_{g(\mathcal{A},k)}$ a subsequence $p_{l_0} \ldots p_{l_{k_1}}$ such that $\rho(p_{l_0}) \ldots \rho(p_{l_{k_1}})$ have all the same target state, with:

$$k_1 := \frac{g(\mathcal{A}, k)}{|Q|} = (e \times k + 1) \times 2^{c(\mathcal{A})} \times c(\mathcal{A})!$$

Let us define $u_0 := p_{l_0}, \ldots, u_{k_1} := p_{l_{k_1}}$ To extract a second subsequence we use a function test(p) defined on the positions of $Pos(\rho)$ and such that for all $p \in Pos(\rho)$:

$$\text{test}(p) = \left\{ (p', \pi) \begin{array}{|l} p' \prec p \preccurlyeq p'\pi \\ \exists \pi' \text{ s.t. } (\pi \neq \pi') \text{ or } (\pi' \neq \pi) \text{ is a constraint of } \rho(p') \end{array} \right\}$$

With this function test(p), we associate to each position $p \in Pos(\rho)$ a set of positions cr(p) defined by:

$$\text{cr}(p) := \{(p'\pi)/p \mid (p', \pi) \in \text{test}(p)\}$$

The quotient $(p'\pi)/p$ of two positions is defined by: $pp'/p := p'$. The figure 3 illustrates the definition of cr(p). Note that if $(p', \pi) \in \text{test}(p)$, then $(p'\pi)/p$ is well defined.

Fact 1. For all $p \in Pos(\rho)$, cr(p), $|\text{cr}(p)| \leq c(\mathcal{A})$

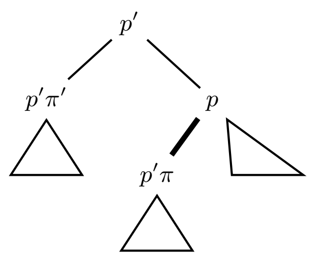

Fig. 3. The bold branch is in cr(p).

We see that for all $p \in Pos(\rho)$, cr(p) is included in the set of suffixes of positions π and π' such that $(\pi \neq \pi')$ is a atomic constraint occurring in one of the rules of Δ. Thus the number of distinct sets cr(p) for $p \in Pos(\rho)$ is smaller than $2^{c(\mathcal{A})}$. We can extract a subsequence $u_{l'_0} \ldots u_{l'_{k_2}}$ of positions from $u_0 \ldots u_{k_1}$ such that $\text{cr}(u_{l'_0}) = \ldots = \text{cr}(u_{l'_{k_2}})$, with:

$$k_2 := \frac{k_1}{2^{c(\mathcal{A})}} = (e \times k + 1) \times c(\mathcal{A})!$$

We note $v_0 := u_{l'_0}, \ldots, v_{k_2} := u_{l'_{k_2}}$. Then we are going to show that we can finally extract from $v_0 \ldots v_{k_2}$ another subsequence corresponding to the one in Lemma 10. This is a consequence of the following intermediate Lemma 11. Some additional definitions and notations are used in this Lemma 11. The *dependency degree* of a subsequence $v_{i_0} \dots v_{i_m}$ of $v_0 \dots v_{k_2}$ is:

$$\mathrm{dep}(v_{i_0} \dots v_{i_m}) := |\{\beta \in \mathrm{cr}(v_0) \mid t|_{v_{i_0}\beta} = \dots = t|_{v_{i_m}\beta}\}|$$

where $t \in \mathcal{T}(\mathcal{F})$ is the term associated[3] to ρ.
Let $f(n)$ be an integer function recursively defined on the interval $[0 \dots c(\mathcal{A})]$ by:

$$\begin{aligned} f(c(\mathcal{A})) &= k \\ f(n) &= (c(\mathcal{A}) - n) \times (f(n+1) + 1) + k - 1 \ \text{ for } n < c(\mathcal{A}) \end{aligned}$$

Lemma 11. *If for all $0 \leq j \leq k_2$, the cardinal of the set $\{j' \mid k_2 \geq j' > j,\ \rho[\rho|_{v_{j'}}]_{v_j}$ has no close equality$\}$ is smaller than k then for all $0 \leq n \leq c(\mathcal{A})$, there exists a subsequence $v_{i_0} \dots v_{i_{f(n)}}$ of $v_0 \dots v_{k_2}$ such that $\mathrm{dep}(v_{i_0} \dots v_{i_{f(n)}}) \geq n$.*

Proof. We assume that the hypothesis of Lemma 11 is true and we prove the conclusion by induction on n.
For $n = 0$, by definition of the function dep, for every subsequence $v_{i_0} \dots v_{i_m}$ of $v_0 \dots v_{k_2}$, we have $\mathrm{dep}(v_{i_0} \dots v_{i_m}) \geq 0$. Thus on this case, it is sufficient to show that $f(0) \leq k_2$. Let $F(n) = f(c(\mathcal{A}) - n)$ for all $0 \leq n \leq c(\mathcal{A})$.

$$\begin{aligned} F(0) &= k \\ F(n) &= n(F(n-1) + 1) + k - 1 \ \text{ for } 1 \leq n \leq c(\mathcal{A}) \end{aligned}$$

Developing,

$$\begin{aligned} F(n) &= n! \times (F(0) + 1) + k \times n! \sum_{i=1}^{n} \frac{1}{i!} - 1 \\ &\leq k \times n! + n! + k \times n! \times (e - 1) - 1 \\ &\leq n! \times (k \times e + 1) \end{aligned}$$

Thus,

$$\begin{aligned} f(0) &= F(c(\mathcal{A})) \\ &\leq c(\mathcal{A})! \times (k \times e + 1) \\ &\leq (e \times k + 1) \times c(\mathcal{A})! \\ &\leq k_2 \end{aligned}$$

For $n+1$, assume that the property is true for $n < c(\mathcal{A})$. By induction hypothesis, we have a subsequence $v_{i_0} \dots v_{i_{f(n)}}$ extracted from $v_0 \dots v_{k_2}$ such that $\mathrm{dep}(v_{i_0} \dots v_{i_{f(n)}}) \geq n$. Moreover, by the hypothesis of Lemma 11, for at least $f(n) - (k-1) = (c(\mathcal{A}) - n) \times (f(n+1) + 1)$ positions w among $v_{i_1} \dots v_{i_{f(n)}}$, $\rho[\rho|_w]_{v_{i_0}}$ has a close equality. We let:

$$k_3 = (c(\mathcal{A}) - n) \times (f(n+1) + 1)$$

and we let $w_1 \dots w_{k_3}$ be the above positions w, assuming that $w_1 \dots w_{k_3}$ is a subsequence of $v_{i_1} \dots v_{i_{f(n)}}$. By definition of close equalities, for all j such that $1 \leq j \leq k_3$, there exists $\beta_j \in \mathrm{cr}(v_{i_0}) = \mathrm{cr}(v_0)$, there exists $v \prec v_{i_0}$ and $(\pi \neq \pi')$ an atomic constraint in $\rho(v)$ such that (we only consider one case because of the symmetry):

$$v_{i_0}\beta_j = v\pi' \tag{1}$$

$$t|_{v_{i_0}\beta_j} \neq t|_{v\pi} \tag{2}$$

$$t|_{w_j\beta_j} = t|_{v\pi} \tag{3}$$

[3] the term associated to the run ρ is the term on which the ADC $\mathcal{A}$ makes the run ρ.

Lets recall that $t \in \mathcal{T}(\mathcal{F})$ is the term associated to the run ρ. The construction of β_j is depicted on figure 4.

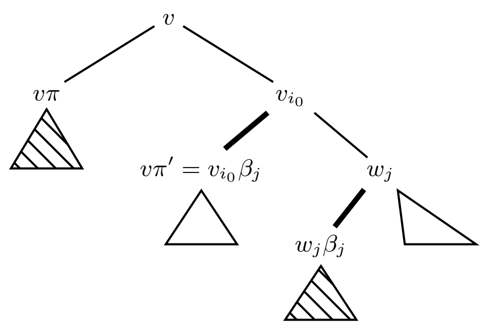

Fig. 4. Definition of β_j, proof of Lemma 11.

By definition of $\mathrm{dep}(v_{i_0} \dots v_{i_{f(n)}}) \geq n$, there exists a subset $E \subseteq \mathrm{cr}(v_0) = \mathrm{cr}(v_{i_0})$ such that:

$$|E| = n \tag{4}$$

$$\text{for all } \beta \in E,\ t|_{v_{i_0}\beta} = \dots = t|_{v_{i_{f(n)}}\beta} \tag{5}$$

In particular, for all $\beta \in E$, $t|_{v_{i_0}\beta} = t|_{w_1\beta} \dots = t|_{w_{k_3}\beta}$. Hence, $\{\beta_1 \dots \beta_{k_3}\} \cap E = \emptyset$ by (2) and (3). Moreover, according to the above fact 1, $|\mathrm{cr}(v_0)| \leq c(\mathcal{A})$. This implies that there are at most $c(\mathcal{A}) - n$ distinct positions among $\beta_1 \dots \beta_{k_3}$. Thus there exists: $1 \leq j_0 < \dots < j_{f(n+1)} \leq k_3$ such that $\beta_{j_0} = \dots = \beta_{j_{f(n+1)}}$, because $\frac{k_3}{c(\mathcal{A})-n} = f(n+1)+1$. Let β' this unique position. By construction:

$$t|_{w_{j_0}\beta'} = \dots = t|_{w_{j_{f(n+1)}}\beta'}$$

Let us recall that by definition of E, $\beta' \notin E$, hence:

$$\mathrm{dep}(w_{j_0} \dots w_{j_{f(n+1)}}) > \mathrm{dep}(v_{i_0} \dots v_{i_{f(n)}}) \geq n$$

This completes the proof of Lemma 11 because $w_{j_0} \dots w_{j_{f(n+1)}}$ is a subsequence of $v_0 \dots v_{k_2}$.
(end of the proof of Lemma 11) $\square$

Now, we have to finish the proof of Lemma 10. We will show that the hypothesis of Lemma 11 cannot be true. Assume it is true. Thus, for $n = c(\mathcal{A})$ and $f(n) = k$, there exists a subsequence $v_{i_0} \dots v_{i_k}$ of $v_0 \dots v_{k_2}$ such that $\mathrm{dep}(v_{i_0} \dots v_{i_k}) \geq c(\mathcal{A})$. But, by the above fact, $|\mathrm{cr}(v_0)| \leq c(\mathcal{A})$, thus by definition of $\mathrm{dep}(v_{i_0} \dots v_{i_k})$ we have:

$$\text{for all } \beta \in \mathrm{cr}(v_0),\ t|_{v_{i_0}\beta} = \dots = t|_{v_{i_k}\beta}$$

Assume now that one of the pumping $\rho[\rho|_{v_{i_j}}]_{v_{i_0}}$ for $1 \leq j \leq k$ has a close equality. This means that there exists $v \prec v_{i_0}$ and $(\pi \neq \pi')$, an atomic constraint in $\rho(v)$ such that, $v \prec v_{i_0} \preccurlyeq v\pi'$ and $t|_{v\pi} = t|_{v_{i_j}\beta}$. The position $\beta := (v\pi')/v_{i_0} \in \mathrm{cr}(v_{i_0})$ is such that $t|_{v\pi} \neq t|_{v_{i_0}\beta}$ and with $t|_{v\pi} = t|_{v_{i_j}\beta}$ this contradicts $t|_{v_{i_0}\beta} = t|_{v_{i_j}\beta}$ (see figure 5).

Thus for all $1 \leq j \leq k$, the pumping $\rho[\rho|_{v_{i_j}}]_{v_0}$ does not have any close equality. This completes the proof of Lemma 10. $\square$

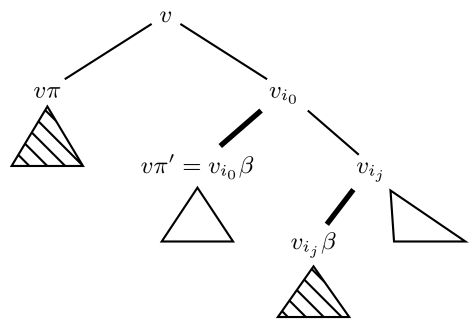

Fig. 5. Proof of Lemma 10.

4.2 Pumping without creating equalities

Definition 5. *$\mathcal{M}$ is the predicate (defined relatively to an ADC $\mathcal{A}$ and an ordering $\gg$) which holds true on a run ρ of $\mathcal{A}$, a position p of ρ and an integer k iff there exists k runs $\rho|_p \gg \rho_k \gg \ldots \gg \rho_1$ such that $\rho(p), \rho_1(\Lambda), \ldots, \rho_k(\Lambda)$ have the same target state and for every $1 \leq i \leq k$ the pumping $\rho[\rho_i]_p$ does not contain any close equality.*

We list without proof two obvious consequences of Definition 5.

Lemma 12. *If $k \geq k'$ then $\mathcal{M}(\rho, p, k)$ implies $\mathcal{M}(\rho, p, k')$.*

Lemma 13. *If a run ρ is such that $\mathcal{M}(\rho, \Lambda, k)$ for some $k \geq 1$, then there exists a run $\rho' \ll \rho$ such that the target states of $\rho'(\Lambda)$ and $\rho(\Lambda)$ are the same.*

Let

$$h(\mathcal{A}, k) = (d(\mathcal{A}) + 1) \times n(\mathcal{A}) \times \Big[k + g(\mathcal{A}, k + 2d(\mathcal{A}) \times n(\mathcal{A}))\Big]$$

where $n(\mathcal{A})$ is the maximal number of atomic constraints occurring in a rule of $\mathcal{A}$ and $d(\mathcal{A})$ is the maximal length $|\pi|$ or $|\pi'|$ among every atomic constraints $\pi \neq \pi'$ in the transitions rule of $\mathcal{A}$.

The following propagation lemma is the crux part of our proof. (It is also very technical to prove). It explains how to get rid of far equalities, if we have enough pumpings which do not create close equalities. The underlying intuitive idea behind Lemma 14 is the following. If we assume $h(\mathcal{A}, k)$ pumpings below p, which do not create close equalities (it will be possible to construct such pumpings thanks to Lemma 10), either one of them yields a run, and we completed our goal, or each of them contains a far equality. However, all these far equalities give us some information on the structure of the original run, and we are going to take advantage of this to design new other pumpings, which, combined with the original ones, ensure again $h(\mathcal{A}, k)$ pumpings below $p' < p$ each of them not containing equalities below p'. This allows to prime an induction: we can construct pumpings such that $\rho|_{p'}[\rho_i]$ is a run, provided that $\rho|_p[\rho_j]$ is a run. Eventually, we will have $p' = \Lambda$ and hence a pumping which is a run.

Lemma 14 (Propagation lemma). *Let ρ be a run of $\mathcal{A}$, $p \in Pos(\rho)$ and k be an integer such that $k^2 \geq h(\mathcal{A}, k)$. If $\mathcal{M}(\rho, p, h(\mathcal{A}, k))$ is true, then one of the following properties holds:*

1. *there is a run ρ' such that $\rho|_p \gg \rho'$ and $\rho[\rho']_p$ is a run*

2. there exists a position p' such that $|p'| < |p|$ and $\mathcal{M}(\rho, p', h(\mathcal{A}, k))$ is true.

We shall show below (page 20) that such an integer k exists, and depends on $\mathcal{A}$.

Proof. Assume $\mathcal{M}(\rho, p, h(\mathcal{A}, k))$ is true. This means that we have $h(\mathcal{A}, k)$ runs $\rho_1, \ldots, \rho_{h(\mathcal{A},k)} \in \mathcal{T}(\Delta)$ such that $\rho|_p \gg \rho_i$ and $\rho[\rho_i]_p$ does not create a close equality, for $1 \leq i \leq h(\mathcal{A}, k)$, following Definition 5 of $\mathcal{M}()$. If we are not in the first case of the lemma, then for each $1 \leq i \leq h(\mathcal{A}, k)$, $\rho[\rho_i]_p$ contains far equalities.

For each index $j \leq h(\mathcal{A}, k)$, let γ_j be a maximal position w.r.t. prefix ordering such that (γ_j, π, π') is a (far) equality of $\rho[\rho_j]_p$, see figure 6. Let E be the set of triples (γ_j, π, π'). We have $|E| = h(\mathcal{A}, k)$. Indeed, having two identical far equalities $(\gamma_j, \pi, \pi') = (\gamma_{j'}, \pi, \pi')$ for two distinct pumpings $\rho[\rho_j]_p$ and $\rho[\rho_{j'}]_p$ would mean an equality of two different terms to the same subterm.

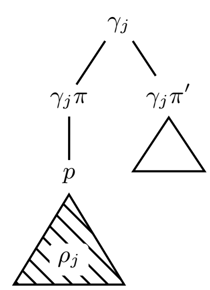

Fig. 6. The far equality (γ_j, π, π') in $\rho[\rho_j]_p$.

Moreover, the number of distinct first components of elements of E is:

$$\begin{aligned} |\{\gamma \mid \exists \pi, \pi' \ (\gamma, \pi, \pi') \in E\}| &\geq \tfrac{|E|}{n(\mathcal{A})} \\ &\geq (d(\mathcal{A}) + 1) \times \Big[k + g(\mathcal{A}, k + 2d(\mathcal{A}) \times n(\mathcal{A}))\Big] \end{aligned} \tag{6}$$

Note that every position γ_j is a prefix of p (since ρ and ρ_i are runs) hence the set of first components of E can be totally ordered by the prefix ordering.
Let u_i, $1 \leq i \leq (d(\mathcal{A}) + 1) \times [k + g(\mathcal{A}, k + 2d(\mathcal{A}) \times n(\mathcal{A}))]$, be a strictly decreasing sequence (w.r.t. the prefix ordering) of first components of elements in E.

We are going to show that $\mathcal{M}(\rho, u_i, k^2)$ is true for some i, which implies the second case of Lemma 14 by hypothesis and by Lemma 12.
First, we extract from the sequence (u_i) a subsequence (p_i) of length $k_1 := k + g(\mathcal{A}, k + 2d(\mathcal{A}) \times n(\mathcal{A}))$ defined by:

$$p_i = u_{(d(\mathcal{A})+1)\times i} \text{ for all } 1 \leq i \leq k_1 \tag{7}$$

This ensures that two positions p_i and p_j are distant enough.
To each integer $1 \leq i \leq k_1$, we can associate a unique index $\nu(i) \leq h(\mathcal{A}, k)$ defined by $p_i = \gamma_{\nu(i)}$. By construction, for every equality (γ, π, π') of the pumping $\rho[\rho_{\nu(i)}]_p$ one has $\gamma \preccurlyeq p_i$ (p_i is itself one of these γ which has been chosen to be maximal).

Now, we consider any given pumping $\rho[\rho_{\nu(m)}]_p$ for $k_1 - k + 1 \leq m \leq k_1$ (i.e. p_m is one of the k smallest positions p_i) and we show that there is one position $p_{i_{m,0}}$, $i_{m,0} \leq k_1 - k$, and k other pumpings on $\rho[\rho_{\nu(m)}]_p$ whose equalities are far from $p_{i_{m,0}}$. For sake of simplicity, we note $\rho'_m := \rho[\rho_{\nu(m)}]_p$. Note that, by construction, for each $k_1 - k + 1 \leq m \leq k_1$, $\rho'_m|_{p_{k_1-k}}$ is a run. We shall apply Lemma 10 to these runs in order to find appropriate pumpings.
To each $1 \leq i \leq k_1$ we can associate some positions π_i and π'_i such that $(p_i, \pi_i, \pi'_i) \in E$ (i.e. it is a far equality of some $\rho[\rho_{\nu(i)}]_p$). Moreover, with this construction, by definition of far equalities, for each i, we have either $p_i\pi_i \prec p$ or $p_i\pi'_i \prec p$ (p is from the hypotheses of Lemma 14). By symmetry, we assume that we are in the first case for all i. Note that by construction of (p_i), and by definition of $d(\mathcal{A})$, we have:

$$p_1\pi_1 \succ p_1 \succ p_2\pi_2 \succ p_2 \succ \ldots \succ p_m \tag{8}$$

The situation is depicted in figure 7. The equality (p_i, π_i, π'_i) is a far equality, hence, for every

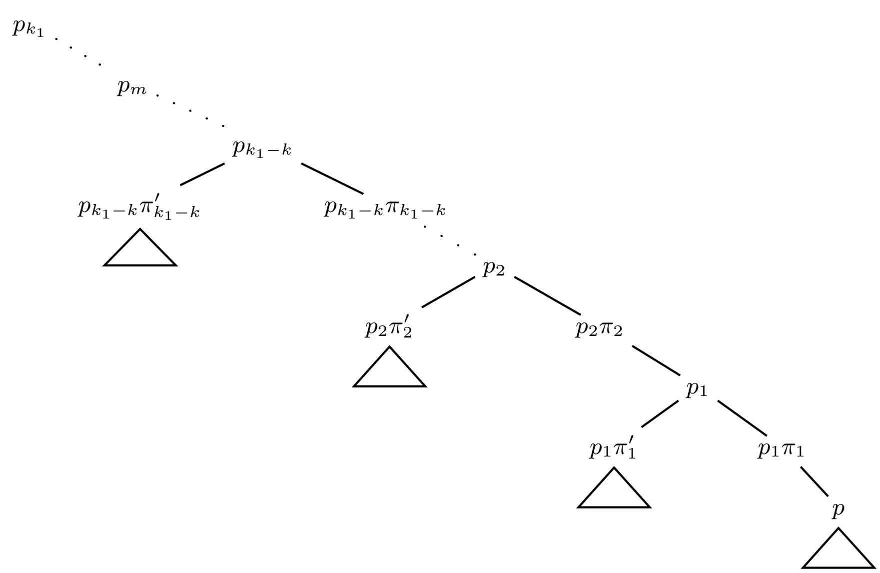

Fig. 7. Proof of Lemma 14.

i, p_i, $p_i\pi_i$ and $p_i\pi'_i$ are indeed positions of ρ'_i and $\rho'_i|_{p_i\pi_i} = \rho|_{p_i\pi'_i}$. Following (8), $\rho|_{p_l\pi'_l}$ is a subterm of $\rho'_i|_{p_i\pi'_i}$ for $l > i$. It follows that the terms $\rho|_{p_i\pi'_i}$ are pairwise distinct.

Hence, we can apply Lemma 10 to the run $\rho'_m|_{p_{k_1-k}}$ and the positions $p_1\pi'_1, \ldots, p_{k_1-k}\pi'_{k_1-k}$ (of ρ). Note in particular that $k_1 - k = g(\mathcal{A}, k + 2d(\mathcal{A}) \times n(\mathcal{A}))$ and that $\rho'_m|_{p_{k_1-k}\pi'_{k_1-k}} \gg \ldots \gg \rho'_m|_{p_1\pi'_1}$. This yields a subsequence $(p_{i_{m,j}}\pi'_{i_{m,j}})$, with $0 \leq j \leq k + 2d(\mathcal{A}) \times n(\mathcal{A})$, such that every pumping $\rho''_{m,j} := \rho'_m[\rho'_m|_{p_{i_{m,j}}\pi'_{i_{m,j}}}]_{p_{i_{m,0}}\pi'_{i_{m,0}}}$ does not contain close equalities (note that these pumpings $\rho''_{m,j}$ are pairwise distinct).

The pumpings $\rho''_{m,j}$ may though contain some far equalities. In the following table, we give (upper bounds for) the number of these far equalities, w.r.t. the positions $p_{i_{m,0}}$ and $p_{i_{m,0}}\pi'_{i_{m,0}}$. See also figure 8 for a picture of the three situations.

Equalities	Max. number
(γ, π, π') far from $p_{i_{m,0}}\pi'_{i_{m,0}}$ and $p_{i_{m,0}} \prec \gamma$	$d(\mathcal{A}) \times n(\mathcal{A})$
(γ, π, π') far from $p_{i_{m,0}}\pi'_{i_{m,0}}$ and close to $p_{i_{m,0}}$	$d(\mathcal{A}) \times n(\mathcal{A})$
(γ, π, π') far from $p_{i_{m,0}}\pi'_{i_{m,0}}$ and far from $p_{i_{m,0}}$	$\lvert p_{i_{m,0}}\rvert \times n(\mathcal{A})$

For the first two lines of this table, there are at most $d(\mathcal{A})$ possible positions for γ and at most $n(\mathcal{A})$ possible equalities for each of these positions. For the last line, the maximal number of positions is $|p_{i_{m,0}}|$. Note that every equality in one of the pumpings $\rho'_m[\rho'_m|_{p_{i_{m,j}}\pi'_{i_{m,j}}}]_{p_{i_{m,0}}\pi'_{i_{m,0}}}$ is registered in this array. Thus, there exists at least k pumpings of the form $\rho'_m[\rho'_m|_{p_{i_{m,j}}\pi'_{i_{m,j}}}]_{p_{i_{m,0}}\pi'_{i_{m,0}}}$ every equality of which is far from $p_{i_{m,0}}$.

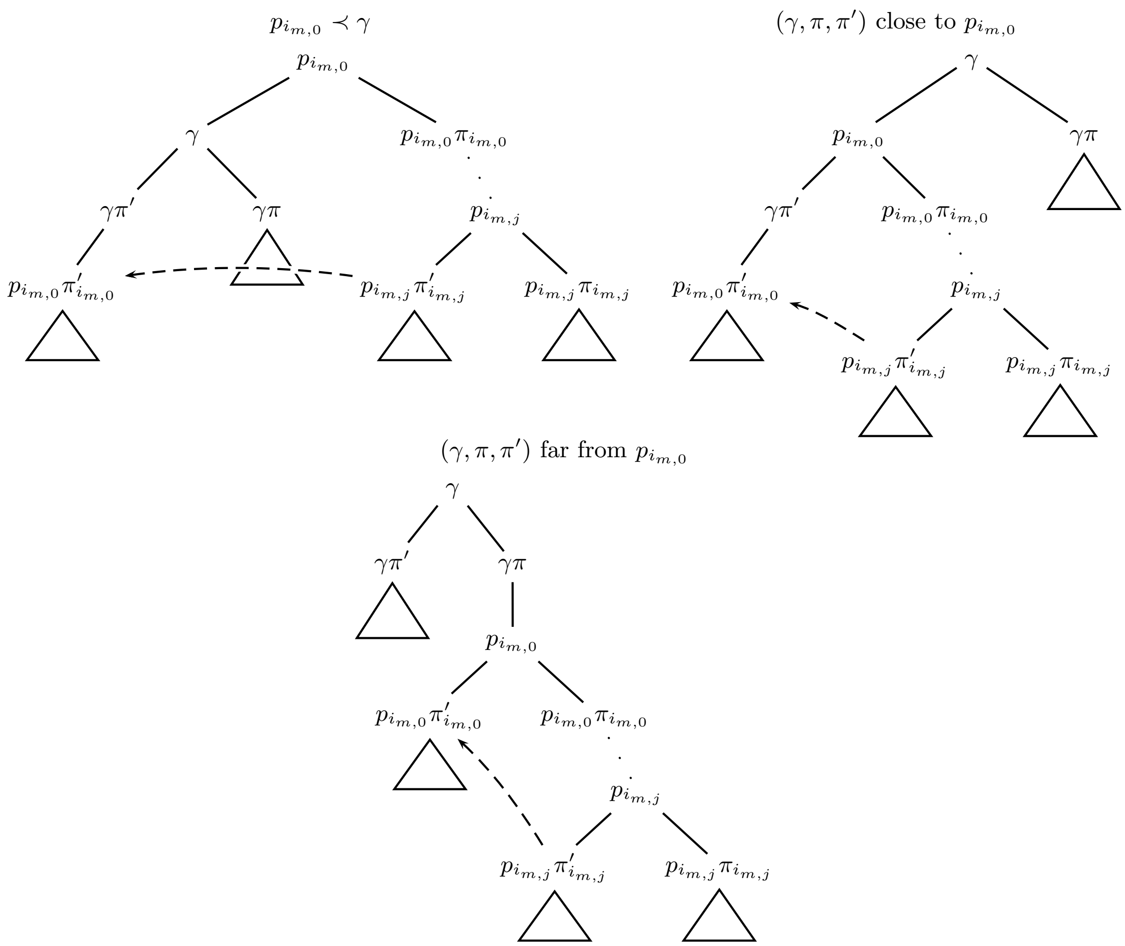

Fig. 8. (γ, π, π') far from $p_{i_{m,0}}\pi'_{i_{m,0}}$.

Every equality in ρ'_m itself is also far from $p_{i_{m,0}}$ since:

1. the first component of such each an equality is one of $p_{k_1-k+1}, \ldots, p_{k_1}$

2. $p_{k_1} \prec p_{k_1-k+1} \prec p_{i_{m,0}}$
3. the distance between p_{k_1} and $p_{i_{m,0}}$ is at least $d(\mathcal{A})+1$ ($|p_{i_{m,0}}| - |p_{k_1}| \leq d(\mathcal{A})+1$).

To summarise, we have k possible pumpings $\rho'_m \ll \rho$ and for each of them we have k other pumpings $\rho''_{m,j} \ll \rho'_m$ ($j \leq k$) such that every equality in a $\rho''_{m,j}$ is far from some position $p_{i_{m,0}} \prec p$. By the remarks above, the $\rho''_{m,j}$ are pairwise distinct. Let p' be the largest of the above $p_{i_{m,0}}$. With the remark that all of these pumping are only replacement at some positions bigger than p', we proved $\mathcal{M}(\rho, p', k^2)$ thus $\mathcal{M}(\rho, p', h(\mathcal{A}, k))$, by Lemma 12. (end of proof of Lemma 14)□

We initiate the process with Lemma 10 and use the propagation lemma 14 to push the position under which no equality is created, up to the root of the tree. With simple sufficient conditions for the inequality $k^2 \geq h(\mathcal{A}, k)$, this yields the following lemma 15, where $cs(\mathcal{A})$ is the global size of constraints in the transition rules of $\mathcal{A}$.

Lemma 15. *Let $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ be an ADC . There exists two constants γ and δ independent from $\mathcal{A}$ such that if $\mathcal{M}\Big(\rho, p, \gamma \times |Q|^2 \times 2^{\delta . cs(\mathcal{A})^2 . \ln\left(cs(\mathcal{A})\right)}\Big)$ is true for some position p of a run ρ of $\mathcal{A}$ then there is a run of $\mathcal{A}$ $\rho' \ll \rho$ such that $\rho(\Lambda)$ and $\rho'(\Lambda)$ have the same target state.*

Proof. We shall use Lemma 14. Hence, we need an integer k such that:

$$k^2 \geq h(\mathcal{A}, k) \tag{9}$$

For sake of simplicity, we write c, n and d respectively for $c(\mathcal{A})$, $n(\mathcal{A})$ and $d(\mathcal{A})$. We assume that $d, n \geq 1$.

$$\begin{aligned}
h(\mathcal{A}, k) &\leq (d+1).n \times [k + g(\mathcal{A}, k + 2dn)] \\
&\leq (d+1).n \times [k + |Q|.2^c.c! \times (ek + 2edn + 1)] \\
&\leq (d+1).n.(e.|Q|.2^c.c! + 1).k + (d+1).n.|Q|.2^c.c! \times (2edn + 1) \\
&\leq \alpha.|Q|.2^{\beta.c.\ln(c)}.dn.k + \alpha.|Q|.2^{\beta.c.\ln(c)}.d^2.n^2 \quad \text{where } \alpha \text{ and } \beta \text{ are constants independent of } \mathcal{A} \\
&\leq m.k + m.dn \quad \text{where } m := \alpha.|Q|.2^{\beta.c.\ln(c)}.dn
\end{aligned}$$

One can check that for (9), it is sufficient to have $k = m + dn$. Therefore,

$$\begin{aligned}
h(\mathcal{A}, k) = h(\mathcal{A}, m + dn) &= m^2 + 2mdn \\
&= \alpha^2.|Q|^2.2^{2\beta.c.\ln(c)}.d^2n^2 + 2\alpha.|Q|.2^{\beta.c.\ln(c)}.d^2n^2 \\
&\leq 2\alpha^2|Q|^2.2^{2\beta.c.\ln(c)}.d^2n^2 \\
&\leq 2\alpha^2|Q|^2.2^{2\beta.dn.\ln(dn)}.d^2n^2 \quad \text{because } c(\mathcal{A}) \leq d(\mathcal{A}) \times n(\mathcal{A}) \\
&\leq 2\alpha^2|Q|^2.2^{4\beta.cs(\mathcal{A})^2.\ln\left(cs(\mathcal{A})\right)}.cs(\mathcal{A})^4 \quad \text{because } d(\mathcal{A}), n(\mathcal{A}) \leq cs(\mathcal{A}) \\
&\leq \gamma|Q|^2 \times 2^{\delta \times cs(\mathcal{A})^2 \times \ln\left(cs(\mathcal{A})\right)} \quad \text{with } \gamma = 2\alpha^2 \text{ and } \delta = 4\beta + 2
\end{aligned}$$

The rest follows by induction on the depth of ρ, using the propagation Lemma 14 and the Lemmas 12 and 13. □

5 Emptiness Decision for ADC

In this section we present the following result:

Theorem 1. *There is an algorithm which decides the emptiness of an ADC $\mathcal{A} = (Q, Q^{\mathrm{f}}, \Delta)$ and which runs in time $\mathrm{O}\Big(\big(|Q| \times |\Delta|\big)^{P(cs(\mathcal{A}))}\Big)$ where P is a polynomial.*

5.1 Algorithm

We use a marking algorithm in which each state is marked with some successful runs yielding the state. This generalises the usual marking algorithm for finite bottom-up tree automata: we do not keep only the information that a state is inhabited but also keep witnesses of this fact. The witnesses are used to check the disequality constraints higher up in the run.

To choose the witnesses runs which mark the states and ensure the termination of the algorithm, we use a sufficient condition for the above $\mathcal{M}$ predicate. In the algorithm, we use the set $C(\mathcal{A})$ of suffixes of positions π, π' in an atom $\pi \neq \pi'$ occurring in a constraint of transition rule of $\mathcal{A}$. Note that $c(\mathcal{A}) = |C(\mathcal{A})|$. We use also a bound evaluated in the proof of Lemma 15:

$$b(\mathcal{A}) := \gamma \times |Q|^2 \times 2^{\delta . cs(\mathcal{A})^2 . \ln\left(cs(\mathcal{A})\right)} \tag{10}$$

We assume moreover that the constants γ and δ are such that:

$$b(\mathcal{A}) > |Q| \times |\mathcal{F}| \tag{11}$$

Emptiness decision algorithm.

Start with a mapping which associates each state q with an empty set E_q^0 then saturate the states E_q^0 using the rule: $\frac{\{\rho_1,\ldots,\rho_n\} \in \bigcup_{i=0}^m \bigcup_{q\in Q} E_q^i}{r(\rho_1,\ldots,\rho_n) \in E_{q_0}^{m+1}}$
under the conditions:
1. $r(\rho_1, \ldots, \rho_n)$ is a run,
2. the target state of r is q_0,
3. for every $p \in Pos(\rho) \setminus C(\mathcal{A})$, with $|p| \leq d(\mathcal{A})+1$, there exists no sequence of length $b(\mathcal{A})$ of runs of $\bigcup_{i=0}^m \bigcup_{q\in Q} E_q^i$, $\rho|_p \gg \rho'_{b(\mathcal{A})} \gg \ldots \gg \rho'_1$ such that $\rho(p), \rho'_1(\Lambda), \ldots, \rho'_{b(\mathcal{A})}(\Lambda)$ have the same target state and for every $1 \leq j \leq b(\mathcal{A})$, the pumping $\rho[\rho'_j]_p$ does not contain any close equality.

We consider only fair executions of the algorithm: we assume that for each runs $\rho_1, \ldots, \rho_n$ constructed and each $r \in \Delta$, $r(\rho_1, \ldots, \rho_n)$ is eventually checked. We also also assume some marking of the runs checked which prevents the algorithm to check the same run twice.

We have to prove on one hand that the saturated set $E^* := \bigcup_{m\geq 0} \bigcup_{q\in Q} E_q^m$ contains an accepting run iff $\mathcal{A}$ accepts at least one tree (correctness, completeness) and on the other hand that E^* is computed with the expected complexity.

5.2 Correctness and completeness

Lemma 16 (Correctness). *If E^* contains an accepting run then $L(\mathcal{A})$ is not empty.*

Proof. Immediate by the condition "1. $r(\rho_1, \ldots, \rho_n)$ is a run" in the emptiness decision algorithm. □

Lemma 17 (Completeness). *If E^* does not contain any accepting run then $L(\mathcal{A})$ is empty.*

Proof. Assume that $\mathcal{A}$ accepts at least one ground term, and let ρ be an accepting run of $\mathcal{A}$, minimal w.r.t. $\gg$. We prove that $\rho \in E^*$ by contradiction.

Assume that $\rho \notin E^*$, and let μ be a subterm of ρ, minimal w.r.t. the subterm ordering in the set of subterms of ρ which do not belong to E^*.

We can first show that have $depth(\mu) > 0$. Indeed, if $depth(\mu) = 0$, every run $\mu' \ll \mu$ has the form $c \to q$. By (11) the number of such runs is smaller that $b(\mathcal{A})$, hence the condition 3 of the emptiness decision algorithm must be true for μ, and therefore $\mu \in E^*$, a contradiction.

Hence, let $\mu = r(\mu_1, \ldots, \mu_n)$. By minimality hypothesis, $\mu_1, \ldots, \mu_n \in E^*$. Since $\mu \notin E^*$, the condition 3 of the emptiness decision algorithm is not true, which means that there exists a position $p \in Pos(\mu) \setminus C(\mathcal{A})$, with $|p| \leq d(\mathcal{A}) + 1$, and a sequence $\mu|_p \gg \mu'_{b(\mathcal{A})} \gg \ldots \gg \mu'_1$ of runs of E^* such that $\mu(p), \mu'_1(\Lambda), \ldots, \mu'_{b(\mathcal{A})}(\Lambda)$ have the same target state and for every $1 \leq j \leq b(\mathcal{A})$, the pumping $\mu[\mu'_j]_p$ does not contain any close equality.

Let $p' \in Pos(\rho)$ be the position such that $\mu = \rho|_{p'}$. Since $p \notin C(\mathcal{A})$, and by definition of close equalities page 4, for all $1 \leq j \leq b(\mathcal{A})$, the pumping $\rho[\mu'_j]_{p'.p}$ does not contain any close equality. By Lemma 15, this contradicts the minimality of ρ w.r.t. $\ll$. $\square$

5.3 Termination and complexity

In order to show the termination of the emptiness decision algorithm and to give a complexity bound, we need an additional argument: a generalization of König's theorem for bipartite graphs to hypergraphs. Let us first define a notion of dependency in hypergraphs:

Definition 6. *Let S be a set and n, k be integers. The n-uples $\overline{s_1}, \ldots, \overline{s_k}$ of elements in S are* independent *iff there is a set $I \subseteq \{1, \ldots, n\}$ such that*

- $\forall i \in I, s_{1,i} = \ldots = s_{k,i}$
- $\forall i \notin I, \forall j \neq j', s_{j,i} \neq s_{j',i}$

Now, we have the analogue of König's theorem:

Theorem 2 (B. Reed, private communication). *Let S be a set and K, n be integers. Let $G \subseteq S^n$. If every subset $G_1 \subseteq G$ of independent elements has a cardinal $|G_1| \leq K$, then $|G| \leq K^n \times n!$*

Proof. We prove the result by induction on n.
For $n = 1$, G itself is a set of independent elements, hence $|G| \leq K$.
Assume now that the property holds for $n - 1$. Consider the graph $H(G)$ whose vertices are the elements of G and such that there is an edge (g, g') iff there is a component $i \in [1..n]$ such that $g_i = g'_i$. Any stable subset G_1 of G is independent, hence $|G_1| \leq K$ (a subset G_1 of G is stable if $G_1 \times G_1$ contains no edge of $H(G)$).

Now, let V be the maximal number of edges sharing some vertex v_0 in $H(G)$ (maximal neighbourhood). We construct a stable set G_1 whose cardinal is $|G_1| \geq \frac{|G|}{V+1}$ as follows:

> Initially, $G_2 = G$ and $G_1 = \emptyset$.
> Repeat the following:
>   Choose a vertex v in G_2 and put it in G_1
>   remove v from G_2 as well as all vertices w such that $(v, w) \in H(G)$
> until G_2 is empty.

Since, at each step, we remove at most $V + 1$ elements from G_2, we have at least $\frac{|G|}{V+1}$ steps, and, at each step, we add an element in G_1. Hence $|G_1| \geq \frac{|G|}{V+1}$. Moreover, G_1 is stable, by construction.

Now, $K \geq |G_1| \geq \frac{|G|}{V+1}$, hence $V \geq \frac{|G|}{K} - 1$. Assume, by contradiction, that $|G| > K^n \times n!$, then $V > \frac{K^n \times n!}{K} - 1 = K^{n-1} \times n! - 1$. Now, there are at least $K^{n-1} \times n!$ edges departing from v_0 in $H(G)$. Hence there is an index $j \in [1..n]$ such that, for at least $K^{n-1} \times (n-1)!$ vertices v in $H(G)$, $v_0^j = v^j$. Let $G' = \{(v_1, ..., v_{j-1}, v_{j+1}, \ldots, v_n) | (v_1, \ldots, v_{j-1}, v_0^j, v_{j+1}, \ldots, v_n) \in G\}$ every subset G'_1 of G' of independent elements has a cardinal smaller or equal to K and $|G'| \geq K^{n-1} \times (n-1)! + 1$, which contradicts the induction hypothesis. $\square$

Independence of runs Let us denote $C(\mathcal{A}) = \{\pi_1, \ldots, \pi_{c(\mathcal{A})}\}$ and let ρ be a run of $\mathcal{A}$. Then Check(ρ) is the tuple $(t_1, \ldots, t_{c(\mathcal{A})}) \in (\mathcal{T}(\mathcal{F}) \cup \{\perp\})^{c(\mathcal{A})}$ such that $t_i = \perp$ if $\pi_i \notin Pos(\rho)$ and t_i is the term of $\mathcal{T}(\mathcal{F})$ associated to $\rho|_{\pi_i}$ otherwise. We say that the runs $\rho_1, \ldots, \rho_k$ are independent if Check$(\rho_1), \ldots,$ Check(ρ_k) are independent.

Lemma 18. *Let ρ be a run of the ADC $\mathcal{A}$ and $p \in Pos(\rho)$, let $k > b(\mathcal{A})$ and $\gg$ be a total ordering. If there are k runs $\rho_1, \ldots, \rho_k$ such that $\rho_1 \ll \ldots \ll \rho_k \ll \rho|_p$, and $\rho_1(\Lambda), \ldots, \rho_k(\Lambda)$ and $\rho(p)$ have the same target state, and $\rho_1, \ldots, \rho_k$, $\rho|_p$ are independent, then there are at least $k - c(\mathcal{A}) \times cs(\mathcal{A})$ different pumpings $\rho[\rho_i]_p$ (with $i \leq k$) without close equalities.*

Note that the totality of $\gg$ is sufficient for this lemma.

Proof. Let t, $t_1, \ldots, t_k$ be the terms associated to respectively ρ, $\rho_1, \ldots, \rho_k$. By hypothesis, we have a subset $I \subseteq \{1, \ldots, c(\mathcal{A})\}$ such that:

- for each $i \in I$, for each $j \leq k$, $t_j|_{\pi_i} = t|_{p.\pi_i}$
- for each $i \notin I$, for ever $j \neq l \leq k$, $t_j|_{\pi_i} \neq t_l|_{\pi_i}$

Let $J \subseteq \{1, \ldots, k\}$ be the set of all indices j such that $\rho[\rho_j]_p$ contains a close equality. We associate with each $j \in J$:

- a close equality (p_j, α_j, β_j) of $\rho[\rho_j]_p$,
- the integer $i(j)$ such that $C(\mathcal{A}) \ni \pi_{i(j)} = p_j\alpha_j \setminus p$ (by construction, $p_j\alpha_j \succ p$ or $p_j\beta_j \succ p$, by symmetry, we may assume that $p_j\alpha_j \succ p$)

We note $p = p_j.p'_j$ and $\alpha_j = p'_j.\pi_{i(j)}$. The mapping $j \mapsto (p_j, \alpha_j, \beta_j, i(j))$ has an image of cardinality at most $c(\mathcal{A}) \times cs(\mathcal{A})$ (for a fixed p). Indeed, $\pi_{i(j)}$, α_j, β_j uniquely define p'_j, and p'_j, p determine p_j; and there are at most $cs(\mathcal{A})$ possible values for $i(j)$ and less than $cs(\mathcal{A})$ possible values for (α_j, β_j)

By contradiction, assume that $|J| \geq c(\mathcal{A}) \times cs(\mathcal{A}) + 1$. By a pigeon hole principle, there are two indices $j, j' \in J$ such that $p_j = p_{j'}$, $\alpha_j = \alpha_{j'}$, $\beta_j = \beta_{j'}$, and $i(j) = i(j')$. Hence,

$$t_j|_{\pi_{i(j)}} = t|_{p_j.\beta(j)} = t|_{p_{j'}.\beta(j')} = t_{j'}|_{\pi_{i(j')}} \tag{12}$$

Hence, $i(j) = i(j') \in I$, and by definition of I, it follows that $t_j|_{\pi_{i(j)}} = t|_{p.\pi_{i(j)}}$. Therefore, by (12), $t|_{p.\pi_{i(j)}} = \rho_{p_j.\beta(j)}$, and, noting that $t|_{p.\pi_{i(j)}} = t_{p_j.\alpha_j}$, it means that ρ contains a close equality, which is a contradiction. $\square$

Bound on the number of steps. Let $G^* \subseteq E^*$ be the subset of runs occurring as strict subterms of runs of E^* at positions not in $C(\mathcal{A})$.

$$G^* := \{\rho|_p \mid \rho \in E^*,\ p \in Pos(\rho) \setminus C(A)\}$$

Note that

$$|E^*| \leq |\Delta|^{c(\mathcal{A})} \times |G^*|^{c(\mathcal{A})\times\alpha} \tag{13}$$

where α is the maximal arity of a function symbol of $\mathcal{F}$. Indeed constructing a run $\rho \in E^*$, we shall fix the labels at positions in $C(\mathcal{A}) \cap Pos(\rho)$ (this can be done in at most $|\Delta|^{c(\mathcal{A})}$), and then choose at most $|G^*|^{c(\mathcal{A})\times\alpha}$ subruns belonging to G^*.

According to the condition 3 of the emptiness decision algorithm and to Lemma 18, for each state $q \in Q$, the set $G^* \cap \bigcup_{m\geq 0} E_q^m$ cannot contain more than $b(\mathcal{A}) + c(\mathcal{A}) \times cs(\mathcal{A}) + 1$ independent runs. By Theorem 2, and by definition of independence of runs, this means that

$$|G^* \cap \bigcup_{m\geq 0} E_q^m| \leq (b(\mathcal{A}) + c(\mathcal{A}) \times cs(\mathcal{A}) + 1)^{c(\mathcal{A})} \times c(\mathcal{A})!$$

Hence, with the remark that $G^* \subseteq E^*$,

$$|G^*| \leq |Q| \times (b(\mathcal{A}) + c(\mathcal{A}) \times cs(\mathcal{A}) + 1)^{c(\mathcal{A})} \times c(\mathcal{A})! \tag{14}$$

Every term considered by the emptiness decision algorithm, (and checked for conditions 1, 2 and 3) has the form $r(\rho_1, \ldots, \rho_n)$ where $r \in \Delta$ and $\rho_1, \ldots, \rho_n \in E^*$. This ensures the termination of the algorithm and together with (13) and (14), this gives the following bound for the number of steps for emptiness decision, for some polynomial P_1:

$$\begin{array}{rll}
\#\text{ of steps} & \leq |\Delta| \times |E^*|^\alpha & \\
& \leq |\Delta|^{c(\mathcal{A}).\alpha+1} \times |G^*|^{c(\mathcal{A}).\alpha^2} & (13) \\
& \leq |\Delta|^{c(\mathcal{A}).\alpha+1} \times |Q|^{c(\mathcal{A}).\alpha^2} \times (b(\mathcal{A}) + c(\mathcal{A}) \times cs(\mathcal{A}) + 1)^{c(\mathcal{A})^2.\alpha^2} \times (c(\mathcal{A})!)^{c(\mathcal{A}).\alpha^2} & (13) \\
& \leq |\Delta|^{c(\mathcal{A}).\alpha+1} \times |Q|^{c(\mathcal{A}).\alpha^2} \times & \\
& \quad (\gamma \times |Q|^2 \times 2^{\delta.cs(\mathcal{A})^2.\ln\left(cs(\mathcal{A})\right)} + c(\mathcal{A}) \times cs(\mathcal{A}) + 1)^{c(\mathcal{A})^2.\alpha^2} \times (c(\mathcal{A})!)^{c(\mathcal{A}).\alpha^2} & (10) \\
& \leq (|Q| \times |\Delta|)^{P_1(cs(\mathcal{A}))} &
\end{array}$$

In the last step, we assume that $|Q| \times |\Delta| \geq 2$ and we use the inequality:

$$c(\mathcal{A}) \leq d(\mathcal{A}) \times n(\mathcal{A}) \leq cs(\mathcal{A})^2$$

Cost of one inference step. Now, we have to estimate the cost of each inference step. The choice of one transition rule r in the algorithm is done among the set Δ, thus this (deterministic) choice is performed in time at most $|\Delta|$. For each new candidate $r(\rho_1, \ldots, \rho_n)$, the sons are already in the set E^*. Hence, for checking that $r(\rho_1, \ldots, \rho_n)$ is a run, it is sufficient to check that the disequality constraints of r are satisfied. Of course, we assume that identical subterms are shared, and therefore disequalities are checked in constant time. Hence, verifying the condition 1 of the algorithm is performed in time at most $cs(\mathcal{A})$.

Finally, we need to estimate the cost of checking the condition 3 on a run ρ. The number of position $p \in Pos(\rho) \setminus C(\mathcal{A})$ to consider is at most $\alpha^{d(\mathcal{A})+1}$ (we recall that α is the maximal arity of a function symbol of $\mathcal{F}$). Let q be the target state of the rule $\rho(p)$. To check the non existence of a sequence of runs $\rho|_p \gg \rho'_{b(\mathcal{A})} \gg \ldots \gg \rho'_1$ like in condition 3, we check all the possible ρ'_j individually. To bound the number of possible candidates for ρ'_j, $j \leq b(\mathcal{A})$, we add the additional requirement that $\gg$ is such that:

$$\rho \gg \rho' \text{ implies } d(\rho) > d(\rho') \tag{15}$$

With the condition (15) above, only the runs which are already in $\bigcup_{n\geq 0} E_q^n$, are possible candidates for ρ'_j, $j \leq b(\mathcal{A})$, which means that the number of runs to check is smaller than $|E^*|$. For each pumping $\rho[\rho'_j]_p$, verifying whether or not it creates a close equality requires time at most $d(\mathcal{A}) \times cs(\mathcal{A})$. Indeed, it is sufficient to check that for each position $p' \preccurlyeq p$, $p' \in Pos(\rho[\rho'_j]_p)$, at distance at most $d(\mathcal{A})$ from p (i.e. such that $|p| - |p'| \leq d(\mathcal{A})$), and for each disequality $\pi \neq \pi'$ in the constraint of the rule at position p' in $\rho[\rho'_j]_p$, with $p \prec p'\pi$ or $p \prec p'\pi'$, one has $t[t'_j]_p|_{p'\pi} \neq t[t'_j]_p|_{p'\pi'}$, where t and t'_j are the terms associated respectively to ρ and ρ'_j. As above, we assume that the verification of the inequalities is performed in constant time.

Hence the condition 3 can be checked in time at most, for some polynomial P_2:

$$\alpha^{d(\mathcal{A})+1} \times |E^*| \times d(\mathcal{A}) \times cs(\mathcal{A}) \leq (|Q| \times |\Delta|)^{P_2(cs(\mathcal{A}))}$$

where $|E^*|$ is bounded as above.

With the above remarks, the cost of one inference step is therefore smaller than, for some polynomial P_3:

$$|\Delta| \times cs(\mathcal{A}) \times (|Q| \times |\Delta|)^{P_2(cs(\mathcal{A}))} \leq (|Q| \times |\Delta|)^{P_3(cs(\mathcal{A}))}$$

Together with the bound on the number of steps, we get the complexity in Theorem 1.

5.4 Ordering

It still remains to exhibit an ordering $\gg$ which satisfies all our requirements:

Lemma 19. *There is an ordering $\gg$ which is monotonic, well-founded, total on $T(\Delta)$ and such that, if $depth(\rho) > depth(\rho')$ then $\rho \gg \rho'$.*

Proof. Consider the following interpretation of a term t: $I(t)$ is the triple $(depth(t), M(t), t)$ where $M(t)$ is the multiset of strict subterms of t.
Triples are ordered with the lexicographic composition of:

1. the ordering on natural numbers,
2. the multiset extension of $\gg$,
3. a lexicographic path ordering extending a total precedence.

$\gg$ itself is defined as $u \gg t$ iff $I(u) > I(t)$.

First, we should explain why the definition of the ordering itself is well-founded: $\gg$ is defined recursively, using its multiset extension. However, while defining $\gg$ on t, we use the multiset extension of $\gg$ on strict subterms of t and the subterm ordering is well-founded.

If $d(\rho) > d(\rho')$, then $\rho \gg \rho'$, simply because $d(\rho)$ is the first component of $I(\rho)$.

$\gg$ is monotonic. Assume $\rho_1 \gg \rho_2$ *i.e.* $I(\rho_1) > I(\rho_2)$. Then $d(\rho_1) \geq d(\rho_2)$ by definition of the lexicographic composition. Next,

$$d(\delta(t_1, \ldots, t_i, \rho_j, t_{i+1}, \ldots, t_n)) = \begin{cases} d(\rho_j) + 1 & \text{if } d(\rho_j) \geq \max(d(t_k)) \\ 1 + d(t_k) > d(\rho_j) & \text{for some } k \text{, otherwise} \end{cases}$$

In any case, $d(\delta(t_1, \ldots, t_i, \rho_1, t_{i+1}, \ldots, t_n)) \geq d(\delta(t_1, \ldots, t_i, \rho_2, t_{i+1}, \ldots, t_n))$.

$I(\rho_1) > I(\rho_2)$ implies either $d(\rho_1) > d(\rho_2)$ or $M(\rho_1) \geq M(\rho_2)$. Moreover,

$$M(\delta(t_1, \ldots, t_i, \rho_j, t_{i+1}, \ldots, t_n)) = \bigcup_{i=1}^{n} M(t_i) \cup \{\{t_1, \ldots, t_n\}\} \cup \{\{\rho_j\}\} \cup M(\rho_j)$$

If $d(\rho_1) > d(\rho_2)$, then there is a strict subterm ρ_3 of ρ_1 such that, for every strict subterm ρ_4 of ρ_2, $d(\rho_3) > d(\rho_4)$, which implies $\rho_3 \gg \rho_4$. Then, by definition of the multiset ordering, $\{\{\rho_3\}\} > M(\rho_2)$ and hence $M(\rho_1) > M(\rho_2)$. It follows that, in any case,

$$\rho_1 \gg \rho_2 \Rightarrow M(\rho_1) \geq M(\rho_2)$$

then
$$\begin{aligned} M(\delta(\ldots, \rho_1, \ldots)) &\geq \textstyle\bigcup_{i=1}^{n} M(t_i) \cup M(\rho_2) \cup \{\{t_1, \ldots, t_n\}\} \cup \{\{\rho_1\}\} \\ &> \textstyle\bigcup_{i=1}^{n} M(t_i) \cup M(\rho_2) \cup \{\{t_1, \ldots, t_n\}\} \cup \{\{\rho_2\}\} \\ &= M(\delta(\ldots, \rho_2, \ldots)) \end{aligned}$$

This suffices to guarantee the monotonicity.

$\gg$ is well-founded. By structural induction on t, there is no infinite strictly decreasing sequence starting with t.
If t is a constant, then $I(t) = (1, \emptyset, t)$ and t is minimal.
If this is true for the strict subterms of t, then let $\mathcal{E}$ be the set of terms smaller (w.r.t. $\gg$) than some strict subterm of t. $\gg$ is well-founded on $\mathcal{E}$ by induction hypothesis. Then its multiset extension is well-founded on multisets whose elements are in $\mathcal{E}$.

Then there is no infinite strictly decreasing sequence starting with t by well-foundedness of the lexicographic path ordering and since the lexicographic combination of well-founded orderings is itself well-founded.

$\gg$ contains the strict superterm relation. Because of the first component of the interpretation.

$\gg$ is total. That is the purpose of the last component of the interpretation: the lexicographic path ordering extending a total precedence is total on ground terms. Hence, for any distinct terms ρ_1, ρ_2, either $I(\rho_1) \gg I(\rho_2)$ or $I(\rho_2) \gg I(\rho_1)$. □

As a consequence of Theorem 1 and Proposition 2, the decision of ground reducibility is in DEXPTIME.

Theorem 3. *Ground reducibility of a term t w.r.t. a rewrite system $\mathcal{R}$ can be decided in deterministic time* $\mathrm{O}(2^{P(\|t\|, \|\mathcal{R}\|)})$ *where P is a polynomial.*

6 Lower bound

Theorem 4. *Ground reducibility is EXPTIME-hard, for linear rewrite systems $\mathcal{R}$ and linear terms t, with PTIME reductions.*

The proof is a reduction of the emptiness problem for the intersection of (languages recognized by) k tree automata.The latter is know to be EXPTIME-complete ([7, 14]).
We encode several (parallel) computations (runs) of k given tree automata on the same ground term $t \in \mathcal{T}(\mathcal{F})$ as a term of $s \in \mathcal{T}(\mathcal{F}')$ where $\mathcal{F}'$ is a new alphabet built from $\mathcal{F}$ and the tree automata. This encoding is polynomial. Then, we build a rewrite system $\mathcal{R}$ whose every

ground normal form as the form $g(s)$ where g is a new function symbol and $s \in \mathcal{T}(\mathcal{F}')$ represents successful runs of the k automata $\mathcal{A}_1 \dots \mathcal{A}_k$.
Thus, $L(\mathcal{A}_1 \cap \ldots \cap \mathcal{A}_k)$ is not empty iff the term $g(x)$ $(x \in \mathcal{X})$ is not ground reducible w.r.t. $\mathcal{R}$. Finally, we can conclude the proof of Theorem 4 by checking that the system $\mathcal{R}$ is built in polynomial time w.r.t. the size of the size of the tree automata.

6.1 An EXPTIME complete problem

The formal definition of the EXPTIME-hard problem we consider is the following:

Proposition 3. *T. Frühwirth et. al. [7], H. Seidl [14] The following problem is EXPTIME hard: "given k tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_k$, is $L(\mathcal{A}_1) \cap \ldots \cap L(\mathcal{A}_k)$ empty ?"*

6.2 Representation of runs

Let $\mathcal{A}_1 = (Q_1, Q_1^{\mathrm{f}}, \Delta_1) \ldots \mathcal{A}_k = (Q_k, Q_k^{\mathrm{f}}, \Delta_k)$. We can assume without loss of generality that the sets $Q_1 \dots Q_k$ are pairwise disjoint.

The alphabet $\mathcal{F}'$ is defined as follows:

- $\mathcal{F}' := \mathcal{F} \uplus \{g\} \uplus Q_1 \uplus \ldots \uplus Q_k$
- $g \notin \mathcal{F}$ and g is unary in $\mathcal{F}'$.
- The arity (in $\mathcal{F}'$) of each symbol of $Q_1 \uplus \ldots \uplus Q_k$ is zero.
- The arity (in $\mathcal{F}'$) of each symbol of $f \in \mathcal{F}$ is the arity of f in $\mathcal{F}$ plus k.

We distinguish a subset $\mathcal{S} \subseteq \mathcal{T}(\mathcal{F}')$ which is recursively defined as follows:

- For each constant a in $\mathcal{F}$, each states $q_1 \in Q_1, \ldots, q_k \in Q_k$, $a(q_1, \ldots, q_k) \in \mathcal{S}$.
- For each symbol $f \in \mathcal{F}$, f having arity n in $\mathcal{F}$, each states $q_1 \in Q_1, \ldots, q_k \in Q_k$, and each $t_1, \ldots, t_n \in \mathcal{S}$, $f(q_1, \ldots, q_k, t_1, \ldots, t_n) \in \mathcal{S}$.

Note that the above $a(q_1, \ldots, q_k)$ and $f(q_1, \ldots, q_k, t_1, \ldots, t_n)$ are indeed terms of $\mathcal{T}(\mathcal{F}')$. The terms of $\mathcal{S}$ will be used to represent parallel computations of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ on a term $t \in \mathcal{T}(\mathcal{F})$.

6.3 The rewrite system $\mathcal{R}$

The system $\mathcal{R}$ is expected to reduce any ground term $t \in \mathcal{T}(\mathcal{F}')$ which is not of the form $t = g(s)$ where s represents successful runs of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ on a term $t \in \mathcal{T}(\mathcal{F})$. There can be four (mutually exclusive) reasons for that:

1. g occurs in t at a position which is not Λ.
2. $t = g(s)$ and s contains no g symbols $(s \in \mathcal{T}(\mathcal{F}' \setminus \{g\}))$ but $s \notin \mathcal{S}$.
3. $t = g(s)$ and $s \in \mathcal{S}$ but s contains a transition which is not conform (this means, s does not code runs).
4. $t = g(s)$, $s \in \mathcal{S}$ and s codes n runs but at least one is not successful.

In the following, we enumerate the rules of $\mathcal{R}$ which reduce the ground terms falling in one of the categories. We are only interested in reducibility, which means that the right members of rules of $\mathcal{R}$ are irrelevant for our purpose. Thus, every right member of rule of $\mathcal{R}$ will be one arbitrary constant $q \in \mathcal{F}'$.

1. In this category, we have the following rules:

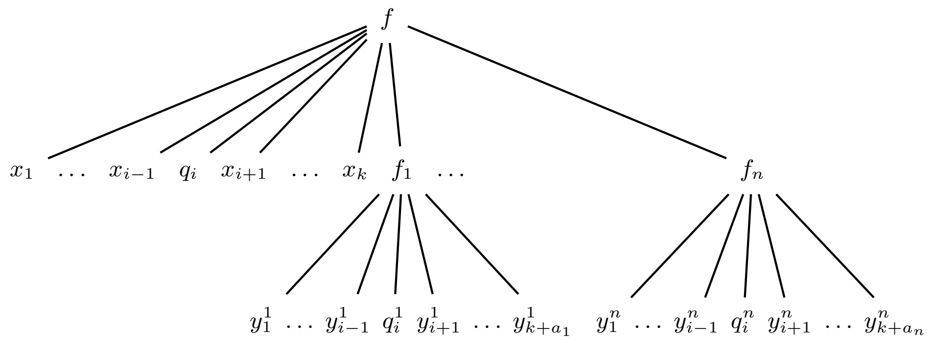

Fig. 9. Rules of category 3 (left members).

[*g cannot occur inside a term*]
$f(x_1, \ldots, x_{i-1}, g(x), x_{i+1}, \ldots, x_{k+n}) \to q$ such that:
- $f \in \mathcal{F}$ and f as arity n in $\mathcal{F}$
- $x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{k+n}$ are distinct variables of $\mathcal{X}$

2. Every rule of the second category has one of the forms:
   (a) [*no state can occur after the first kth positions below an $f \in \mathcal{F}$*]
   $f(x_1, \ldots, x_{i-1}, q', x_{i+1}, \ldots, x_{k+n}) \to q$ such that:
   - $f \in \mathcal{F}$ and f as arity n in $\mathcal{F}$
   - $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+n}$ are distinct variables of $\mathcal{X}$
   - $i > k$
   - $q' \in Q_1 \uplus \ldots \uplus Q_k$

   (b) [*no symbol of the original signature $\mathcal{F}$ can occur in the first kth positions*]
   $f(x_1, \ldots, x_{i-1}, f'(y_1, \ldots, y_{k+n'}), x_{i+1}, \ldots, x_{k+n}) \to q$ such that:
   - $f, f' \in \mathcal{F}$ and their respective arity are n and n' in $\mathcal{F}$
   - $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+n}, y_1, \ldots, y_{k+n'}$ are distinct variables of $\mathcal{X}$
   - $i \leq k$

   (c) [*at position i, one must have a state of Q_i*]
   $f(x_1, \ldots, x_{i-1}, q', x_{i+1}, \ldots, x_{k+n}) \to q$ such that:
   - $f \in \mathcal{F}$ and its arity in $\mathcal{F}$ is n
   - $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+n}$ are distinct variables of $\mathcal{X}$
   - $i \leq k$
   - $q' \in Q_1 \uplus \ldots \uplus Q_{i-1} \uplus Q_{i+1} \uplus \ldots \uplus Q_k$
3. The rules for this category are (see also figure 9):
   [*the subterms must describe transitions of the automata*]
   $$f\left(\begin{array}{r} x_1 \ldots x_{i-1}, q_i, x_{i+1} \ldots x_k,\ f_1(y_1^1 \ldots y_{i-1}^1, q_i^1, y_{i+1}^1 \ldots y_{k+a_1}^1), \\ \vdots \qquad\qquad \\ f_n(y_1^n \ldots y_{i-1}^n, q_i^n, y_{i+1}^n \ldots y_{k+a_n}^n) \end{array}\right) \to q$$
   such that:
   - $f, f_1, \ldots, f_n \in \mathcal{F}$ and their respective arities in $\mathcal{F}$ are n and $a_1, \ldots, a_n$
   - $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k,\ y_1^1, \ldots, y_{i-1}^1, y_{i+1}^1, \ldots, y_{k+a_1}^1,$
     $y_1^n, \ldots, y_{i-1}^n, y_{i+1}^n, \ldots, y_{k+a_n}^n$ are distinct variables of $\mathcal{X}$
   - $i \leq k$
   - $q_i, q_i^1, \ldots, q_i^n \in Q_i$

   - $f(q_i^1, \ldots, q_i^n) \to q_i \notin \Delta_i$
4. Finally, in the last category, we have:
   [*at the top of the term, we want final states*]
   $g(f(x_1, \ldots, x_{i-1}, q', x_{i+1}, \ldots, x_{k+n})) \to q$ such that:
   - $f \in \mathcal{F}$ and f as arity n in $\mathcal{F}$
   - $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k+n}$ are distinct variables of $\mathcal{X}$
   - $i \leq k$
   - $q' \in (Q_1 \setminus Q_1^{\mathrm{f}}) \uplus \ldots \uplus (Q_k \setminus Q_k^{\mathrm{f}})$

Size of $\mathcal{R}$*.* First of all, note that the system $\mathcal{R}$ is linear.

Now, we need to evaluate its size. It will be expressed in term of k, of the number of states of the automata $\mathcal{A}_1, \ldots, \mathcal{A}_k$, of the initial number of function symbols $|\mathcal{F}|$ and finally of the maximal arity α of a function symbol in $\mathcal{F}$: $\alpha := \max\{\text{arity of } f \mid f \in \mathcal{F}\}$

The biggest rule of $\mathcal{R}$ belongs to the category 3 and its size is: $k + 2 \times \alpha + 2$

The number of rules in each category is summarised below:

Cat.	Number of rules
1	$\lvert\mathcal{F}\rvert \times (k + \alpha)$
2	$\lvert\mathcal{F}\rvert \times \alpha \times \sum_{i=1}^{k} \lvert Q_i\rvert + \lvert\mathcal{F}\rvert \times k \times (\lvert\mathcal{F}\rvert + \alpha) + \lvert\mathcal{F}\rvert \times k \times \sum_{i=1}^{k} \lvert Q_i\rvert$
3	$\lvert\mathcal{F}\rvert \times k \times \sum_{i=1}^{k} \lvert Q_i\rvert \times (\lvert\mathcal{F}\rvert \times (k + \alpha) \times \sum_{i=1}^{k} \lvert Q_i\rvert)^{\alpha}$
4	$\lvert\mathcal{F}\rvert \times k \times \sum_{i=1}^{k} \lvert Q_i\rvert$

Thus, the size of $\mathcal{R}$ is polynomial in the (sum of) sizes of the given tree automata.
On the other hand, it is clear that the construction of $\mathcal{R}$ does not require a time bigger than the size of this system.
Altogether, this proves Theorem 4.

7 Conclusion

We proved that ground reducibility is EXPTIME-complete for both the linear and the non-linear case. This closes a pending question. However, we do not claim that this result in itself gives any hint on how to implement a ground reducibility test. As we have seen, it is not tractable in general. A possible way to implement these techniques as efficiently as possible was suggested in [1]. In the average, some algorithms may behave well. In any case, we claim that tree automata help both in theory and in practice.

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