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Erratum

Ground State of N Coupled Nonlinear Schrodinger Equations in $\mathbb{R}^n$, $n \leq 3$

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Certain statements in [1] need to be reformulated. The reason is that the infimum in Lemma 3 on p. 636 and the infimum c' on p. 642 may not be finite. Throughout the whole paper [1] due to physical considerations, the coupling constants β_{ij}'s satisfy $\beta_{ij} = \beta_{ji}$, for $i \neq j$. In [1], Theorem 2 should be restated as follows:

Theorem 2. *There exists $\beta_0 > 0$ depending on λ_j's, μ_j's, n and N such that if $0 < \beta_{ij} < \beta_0$, $\beta_{ij} = \beta_{ji}$, $\forall i \neq j$ and the matrix Σ (defined at (1.9) of [1]) is positively definite, then there exists a ground state solution $(u_1^0, \ldots, u_N^0)$. All u_j^0's are positive, radially symmetric and strictly decreasing.*

Theorem 3 of [1] should also be restated as follows:

Theorem 3. *There exists $\beta_0 > 0$ depending on λ_j's, μ_j's, n and N such that if the matrix Σ is positively definite, $\beta_{ij} = \beta_{ji}$, $\forall i \neq j$ and*

$$\beta_{i_0 j} < 0, \quad \forall j \neq i_0, \quad \text{and} \quad 0 < \beta_{ij} < \beta_0, \quad \forall i \neq i_0, j \in \{i, i_0\},$$

for some $i_0 \in \{1, \ldots, N\}$, then the ground state solution to (1.2) doesn't exist.

The reason for this correction is that the current form of Lemma 3 is incorrect. We now modify the statement by setting

$$E_\lambda^1[u] = \frac{1}{4} \int_{R^n} (|\nabla u|^2 + \lambda u^2). \tag{0.1}$$

Then the revised Lemma 3 can be stated as follows:

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Lemma 3. $\inf_{u\in N'_{\lambda,\mu}} E^1_\lambda[u]$ *is attained only by* $w_{\lambda,\mu}$.

The proof is similar by noting that

$$\int_{R^n}(|\nabla u_0|^2+\lambda u_0^2)=2<\nabla E^1_\lambda[u_0],u_0>.$$

For the proof of Theorem 1, we replace $I_{\lambda_j,\mu_j}[u_j]$ by $E^1_{\lambda_j}[u_j]$ and note that if c is attained by some $(u_1^0,\ldots,u_N^0)\in\mathbf{N}$, then $(u_1^0,\ldots,u_N^0)$ satisfies (1.2). In fact, let

$$G_j[\mathbf{u}]=\int_{R^n}\left(|\nabla u_j|^2+\lambda_j u_j^2-\mu_j u_j^4\right)-\sum_{i\neq j}\int_{R^n}\beta_{ij}u_i^2u_j^2.$$

Then there are Lagrange multipliers $\alpha_1,\ldots,\alpha_N$ such that

$$\nabla E+\sum_{j=1}^N\alpha_j\nabla G_j=0,$$

which implies that

$$\sum_{j=1}^N\alpha_j\beta_{ij}\int_{R^n}(u_i^0)^2(u_j^0)^2=0. \tag{0.2}$$

Since $(u_1^0,\ldots,u_N^0)\in\mathbf{N}$, we have

$$\sum_{i\neq j}|\beta_{ij}|\int_{R^n}(u_i^0)^2(u_j^0)^2<\int_{R^n}\beta_{jj}(u_j^0)^4,$$

which implies that the matrix $(\int_{R^n}\beta_{ij}(u_i^0)^2(u_j^0)^2)$ is diagonally dominant, and hence from (0.2), we deduce that $\alpha_1=\cdots=\alpha_N=0$. The rest is the same as in [1].

For the proof of Theorem 2, we remark that

$$c=\inf_{\mathbf{u}\in\mathbf{N}}E[\mathbf{u}]=\inf_{\mathbf{u}\in\mathbf{N}}E^1[\mathbf{u}]\geq\inf_{\mathbf{u}\in\mathbf{N}'}E^1[\mathbf{u}]:=c' \tag{0.3}$$

and replace $E[u_1,\ldots,u_N]$ by $E^1[u_1,\ldots,u_N]$ in the rest of the proof, where E^1 is defined by

$$E^1[\mathbf{u}]=\frac{1}{4}\sum_{j=1}^N\int_{R^n}(|\nabla u_j|^2+\lambda_j u_j^2). \tag{0.4}$$

As in our paper, we can show that a minimizer $(u_1,\ldots,u_N)$ of c' exists. Since $\beta_{ij}<\beta_0$, by the same proof as those of Lemma 2.1 of [2], we infer that

$$C_1\leq\int_{R^n}u_j^4\leq C_2,\ j=1,\ldots,N, \tag{0.5}$$

where C_1 and C_2 are positive constants depending on n,N,λ_j,β_{ij}.

We now claim that $(u_1, \dots, u_N) \in \mathbf{N}$. To this end, let $(\sqrt{t_1}u_1, \dots, \sqrt{t_N}u_N) \in \mathbf{N}$, where each $t_j > 0$. Then $(\sqrt{t_1}, \dots, \sqrt{t_N})$ satisfies

$$\int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2) = t_j \int_{R^n} \mu_j u_j^4 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \int_{R^n} t_i \beta_{ij} u_i^2 u_j^2, \quad j = 1, \dots, N.$$

Consequently,

$$\sum_{j=1}^{N} \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2) = \sum_{j=1}^{N} t_j \left(\int_{R^n} \mu_j u_j^4 + \sum_{\substack{i=1 \\ i \neq j}}^{N} \int_{R^n} \beta_{ij} u_i^2 u_j^2 \right). \tag{0.6}$$

Here we have used the fact that $\beta_{ij} = \beta_{ji}$.

Due to $(\sqrt{t_1}u_1, \dots, \sqrt{t_N}u_N) \in \mathbf{N} \subset \mathbf{N}^{'}$, we have

$$c^{'} \leq E^1[\sqrt{t_1}u_1, \dots, \sqrt{t_N}u_N],$$

and hence

$$\sum_{j=1}^{N} (t_j - 1) \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2) \geq 0,$$

i.e.

$$\sum_{j=1}^{N} \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2) \leq \sum_{j=1}^{N} t_j \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2). \tag{0.7}$$

Substituting (0.6) into the left-hand side of (0.7), and regrouping all the terms, we obtain

$$\sum_{j=1}^{N} t_j \left[\int_{R^n} u_j^4 + \sum_{i \neq j} \beta_{ij} \int_{R^n} u_i^2 u_j^2 - \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2) \right] \leq 0.$$

Each of the terms above are nonnegative. Since $(u_1, \dots, u_N) \in \mathbf{N}^{'}$ and each $t_j > 0$, we obtain that

$$\int_{R^n} u_j^4 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j^2 = \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2), \quad \forall j = 1, \dots, N.$$

Therefore, $(u_1, \dots, u_N) \in \mathbf{N}$ and hence $(u_1, \dots, u_N)$ also attains c. By the same proof of Lemma 2.2 of [2], $(u_1, \dots, u_N)$ is a critical point of $E[\mathbf{u}]$. The rest of proof then follows. (It is remarkable that this argument has been used in the proof of Lemma 2.2 in [2].) Actually, we have shown that

$$\inf_{\mathbf{u} \in \mathbf{N}} E[\mathbf{u}] = \inf_{\mathbf{u} \in \mathbf{N}} E^1[\mathbf{u}] = \inf_{\mathbf{u} \in \mathbf{N}^{'}} E^1[\mathbf{u}]. \tag{0.8}$$

The main idea for the proof of Theorem 3 remains unchanged. Here we modify the proof of Theorem 3 as follows: By (0.8), (6.6) can be replaced by

$$E_*^1[u_2, \dots, u_N] \geq \inf_{(u_2, \dots, u_N) \in N_1} E_*^1[u_2, \dots, u_N] = \inf_{(u_2, \dots, u_N) \in N_1} E'[u_2, \ dots, u_N] = c_1, \tag{0.9}$$

where

$$E_*^1[u_2, \cdots, u_N] = \frac{1}{4} \sum_{j=2}^{N} \int_{R^n} (|\nabla u_j|^2 + \lambda_j u_j^2).$$

Besides, the revised Lemma 3 may imply

$$E_{\lambda_1}^1[u_1] \geq E_{\lambda_1}^1[w_{\lambda_1,\mu_1}]\,. \tag{0.10}$$

Thus by (0.8)–(0.10), we have

$$\inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] = \inf_{\mathbf{u}\in\mathbf{N}} E^1[\mathbf{u}] \geq E_{\lambda_1}^1[w_{\lambda_1,\mu_1}] + c_1\,. \tag{0.11}$$

However, by (6.10),

$$\inf_{\mathbf{u}\in\mathbf{N}} E[\mathbf{u}] \leq I_{\lambda_1,\mu_1}[w_{\lambda_1,\mu_1}] + c_1 < E_{\lambda_1}^1[w_{\lambda_1,\mu_1}] + c_1\,,$$

which may contradict (0.11). Therefore, we may complete the proof of Theorem 3.

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