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Ground State Representation of the Infinite One-dimensional Heisenberg Ferromagnet

II. An Explicit Plancherel Formula

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Abstract. In its ground state representation, the infinite, spin 1/2 Heisenberg chain provides a model for spin wave scattering, which entails many features of the quantum mechanical N-body problem. Here, we give a complete eigenfunction expansion for the Hamiltonian of the chain in this representation, for *all* numbers of spin waves. Our results resolve the questions of completeness and orthogonality of the eigenfunctions given by Bethe for finite chains, in the infinite volume limit.

1. Introduction

Let H be the self adjoint Hamiltonian corresponding to the ground state representation of the spin 1/2, infinite one-dimensional Heisenberg ferromagnet with nearest neighbor interactions. The operator H is reduced by a spin-wave number operator, and H restricted to the N spin-wave sector is unitarily equivalent in a natural way to a second difference operator $-\Delta_N$ with "sticky" boundary conditions acting in an l^2-space.

The purpose of this article is to prove the completeness of an *explicit* eigenfunction expansion of $-\Delta_N$, for *all* N i.e. *all* numbers of spin-waves. This result was announced in [1]. In addition, using the generalized eigenfunctions for $-\Delta_N$, we construct a complete set of commuting self adjoint projections $\{E_\beta(\Delta)\}$ which reduce $-\Delta_N$. Here the subscript β called the binding, describes the manner in which the N-spin waves are bound together into "complexes" (in Bethe's terminology [2]), and Δ is a Borel subset of a torus whose dimension depends on the number of complexes comprising β. Any two projective $E_\beta(\Delta)$, $E_{\beta'}(\Delta')$ are orthogonal for β and β' distinct or if $\beta=\beta'$, for Δ and Δ' disjoint.

In fact, the projections $\{E_\beta(\Delta)\}$ were already obtained in [4] in a slightly different representation by considering the thermodynamic limit and utilizing the Bethe solution in [2] for the finite volume eigenfunctions. But the questions of

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completeness and mutual orthogonality of these projections were left open. Here, we settle these questions by working directly with the infinite volume expressions.

The original motivation for working out the eigenfunctions expansion was to determine the dynamics of the ground state representation for the Heisenberg chain in as much detail as possible (cf. [3, 6]). Our results do shed some light on the Bethe solution and hopefully they will be of utility in other situations, e.g. the Heisenberg chain at non-zero temperatures.

In itself the Heisenberg chain provides an interesting example of the interplay between certain "phase factors" in the generalized eigenfunctions and the "Plancherel" measure associated with these generalized eigenfunctions. Roughly speaking, in the N-spin wave sector there is a particular family of generalized eigenfunctions $\{\psi_U\}$ parameterized by $z=(z_1, \ldots, z_N) \in \mathbb{C}^N$, $|z_i|=1$, $i=1, \ldots, n$ which describe the case where all N spin waves are free (i.e. not bound into complexes); the appropriate Plancherel measure for these functions is (essentially) Lebesgue measure on the N-torus. Now ψ_U extends analytically away from the torus to a family parameterized by $z \in \mathbb{C}^N$. For $z \in \mathbb{C}^N$ fixed, ψ_U, which will be a function defined on a subset of z^N, will not be "tempered" unless z belongs, to one of a set of algebraic varieties indexed by β contained in $\mathbb{C}^N$. Corresponding to the β variety (and the restriction of $\{\psi_U | z \in \mathbb{C}^N\}$ to this variety), which we call $\{\psi_\beta\}$, there is a Plancherel measure density μ_β determined by the residues of some "phase factors" of the analytic continuation of $\bar{\psi}_U$ evaluated on the variety. The totality of the $\{\psi_\beta\}$ with the measures μ_β, provide the complete Plancherel formula.

We add that our results are rather different in character from those of Yang and Yang, who proved that the ground state of a finite Heisenberg chain for fixed magnetization is given by the Bethe solution [8]. Our results should be compared with those of Wortis [7], who obtained the two spin-wave Green's function in one, two and three lattice dimensions, from which he was able to deduce a bound state structure in the two spin-wave case.

In Section 2.1, $-\Delta_N$ and the l^2-space in which it acts are defined. Section 2.2 contains the definition of the generalized eigenfunctions and introduces the notation used throughout the proofs. Section 2.3 delineates the important analytic properties of the generalized eigenfunctions. (For lack of space, Theorem 2.3.2 is not proved here. Its proof is contained in [5].) In Section 2.4, the proof of completeness is given. Finally, Section 3.1 summarizes the properties of the projection $\{E_\beta(\Delta)\}$, and Section 3.2 presents the Plancherel formula for the generalized eigenfunctions.

2.

2.1. The Hamiltonian

Let N be a positive integer. Then $\hat{\mathbb{Z}}^N$ will denote the set of N-tuples $\boldsymbol{m}=(m_1, \ldots, m_N)$ of integers such that $m_1 < m_2 < \ldots < m_N$ and $l^2(\hat{\mathbb{Z}}^N)$ will denote the Hilbert space of square summable sequences indexed by $\hat{\mathbb{Z}}^N$. Δ_N will denote the self-adjoint operator on $l^2(\hat{\mathbb{Z}}^N)$ which is defined as follows [2]:

$$\begin{aligned}\Delta_N f(m_1, m_2, \ldots, m_N) = \tfrac{1}{2} \sum_{i=1}^{N} (f(m_1, \ldots, m_{i-1}, m_i+1, \ldots, m_N) \\ + f(m_1, \ldots, m_{i-1}, m_i-1, \ldots, m_N) - 2f(m_1, \ldots, m_i, \ldots, m_N))\end{aligned}$$

providing the m_i's are *not* neighboring. If just two of the m_i's are neighboring e.g. $m_{k+1}=m_k+1$, then

$$\begin{aligned}\Delta_N f(m_1,\dots,m_N)=\tfrac{1}{2}\sum_{i\neq k,k+1}&(f(m_1,\dots,m_{i-1},m_i+1,\dots,m_N)\\&+f(m_1,\dots,m_{i-1},m_i-1,\dots,m_N)-2f(m_1,\dots,m_i,\dots,m_N))\\&+\tfrac{1}{2}f(m_1,\dots,m_k-1,m_k+1,\dots,m_N)\\&+\tfrac{1}{2}f(m_1,\dots,m_k,m_k+2,\dots,m_N)-f(m_1,\dots,m_k,\dots,m_N),\end{aligned}$$

etc. We might say that Δ_N is the discrete Laplacian on $l^2(\hat{\mathbb{Z}}^N)$ with a "sticky" boundary condition because the random walk of N-particles on $\mathbb{Z}$ determined by Δ_N is such that two adjacent particles may stick together for an arbitrary finite (integral) time with positive probability.

Remark 2.1.1. Observe that the above expressions defining Δ_N make sense for *all* functions on $\hat{\mathbb{Z}}^N$. We will continue to use this symbol for the extended operator.

Let $\mathscr{H}=\mathbb{C}\oplus\left(\bigoplus_{N=1}^{\infty} l^2(\hat{\mathbb{Z}}^N)\right)$. Then it is shown in ([2, 4]), that the ground state Heisenberg Hamiltonian is equivalent to $H=I\oplus\left(\bigoplus_{N=1}^{\infty}-\Delta_N\right)$. The (sub-) spaces $l^2(\hat{\mathbb{Z}}^N)$ are called the N spin-wave sectors.

2.2. *The Definition of the ψ-Functions*

Let N be a fixed positive integer fixed for the remainder of this section. Let $(n_1,n_2,\dots,n_N)$ be an N-tuple of non-negative integers such that $\sum_{k=1}^{N} kn_k=N$. Let j be a positive integer such that $n_j\neq 0$ and k a positive integer such that $1\leqq k\leqq n_j$. Let $N_{jk}=\sum_{l=1}^{j-1} ln_l+(k-1)j$ and let $I_{jk}=\{N_{jk}+1,\dots,N_{jk}+j\}$. The partition $\beta=\{I_{jk}\}$ is called the *standard N-binding of type* $(n_1,\dots,n_N)$. The set I_{jk} is called a *complex* of the binding β. Usually we will denote β by $(n_1,\dots,n_N)$. The collection of standard N-bindings will be denoted by $\mathscr{B}_N$. $\mathscr{C}_\beta$ will denote the pairs of integers (jk) [sometimes written (j,k)] such that $n_j\neq 0$, $1\leqq k\leqq n_j$. We write $(jk)<(j'k')$ if either $j=j'$ and $k<k'$ or $j<j'$.

We shall now define some partitions related to β when $n_1\neq 0$. If $n_1\geqq 2$ and $2\leqq k_0\leqq n_1$, then $\beta\wedge(1k_0)$ will denote the partition $\{I'_{jk}:(jk)\in\mathscr{C}_\beta-\{(1,1)\}\}$ where $I'_{jk}=I_{jk}$ if $(jk)\neq(1k_0)$ and $I'_{1k_0}=\{1\}\cup I_{1k_0}$. β_1 will denote the *standard* N-partition determined by $(n_1-2,\ n_2+1,\ n_3,\dots,n_N)$. If $j_0>1$, $(j_0k_0)\in\mathscr{C}_\beta$, then $\beta\wedge(j_0k_0)$ will denote the partition $\{I'_{jk}:(jk)\in\mathscr{C}_\beta-\{(11)\}\}$ where $I'_{jk}=I_{jk}$ if $(jk)\neq(j_0k_0)$ and $I'_{j_0k_0}=\{1\}\cup I_{j_0k_0}$. β_{j_0} will denote the *standard* binding $(n_1-1,\dots,n_{j_0-1},\ n_{j_0}-1,\ n_{j_0+1}+1,\ n_{j_0+2},\dots,n_N)$. Observe that $\beta\wedge(j_0k_0)$ is *not* a standard binding. $\bar{\beta}$ will denote the standard $N-1$ binding $(n_1-1,n_2,\dots,n_N)$.

$\boldsymbol{S}_N$ will denote the group of permutations of $\{1,2,\dots,N\}$. Then we define:

$$\mathscr{P}_\beta=:\{P\in\boldsymbol{S}_N:P(N_{jk}+1)<\dots<P(N_{jk}+j),\ (jk)\in\mathscr{C}_\beta\}$$

and

$$\mathscr{P}^\beta=:\{P\in\boldsymbol{S}_N:P(N_{jk}+1)>\dots>P(N_{jk}+j),\ (jk)\in\mathscr{C}_\beta\}.$$

Eigenfunctions for $-\Delta_N$ will be sums of functions indexed by $\mathscr{P}_\beta$. $\mathscr{P}_{\beta\wedge(j_0k_0)}$ is defined to be the set of permutations which preserve the natural order of I'_{jk}, $(jk)\in\mathscr{C}_\beta-\{(11)\}$ and $\mathscr{P}^{\beta\wedge(j_0k)}$ is defined to be the set of permutations which reverse their order. If $\boldsymbol{z}=(z_1,\ldots,z_N)\in\mathbb{C}^N$ and $\boldsymbol{m}=(m_1,\ldots,m_N)\in\hat{\mathbb{Z}}^N$, then $z^{\pm\boldsymbol{m}}$ will denote $z_1^{\pm m_1}z_2^{\pm m_2}\ldots z_N^{\pm m_N}$. If $\boldsymbol{m}\in\hat{\mathbb{Z}}^N$ and $P\in\boldsymbol{S}_N$, then $P(\boldsymbol{m})$ will denote $(m_{P(1)},\ldots,m_{P(N)})$ and $P(z)=(z_{P(1)},\ldots,z_{P(N)})$.

The symbol t will denote the fractional linear transformation $t(z)=:(2z-1)/z$. *This fractional linear transformation will play a very important role in this chapter.* Note that $t^l(z)=((l+1)z-l)/(l_z-(l-1))$.

To each $\beta=(n_1,\ldots,n_N)\in\mathscr{B}_N$ we associate an r-dimensional complex variable $z_\beta=:\{z_{jk},(jk)\in\mathscr{C}_\beta\}$ where $r=\sum_{k=1}^{N}n_k$ and where $z_{jk}=z_{N_{jk}+j}$. If $n_1\neq 0$, the symbol z_{β_c} will denote the $(r-1)$-dimensional complex variable $\{z_{jk},(jk)\in\mathscr{C}_\beta-\{(11)\}\}$. Thus $z_\beta=(z_{11},z_{\beta_c})$.

Let $\alpha,\gamma\in\{1,\ldots,N\}$. Then we define:

$$e^{-i\varphi_{\alpha\gamma}}=:-\frac{(2z_\gamma-z_\alpha z_\gamma-1)}{2z_\alpha-z_\alpha z_\gamma-1}$$

and

$$e^{i\varphi_{\alpha\gamma}}=(e^{-i\varphi_{\alpha\gamma}})^{-1}.$$

These functions are called *phase factors* and they also will play an important role in our discussions. For $P\in\mathscr{P}_\beta$, we define:

$$e^{-i\varphi_P}=:\prod_{\substack{1\leqq\alpha<\gamma\leqq N\\ P(\gamma)<P(\alpha)}}e^{-i\varphi_{\alpha\gamma}}.$$

For $P\in\mathscr{P}^\beta$, we define:

$$e^{-i\varphi_P}=\prod_{\substack{1\leqq\alpha<\gamma\leqq N:\\ \alpha\perp\gamma\\ P(\gamma)<P(\alpha)}}e^{i\varphi_{\alpha\gamma}}$$

where $\alpha\perp\gamma$ means that α and γ belong to different complexes of β. If we replace $z_{N_{jk}+l}$ by $t^{j-l}(z_{jk})$, $(jk)\in\mathscr{C}_\beta$, $1\leqq l\leqq j$ in the above, we shall denote these functions of z_β by $e^{-i\varphi_{\beta P}}$ and $e^{i\varphi^\beta_P}$ respectively. The function $e^{-i\varphi_{\beta\wedge(j_0k_0)P}}$, $P\in\mathscr{P}_{\beta\wedge(j_0k_0)}$ and $e^{i\varphi^{\beta\wedge(j_0k_0)}P}$, $P\in\mathscr{P}^{\beta\wedge(j_0k_0)}$ are defined in a similar way.

We can now define the central family of functions of this paper. Let $\beta\in\mathscr{B}_N$, $\boldsymbol{m}\in\hat{\mathbb{Z}}^N$, and define:

$$\psi_\beta(z_\beta;\boldsymbol{m})=:\sum_{P*\mathscr{P}_\beta}z^{P(\boldsymbol{m})}e^{-i\varphi_{\beta P}}$$

where:

$$z_{n_{jk}+l}=t^{j-l}(z_{jk}),\qquad 1\leqq l\leqq j,\ (jk)\in\mathscr{C}_\beta. \tag{2.2.1}$$

Remark 2.2.1. A priori $\psi_\beta(z_\beta;\boldsymbol{m})$ is only defined for z_β in the Zariski open set $\mathbb{C}^r-\bigcup_{\substack{\alpha,\gamma\in I_N:\\ \alpha<\gamma}}\{z_\beta\in\mathbb{C}^r:2z_\alpha-z_\alpha z_\gamma-1=0,$ where 2.2.1 is assumed to hold$\}$. We say a statement holds generically in z_β if it holds for z_β in this open dense set. In this section all statements involving z_β will be assumed to be generic statements. One of our main technical problems will be to show that $\psi_\beta(z_\beta; \boldsymbol{m})$ and related functions can be defined on somewhat larger sets of z_β.

Remark 2.2.2. Let $U = :(N, 0, \ldots, 0)$ where U stands for "unbound". Observe that $\psi_\beta(z_\beta; \boldsymbol{m})$, as a function of z_β, is the restriction of $\psi_U(\cdot; \boldsymbol{m})$ to the r dimensional variety defined by the $N-r$ equations:

$$2z_{N_{jk}+l} - z_{N_{jk}+l+1} z_{N_{jk}} + l - 1 = 0, \quad (jk) \in \mathscr{C}_\beta,\ 1 \leqq l \leqq j-1.$$

The reason why the ψ_β functions are of central importance is contained in the following theorem.

Theorem 2.2.1. *$\psi_\beta(z_\beta; \cdot)$ is a generalized eigenfunction for $-\Delta_N$ with eigenvalue*

$$\varepsilon_\beta(z_\beta) = \sum_{l=1}^{N} [1 - \tfrac{1}{2}(z_l + z_l^{-1})] = -\tfrac{1}{2} \sum_{jk} \frac{j(z_{jk}-1)^2}{(jz_{jk} - j + 1)}$$

where it is assumed that 2.2.1 holds, i.e. $-\Delta_N \psi_\beta = \varepsilon_\beta \psi_\beta$.

Proof. These are simply Bethe's eigenfunctions in the infinite volume limit written in terms of the variables $z_l = e^{ik_l}$, $l = 1, \ldots, N$. See [2, 4] for the proof. □

Note that once it is established that $\psi_U(z_U; \cdot)$, $|z_1| = \ldots = |z_N| = 1$, is a family of (generalized) eigenfunctions for $-\Delta_N$ it follows by analytic continuation and Remark 2.2.2 that $\psi_\beta(z_\psi; \cdot)$, $\beta \in \mathscr{B}_N$, z_β generic, is an eigenfunction for $-\Delta_N$ with eigenvalue $\varepsilon_\beta(z_\beta)$.

In the proof of completeness we shall need several other *ψ-functions.* They are defined as follows:

$$\psi^\beta(z_\beta; \boldsymbol{m}) = : \sum_{P \in \mathscr{P}^\beta} z^{-P(\boldsymbol{m})} e^{i\varphi_P^\beta};$$

$$\psi_\beta((j,k); z_\beta; \boldsymbol{m}) = : \sum_{\substack{P \in \mathscr{P}_\beta \\ P(N_{jk}+1) = 1}} z^{P(\boldsymbol{m})} e^{-i\varphi_{\beta P}}, \qquad (jk) \in \mathscr{C}_\beta;$$

$$\psi_\beta(j; z_\beta; \boldsymbol{m}) = : \sum_{k=1}^{n_j} \psi_\beta((j,k); z_\beta; \boldsymbol{m}),$$

j such that $n_j \neq 0$. In the above definitions it is always assumed that $\beta = (n_1, \ldots, n_N)$ and that 2.2.1 holds. Now suppose $n_1 \neq 0$ and let $(j_0 k_0)$ be as in the beginning of this section. We then define:

$$\psi^{\beta \wedge (j_0 k_0)}(z_{\beta_c}; \boldsymbol{m}) = : \sum_{P \in \mathscr{P}^{\beta \wedge (j_0 k_0)}} z^{-P(\boldsymbol{m})} e^{i\varphi_P^{\beta \wedge (j_0 k_0)}}$$

$$\psi_{\beta \wedge (j_0 k_0)}((1,1); z_{\beta_c}; \boldsymbol{m}) = \sum_{\substack{P \in \mathscr{P}_{\beta \wedge (j_0 k_0)}: \\ P(1) = 1}} z^{P(\boldsymbol{m})} e^{-i\varphi_{\beta \wedge (j_0 k_0) P}}$$

where it is assumed that 2.2.1 holds for $(j'k) \neq (1,1)$ and $z_1 = t^{j_0}(z_{j_0 k_0})$.

We now state and prove a simple lemma which will imply important change of variable and symmetry properties for various ψ-functions.

Lemma 2.2.2. *Let P_0, P and $P' \in \boldsymbol{S}_N$. Then*

$$(e^{-i\varphi_P} e^{i\varphi_{P'}})(P_0 z) = (e^{-i\varphi_{PP_0^{-1}}} e^{i\varphi_{P'P_0^{-1}}})(z). \tag{2.2.2}$$

Proof. Note that (2.2.2) implies (and is implied by) by setting P' and P respectively equal to the identity, that:

$$e^{-i\varphi_P}(P_0 z) = (e^{-i\varphi_{PP_0^{-1}}} e^{-i\varphi_{P_0^{-1}}})(z) \tag{2.2.3}$$

$$e^{i\varphi_{P'}}(P_0 z) = (e^{i\varphi_{P'P_0^{-1}}} e^{-i\varphi_{P_0^{-1}}})(z). \tag{2.2.4}$$

A direct calculation shows that (2.2.3) and (2.2.4) hold if $P_0 = (k, k+1) = P_0^{-1}$, $1 \leqq k < N$, is the transposition which interchanges k and $k+1$. Next observe that if (2.2.3) and (2.2.4) have been established for $P_0 = P_1$ and P_2, then they hold for $P_0 = P_1 P_2$. For example,

$$e^{-i\varphi_P}((P_1 P_2)z) = e^{-i\varphi_P}(P_2(P_1 z)) = e^{-i\varphi_{PP_2^{-1}}}(P_1 z) e^{i\varphi_{P_2^{-1}}}(P_1 z) = e^{-i\varphi_{PP_2^{-1}P_1^{-1}}}(z)$$
$$\cdot e^{i\varphi_{P_1^{-1}}}(z) e^{i\varphi_{P_2^{-1}P_1^{-1}}}(z) e^{-i\varphi_{P_1^{-1}}} = (e^{-i\varphi_{PP_0^{-1}}} e^{i\varphi_{P_0^{-1}}})(z).$$

Since every element of S_N can be written as a product of transposition of the form $(k, k+1)$, the lemma is proved. □

Let $\beta \in \mathscr{B}_N$ and $(j_0 k_0) \in \mathscr{C}_\beta$. If the complexes of β are $\{I_{jk}, (jk) \in \mathscr{C}_\beta\}$ and the complexes of $\beta_{j_0} = (n_1^0, \ldots, n_N^0)$ are $\{I_{jk}^0, (jk) \in \mathscr{C}_{\beta_j}\}$, we define a permutation $P_{j_0 k_0} \in S_N$ as follows:

$$P_{j_0 k_0} : \{1\} \cup I_{j_0 k_0} \to I^0_{(j_0+1)1}$$
$$P_{j_0 k_0} : I_{jk} \to I^0_{jk}, \quad \text{if} \quad j \neq j_0 \quad \text{or} \quad 1$$
$$P : I_{j_0 k} \to I^0_{j_0 k}, \quad \text{if} \quad j_0 \neq 1, k > k_0$$
$$P : I_{1k} \to I^0_{1k-1} \quad \text{if} \quad j_0 \neq 1$$
$$P : I_{1k} \to I^0_{1(k-1)} \quad \text{if} \quad j_0 \neq 1, k < k_0$$
$$P : I_{1k} \to I_{1(k-2)} \quad \text{if} \quad j_0 = 1, k > k_0$$

and P preserves the internal order of each of the above sets. The first consequence of Lemma 2.2.2 is the following:

Corollary 2.2.3. (Change of variable formula). *Let β, $(j_0 k_0)$ and $P_{j_0 k_0}$ be as above. Define $z'_{\beta_{j_0}}$ as follows:*

$$z'_{j'k'} = z_{P^{-1}_{j_0 k_0}(N^0_{j'k'} + j')}, (j'k') \in \mathscr{C}_{\beta_{j_0}}$$

where $z_{jk} = z_{n_{jk} + j}$. Then:

$$\psi_{\beta_{j_0}}(((j_0+1), 1); z'_{\beta_{j_0}}; \boldsymbol{m}) \psi^{\beta_{j_0}}(z_{\beta'_{j_0}}; \boldsymbol{m}')$$
$$= \psi_{\beta \wedge (j_0 k_0)}((1, 1); z_{\beta_c}; \boldsymbol{m}) \psi^{\beta \wedge (j_0 k_0)}(z_{\beta_c}; \boldsymbol{m}').$$

Proof. The proof is straightforward.

The second consequence of Lemma 2.2.2 will pertain to various important symmetry properties of products of various ψ-functions. Let $\beta = (n_1, n_2, \ldots, n_N) \in \mathscr{B}_N$ and let j_0 be such that $n_{j_0} \geqq 2$. Let $S \in S_{n_{j_0}}$ and define $S z_\beta \equiv z'_\beta$ as follows:

$$z'_{jk} = z_{jk}, \quad \text{if} \quad j \neq j_0, 1 \leqq k \leqq n_j;$$
$$z'_{j_0 k} = z_{j_0 S(k)}, \quad \text{if} \quad 1 \leqq k \leqq n_{j_0}.$$

If k is such that $2 \leqq k \leqq n_{j_0}$, let $S_{j_0 k}$ denote the transposition $(1k)$.

Corollary 2.2.4. (a) *Let β, j_0 and S_{j_0k} be as above. Then*

$$\psi_\beta((j_0,k);z_\beta;\boldsymbol{m})\psi^\beta(z_\beta;\boldsymbol{m})$$
$$=\psi_\beta((j_0,1);S_{j_0k}z_\beta;\boldsymbol{m})\psi^\beta(S_{j_0k}z_\beta;\boldsymbol{m}').$$

(b) *If $S\in S_{n_{j_0}}$, then:*

$$\psi_\beta(Sz_\beta;\boldsymbol{m})\psi^\beta(Sz_\beta;\boldsymbol{m}')=\psi_\beta(z_\beta;\boldsymbol{m})\psi^\beta(z_\beta;\boldsymbol{m}').$$

Proof. The proof again is straightforward.

We next introduce some complex differential forms. We define:

$$\mu_j(z)dz=:\frac{1}{2\pi i}(-1)^{j-1}[(j-1)!]^2\prod_{l=1}^{j-1}\left[\frac{z-1}{lz-(l-1)}\right]^2\frac{j}{(jz-(j-1))}dz$$

for $j=1,2,\ldots$. Let $\beta=(n_1,n_2,\ldots,n_N)$. Then we define:

$$\mu_\beta(z_\beta)dz_\beta=\prod_{(jk)\in\mathscr{C}_\beta}\mu_j(z_{jk})dz_{jk}.$$

The form $\mu_\beta dz_\beta$, restricted to the "physical" β contour (see below), is the "Plancherel measure" for the β-sector of the eigenfunction expansion.

Now suppose $n_1\geqq 1$ and (j_0k_0) is as in the beginning of this section. We then define:

$$\mu_{\beta_c}(z_{\beta_c})dz_{\beta_c}=:\prod_{(jk)\in\mathscr{C}_\beta-\{(1,1)\}}\mu_j(z_{jk})dz_{jk}$$

and

$$\mu_{\beta\wedge(j_0k_0)}(z_{\beta_c})dz_{\beta_c}=:\frac{-j_0(j_0+1)(z_{j_0k_0}-1)^2\mu_{\beta_c}dz_{\beta_c}}{((j_0+1)z_{j_0k_0}-j_0)(j_0z_{j_0k_0}-(j_0-1))}. \tag{2.2.5}$$

Proposition 2.2.5. *Let β, (j_0k_0) and $z'_{\beta_{j_0}}$ be as in Corollary 2.2.3. Then:*

$$\mu_{\beta_{j_0}}(z'_{\beta_{j_0}})=\mu_{\beta\wedge(j_0k_0)}(z_{\beta_c}).$$

Proof. This is just a direct substitution. □

We will now introduce some contours in $\mathbb{C}^r$. Let j and k be positive integers and define: $\Gamma(j)=\Gamma(jk)=:\{z_{jk}:|jz_{jk}-(j-1)|=1\}$ and for $j\geqq 2$, we define $\hat{\Gamma}(j)=\hat{\Gamma}(jk)=:\{z_{jk}:|(j-1)z_{jk}-(j-2)|=1\}$. Note that $\hat{\Gamma}(j)=\Gamma(j-1)$. Both the Γ and $\hat{\Gamma}$ contours are assumed to be oriented in the counterclockwise direction. If $\beta=(n_1,\ldots,n_N)$, define:

$$\Gamma_\beta=:\prod_{(jk)\in\mathscr{C}_\beta}\Gamma(jk).$$

The following contours will be needed for certain proofs in Sections 2.4 and 2.5. If $n_1\geqq 1$, define:

$$\Gamma_{\beta_c}=\prod_{(jk)\in\mathscr{C}_\beta-\{(1\,1)\}}\Gamma(jk).$$

If $j_0 \geqq 2$, $n_{j_0} \neq 0$ and $1 \leqq k_0 \leqq n_{j_0}$, define:

$$\hat{\Gamma}_\beta(j_0 k_0) = \prod_{\substack{(jk)\in\mathscr{C}_\beta:(jk)<(j_0 1)\\ \text{or}\\ (j_0k_0)<(jk)}} \Gamma(jk) \times \prod_{k=1}^{k_0} \hat{\Gamma}(j_0 k).$$

Remark 2.2.3. Observe that if we use the change of variable of Corollary 2.2.3 that the contour Γ_{β_c} goes over to $\hat{\Gamma}_{\beta_{j_0}}(j_0+1)1)$.

The following theorem will indicate why the $\{\Gamma_\beta\}$ contours are called *physical* contours.

Theorem 2.2.6. (a) *When $\mu_\beta dz_\beta$ is restricted to Γ_β, it is a positive measure.* (b) $\psi_\beta(z_\beta; \boldsymbol{m})$ *is bounded as a function on $\mathbb{Z}^N$ for each (generic) $z_\beta \in \Gamma_\beta$.* (c) $\overline{\psi_\beta(z_\beta; \boldsymbol{m})} = \psi^\beta(z_\beta; \boldsymbol{m})$, $z_\beta \in \Gamma_\beta$. (d) ε_β *is real and positive on* Γ_β.

Proof. a) Under the substitution of variable $jz_{jk} - j + 1 = e^{ik_{jk}}$, $0 \leqq k_{jk} \leqq 2\pi$, which provides a parameterization of $\Gamma(jk)$, we get that

$$\mu_j(z_{jk})dz_{jk} = \frac{1}{2\pi}((j-1)!)^2 \prod_{l=1}^{j-1} \left|\frac{e^{ik_{jk}}-1}{le^{ik_{jk}}+j-l}\right|^2 dk_{jk}$$

which is manifestly positive. The quantity $\mu_\beta(z_\beta)dz_\beta$ is simply a product of such expressions.

b) It suffices to show that each term $\boldsymbol{z}^{P(\boldsymbol{m})}e^{-i\varphi_P}$, $P\in\mathscr{P}_\beta$, in ψ_β is bounded and hence $\boldsymbol{z}^{P(\boldsymbol{m})}$ is bounded, for generic z_β. Now $\boldsymbol{z}^{P(\boldsymbol{m})}$ is a product of factors of the form $z_{N_{jk}+1}^{m_{i_1}} \dots z_{N_{jk}+j}^{m_{i_j}}$ with $m_{i_r} < m_{i_s}$ for $r<s$, hence it suffices to show boundedness for such a factor. To simplify notation, we assume $N_{jk}=0$ and replace $(m_{i_1}, \dots, m_{i_j})$ by $(m_1, \dots, m_j)$. For $z_{j-l} = t^l(z_j)$ with $z_j \in \Gamma(j)$, one can verify that $|z_{l+1}z_{j-l}| = 1$, $0 \leqq l < j$, and that $z_{j-l} \in \Gamma(j-2l)$ for $2l<j$ so that $|z_{j-l}| \leqq 1$ for $2l<j$. If for example j is even, we have $|z_1^{m_1} \dots z_j^{m_j}| = |(z_1z_j)^{m_1}(z_2z_{j-1})^{m_2} \dots (z_{j/2}z_{j/2+1})^{m_{j/2}} z_j^{m_j-m_1} \dots z_{j/2+1}^{m_{j/2+1}-m_{j/2}}| \leqq 1$. One can give an analogous argument for j odd, keeping in mind that for j odd, $|z_{(j+1)/2}| = 1$.

c) For β fixed, define the permutation $*$ by $*i = N_{jk}+j-l+1$ for $i = N_{jk}+l \in I_{jk}$. If $i<j$ are in distinct complexes then so are $*i$, $*j$ with $*i<*j$ and if $i<j$ and in the same complex then $*i>*j$. Note that $** = $ identity. Now it is easy to check that for $z_\beta \in \Gamma_\beta$, $\bar{z}_i = z_{*i}^{-1}$, and that

$$\overline{e^{-i\varphi_{ij}}} = e^{i\varphi_{*i*j}}$$

for i, j in distinct complexes. If we let Q be the composition $Q = P^*$ and observe that $\mathscr{P}^\beta = \mathscr{P}_{\beta *}$, then

$$\overline{\psi_\beta} = \sum_{P\in\mathscr{P}_\beta} z_{*1}^{-m_{P(1)}} z_{*2}^{-m_{P(2)}} \dots z_{*N}^{-m_{P(N)}} \prod_{\substack{i<j\\ P(i)>P(j)}} e^{i\varphi_{*i*j}}$$

$$= \sum_{Q^*\in\mathscr{P}_\beta} \boldsymbol{z}^{-Q(\boldsymbol{m})} \prod_{\substack{*i<*j\\ Q(*i)>Q(*j)\\ *i\perp *j}} e^{i\varphi_{*i*j}} = \sum_{Q\in\mathscr{P}^\beta} \boldsymbol{z}^{-Q(\boldsymbol{m})} \prod_{\substack{i<j\\ Q(i)>Q(j)\\ i\perp j}} e^{i\varphi_{ij}} \equiv \psi^\beta.$$

d) This follows from the variable substitution of part a). □

2.3. *The Structure of the Denominators of the ψ-Functions*

It is the purpose of this section to state a theorem which says that certain troublesome factors in the denominators of the phase factors $e^{-i\varphi_{\beta P}}$ and $e^{i\varphi_\beta^P}$ are not present in the ψ-functions i.e. certain cancellations take place in the sums defining the ψ-functions. The proof which is purely algebraic and rather lengthy will appear elsewhere. (It is also contained in the preprint [5].)

$\beta=(n_1,\ldots,n_N)\in\mathscr{B}_N$ will be fixed throughout this section. All pairs (jk), $(j'k')$, etc. will be assumed to be in $\mathscr{C}_\beta$. If $l\in\mathbb{Z}$, then:

$$Q_\beta(l;(j,k),(j',k');z_\beta)=:(l-1)z_{jk}z_{j'k'}+(l-1)\\ -(l-2)z_{jk}-lz_{j'k'}.$$

When no confusion should arise we suppress β. It is clear that these are irreducible polynomials in $\mathfrak{P}_r$, the ring of polynomials in r complex variables. A direct calculation shows that:

$$e^{-i\varphi_{N_{jk}+l,\,N_{j'k'}+l'}}=\frac{Q(j'-j-(l'-l)+2;(j,k),(j',k'))}{Q(j'-j-(l'-l);(j,k),(j',k'))}$$

when 2.2.1 holds. Thus the phase factors $e^{-i\varphi_{\beta P}}$ and $e^{i\varphi_\beta}$ are *rational* functions in z_β whose denominators are products of the irreducible polynomials $\{Q(l;(j,k),(j',k'))\}$.

If $P\in\mathscr{P}_\beta$ and 2.2.1 holds, then

$$z^{P(\boldsymbol{m})}=\prod_{(jk)}\left(\frac{jz_{jk}-(j-1)}{(j-1)z_{jk}-(j-2)}\right)^{m_{P(N_{jk}+1)}}\cdots\left(\frac{2z_{jk}-1}{z_{jk}}\right)^{m_{P(N_{jk}+j-1)}}z_{jk}^{m_{P(N_{jk}+j)}}.$$

Thus $z^{P(\boldsymbol{m})}$ is a *polynomial* in z_{jk} which *vanishes* when $z_{jk}=(l-1)/l$, $1\leqq l\leqq j$ because $P(N_{jk}+1)<\ldots<P(N_{jk}+j)$ and $m_1<\ldots<m_N$. A similar statement can be made about $z^{-P(\boldsymbol{m})}$, $P\in\mathscr{P}^\beta$. We summarize these remarks in the following.

Lemma 2.3.1. *The ψ functions can be written as follows:*

$$1.\ \psi_\beta(j_0k_0;z_\beta;\boldsymbol{m})=\frac{P_{\beta_{j_0k_0}}}{Q_{\beta_{j_0k_0}}}$$

where:

$$Q_{\beta_{j_0k_0}}=\prod_{\substack{(jk)<(j'k')\\ 1-j\leqq l\leqq j'-1}}[Q(j'-j-l;(jk),(j',k'))]^{e((j_0,k_0);(j,k),(j',k');l)}$$

and where the e's are non-negative integers, $P_{\beta j_0k_0}$ is a polynomial in z_β which vanishes when $z_{jk}=(l-1)/l$, $1\leqq l\leqq j$, $(jk)\in\mathscr{C}_\beta$ and $(P_{\beta_{j_0k_0}},Q_{\beta_{j_0k_0}})=1$ (i.e. $P_{\beta_{j_0k_0}}$ and $Q_{\beta_{j_0k_0}}$ are relatively prime in $\mathfrak{P}_r$). A similar statement can be made about $\psi_\beta(j_0;z_\beta;\boldsymbol{m})$, $n_{j_0}\neq 0$ and $\psi_\beta(z_\beta;\boldsymbol{m})$ where we write the exponents as $e(j_0;(j,k),(j',k');l)$ and $e((j,k);(j',k');l)$ respectively.

2. *ψ^β can be written as follows:*

$$\psi^\beta(z_\beta;\boldsymbol{m})=\psi^\beta(z_\beta)=\frac{P^\beta}{Q^\beta}$$

where:

$$Q^{\beta}=\prod_{\substack{(jk)<(j'k')\\ 1-j\leqq l\leqq j'-1}}[Q(j'-j-l+2;(j,k),(j',k'))]^{f((j,k),(j',k');l)}$$

and where the f's are non-negative integers, P^{β} is a polynomial in z_{β} which vanishes when $z_{jk}=(l-1)/l$, $1\leqq l\leqq j$, $(jk)\in\mathscr{C}_{\beta}$, and $(P^{\beta},Q^{\beta})=1$.

We now state the theorem mentioned in the beginning of this section.

Theorem 2.3.2. 1) *Assume $n_1\neq 0$. Then $e(1,1);(j,k),(j',k');l)=0$ if either $(j,k)=(1,1)$, $0\leqq l\leqq j'-1$ or $(11)<(jk)$, $j'\equiv j(\mathrm{mod}\,2)$ and $l=(j'-j)/2-1$;*

2) *Assume $n_{j_0}\neq 0, j_0>1$. Then $e((j_0,1);(j,k),(j',k');l)=0$ if* (a) *$(jk),(j'k')\neq(j_0,1)$ or $(jk)=(j_0,1)$, $j'=j_0$, and $j'\equiv j(\mathrm{mod}\,2)$, $l=(j'-j)/2-1$; or* (b) *$j<j_0$, $(j-j_0)\equiv 1(\mathrm{mod}\,2)$, $j'=j_0$, $l=(j_0-j-1)/2$; or* (c) *$j_0<j'$, $(j'-j_0)\equiv(\mathrm{mod}\,2)$, $j=j_0$, $l=(j'-j_0-3)/2$;*

3) *Assume $n_{j_0}\neq 0$. Then $e(j_0;(j,k),(j',k');l)=0$ if* (a) *$j'\equiv j(\mathrm{mod}\,2)$, $l=(j'-j)/2)-1$; or* (b) *$j<j_0$, $(j-j_0)\equiv 1(\mathrm{mod}\,2)$, $j'=j_0$, $l=(j_0-j-1)/2$; or* (c) *$j_0<j'$, $(j'-j_0)\equiv 1(\mathrm{mod}\,2)$, $j=j_0$, $l=j'-j_0-3)/2$;*

4) *The same statement holds for $e(jk,j'k';l)$ as hold for $e(j_0;jk,j'k';l)$.*

5) *$f((j,k),(j',k');l)=0$ if* (a) *$j'\equiv j(\mathrm{mod}\,2)$, $l=(j'-j/2)+1$; or* (b) *$j<j_0$, $(j-j_0)\equiv 1(\mathrm{mod}\,2)$, $j'=j_0$, $l=j_0-j+3/2$; or* (c) *$j_0<j'$, $(j'-j_0)\equiv 1(\mathrm{mod}\,2)$, $j=j_0$, $l=j'-j_0+1/2$; or* (d) *if $n_1\neq(jk)=(1,1)$, $1\leqq l\leqq j'-1$. Moreover $f((1,1);(j',k');0)\leqq 1$.*

2.4. Integrability of the ψ-Functions

In this section we establish the existence of various integrals involving ψ-functions, the differential form μdz and the Γ contours. Fix $\beta=(n_1,\ldots,n_N)\in\mathscr{B}_N$. Pairs of integers (jk), $(j'k')$, etc. will be assumed to be in $\mathscr{C}_{\beta}$ unless stated otherwise. If Γ_1 is a Γ-contour (see Sect. 3.1), then $\mathring{\Gamma}_1=:\Gamma_1-\bigcup_{(jk)<(j'k')}\{z_{jk}=z_{j'k'}=1\}$. It is clear that $\Gamma_1-\mathring{\Gamma}_1$ has Lebesgue measure 0 in Γ_1 where Γ_1 is viewed as a real r dimensional submanifold of $\mathbb{C}^r$.

Lemma 2.4.1. a) *$Q_{\beta}(j'-j-l;(j,k),(j',k');z_{\beta})\neq 0$ for $z_{\beta}\in\mathring{\Gamma}_{\beta}$ unless $j'\equiv j(\mathrm{mod}\,2)$ and $l=(j'-j)/2-1$.*

b) *Suppose $n_1\neq 0$ and j_0 is as in the beginning of Section 2.2. Suppose the triple $\{(j,k),(j',k'),l\}$ where $(j,k),(j',k')\in\mathscr{C}_{\beta_{j_0}}$, does not satisfy:*

i) *$(jk),(j'k')\neq(j_0+1,1)$, $j'\equiv j(\mathrm{mod}\,2)$ or $j=j'=j_0+1$, $k=1$, and, in both cases, $l=(j'-j)/2-1$; or*

ii) *$j<j_0+1$, $j\equiv j_0(\mathrm{mod}\,2)$, $j'=j_0+1$ and $l=(j_0-j)/2$; or*

iii) *$j=j_0+1$, $j_0+1\lesssim j'$, $j'\equiv j_0(\mathrm{mod}\,2)$, and $l=(j-j_0-4)/2$; then $Q_{\beta_{j_0}}(j'-j-l;(j,k),(j',k');z_{\beta_{j_0}})\neq 0$ on $\hat{\Gamma}_{\beta_{j_0}}(j_0+1,k_0)$, $1\leqq k_0\leqq n_{j_0+1}+1$.*

c) *Suppose $n_1\neq 0$ and j_0 is as in* b) *above. Suppose the triple $\{(j,k),(j',k'),l\}$, where $(j,k)(j',k')\in\mathscr{C}_{\beta_{j_0}}$, does not satisfy: $j'\equiv j(\mathrm{mod}\,2)$ and $l=(j'-j)/2-1$ or conditions* b(ii), b(iii), *then $Q_{\beta_{j_0}}(j'-j-l;\ (j,k),(j',k');\ l;\ z_{\beta_{j_0}})\neq 0$ for $z_{\beta_{j_0}}\in\mathring{\Gamma}_{\beta_{j_0}}(j_0+1,k_0)$, $1\leqq k_0\leqq n_{j_0+1}+1$ or $\mathring{\hat{\Gamma}}_{\beta_{j_0}}$.*

Proof. We only prove a). The remainder of the proof uses exactly the same technique. Observe first of all that $Q_{\beta}(j'-j-l;\ (j,k),(j',k');\ z_{\beta})=0$, $z_{\beta}\in\mathring{\Gamma}_{\beta}$ iff $z_{jk}=t^{j-j-l-1}(z_{j'k'})$, $z_{j'k'}\in\Gamma(j')-\{1\}$. The latter occurs iff $j=j'-2(j'-j-l-1)$ i.e. $l=(j'-j)/2-1$ since $T^l:\Gamma(j)\xrightarrow{\text{onto}}\Gamma(j-2l)$ for $l\in\mathbb{Z}$, j is a positive integer. □

Corollary 2.4.2. 1) *The functions* $\psi_\beta(1,1; z_\beta; \boldsymbol{m})$, $\psi_\beta(j_0; z_\beta; \boldsymbol{m})$, $n_{j_0} \neq 0$, $\psi_\beta(z_\beta; \boldsymbol{m})$ *and* $\psi^\beta(z_\beta; \boldsymbol{m})$ *are continuous as functions of* z_β *on* $\mathring{\hat{\Gamma}}_\beta$;

2) *Let* j_0 *be as in Lemma 2.4.1b. Then* $\psi_{\beta_{j_0}}(j_0+1, 1); z_{\beta_{j_0}}; \boldsymbol{m})$, $\psi_{\beta_{j_0}}(j_0+1; z_{\beta_{j_0}}; \boldsymbol{m})$ *and* $\psi^{\beta_{j_0}}(z_{\beta_{j_0}}; \boldsymbol{m})$ *are continuous functions of* z_β *on* $\mathring{\Gamma}_{\beta_{j_0}}((j_0+1)k_0)$, $1 \leqq k_0 \leqq n_{j_0+1}+1$;

3) *Let* j_0 *be as in* 2). *Then* $\psi_{\beta_{j_0}}(j_0+1; z_{\beta_{j_0}}; \boldsymbol{m})$ *and* $\psi^\beta(z_{\beta_{j_0}}; \boldsymbol{m})$ *are continuous on* $\mathring{\Gamma}_{\beta_{j_0}}$.

Proof. This follows immediately from Theorem 2.3.2 and Lemma 2.4.1.

Lemma 2.4.3. *The same statements hold as in Corollary 2.4.2 if we replace "continuous" by "bounded".*

Proof. Here we prove only the boundedness of ψ_β on $\mathring{\Gamma}_\beta$; the proof of boundedness of the other functions on the various contours is entirely analogous and so will be omitted. It is convenient to make the change of variable $x_{jk} = (1-z_{jk})^{-1}$ and to denote by x_β the totality of these transformed binding variables. Then the set $\Gamma(jk)$ is mapped to $\gamma_{jk} = \{x \in \mathbb{C} | \operatorname{re} x = j/2\}$. The point $z_{jk} = 1$ is mapped to ∞. Under this change of variables, we have that if

$$z_{N_{jk}+j-p} = t^p(z_{jk}), \qquad z_{N_{j'k'}+j'-q} = t^q(z_{j'k'}),$$

then

$$z_{N_{jk}+j-p} = \frac{x_{jk}-(p+1)}{x_{jk}-p}, \qquad z_{N_{j'k'}+j-q} = \frac{x_{j'k'}-(q+1)}{x_{j'k'}-q}$$

and, in terms of x_β, the phase factors are given by

$$\exp - i\varphi_{N_{jk}+j-p, N_{j'k'}+j'-q} = \frac{x_{jk}-x_{j'k'}+(q-p+1)}{x_{jk}-x_{j'k'}+(q-p-1)} = 1 + \frac{2}{x_{jk}-x_{j'k'}+(q-p-1)}.$$

Let $\gamma_\beta = \prod_{(j,k)} \gamma_{jk}$ and let $\mathfrak{F}$ be the ring of complex valued functions defined on γ_β, $\mathfrak{F} = \{F | \mathscr{P}(\partial/\partial x_{11}, \partial/\partial x_{12}, \ldots) F$ is uniformly bounded on γ_β for each polynomial $\mathscr{P}\}$. By the above representation, we see that a given phase factor, regarded as a function of x_β, is in $\mathfrak{F}$ provided it is not singular i.e. $j/2 - j'/2 + (q-p-1) \neq 0$. Second, we note that the product of a singular phase factor and a "mollifying" phase factor, i.e. one which has zeros on γ_β, of the form

$$\left(\frac{x_{jk}-x_{j'k'}+q-p+1}{x_{jk}-x_{j'k'}+(q-p-1)}\right)\left(\frac{x_{jk}-x_{j'k'}+q-p-1}{x_{jk}-x_{j'k'}+(q-p-3)}\right)$$

with $j/2 - j'/2 + (q-p-1) = 0$ is, in lowest terms, a function in $\mathfrak{F}$. Thirdly we note that if a phase factor is singular, then its reciprocal is not singular on γ_β and is in $\mathfrak{F}$. Finally we note that $z^{P(\boldsymbol{m})}$ written in terms of x_β also is in $\mathfrak{F}$.

Now in terms of x_β, ψ_β may be expressed as

$$\psi_\beta = \sum_{P \in \mathscr{P}_\beta} z^{P(m)} e^{-i\varphi_{\beta P}} = \left(\prod_{ii'}{}' e^{-i\varphi_{ii'}}\right) F$$

where $\prod'$ denotes a product over the unmollified singular phase factors of ψ_β, and F is a sum of products which are schematically of the form

$(z^{P(\boldsymbol{m})}) \times$ (products of non-singular phase factors) $\times$ (products of a singular and corresponding mollifying phase factor) $\times$ (products of reciprocals of singular phase factors).

Hence F is in $\mathfrak{F}$. But in addition we know by Corollary 3.3.2 that F vanishes at the singularities of the singular phase factors to the appropriate orders so that ψ_β is bounded, at least for the x_{jk} variables bounded away from ∞.

Consider a single singular phase factor $e^{-i\varphi_{ii'}}$. We have that

$$F_1 \equiv e^{-i\varphi_{ii'}} F = F + \frac{2}{x_{jk} - x_{j'k'} + (q-p-1)} F$$

is in $\mathfrak{F}$ by Lemma A.1 of the Appendix and that F_1 vanishes to the appropriate orders at the singular points of the remaining singular phase factors to accommodate these remaining phase factors. The argument may be iterated, one singular phase factor at a time to conclude that ψ_β itself is in $\mathfrak{F}$ and, in particular, is uniformly bounded. It follows that ψ_β, regarded again as a function of z_β, is uniformly bounded on $\mathring{\Gamma}_\beta$, which is what we wished to show.

Corollary 2.4.4. 1) $\psi_\beta \psi^\beta \mu_\beta dz_\beta$, $\psi_\beta(j)\psi^\beta \mu_\beta dz_\beta$, $n_j \neq 0$, *and* $\psi_\beta((11))\psi^\beta \mu_\beta dz_\beta$, $n_1 \neq 0$ *are integrable on* Γ_β *and*

$$\int_{\Gamma_\beta} \psi_\beta(1)\psi^\beta \mu_\beta dz_\beta = n_1 \int_{\Gamma_\beta} \psi_\beta((1,1))\psi^\beta \mu_\beta dz_\beta \; ; \tag{2.4.1}$$

2) *Let j_0 be as in the beginning of Section 2.2. Then* $\psi_{\beta_{j_0}}((j_0+1),1)\psi^{\beta_{j_0}}\mu_{\beta_{j_0}} dz_{\beta_{j_0}}$ *and* $\psi_{\beta_{j_0}}(j_0+1)\psi^{\beta_{j_0}}\mu_{\beta_{j_0}} dz_{\beta_{j_0}}$ *are integrable* $\hat{\Gamma}_{\beta_{j_0}}((j_0+1),k_0)$, $1 \leqq k_0 \leqq n_{j_0+1}+1$, *and*

$$\int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),(n_{j_0+1}+1))} \psi_{\beta_{j_0}}((j_0+1,1))\psi^{\beta_{j_0}}\mu_{\beta_{j_0}} dz_{\beta_{j_0}}$$
$$= \frac{1}{n_{j_0+1}+1} \int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),(n_{j_0}+1))} \psi_{\beta_{j_0}}(j_0+1)\psi^{\beta_{j_0}}\mu_{\beta_{j_0}} dz_{\beta_{j_0}} \; ; \tag{2.4.2}$$

3) $\psi_{\beta_{j_0}}(j_0+1)\psi^{\beta_{j_0}}\mu_{\beta_{j_0}} dz_{\beta_{j_0}}$ is integrable on $\Gamma_{\beta_{j_0}}$;
4) *Let (j_0, k_0) be as in the beginning of Section 2.2. Then*

$$\psi_{\beta \wedge (j_0 k_0)}(1,1)\psi^{\beta \wedge (j_0 k_0)}\mu_{\beta \wedge (j_0 k_0)} dz_{\beta_c}$$

is integrable on $\Gamma_{\beta \wedge (j_0, k_0)}$ *and*

$$\int_{\Gamma_{\beta \wedge (j_0 k_0)}} \psi_{\beta \wedge (j_0 k_0)}(1,1)\psi^{\beta \wedge (j_0,k_0)}\mu_{\beta \wedge (j_0,k_0)} dz_{\beta_c}$$
$$= \int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),1)} \psi_{\beta_{j_0}}((j_0+1),1)\psi^{\beta_{j_0}}\mu_{\beta_{j_0}} dz_{\beta_{j_0}} . \tag{2.4.3}$$

Proof. The integrability statements in 1), 2) and 3) follow directly from Corollary 2.4.2, Lemma 2.4.3 and the fact that the ψ-functions vanish when $z_{jk} = (l-1)/l$, $1 \leqq l \leqq j$ (Lemma 2.4.1). The integrability statement in 4) and 2.4.3 follow from the change of variable formulas i.e. Corollary 2.2.3 and the fact that $\Gamma_{\beta \wedge (j_0 k_0)}$ goes to $\hat{\Gamma}_{\beta_{j_0}}((j+1),1)$ under the change of variables. The identities 2.4.1 and 2.4.2 follow from the symmetry relations in Corollary 2.2.4. □

Finally we show that we are able to distort some of the Γ contours.

Lemma 2.4.5. *Let j_0 be such that $n_{j_0} \neq 0$ and if $j_0 = 1$, then it is assumed that $n_1 \geqq 2$. Then:*

$$\int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),1)} \psi_{\beta_{j_0}}((j_0+1),1)\psi^{\beta_{J_0}}\mu_{\beta_{J_0}} dz_{\beta_{J_0}}$$
$$= \int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),(n_{J_0+1}+1))} \psi_{\beta_{J_0}}(j_0+1)1)\psi^{\beta_{J_0}}\mu_{\beta_{J_0}} dz_{\beta_{J_0}} \tag{2.4.4}$$

and:

$$\int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),(n_{J_0+1}+1))} \psi_{\beta_{j_0}}(j_0+1)\psi^{\beta_{J_0}}\mu_{\beta_{j_0}} dz_{\beta_{J_0}}$$
$$= \int_{\hat{\Gamma}_{\beta_{j_0}}} \psi_{\beta_{J_0}}(j_0+1)\psi^{\beta_{J_0}}\mu_{\beta_{J_0}} dz_{\beta_{J_0}}. \tag{2.4.5}$$

Proof. To prove 2.4.4 one proceeds by steps i.e. one shows that:

$$\int_{\hat{\Gamma}_{\beta_{j_0}}(j_0+1),1} = \ldots = \int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),2)} = \ldots = \int_{\hat{\Gamma}_{\beta_{j_0}}(j_0+1),(n_{j_0+1}+1))} \ldots .$$

To prove 2.4.5 one also proceeds by steps i.e. one shows that:

$$\int_{\hat{\Gamma}_{\beta_{j_0}}((j_0+1),(n_{j_0+1}+1))} = \ldots = \int_{\mathring{\Gamma}_{\beta_{j_0}}((j_0+1),)} \ldots = \int_{\hat{\Gamma}_{\beta_{j_0}}} \ldots .$$

Observe that each of the integrals is defined by Corollary 2.4.4. Each of the above steps uses essentially the same argument so we will only discuss the case:

$$\int_{\hat{\Gamma}_{\beta_{j_0}}(j_0+1)} \psi_{\beta_{j_0}}(j_0+1)\psi^{\beta_{J_0}}\mu_{\beta_{j_0}} dz_{\beta_{J_0}}$$
$$= \int_{\hat{\Gamma}_{\beta_{j_0}}} \psi_{\beta_{J_0}}(j_0+1)\psi^{\beta_{J_0}}\mu_{\beta_{J_0}} dz_{\beta_{J_0}}. \tag{2.4.6}$$

Fix $z_{j_1k_1} \in \Gamma(j_1) - \{1\}$, for $(j_1k_1) \neq (j_0+1)1 \in \mathscr{C}_{\beta_{J_0}}$. Then consider $F = \psi_{\beta_{j_0}}(j_0+1; z_\beta, \boldsymbol{m}) \times \psi^{\beta^J}(z_\beta; \boldsymbol{m}')\mu_{\beta_{j_0}}$ as a function of $z_{(j_0+1)1} =: z$. To establish 2.4.6 it is sufficient to show that F is analytic in closed region bounded by $\mathring{\Gamma}((j_0+1)1)$ and $\Gamma((j_0+1)1)$. In view of Theorem 2.2.2 it is sufficient to show:

a) The set $\{z \in \mathbb{C}: Q_{j'-(j_0+1)-l}(z, z_{j'k'}) = 0$ for some $z_{j'k'} \in \Gamma(j') - \{1\}\}$, where $Q_l(z,w) = (l-1)zw + (l-1) - (l-2)z - lw$, does not meet the region bounded by $\Gamma(j_0+1)$ and $\Gamma(j_0+1)$ if the triple (j', k', l) is excluded when either $j' \equiv (j_0+1) \pmod 2$, $l = (j'-j_0)/2 - 3/2$ or $j \equiv j_0 (\mathrm{mod}\, 2)$, $l = (j'-j_0-4)/2$.

b) The set $\{z \in \mathbb{C}: Q_{j_0+1-j-l}(z_{jk}, z) = 0$ for some $z_{jk} \in \Gamma(j) - \{1\}\}$ does not meet the region bounded by $\Gamma(j_0+1)$ and $\Gamma(j_0+1)$ if the triple (j,k,l) is excluded when either $j \equiv (j_0+1) \pmod 2$, $l = (j_0+1-j)/2 - 1$ or $j \equiv j_0 (\mathrm{mod}\, 2)$, $l = (j_0 - j)/2$.

We will only treat Case a). Case b) is handled in the same way. Arguing as in the proof of Lemma 2.4.1, we see that $z \in \{z \in \mathbb{C}: Q_{j'-(j_0+1)-l}(z, z_{j'k'}) = 0$ for some $z_{j'k'} \in \Gamma(j') - \{1\}\}$ implies $z = t^{j'-j_0-l-2}(z_{j'k'})$ for some $z_{j'k'} \in \Gamma(j') - \{1\}$ which in turn implies $z \in \Gamma(j' - 2(j_0 - l - 2)) - \{1\} = \Gamma(2j_0 - j' + 2l - 4) - \{1\}$. But this point is in $\Gamma(j_0+1)$ or $\mathring{\Gamma}(j_0+1)$ only if $2j_0 - j' + 2l - 4 = j_0$ or $j_0 + 1$ which proves a). □

2.5. A Completeness Relation for the ψ_β's

We will use the notation established in Section 2.2. In addition if $\beta=(n_1,\dots,n_N)\in\mathscr{B}_N$ then $\beta!$ will denote $n_1!\dots n_N!$. In many of the formulas we will suppress z_β. The main computation of this chapter can now be given.

Theorem 2.5.1. *Let $\beta=(n_1,\dots,n_N)\in\mathscr{B}_N$ be such that $n_1\geqq 1$ and suppose $m_2\leqq m_2'$. Then:*

$$\begin{aligned}&\frac{1}{\beta!}\int_{\Gamma_\beta}\psi_\beta(1;\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\mu_\beta dz_\beta\\ &=-\sum_{\substack{j_0:n_{j_0}\neq 0,\\ j_0\geqq 2\text{ and}\\ j_0=1\text{ if }n_1\geqq 2}}\frac{1}{\beta_{j_0}!}\int_{\Gamma_\beta}\psi_{\beta_{j_0}}(j_0+1;\boldsymbol{m})\psi^{\beta_{j_0}}(\boldsymbol{m}')\mu_{\beta_{j_0}}dz_{\beta_{j_0}}\\ &\quad+\delta_{m_1m_1'}\frac{1}{\hat\beta!}\int_{\Gamma_{\hat\beta}}\psi_{\hat\beta}(\hat{\boldsymbol{m}})\psi^{\hat\beta}(\hat{\boldsymbol{m}}')\mu_{\hat\beta}dz_{\hat\beta}\end{aligned}\tag{2.5.1}$$

where $\hat{\boldsymbol{m}}=(m_2,\dots,m_M)$ and $\hat{\boldsymbol{m}}'=(m_2',\dots,m_N')$.

Proof. From Corollary 2.4.4. (1), we see that the integrals in 2.5.1 exist and

$$\begin{aligned}\frac{1}{\beta!}\int_{\Gamma_\beta}\psi_\beta(1;\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\,\mu_\beta dz_\beta&=\frac{1}{(n_1-1)!n_2!\dots n_N!}\int_{\Gamma_\beta}\psi_\beta((1,1);\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\mu_\beta dz_\beta\\ &=\frac{-1}{(n_1-1)!n_2!\dots n_N!}\int_{\Gamma_\beta}\left[\frac{1}{2\pi i}\int_{|z_{11}|=1}\psi_\beta((1,1);\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\frac{dz_{11}}{z_{11}}\right]\mu_{\beta_c}dz_{\beta_c}.\end{aligned}$$

If we keep $z_{\beta_c}\in\Gamma_{\beta_c}-\bigcup_{(1,1)<(jk)}\{z_{jk}=1\}$ fixed, then it follows from Theorem 2.3.2 (5d) that $\psi_\beta((1,1);\boldsymbol{m})\psi^\beta(\boldsymbol{m}')1/z_{11}$ viewed as a function of z_{11} has simple poles in the exterior of $|z_{11}|=1$ at $z_{11}=t^{j_0}(z_{j_0k_0})$, for $(1,1)<(j_0k_0)$, $(j_0k_0)\in\mathscr{C}_\beta$, and possibly at $z_{11}=\infty$.

We will now compute the residues. Fix $(j_0k_0)>((1,1))$. Since $\psi_\beta((1,1);z_\beta;\boldsymbol{m})=z_{11}^{m_1}\psi_{\beta_c}(z_{\beta_c};\hat{\boldsymbol{m}})$ it is regular at $z_{11}=t^{j_0}(z_{j_0k_0})$. Moreover,

$$\begin{aligned}&z_{11}^{-1}\psi^\beta(\boldsymbol{m}')\\ &=z_{11}^{-1}\left\{\sum_{\substack{P\in\mathscr{P}^\beta:\\ P(N_{j_0k_0}+1)<P(1)}}z^{-P(\boldsymbol{m}')}e^{i\varphi_P^\beta}+\sum_{\substack{P\in\mathscr{P}^\beta:\\ P(1)<P(N_{j_0k_0}+1)}}z^{-P(\boldsymbol{m}')}e^{i\varphi_P}\right\}\\ &=z_{11}^{-1}\prod_{l=1}^{j_0}e^{i\varphi^\beta_{1N_{j_0}+l}}\left(\sum_{P\in\mathscr{P}^{\beta\wedge(j_0k_0)}}e^{-P(\boldsymbol{m}')}e^{i\varphi_P^\beta}\right)\\ &\quad+z_{11}^{-1}\left(\sum_{\substack{P\in\mathscr{P}^\beta:\\ P(1)<P(N_{j_0k_0}+1)}}z^{-P(m')}e^{i\varphi_P^\beta}\right)\end{aligned}$$

where

$$e^{i\varphi_P^\beta}=\prod_{\substack{1<\alpha<\gamma\\ \alpha\perp\gamma\\ P(\gamma)<P(\alpha)}}e^{i\varphi_{\alpha\gamma}}$$

and where 2.2.1 holds for $(jk)=(j_0k_0)$. Observe that when $z_{11}=t^{j_0}(z_{j_0k_0})$ for $e^{i\varphi^\beta_P} = e^{i\varphi_P^{\beta\wedge(j_0k_0)}}$. Both functions within the parentheses of the last expression are regular at $z_{11}=t^{j_0}(z_{j_0k_0})$ and hence:

$$\operatorname*{Res}_{t^{j_0}(z_{j_0k_0})} (\psi_\beta((1,1);\boldsymbol{m})\psi^\beta(\boldsymbol{m}')z_{11}^{-1})$$

$$=\psi_{\beta\wedge(j_0k_0)}((1,1);\boldsymbol{m})\psi^{\beta\wedge(j_0k_0)}(\boldsymbol{m}') \operatorname*{Res}_{t^{j_0}(z_{j_0k_0})}\left(z_{11}^{-1}\prod_{l=1}^{j_0} e^{i\varphi_1^\beta N_{j_0k_0}+l}\right). \tag{2.5.2}$$

Using the fact that

$$\prod_{s=1}^{j_0} e^{i\varphi_1 N_{j_0k_0}+s} = \frac{Q_2(z_{11},z_{j_0k_0})Q_1(z_{11},z_{j_0k_0})}{Q_{j_0+1}(z_{11},z_{j_0k_0})Q_{j_0}(z_{11},z_{j_0k_0})}$$

for $l=1$, we see that:

$$\operatorname*{Res}_{t^{j_0}(z_{j_0k_0})}\left(z_{11}^{-1}\prod_{l=1}^{j_0} e^{i\varphi_1^\beta N_{j_0k_0}+l}\right)$$

$$= \lim_{z_{11}\to t^{j_0}(z_{j_0k_0})}\left\{\frac{z_{11}-t^{j_0}(z_{j_0k_0})}{z_{11}}\prod_{l=1}^{j_0} e^{i\varphi_1^\beta N_{j_0k_0}+l}\right\}$$

$$= \lim_{z_{11}\to t^{j_0}(z_{j_0k_0})}\left(\frac{(z_{11}-t^{j_0}(z_{j_0k_0}))Q_2(z_{11},z_{j_0k_0})Q_1(z_{11},z_{j_0k_0})}{Q_{j_0+1}(z_{11},z_{j_0k_0})Q_{j_0}(z_{11},z_{j_0k_0})}\right) \tag{2.5.3}$$

We now examine the pole at ∞. We write:

$$\psi^\beta(\boldsymbol{m}')=z_{11}^{-m'_1}\psi^{\beta_c}(\hat{\boldsymbol{m}}')+\sum_{\substack{P\in\mathscr{P}^\beta\\ P(1)>1}} z^{-P(\boldsymbol{m}')}e^{i\varphi_P^\beta}$$

where

$$z_{11}^{-m'_1}\psi^{\beta_c}(\hat{\boldsymbol{m}}')=:\sum_{\substack{P\in\mathscr{P}^\beta\\ P(1)=1}} z^{-P(\boldsymbol{m}')}e^{i\varphi_P^\beta}$$

If $P(1)>1$ observe that $m_1=m'_{P(1)}\leqq m_1-m'_2\leqq -1$ and $(1/z_{11})\psi_\beta((1,1);\boldsymbol{m})z^{-P(\boldsymbol{m}')}e^{i\varphi_\beta^P} = z_{11}^{m_1-m'_{P(1)}-1}\cdot F$ where F is regular at $z_{11}=\infty$ (note $e^{i\varphi_P^\beta}$ is regular at ∞ for all $P\in\mathscr{P}^\beta$) and then

$$z_{11}^{-1}\psi_\beta((1,1);\boldsymbol{m})\left(\sum_{\substack{P\in\mathscr{P}^\beta\\ P(1)>1}} z^{-P(\boldsymbol{m}')}e^{i\varphi_P^\beta}\right)$$

is regular at ∞. Hence, recalling that $\psi_\beta((1,1);\boldsymbol{m})=z_{11}^{m_1}\psi_{\beta_c}(\hat{\boldsymbol{m}}')$, we see that:

$$\operatorname*{Res}_{z_{11}=\infty} (z_{11}^{-1}\psi_\beta((1,1);\boldsymbol{m})\psi^\beta(\boldsymbol{m}'))=-\delta_{m_1m'_1}\psi_{\beta_c}(\hat{\boldsymbol{m}})\psi^{\beta_c}(\hat{\boldsymbol{m}}'). \tag{2.5.4}$$

Combining 2.5.2, 2.5.3, 2.5.4, 2.2.5 and the first part of Corollary 2.4.4 (Part 4), we have that:

$$\frac{1}{\beta!}\int_{\Gamma_\beta}\psi_\beta(1;\boldsymbol{m})\psi^\beta\mu_\beta dz_\beta=-\sum_{\substack{j_0>1,\,n_{j_0}\neq 0\\ \text{and}\\ j_0=1 \text{ if } n_1\geqq 2}}\frac{1}{(n_1-1)!n_2!\dots n_N!}\sum_{\substack{1\leqq k_0\leqq n_{j_0},\,j_0>2\\ \text{and}\\ 2\leqq k\leqq n_1,\,j_0=1}}$$

$$\cdot\int_{\Gamma_\beta}\psi_{\beta\wedge(j_0k_0)}((1,1);\boldsymbol{m})\psi^{\beta\wedge(j_0k_0)}(\boldsymbol{m}')\mu_{\beta\wedge(j_0k_0)}dz_{\beta_c}$$

$$+\delta_{m_1m_1'}\frac{1}{\hat{\beta}!}\int_{\Gamma_{\hat\beta}}\psi_{\hat\beta}(\hat{\boldsymbol{m}})\psi^{\hat\beta}(\hat{\boldsymbol{m}}')\mu_{\hat\beta}dz_{\hat\beta}$$

where we have used the obvious identity:

$$\int_{\Gamma_{\beta_c}}\psi_{\beta_c}(\hat{\boldsymbol{m}})\psi^{\beta_c}(\hat{\boldsymbol{m}}')\mu_{\beta_c}dz_{\beta_c}=\int_{\Gamma_{\hat\beta}}\psi_{\hat\beta}(\hat{\boldsymbol{m}})\psi^{\hat\beta}(\hat{\boldsymbol{m}}')\mu_{\hat\beta}dz_{\hat\beta}.$$

We now apply the second half of Corollary 2.4.4 (Part 4) to the integrals in the sum and we obtain:

$$\frac{1}{\beta!}\int_{\Gamma_\beta}\psi_\beta(1;\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\mu_\beta dz_\beta$$

$$=-\sum_{\substack{j_0:j_0\geqq 2,\,n_{j_0}\neq 0\\ \text{and}\\ j_0=1\text{ if } n_1\geqq 2}}\frac{n_{j_0+1}+1}{\beta_{j_0}!}\int_{\hat\Gamma_{\beta_{j_0}}((j_0+1),1)}\psi_{\beta_{j_0}}((j_0+1),1;\boldsymbol{m})\psi^{\beta_{j_0}}(\boldsymbol{m}')\mu_{\beta_{j_0}}dz_{\beta_{j_0}}$$

$$+\delta_{m_1m_1'}\frac{1}{\hat\beta!}\int_{\Gamma_{\hat\beta}}\psi_{\hat\beta}(\hat{\boldsymbol{m}})\psi^{\hat\beta}(\hat{\boldsymbol{m}}')\mu_{\hat\beta}dz_{\hat\beta}. \tag{2.5.5}$$

However using Lemma 2.4.5 and Corollary 2.4.4 Parts 2 and 3, we see that:

$$(n_{j_0+1}+1)\int_{\hat\Gamma_{\beta_{j_0}}((j_0+1))}\psi_{\beta_{j_0}}(((j_0+1),1);\boldsymbol{m})\psi^{\beta_{j_0}}(\boldsymbol{m}')\mu_{\beta_{j_0}}dz_{\beta_{j_0}}$$

$$=\int_{\Gamma_{\beta_{j_0}}}\psi_{\beta_{j_0}}(j_0+1;\boldsymbol{m})\psi^{\beta_{j_0}}(\boldsymbol{m}')\mu_{\beta_{j_0}}dz_{\beta_{j_0}}.$$

This combined with 2.5.5 yields 2.5.1 and the theorem is proved. □

We can now easily prove the completeness relation for the ψ_β's.

Theorem 2.5.2. *Let N be a positive integer. Then:*

$$\delta_{\boldsymbol{mm}'}=\sum_{\beta\in\mathscr{B}_N}\frac{1}{\beta!}\int_{\Gamma_\beta}\psi_\beta(\boldsymbol{m})\psi^{\beta(\boldsymbol{m}')}\mu_\beta dz_\beta. \tag{2.5.6}$$

Proof. First of all the integrals on the right hand side exist by Corollary 2.4.4, Part 1. Secondly it is sufficient to prove 2.5.6 under the assumption that $m_2\leqq m_2'$. This follows from the fact (Theorem 2.2.6) that

$$\overline{\int_{\Gamma_\beta}\psi_\beta(\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\mu_\beta dz_\beta}=\int_{\Gamma_\beta}\psi_\beta(\boldsymbol{m}')\psi^\beta(\boldsymbol{m})\mu_\beta dz_\beta.$$

We will use induction on N to establish 2.5.6. For $N=1$, there is only one binding, namely $U=(1)$ and $\psi_U(z;m)=z^m$ and 2.5.6 is just the orthonormality of z^m, $m\in\mathbb{Z}$, on the unit circle with respect to $\mu_U dz_U=(1/2\pi i)(dz/z)$. We now assume that 2.5.6 has been established for $N-1$. Let $\mathscr{K}=\{\beta=(n_1,\ldots,n_N)\in\mathscr{B}_N: n_1\geqq 1\}$ and let $\mathscr{K}_0=\{\beta\in\mathscr{B}_N:\beta\neq U\}$. Observe that the mapping $\beta\to\hat{\beta}$ of $\mathscr{K}$ into $\mathscr{B}_{N-1}$ is a bijection. We can rewrite 2.5.6 as follows:

$$\sum_{\beta\in\mathscr{K}}\frac{1}{\beta!}\int_{\Gamma_\beta}\psi_\beta(1;\boldsymbol{m})\psi^\beta(\boldsymbol{m}')\mu_\beta dz_\beta$$

$$=\delta_{\boldsymbol{mm}'}-\sum_{\beta'\in\mathscr{K}_0}\frac{1}{\beta!}\sum_{\substack{j:j\geqq 1,\\ n_{j+1}\neq 0}}\int_{\Gamma_{\beta'}}\psi_{\beta'}(j+1;\boldsymbol{m})\psi^{\beta'}(\boldsymbol{m}')\mu_{\beta'}dz_{\beta'}\,. \tag{2.5.7}$$

Using Theorem 2.5.1, the induction hypothesis and the above bijection between $\mathscr{K}$ and $\mathscr{B}_{N-1}$ we see that the left hand side of 2.5.7 can be rewritten as follows:

$$\delta_{m_1m_1'}\delta_{\boldsymbol{mm}'}-\sum_{\beta\in\mathscr{K}}\sum_{\substack{j:n_j\neq 0,\,j\geqq 2\\ \text{and}\\ j=1\text{ if }n_1\geqq 2}}\frac{1}{\beta_j!}\int_{\Gamma_{\beta_j}}\psi_{\beta_j}(j+1;\boldsymbol{m})\psi^{\beta_j}(\boldsymbol{m}')\mu_{\beta_j}dz_{\beta_j}\,.$$

But given $\beta'=(n_1',\ldots,n_N')\in\mathscr{K}_0$ and j such that $n_{j+1}'\neq 0$, there exists a unique $\beta\in\mathscr{K}$ such that $\beta_j=\beta'$ and thus $\psi_{\beta_j}(j+1)=\psi_{\beta'}(j+1)$ and $\psi^{\beta_j}=\psi^{\beta'}$. In fact if $j\geqq 2$, $\beta=(n_1'+1,\ n_2',\ldots,n_{j-1}',\ n_j'+1,\ n_{j+1}'-1,\ n_{j+2}',\ldots,n_N')$ and if $j=1$, $\beta=(n_1'+2,\ n_2'-1,\ n_3',\ldots,n_N')$. Conversely given $\beta\in\mathscr{K}$ and j such that $j\geqq 2$, $n_j\neq 0$ or $j=1$ if $n_1\geqq 2$, there exists a unique $\beta'\in\mathscr{K}_0$ such that $\beta_j=\beta'$. In fact if $j\geqq 2$, $\beta'=(n_1-1,n_2,\ldots,n_{j-1},n_j-1,n_{j+1}+1,n_{j+2},\ldots,n_r)$ or if $j=1$, $n_1\geqq 2$, $\beta'=(n_1-2,n_2+1,n_3,\ldots,n_N)$. This establishes 2.5.7 and hence 2.5.6.

3. Plancherel Theory for the Heisenberg Chain

3.1. Definition of the Operators $\{E_\beta(\Delta)\}$ and Their Properties

Let $\hat{\Gamma}_\beta$ be the subset of Γ_β defined by $\hat{\Gamma}_\beta=\{z_\beta\in\Gamma_\beta|0\leqq\arg(jz_{jk}-(j-1))\leqq\arg(jz_{jk'}-(j-1))\leqq 2\pi$ if $k<k'\}$ and let Δ be a Borel subset of $\hat{\Gamma}_\beta$. We define $E_\beta(\Delta)$ as the operatora with kernel

$$E_\beta(\Delta)(\boldsymbol{m},\boldsymbol{m}')=\int_\Delta\psi_\beta(z_\beta,\boldsymbol{m})\psi^\beta(z_\beta,\boldsymbol{m}')\mu_\beta dz_\beta\,.$$

Theorem 3.1.1. i) *The operators $\{E_\beta(\Delta)\}$ are selfadjoint projections which reduce $-\Delta_N$*

ii) *The $\{E_\beta(\Delta)\}$ satisfy the orthogonality relations*

$$E_\beta(\Delta)E_{\beta'}(\Delta')=0 \tag{3.1.1}$$

for β distinct from β', or $\beta=\beta'$ but Δ and Δ' disjoint.

iii) *The $\{E_\beta(\Delta)\}$ are complete in the sense*

$$\sum_{\beta\in\mathscr{B}_N}E_\beta(\hat{\Gamma}_\beta)=\mathbb{1}\,. \tag{3.1.2}$$

Proof. It is clear from their definition that the $E_\beta(\Delta)$'s reduce $-\Delta_N$. The completeness is simply a restatement of Theorem 2.5.2, if we take into account the facts that there are $n_j!$ orderings of $\arg(jz_{jk}-(j-1))$, $k=1,2,\ldots,n_j$, and that $\psi_\beta\psi^\beta$ is symmetric under interchange of j-complex binding variables (Corollary 2.2.4). Note that the completeness relation (3.1.2), along with the positivity of $E_\beta(\Delta)$ (cf. Theorem 2.2.6) implies that $E_\beta(\Delta)$ is bounded in norm by one. That $E_\beta(\Delta)$ is actually a projection is an immediate consequence of the orthogonality relation (3.1.1) and the completeness 3.1.1. It thus remains to prove the orthogonality.

Now $E_\beta(\cdot)$ is (weakly) absolutely continuous with respect to Lebesgue measure in $\hat{\Gamma}_\beta$, as follows from its definition. It therefore suffices to prove the orthogonality in the special case where Δ and Δ' are both open and both of non-zero distances from a finite number of analytic sets of codimension one or higher in $\hat{\Gamma}_\beta$ and $\hat{\Gamma}_{\beta'}$ respectively. These analytic sets will be described in the context of the proof.

For an arbitrary β, let $|dz_\beta|$ denote Lebesgue measure in $\hat{\Gamma}_\beta$ and let $\|\cdot\|_\beta$ be the norm in $L^2(\hat{\Gamma}_\beta,|dz_\beta|)$.

Lemma 3.1.2. *Let $f\in L^2(\hat{\Gamma}_\beta,|dz_\beta|)$. Then the following estimate holds:*

$$\left\|\int_\Delta \psi_\beta(z_\beta,\cdot)f(z_\beta)\mu_\beta(z_\beta)dz_\beta\right\|_{l^2(\hat{\mathbb{Z}}^N)}\leqq C(\beta,\Delta)\|f\|_\beta$$

where $C(\beta,\Delta)$ is finite for Δ of non-zero distance from a finite number of analytic sets of codimension one in $\hat{\Gamma}_\beta$.

Proof. We consider in detail only the case where β consists of a single complex, $\beta=(0,0,\ldots,1)$. We have that

$$\begin{aligned}\hat{f}(\boldsymbol{m})&=\int_\Delta \psi_\beta(z_\beta,\boldsymbol{m})f(z_\beta)\mu_\beta(z_\beta)dz_\beta\\&=\int_\Delta (z_1z_2\ldots z_N)^{m_1}z_2^{m_2-m_1}\ldots z_N^{m_N-m_1}f(z_N)\mu_N(z_N)dz_N\end{aligned}$$

where $z_i=t^{N-i}(z_N)$ and $z_1z_2\ldots z_N=Nz_N-(N-1)\equiv z$ is of unit modulus. In terms of the variable z the integral may be written

$$\begin{aligned}\hat{f}(\boldsymbol{m})&=\int_{\hat{\Delta}} z^{m_1}(z_2(z))^{m_2-m_1}\ldots(z_N(z))^{m_N-m_1}f(z_N(z))\frac{dz}{N}\\&\equiv\tilde{f}(m_1,m_2-m_1,\ldots,m_N-m_1)\end{aligned}$$

where $\hat{\Delta}$ is the image of Δ under the transformation $z_N\to z$, and so

$$\begin{aligned}\|\hat{f}\|^2_{l_2(\hat{\mathbb{Z}}^N)}&=\sum_{0<n_2<n_3<\ldots<n_N}\sum_{n_1}|\tilde{f}(n_1,n_2,\ldots,n_N)|^2\\&\leqq 2\pi\sum_{0<n_2<\ldots<n_N}\int_{\hat{\Delta}}|(z_2(z))^{n_2}\ldots(z_N(z))^{n_N}|^2\,|f(z_N(z))|^2\mu_N^2(z_N(z))\frac{|dz|}{N^2}\\&\leqq 2\pi\sup_\Delta\left|\frac{\mu_N(z_N)}{N}\right|^2\sum_{0<n_2<\ldots<n_N}\sup_\Delta|z_2^{m_2}\ldots z_N^{n_N}|^2\,\|f\|^2_\beta\equiv C^2(\Delta)\|f\|^2_\beta,\end{aligned}$$

for Δ bounded away from $z_N=1$ and $z_N=(N-2)/N$ (if N is even). For such a Δ,

$$\frac{\mu_N(z_N)}{N}=\frac{1}{2\pi i}\frac{(-1)^{N-1}((N-1)!)^2}{(Nz_N-(N-1))}\prod_{k=1}^{N-1}\left(\frac{z_N-1}{kz_N-(k-1)}\right)^2$$

is bounded, $z_N\in\Delta$, and

$$\sum_{0<n_2<\ldots<n_N}\sup_{\Delta}|z_2^{n_2}\ldots z_N^{n_N}|^2=\sum_{0<n_2<\ldots<n_N}\sup_{\Delta}|z_2^{n_2}\ldots z_{N-1}^{n_{N-1}}|^2|z_N|^{2n_N}$$
$$\leqq\sum_{0<n_N}\sup_{\Delta} n_N^{N-2}|z_N|^{2n_N}<\infty$$

is bounded as well. (Recall that $|z_N|<1$ for z_N away from 1 and $z_k z_{j-k+1}$ is of modulus 1 with $|z_k|\leqq 1$ for $2k\leqq j+1$, so that $|z_2^{n_2}\ldots z_{N-1}^{n_N-1}|\leqq 1$ for $n_2<n_3<\ldots<n_{N-1}$.) This proves the lemma in the case where β consists of a single complex.

In the case where β consists of more than one complex an analogous argument can be carried out for each term of ψ_β. In this case however the $z_{jk}=1$ and $z_{jk}=(j-2)/j$ codimension one analytic sets should be supplemented by those analytic sets of codimension one corresponding to the singularities of the singular phase factors in ψ_β, which the set Δ should avoid. In this case, $C(\beta,\Delta)$ will take into account as well the L^∞-norms of the $e^{-i\varphi_\beta}$ and the sum over the allowable permutations in $\mathscr{P}_\beta$. This concludes the proof of Lemma 3.1.2.

Now let $f\in L^2(\hat{\Gamma}_\beta)$, $g\in L^2(\hat{\Gamma}_{\beta'})$ and define the form

$$A_t(f,g)=\sum_{\boldsymbol{m}}\int_{\Delta}\mu_\beta(z_\beta)dz_\beta$$
$$\cdot\int_{\Delta'}\mu_{\beta'}(z_{\beta'})dz_{\beta'}\bar{f}(z_\beta)\psi^\beta(z_\beta,\boldsymbol{m})\psi_{\beta'}(z_{\beta'},\boldsymbol{m})g(z_{\beta'})e^{it(\varepsilon_\beta(z_\beta)-\varepsilon_{\beta'}(z_{\beta'}))}.$$

The dependence of A_t on β, β', Δ and Δ' will be suppressed since they will be held fixed throughout the argument. In addition to the hypotheses of Theorem 3.1.1, Δ, Δ' are assumed open and of non-zero distance from the analytic sets of Lemma 3.1.2. From Lemma 3.1.2, we have the following result.

Lemma 3.1.3. *The form $A_t(f,g)$ satisfies*

$$(A_t(f,g))|\leqq C(\beta,\Delta)C(\beta',\Delta')\|f\|_\beta\,\|g\|_{\beta'}\,.$$

We next examine the limit $t\to\infty$ of $A_t(f,g)$.

Lemma 3.1.4. *For fixed f, g in $L^2(\hat{\Gamma}_\beta)$, $L^2(\hat{\Gamma}_{\beta'})$ respectively,*

$$\lim_{t\to\infty}A_t(f,g)=0\,.$$

Proof. The idea is to apply the Riemann Lebesgue Lemma. We first note that by Lemma 3.1.3, it suffices to prove Lemma 3.1.4 in the special case when $f\in C_0^\infty(\Delta)$, $g\in C_0^\infty(\Delta')$ with supp$f$ and supp$g$ away from some additional analytic sets of codimension one or higher. These sets will be described below. In this special case we avail ourselves of some distribution theory.

If $|z|\geqq 1$ set

where if $|z|=1$ the convergence is in the sense of distributions. Then in terms of the distribution X,

$$\sum_{m_1<m_2<\ldots<m_N} z_1^{m_1} z_2^{m_2} \ldots z_N^{m_N}$$
$$=2\pi X(z_1)X(z_1 z_2)\ldots X(z_1 z_2 \ldots z_{N-1})\delta(z_1 z_2 \ldots z_N)$$

in the sense of distributions provided $|z_1 z_2 \ldots z_k| \geqq 1$ for $1 \leqq k \leqq N-1$ and $|z_1 z_2 \ldots z_N|=1$. By $\delta(z)$ we mean

$$\frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \frac{-2\varepsilon}{(1-z)^2-\varepsilon^2}.$$

Next, consider the distribution given by

$$\sum_{\boldsymbol{m} \in \mathbb{Z}^{+N}} \psi^{\beta}(z_\beta, \boldsymbol{m}) \psi_{\beta'}(z_{\beta'}, \boldsymbol{m})$$
$$=2\pi \sum_{\substack{P \in \mathscr{P}^\beta \\ Q \in \mathscr{P}_{\beta'}}} (X(z_{P^{-1}(1)}^{-1} z'_{Q^{-1}(1)}) X(z_{P^{-1}(1)}^{-1} z_{P^{-1}(2)}^{-1} z'_{Q^{-1}(1)} z'_{Q^{-1}(2)}) \ldots \tag{3.1.3}$$
$$\ldots X(z_{P^{-1}(1)}^{-1} \ldots z_{P^{-1}(N-1)}^{-1} z'_{Q^{-1}(1)} \ldots z'_{Q^{-1}(N-1)})$$
$$\cdot\, \delta(z_1^{-1} \ldots z_N^{-1} z'_1 \ldots z'_N) e^{i\varphi_P^\beta(z_\beta) - i\varphi_{\beta Q}(z_{\beta'})}).$$

(By z'_i, we mean parameterization by the binding variables of β'.) This identity is justified by the facts that products of the form $z_{N_{jk}+j}^{-1} z_{N_{jk}+j-1}^{-1} \ldots z_{N_{jk}+j-i}^{-1}$, $i<j$, and $z_{N'_{j'k'}+1} z_{N'_{j'k'}+2} \ldots z_{N_{j'k'}+i'}$, $i' \leqq j'$, have moduli greater than or equal to one.

We shall refer to a factor X or δ in a term of the above sum as *singular* if its argument is equal to one for some values of z_β and $z_{\beta'}$ in Δ and Δ' respectively. Let us count the number of singular X's in a given term of the sum. Now actually $z_{N_{jk}+j}^{-1} \ldots z_{N_{jk}+j-i}^{-1}$ has *modulus* strictly greater than one for $z^\beta \in \Delta$ and $i<j-1$ and modulus exactly one for $i=j-1$; $z'_{N_{j'k'}+1} z'_{N_{j'k'}+2} \ldots z'_{N_{j'k'}+i'}$ has *modulus* strictly greater than one for $z_{\beta'} \in \Delta'$ and $i'<j'$ and modulus exactly one for $i'=j'$. It follows that a product $z_{P^{-1}(1)}^{-1} z_{P^{-1}(2)}^{-1} \ldots z_{P^{-1}(i)}^{-1} z'_{Q^{-1}(1)} z'_{Q^{-1}(2)} \ldots z'_{Q^{-1}(i)}$ can have modulus one only if $z_{P^{-1}(1)}^{-1} z_{P^{-1}(2)}^{-1} \ldots z_{P^{-1}(i)}^{-1}$ and $z'_{Q^{-1}(1)} z'_{Q^{-1}(2)} \ldots z'_{Q^{-1}(i)}$ separately have modulus one. This can happen only if $\{P^{-1}(1), P^{-1}(2) \ldots P^{-1}(i)\}$ is *exactly* a union of complexes of β, and $\{Q^{-1}(1), Q^{-1}(2), \ldots, Q^{-1}(i)\}$ is *exactly* a union of complexes in β'. As i ranges from 1 to N, $\{P^{-1}(1), P^{-1}(2), \ldots, P^{-1}(i)\}$, $\{Q^{-1}(1), Q^{-1}(2), \ldots, Q^{-1}(i)\}$ can separately be a union of complexes in β and β' respectively at most r_0 times, where r_0 is the smaller of the number of complexes in β or β'. Furthermore if the number of complexes in β and β' is the same and equal to r, but β and β' are distinct, we will again have $r_0<r$. Thus in the cases $\beta \neq \beta'$ only at most r_0 of the factors of a given term in the sum can be singular, where r_0 is strictly less than the number of complexes in β or β', whichever is greater. This is also true for $\beta=\beta'$ in a slightly weakened sense which we now explain.

Let β and β' be the same binding with r complexes and consider a term in the *rhs* of Equation (3.1.3). The number of singular factors r_0 will certainly be strictly less than r unless; a) the permutations P and Q fill complexes successively in the sense that $P^{-1}(i+1)=P^{-1}(i)-1$ if $P^{-1}(i) \neq N_{jk}+1$ for some jk and analogously $Q^{-1}(i+1)=Q^{-1}(i)+1$ if $Q^{-1}(i) \neq N_{j'k'}+j'$ for some $j'k'$, and b) if $P^{-1}(i)=N_{jk}+1$ then $Q^{-1}(i)$ must equal $N_{jk'}+j$ for some k'. Next suppose that the conditions a) and

b) are satisfied for a given term in Equation (3.1.3). The arguments of all the X's and δ in the term are never simultaneously 1, since if they were we would have $z_{P^{-1}(1)} = z'_{Q^{-1}(1)}, z_{P^{-1}(2)} = z'_{Q^{-1}(2)}, \ldots, z_{P^{-1}(N)} = z'_{Q^{-1}(N)}$ which would contradict the facts that $\operatorname{supp} f \subset \Delta$ is disjoint from $\operatorname{supp} g \subset \Delta'$. Now let $\{h_{\alpha PQ}\}_\alpha$ be a C^∞-partition of unity of $\overline{\Delta \times \Delta'}$ (i.e. each h_α is C^∞ and $\sum_\alpha h_{\alpha PQ} = 1$) so that the argument of some fixed singular X or δ is not equal to one in the support of h_α, for each α. (Hence in the support of h_α at least one singular X or δ is in fact bounded and $h_\alpha X$ or $h_\alpha \delta$, as the case may be, is C^∞.)

Thus in all cases, $\beta \neq \beta'$ or $\beta = \beta'$, we have the following situation; for $f \in C_0^\infty(\Delta)$, $g \in C_0^\infty(\Delta')$,

$$A_t(f,g) = 2\pi \sum_{\substack{P \in \mathscr{P}^\beta \\ Q \in \mathscr{P}_{\beta'}}} \sum_\alpha \int_{\Delta \times \Delta'} dz_\beta dz_{\beta'} \mu_\beta \mu_{\beta'} \bar{f}(\prod X_{PQ}) \cdot \delta h_{\alpha PQ} g e^{i(\varphi_P^\beta - \varphi_{\beta' Q} + \varepsilon_\beta - \varepsilon_{\beta'})} \tag{3.1.4}$$

where the integrand of each term on the right hand side has at most $r-1$ X and δ factors which are singular, where r is the number of complexes in β or β', whichever number is greater. [The product $\prod X_{PQ}$ denotes a product over the X's with appropriately permuted arguments depending on P, Q. The function $h_{\alpha PQ}$ is just unity for $\beta \neq \beta'$ or P, Q not satisfying (a) and (b) above.]

At this point, we impose a further condition on the support of f or g. (See the paragraph following the statement of Lemma 3.1.4.) Suppose β' has the larger number of complexes. We assume that $\operatorname{supp} g$ does not intersect the codimension one or higher sets $\{S_{\alpha PQ}\}$ given by $S_{\alpha PQ} = \{z_{\beta'} | s_{\alpha PQ}(z_{\beta'})$ is not of maximal rank$\}$ where $s_{\alpha PQ}(z_{\beta'})$ is the matrix

$$s_{\alpha PQ}(z_{\beta'}) = \begin{pmatrix} \nabla_{z_{\beta'}} & \varepsilon_{\beta'}(z_{\beta'}) & & \\ \nabla_{z_{\beta'}} & z'_{Q^{-1}(1)} z'_{Q^{-1}(2)} & \cdots & z'_{Q^{-1}(i_1)} \\ \nabla_{z_{\beta'}} & z'_{Q_{-1}(1)} z'_{Q_{-1}(2)} & \cdots & z_{Q_{-1}(i_2)} \\ & \vdots & & \\ \nabla_{z_{\beta'}} & z'_{Q_{-1}(1)} z'_{Q_{-1}(2)} & \cdots & \end{pmatrix}.$$

The symbol $\nabla_{z_{\beta'}}$ denotes the gradient with respect to the $z_{\beta'}$ binding variables and r_0 is the number of singular distributions in the αPQ term ($r_0 < r$ is a function of α, P, Q); $z'_{Q^{-1}(1)} z'_{Q^{-1}(2)} \ldots z'_{Q^{-1}(i_j)}$ denotes the $z_{\beta'}$ factor of the arguments of the singular distributions in Equation (3.1.4). (If β had the larger number of complexes an analogous assumption would instead be made on f. If β and β' have the same number of complexes the assumption could be made on either f or g.) Now the assumption that $\operatorname{supp} g$ is disjoint from $S_{\alpha PQ}$ sets implies that $\varepsilon_\beta(z_\beta) - \varepsilon_\beta(z_{\beta'}) \equiv \Delta\varepsilon$ itself may be regarded as an independent variable of integration for each term in Equation (3.1.4). Performing the integration first with respect to a set of variables independent of $\Delta\varepsilon$ in each term of Equation (3.1.4) (it may be necessary to reduce the partition of unity to make the transformation of variables one-to-one), one obtains a sum of terms of the form $\int F_{\alpha PQ} e^{it\Delta\varepsilon} d(\Delta\varepsilon)$ with $F_{\alpha PQ}$ an L^1-function. By the Riemann-Lebesgue Lemma, each of these terms goes to zero for $t \to \infty$. Thus $A_t(f,g) \to 0$, for $f \in C_0^\infty(\Delta)$, $g \in C_0^\infty(\Delta' - \bigcup_{\alpha PQ} S_{\alpha PQ})$, and so $A_t(f,g) \to 0$, $t \to \infty$, for $f \in L^2(\hat{L}_\beta)$, $g \in L^2(\hat{L}_{\beta'})$ as well. This concludes the proof of Lemma 3.1.4.

The proof or orthogonality, Equation (3.1.2) is now immediate. The kernel for $E_\beta(\Delta)E_{\beta'}(\Delta')$ can be written

$$\begin{aligned}(E_\beta(\Delta)E_{\beta'}(\Delta'))(\boldsymbol{m},\boldsymbol{m}') &= (E_\beta(\Delta)e^{-it\Delta_N}e^{it\Delta_N}E_{\beta'}(\Delta'))(\boldsymbol{m},\boldsymbol{m}')\\ &= \lim_{t\to\infty}(E_\beta(\Delta)e^{-it\Delta_N}e^{it\Delta_N}E_{\beta'}(\Delta'))(\boldsymbol{m},\boldsymbol{m}')\\ &= \lim_{t\to\infty}A_t(\psi^\beta(\cdot,\boldsymbol{m}),\ \psi^{\beta'}(\cdot,\boldsymbol{m}'))=0\end{aligned}$$

by Lemma 3.1.4 and the facts that $\psi_\beta, \psi_{\beta'}$ for fixed $\boldsymbol{m}, \boldsymbol{m}'$ are bounded and hence in $L^2(\hat{\Gamma}_\beta)$ and $L^2(\hat{\Gamma}_{\beta'})$ respectively. Consequently $E_\beta(\Delta)E_{\beta'}(\Delta')$ is zero, which completes the proof of Theorem 3.1.1.

3.2. *Plancherel Theorem*

Let $\mathscr{H}_N = \bigoplus_{\beta\in\mathscr{B}_N} L^2(\hat{\Gamma}_\beta, \mu_\beta(z_\beta)dz_\beta)$. Then Theorem 3.1.1 gives the following Plancherel theorem.

Corollary 3.2.1. *The mapping* $f \to \sum_{\boldsymbol{m}} \psi^\beta(z_\beta, \boldsymbol{m})f(\boldsymbol{m})$ *defines a unitary mapping* U *from* $l^2(\hat{\mathbb{Z}}^{+N})$ *onto* $\mathscr{H}_N$ *such that* $U(-\Delta_N)U^{-1}$ *restricted to* $L^2(\hat{\Gamma}_\beta, \mu_\beta(z_\beta)dz_\beta)$ *is multiplication by* $\varepsilon_\beta(z_\beta)$.

Proof. Theorem 3.1.1 insures that U is an isometry. It remains to show that U is onto $\mathscr{H}_N$.

Let f be a non-zero function of bounded support in $l^2(\hat{\mathbb{Z}}^N)$. Then we claim that $UE_\beta(\Delta_\beta)f$ is zero a.e. outside $\Delta_\beta \subset \Gamma_\beta$ and in $\Gamma_{\beta'}, \beta' \neq \beta$. As in the proof of Theorem 3.1.1, it suffices to consider Borel sets bounded away from certain singular hypersurfaces. If g is C^∞ with support in $\Delta_{\beta'}$, $\beta' \neq \beta$ or $\beta = \beta'$ but $\Delta_{\beta'} \cap \Delta_\beta = \phi$, then

$$\begin{aligned}\langle UE_\beta(\Delta_\beta)f, g\rangle &= \sum_m \iint_{\Delta_\beta\times\Delta_{\beta'}} \mu_\beta\mu_{\beta'}\ dz_\beta dz_{\beta'}\overline{Uf}(z_\beta)\psi^\beta e^{it\Delta_N - it\Delta_N}\psi_{\beta'}g(z_{\beta'})\\ &= A_t(Uf, g) = \lim_{t\to\infty} A_t(Uf, g) = 0\end{aligned}$$

in the notation of Lemma 3.1.3. Since such g's are otherwise arbitrary, this establishes the claim. This argument then implies that

$$UE_\beta(\Delta_\beta)f = \chi_{\Delta_\beta}UE_\beta(\Delta_\beta)f = \chi_{\Delta_\beta}Uf$$

a.e., where χ_{Δ_β} is the characteristic function of Δ_β, regarded as a function on $\bigcup_\beta \hat{\Gamma}_\beta$.

Now by analyticity of the ψ_β's, Uf is a function which vanishes only on a set of Lebesque measure zero in $\hat{\Gamma}_\beta$. Thus finite linear combinations of the form

$$\sum_{\beta,n} a_{\beta,n}\chi_{\Delta_{\beta,n}}Uf = U\sum_{\beta,n} a_{\beta,n}E_\beta(\Delta_{\beta,n})f$$

are clearly dense in $\mathscr{H}_n$, showing that U is onto.

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Appendix. An Elementary Sobolev Inequality

Let $\gamma_i = \{x \in \mathbb{C} | \mathrm{re}\, x = a_i\}$; $\gamma = \prod_{i=1}^{n} \gamma_i$, and let $\mathfrak{F}$ be the set of complex valued functions on γ, $\mathfrak{F} = \{F(x_1, ..., x_n) | \mathscr{P}(\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_n)F$ is uniformly bounded on γ for each polynomial $\mathscr{P}\}$. The function space $\mathfrak{F}$ is a ring under the operations of addition and multiplication.

Lemma A.1. *Let $F \in \mathfrak{F}$ and assume F vanishes along $x_i - x_j = a$ for some i, j with a real. Then the function $\hat{F} = (x_i - x_j - a)^{-1} F$ is uniformly bounded along with its derivatives for $(x_i - x_j - a) \neq 0$ and hence extends to a function in $\mathfrak{F}$.*

Proof. It is no restriction to assume $a = 0$. We consider

$$\left(\frac{\partial}{\partial x_1}\right)^{p_1} \left(\frac{\partial}{\partial x_2}\right)^{p_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n} (x_i - x_j)^{-1} F$$
$$= \left(\frac{\partial}{\partial x_i}\right)^{p_i} \left(\frac{\partial}{\partial x_j}\right)^{p_j} (x_i - x_j)^{-1} \prod_{k \neq i,j} \left(\frac{\partial}{\partial x_k}\right)^{p_k} F .$$

Since $\prod (\partial/\partial x_k)^{p_k} F$ is in $\mathfrak{F}$ and vanishes at $x_i = x_j$, the proof is reduced to showing $(\partial/\partial x_i)^{p_i} (\partial/\partial x_j)^{p_j} (x_i - x_j)^{-1} F$ is uniformly bounded for each p_i, p_j and F in $\mathfrak{F}$ with F vanishing at $x_i = x_j$. We proceed by induction on $p = p_i + p_j$. The case $p = 0$ is a special case of the uniform boundedness of $(\partial/\partial x_i)^p (x_i - x_j)^{-1} F$ which we prove below. If $p \neq 0$ with $p_j \neq 0$ we have

$$\left(\frac{\partial}{\partial x_i}\right)^{p_i} \left(\frac{\partial}{\partial x_j}\right)^{p_j} (x_i - x_j)^{-1} F = \left(\frac{\partial}{\partial x_i}\right)^{p_i} \left(\frac{\partial}{\partial x_j}\right)^{p_j - 1} (x_i - x_j)^{-1} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right) F$$
$$- \left(\frac{\partial}{\partial x_i}\right)^{p_i + 1} \left(\frac{\partial}{\partial x_j}\right)^{p_j - 1} (x_i - x_j)^{-1} F .$$

By the inductive hypothesis and the facts that $\partial/\partial x_i + \partial/\partial x_j)F$ is in $\mathfrak{F}$ and vanishes for $x_i = x_j$, the first term on the right hand side is uniformly bounded. Hence the proof is reduced to showing $(\partial/\partial x_i)^{p_i + 1} (\partial/\partial x_j)^{p_j - 1} \cdot (x_i - x_j)^{-1} F$ is uniformly bounded. But this procedure may be iterated until the proof is reduced to showing

$$\left(\frac{\partial}{\partial x_i}\right)^{p} (x_i - x_j)^{-1} F$$

is uniformly bounded. This term, however, is bounded by

$$\sup_{x} \frac{1}{p+1} \left| \left(\frac{\partial}{\partial x_i}\right)^{p+1} F(x) \right| ,$$

by an application of the following lemma.

Lemma A.2. *Let $f(x)$ be a C^{n+1} function on interval $I \subset \mathbb{R}$ containing 0. Then if $f(0) = 0$,*

$$\left(\frac{d}{dx}\right)^{p} \left(\frac{f(x)}{x}\right) = \frac{p!}{x^{p+1}} \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_p}^x dx_{p+1} \frac{d^{p+1}}{dx^{p+1}} f(x_{p+1})$$

and

$$\sup_{x \in I} \left| \left(\frac{d}{dx}\right)^{p} \frac{f(x)}{x} \right| \leqq \frac{1}{p+1} \sup_{x \in I} \left| \left(\frac{d}{dx}\right)^{p+1} f(x) \right| , \quad p \geqq 0 .$$

Proof. Note first that

$$g_p(x) \equiv \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{p-1}}^x dx_p = \frac{x^p}{p!}, \qquad p \geqq 1,$$

which is easily seen from the facts that $dg_p/dx = g_{p-1}$, $g_1(x) = x$ and $g_p(0) = 0$.

The formula for $d^p/dx^p(f/x)$ holds for $p=0$. Assume that it holds for $p-1$. Then by the formula for $x^p/p!$,

$$\begin{aligned}
\left(\frac{d}{dx}\right)^p \left(\frac{f(x)}{x}\right) &= \frac{d}{dx} \frac{(p-1)!}{x^p} \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{p-1}}^x dx_p \frac{d^p f(x_p)}{dx^p} \\
&= -\frac{p!}{x^{p+1}} \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{p-1}}^x dx_p \frac{d^p f(x_p)}{dx^p} + \frac{(p-1)!}{x^p} \\
&\quad \cdot \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{p-2}}^x dx_{p-1} \frac{d^p f(x)}{dx^p} \\
&= -\frac{p!}{x^{p+1}} \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{p-1}}^x dx_p \frac{d^p f(x_p)}{dx^p} + \frac{p!}{x^{p+1}} \\
&\quad + \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_{p-1}}^x dx_p \frac{d^p f(x)}{dx^p} \\
&= \frac{p!}{x^{p+1}} \int_0^x dx_1 \int_{x_1}^x dx_2 \ldots \int_{x_p}^x dx_{p+1} \frac{d^p f(x_{p+1})}{dx^p}
\end{aligned}$$

which establishes the formula for p and thus the formula holds for all $p \geqq 0$. The estimate follows from this formula and the formula for $x^{p+1}/(p+1)!$.

Remark. This lemma is used in the proof of Lemma 2.5.3. To avail oneself of the lemma, one first makes the transformation $z_\beta \to x_\beta$ defined by $x_{jk} = (1 - z_{jk})^{-1}$ which mapy Γ_β to a γ of the above form.

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