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GROUND STATE SOLUTIONS FOR THE NONLINEAR KLEIN–GORDON–MAXWELL EQUATIONS

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Abstract. In this paper we prove the existence of a ground state solution for the nonlinear Klein–Gordon–Maxwell equations in the electrostatic case.

1. Introduction

In this paper we are interested in studying the following nonlinear Klein–Gordon–Maxwell equations

$$(\mathcal{KGM})\qquad \begin{cases} \Box u + \bigg[|\nabla S - e\mathbf{A}|^2 - \bigg(\dfrac{\partial S}{\partial t} + e\varphi\bigg)^2 + m_0^2\bigg]u - |u|^{p-1}u = 0, \\ \dfrac{\partial}{\partial t}\bigg[\bigg(\dfrac{\partial S}{\partial t} + e\varphi\bigg)u^2\bigg] - \nabla\cdot[(\nabla S - e\mathbf{A})u^2] = 0, \\ \nabla\cdot\bigg(\dfrac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\bigg) = e\bigg(\dfrac{\partial S}{\partial t} + e\varphi\bigg)u^2, \\ \nabla\times(\nabla\times\mathbf{A}) + \dfrac{\partial}{\partial t}\bigg(\dfrac{\partial \mathbf{A}}{\partial t} + \nabla\varphi) = e(\nabla S - e\mathbf{A})u^2, \end{cases}$$

where $e, m_0 > 0$, $1 < p < 5$, $u(x,t) \in \mathbb{R}$, $S(x,t) \in \mathbb{R}$, $(\phi(x,t), \mathbf{A}(x,t)) \in \mathbb{R}\times\mathbb{R}^3$. This system arises in a very interesting physical context: in fact, it provides a "dualistic model" for the description of the interaction between a charged

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relativistic particle of matter and the electromagnetic field that it generates. According to such a model, the matter particle is a solitary wave $u(x,t)e^{iS(x,t)}$ which is solution of a nonlinear field equation, and the interaction with the electromagnetic field described by the gauge potentials $(\phi, \mathbf{A})$ is obtained by coupling the field equation with the Maxwell equations (see [3]).

By the invariance of the system with respect to the group of transformations of Poincaré, in order to find a solitary wave it is sufficient to look for a "standing wave" $u(x)e^{i\omega t}$ (here $\omega \in \mathbb{R}$) and to make it travel by means of a Lorentz transformation. The existence of standing waves for $(\mathcal{KGM})$, which has been proved recently by V. Benci and D. Fortunato in [3] and T. D'Aprile and D. Mugnai in [11], is a consequence of the nonlinear structure of the system. In fact, it is well known that in general wave equations do not possess solitary wave solutions. A typical example is the Klein–Gordon equation

$$\Box\psi + m^2\psi = 0, \quad m \neq 0, \quad \psi(x,t) \in \mathbb{C},$$

whose solutions have a spreading behavior which is time dependent (see [15]).

The characteristic of the solitary waves of preserving their energy density as a localized packet which travels as time goes on, makes the solitary waves behavior similar to that of the particle. Differently from the classical model, where the particle is represented as a dimensionless point, here the particle is endowed with space extension and has finite energy. This fact allows us to avoid the well known problem of the *divergence of the energy* which, in the theory of special relativity, brings to the impossibility of describing the dynamics of the particle (in fact the inertial mass is infinite: see for example [14], [16] and [21]). This is the reason why the solitary waves appear in several mathematical physics contexts, such as classical and quantum field theory, nonlinear optics, fluid mechanics, plasma physics (see e.g. [10], [13], [15], [20], [22]).

Finally, it is quite a remarkable fact that, since $(\mathcal{KGM})$ is invariant with respect to the the Poincaré group of transformations, the model described by $(\mathcal{KGM})$ turns out to be consistent with the basic principles of special relativity theory (see [1] and [4]). As a consequence, the solitary waves experience well known relativistic phenomenona such as length contraction, time dilatation and the equivalence between mass and energy.

In this paper, we are interested in looking for ground state solutions of the electrostatic $(\mathcal{KGM})$, namely for solutions which minimizes the action among all the solutions. The interest in ground states, which has been emphasized in many papers such as the celebrated works of S. Coleman, V. Glaser and A. Martin [9] and of H. Berestycki and P. L. Lions [6], is justified by the fact that they in general exhibit some type of stability. From a physical point of view, the stability of a standing wave is a crucial point to establish the existence of soliton-like solutions.

A first work in this direction is the recent paper of E. Long (see [19]), where the stability properties of the solutions of $(\mathcal{KGM})$ have been investigated, for e sufficiently small.

Consider the system $(\mathcal{KGM})$ in its electrostatic form, namely set $\mathbf{A} = 0$, $\phi(x,t) = \phi(x)$, $u(x,t) = u(x)$ and $S(x,t) = \omega t$:

$$(1.1) \qquad \begin{cases} -\Delta u + [m_0^2 - (\omega + e\phi)^2]u - |u|^{p-1}u = 0 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + e^2u^2\phi = -e\omega u^2 & \text{in } \mathbb{R}^3. \end{cases}$$

Solutions of (1.1), $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$, are critical points of the functional $\mathcal{S}\colon H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{S}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - |\nabla\phi|^2 + [m_0^2 - (\omega + e\phi)^2]u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

We are interested in finding "ground state" solutions of (1.1), that is a solution $(u_0,\phi_0) \in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ which minimizes the functional $\mathcal{S}$ among all the non-trivial solutions of (1.1), namely $\mathcal{S}(u_0,\phi_0) \leq \mathcal{S}(u,\phi)$, for any $(u,\phi) \neq (0,0)$ solution of (1.1).

The main result we provide in this paper is the following

THEOREM 1.1. *The problem* (1.1) *admits a ground state solution if*

(a) $3 \leq p < 5$ *and* $m_0 > \omega$*;*
(b) $1 < p < 3$ *and* $m_0\sqrt{p-1} > \omega\sqrt{5-p}$*.*

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Notation.

- For any $1 \leq s < \infty$, $L^s(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm
$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s.$$
- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the norm
$$\|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 + u^2.$$
- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is completion of $C_0^\infty(\mathbb{R}^3)$ (the compactly supported functions in $C^\infty(\mathbb{R}^3)$) with respect to the norm
$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)}^2 := \int_{\mathbb{R}^3} |\nabla u|^2.$$
- For any $r > 0$, $x \in \mathbb{R}^3$ and $A \subset \mathbb{R}^3$

$$B_r(x) := \{y \in \mathbb{R}^3 \mid |y - x| \leq r\}, \quad B_r := \{y \in \mathbb{R}^3 \mid |y| \leq r\}, \quad A^c := \mathbb{R}^3 \setminus A.$$

2. Preliminary lemmas

The first difficulty in dealing with the functional $\mathcal{S}$ is that it is strongly indefinite, namely it is unbounded both from below and from above on infinite dimensional subspaces. To avoid this indefiniteness, we will use the reduction method.

We need the following:

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which satisfies*

$$-\Delta\phi + e^2u^2\phi = -e\omega u^2 \quad \text{in } \mathbb{R}^3.$$

Moreover, the map $\Phi\colon u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and on the set $\{x \in \mathbb{R}^3 \mid u(x) \neq 0\}$,

$$-\frac{\omega}{e} \leq \phi_u \leq 0. \tag{2.1}$$

Proof. The proof can be found in [3], and [12]. □

Lemma 2.2. *Let $u \in H^1(\mathbb{R}^3)$ and set $\psi_u = (\Phi'[u])[u]/2 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. Then:*

(a) *ψ_u is a solution to the integral equation*

$$\int_{\mathbb{R}^3} e\omega\psi_u u^2 = \int_{\mathbb{R}^3} e(\omega + e\phi_u)\phi_u u^2; \tag{2.2}$$

(b) *it results that $\psi_u \leq 0$.*

Proof. The proof is a consequence of the fact that ψ_u satisfies

$$-\Delta\psi_u + e^2u^2\psi_u = -e(\omega + e\phi_u)u^2,$$

as we know by [12]. □

Set $\Omega = m_0^2 - \omega^2$ and define $I\colon H^1(\mathbb{R}^3) \to \mathbb{R}$ as

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 - e\omega\phi_u u^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}.$$

The functional I is obtained from $\mathcal{S}$ by the reduction method, as in [3]. As one can see, it does not present anymore the strong indefiniteness, and it is strictly connected with our problem, since $(u,\phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if u is a critical point of I and $\phi = \phi_u$.

We will look for a minimizer of the functional I restricted to the its Nehari manifold, namely

$$\mathcal{N} = \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid G(u) = 0\},$$

where

$$G(u) = \langle I'(u), u\rangle = \int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 - 2e\omega\phi_u u^2 - e^2\phi_u^2 u^2 - \int_{\mathbb{R}^3} |u|^{p+1}.$$

In the following lemmas we point out some properties related with the Nehari manifold.

LEMMA 2.3. *There exists a positive constant C such that $\|u\|_{p+1} \geq C$, for all $u \in \mathcal{N}$.*

PROOF. By (2.1), we infer

$$-e\int_{\mathbb{R}^3}(2\omega + e\phi_u)\phi_u u^2 \geq 0.$$

Therefore, by the definition of the Nehari manifold, we get

$$\|u\|_{p+1}^2 \leq C\int_{\mathbb{R}^3}|\nabla u|^2 + \Omega u^2 \leq C\|u\|_{p+1}^{p+1}. \qquad \square$$

LEMMA 2.4. *There exists a positive constant $C > 0$, such that $I(u) \geq C$, for any $u \in \mathcal{N}$.*

PROOF. For any $u \in \mathcal{N}$, we have

$$I(u) = \frac{p-1}{2(p+1)}\int_{\mathbb{R}^3}|\nabla u|^2 + \Omega u^2 - \frac{p-3}{2(p+1)}\int_{\mathbb{R}^3}e\omega\phi_u u^2 + \frac{1}{p+1}\int_{\mathbb{R}^3}e^2\phi_u^2 u^2. \tag{2.3}$$

We have to distinguish two cases. If $3 \leq p < 5$, then, by (2.1), each term in (2.3) is positive and the conclusion follows by Lemma 2.3, supposing $m_0 > \omega$.

Instead, in the case $1 < p < 3$, by (2.1) we have

$$\begin{aligned} I(u) &\geq \frac{p-1}{2(p+1)}\int_{\mathbb{R}^3}|\nabla u|^2 + \Omega u^2 + \frac{p-3}{2(p+1)}\int_{\mathbb{R}^3}\omega^2 u^2 \\ &\geq \frac{p-1}{2(p+1)}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{2(p+1)}\int_{\mathbb{R}^3}[(p-1)m_0^2 - 2\omega^2]u^2. \end{aligned}$$

Assuming that $m_0\sqrt{p-1} > \omega\sqrt{5-p}$, we conclude also in this case. $\square$

LEMMA 2.5. *$\mathcal{N}$ is a C^1 manifold.*

PROOF. For all $u \in H^1(\mathbb{R}^3)$, we have

$$G(u) = 2I(u) + \int_{\mathbb{R}^3}\frac{1-p}{p+1}|u|^{p+1} - \int_{\mathbb{R}^3}e\omega\phi_u u^2 - \int_{\mathbb{R}^3}e^2\phi_u^2 u^2.$$

Let us prove that there exists $C > 0$ such that $\langle G'(u), u\rangle \leq -C$, for all $u \in \mathcal{N}$.

If $u \in \mathcal{N}$, by (2.2)

$$\begin{aligned} \langle G'(u), u\rangle &= \int_{\mathbb{R}^3}(1-p)|u|^{p+1} - \int_{\mathbb{R}^3}4e\phi_u u^2(\omega + e\phi_u + e\psi_u) \\ &= (1-p)\int_{\mathbb{R}^3}|\nabla u|^2 + \Omega u^2 \\ &\quad - \int_{\mathbb{R}^3}e\phi_u u^2[(1-p)(2\omega + e\phi_u) + 4(\omega + e\phi_u + e\psi_u)]. \end{aligned}$$

We have to distinguish two cases. If $3 \leq p < 5$, since $m_0 > \omega$, by Lemma 2.3 and (2.1), we need only to show that

$$(1-p)(2\omega + e\phi_u) + 4(\omega + e\phi_u + e\psi_u) \leq 0.$$

Indeed, since $\phi_u, \psi_u \leq 0$, we have

$$(1-p)(2\omega + e\phi_u) + 4(\omega + e\phi_u + e\psi_u) = 2(3-p)\omega + (5-p)e\phi_u + 4e\psi_u \leq 0.$$

In the case $1 < p < 3$, instead, by (2.1), we have

$$\begin{aligned}\langle G'(u), u\rangle \leq{}& (1-p)\int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 - 2(3-p)\int_{\mathbb{R}^3} e\omega\phi_u u^2 \\ &- (5-p)\int_{\mathbb{R}^3} e^2\phi_u^2 u^2 - 4\int_{\mathbb{R}^3} e^2\phi_u\psi_u u^2 \\ \leq{}& (1-p)\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} [(1-p)m_0^2 + (5-p)\omega^2]u^2.\end{aligned}$$

We get the same conclusion with the additional assumption

$$m_0\sqrt{p-1} > \omega\sqrt{5-p}. \qquad \square$$

According to the definition of [17], we say that a sequence $(v_n)_n$ vanishes if, for all $r > 0$

$$\lim_n \sup_{\xi\in\mathbb{R}^3} \int_{B_r(\xi)} v_n^2 = 0.$$

Lemma 2.6. *Any bounded sequence $(v_n)_n \subset \mathcal{N}$ does not vanish.*

Proof. Suppose by contradiction that $(v_n)_n$ vanishes, i.e. there exists $\overline{r} > 0$ such that

$$\lim_n \sup_{\xi\in\mathbb{R}^3} \int_{B_{\overline{r}}(\xi)} v_n^2 = 0.$$

Then, by [18, Lemma 1.1], we infer that $v_n \to 0$ in $L^s(\mathbb{R}^3)$, for any $2 < s < 6$, contradicting Lemma 2.3. $\square$

The map Φ is continuous for the weak topology in the sense of the following lemma

Lemma 2.7. *If $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$ then, up to subsequences, $\phi_{u_n} \rightharpoonup \phi_{u_0}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. As a consequence $I'(u_n) \to I'(u_0)$ in the sense of distributions.*

Proof. Let $(u_n)_n$ and u_0 be in $H^1(\mathbb{R}^3)$, and assume that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$. As a consequence

$$u_n \rightharpoonup u_0, \quad \text{in } L^s(\mathbb{R}^3), \quad 2 \leq s \leq 6, \tag{2.4}$$

$$u_n \to u_0, \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \leq s < 6. \tag{2.5}$$

We denote by ϕ_n the function ϕ_{u_n}. By the second of (1.1) we have that for any $n \geq 1$

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla \phi_n|^2 &= -e^2 \int_{\mathbb{R}^3} u_n^2 \phi_n^2 - e \int_{\mathbb{R}^3} \omega u_n^2 \phi_n \\ &\leq -e \int_{\mathbb{R}^3} \omega u_n^2 \phi_n \leq C \|u_n\|_{12/5}^2 \|\nabla \phi_n\|_2,\end{aligned}$$

and then we deduce that $(\phi_n)_n$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

We can assume that there exists $\phi_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $\phi_n \rightharpoonup \phi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and, as a consequence,

$$\phi_n \rightharpoonup \phi_0, \quad \text{in } L^6(\mathbb{R}^3), \tag{2.6}$$

$$\phi_n \to \phi_0, \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \leq s < 6. \tag{2.7}$$

If we show that $\phi_0 = \phi_{u_0}$ we have concluded. By the uniqueness of the solution of the second equation in (1.1), we are reduced to prove that

$$-\Delta \phi_0 + e^2 u_0^2 \phi_0 = -e\omega u_0^2$$

in the sense of distributions. So, let $\varphi \in C_0^\infty(\mathbb{R}^3)$ a test function. Since

$$-\Delta \phi_n + e^2 u_n^2 \phi_n = -e\omega u_n^2$$

it is sufficient to show that the following three hold

$$\begin{aligned}\int_{\mathbb{R}^3} (\nabla \phi_n | \nabla \varphi) &\to \int_{\mathbb{R}^3} (\nabla \phi_0 | \nabla \varphi), \\ \int_{\mathbb{R}^3} u_n^2 \phi_n \varphi &\to \int_{\mathbb{R}^3} u_0^2 \phi_0 \varphi, \\ \int_{\mathbb{R}^3} u_n^2 \varphi &\to \int_{\mathbb{R}^3} u_0^2 \varphi.\end{aligned} \tag{2.8}$$

The first is a trivial application of the definition of weak convergence, whereas the third is a consequence of (2.5). As regards the second, observe that

$$\begin{aligned}\int_{\mathbb{R}^3} (u_n^2 \phi_n - u_0^2 \phi_0)\varphi &= \int_{\mathbb{R}^3} (u_n^2 - u_0^2)\phi_n \varphi + \int_{\mathbb{R}^3} (\phi_n - \phi_0) u_0^2 \varphi \\ &\leq C \|\nabla \phi_n\|_2 \bigg(\int_{\mathbb{R}^3} |u_n^2 - u_0^2|^{6/5} |\varphi|^{6/5} \bigg)^{5/6} + \int_{\mathbb{R}^3} (\phi_n - \phi_0) u_0^2 \varphi\end{aligned}$$

and then (2.8) follows by the boundedness of $(\phi_n)_n$, (2.5) and (2.7).

Now we pass to prove the second part of the lemma. Let φ be a test function. We compute:

$$\langle I'(u_n), \varphi \rangle = \int_{\mathbb{R}^3} (\nabla u_n | \nabla \varphi) + \Omega u_n \varphi - 2e\omega \phi_n u_n \varphi - e^2 \phi_n^2 u_n \varphi - |u_n|^{p-1} u_n \varphi,$$

$$\langle I'(u_0), \varphi \rangle = \int_{\mathbb{R}^3} (\nabla u_0 | \nabla \varphi) + \Omega u_0 \varphi - 2e\omega \phi_0 u_0 \varphi - e^2 \phi_0^2 u_0 \varphi - |u_0|^{p-1} u_0 \varphi.$$

Now observe that

$$\int_{\mathbb{R}^3}(\phi_n u_n - \phi_0 u_0)\varphi = \int_{\mathbb{R}^3}\phi_n(u_n-u_0)\varphi + \int_{\mathbb{R}^3}(\phi_n-\phi_0)u_0\varphi$$
$$\le C\|\nabla\phi_n\|_2\bigg(\int_{\mathbb{R}^3}|u_n-u_0|^{6/5}|\varphi|^{6/5}\bigg)^{5/6} + \int_{\mathbb{R}^3}(\phi_n-\phi_0)u_0\varphi = o_n(1)$$

by the boundedness of $(\phi_n)_n$, (2.5) and (2.7). Moreover,

$$\int_{\mathbb{R}^3}(\phi_n^2 u_n - \phi_0^2 u_0)\varphi = \int_{\mathbb{R}^3}\phi_n^2(u_n-u_0)\varphi + \int_{\mathbb{R}^3}(\phi_n^2-\phi_0^2)u_0\varphi$$
$$\le C\|\nabla\phi_n\|_2^2\bigg(\int_{\mathbb{R}^3}|u_n-u_0|^{3/2}|\varphi|^{3/2}\bigg)^{2/3} + \int_{\mathbb{R}^3}(\phi_n^2-\phi_0^2)u_0\varphi = o_n(1)$$

by the boundedness of $(\phi_n)_n$, (2.5) and (2.7). So we have

$$\underbrace{\int_{\mathbb{R}^3}(\nabla u_n|\nabla\varphi)+\Omega u_n\varphi}_{\downarrow} - \underbrace{\int_{\mathbb{R}^3}2e\omega\phi_n u_n\varphi}_{\downarrow} - \underbrace{\int_{\mathbb{R}^3}e^2\phi_n^2 u_n\varphi}_{\downarrow} - \underbrace{\int_{\mathbb{R}^3}|u_n|^{p-1}u_n\varphi}_{\downarrow}$$
$$\int_{\mathbb{R}^3}(\nabla u_0|\nabla\varphi)+\Omega u_0\varphi - \int_{\mathbb{R}^3}2e\omega\phi_0 u_0\varphi - \int_{\mathbb{R}^3}e^2\phi_0^2 u_0\varphi - \int_{\mathbb{R}^3}|u_0|^{p-1}u_0\varphi,$$

and then we conclude that $\langle I'(u_n),\varphi\rangle \to \langle I'(u_0),\varphi\rangle$. □

3. Proof of Theorem 1.1

Let $\sigma = \inf_{u\in\mathcal{N}} I(u)$. By Lemma 2.4, we argue that $\sigma > 0$. Since all the critical points of I are contained in $\mathcal{N}$ and since, by Lemma 2.5, we know that Nehari manifold is a natural constrained for I, if there exists $u_0 \in \mathcal{N}$ such that $I(u_0) = \sigma$, then (u_0, ϕ_{u_0}) is a ground state solution for (1.1).

Let $(u_n)_n \subset \mathcal{N}$ such that $I(u_n) \to \sigma$, as $n \to \infty$. It is easy to see that $(u_n)_n$ is a bounded sequence in $H^1(\mathbb{R}^3)$. By Lemma 2.6, there exists $C > 0$, $\overline{r} > 0$ and a sequence $(\xi_n)_n \subset \mathbb{R}^3$ such that

$$\int_{B_{\overline{r}}(\xi_n)} u_n^2 \ge C.$$

Let $v_n = u_n(\,\cdot\, + \xi_n)$. By the invariance of translations, $(v_n)_n$ is a bounded sequence contained in $\mathcal{N}$ such that

$$\int_{B_{\overline{r}}} v_n^2 \ge C \quad \text{for all } n, \tag{3.1}$$

and, moreover, $I(v_n) \to \sigma$, as $n \to \infty$. Up to a subsequence, there exists $v_0 \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} &v_n \rightharpoonup v_0, && \text{weakly in } H^1(\mathbb{R}^3),\\ &v_n \to v_0, && \text{in } L^s_{\rm loc}(\mathbb{R}^3),\ 1\le s<6,\\ &v_n \to v_0, && \text{a.e. in } \mathbb{R}^3, \end{aligned} \tag{3.2}$$

Denote $\phi_n \equiv \phi_{v_n}$ and $\phi_0 \equiv \phi_{v_0}$. By Lemma 2.7, we know that $\phi_n \rightharpoonup \phi_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, and, as a consequence,

$$\begin{aligned} \phi_n &\to \phi_0, \quad \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \le s < 6, \\ \phi_n &\to \phi_0, \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{3.3}$$

By [22], without lost of generality, we can assume that $(v_n)_n$ is a Palais–Smale sequence for the functional $I_{|\mathcal{N}}$, in particular,

$$\begin{aligned} I(v_n) &\to \sigma, \quad \text{as } n \to \infty, \\ (I_{|\mathcal{N}})'(v_n) &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.4}$$

By (3.4), being $(v_n)_n$ bounded in $H^1(\mathbb{R}^3)$, for suitable Lagrange multipliers l_n, we get

$$o_n(1) = \langle (I_{|\mathcal{N}})'(v_n), v_n\rangle = \langle I'(v_n), v_n\rangle + l_n\langle G'(v_n), v_n\rangle = l_n\langle G'(v_n), v_n\rangle.$$

By Lemma 2.5, we infer that $l_n = o_n(1)$ and, by (3.4),

$$I'(v_n) \to 0,\ \text{as } n \to \infty. \tag{3.5}$$

By (3.1), we infer that $v_0 \neq 0$ (and hence also $\phi_0 \neq 0$). Moreover, by Lemma 2.7 and (3.5), we can conclude that $I'(v_0) = 0$. It remains to prove that $I(v_0) = \sigma$. Observe that, since $(v_n)_n$ is in $\mathcal{N}$, we have

$$I(v_n) = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 - \frac{p-3}{2(p+1)} \int_{\mathbb{R}^3} e\omega\phi_n v_n^2 + \frac{1}{p+1} \int_{\mathbb{R}^3} e^2\phi_n^2 v_n^2.$$

We have to distinguish two cases. If $p \ge 3$, since $\phi_n \le 0$, by the weak lower semicontinuity of the H^1-norm, (3.2), (3.3) and the Lemma of Fatou, we conclude that $I(v_0) = \sigma$. This implies that (v_0, ϕ_0) is a ground state solution. If $1 < p < 3$, by (2.1) and requiring that $m_0\sqrt{p-1} > \omega\sqrt{5-p}$, it is easy to see that

$$\frac{p-1}{2(p+1)}\Omega v_n^2 - \frac{p-3}{2(p+1)}e\omega\phi_n v_n^2 \ge 0,$$

almost everywhere in $\mathbb{R}^3$, and we conclude as before.

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