Nonlinear Differ. Equ. Appl. (2016) 23:52

1021-9722/16/050001-34
published online August 22, 2016
DOI 10.1007/s00030-016-0403-0

Nonlinear Differential Equations and Applications NoDEA

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Ground states of a nonlinear curl-curl problem in cylindrically symmetric media

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Abstract. We consider the nonlinear curl-curl problem $\nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U$ in $\mathbb{R}^3$ related to the Kerr nonlinear Maxwell equations for fully localized monochromatic fields. We search for solutions as minimizers (ground states) of the corresponding energy functional defined on subspaces (defocusing case) or natural constraints (focusing case) of $H(\text{curl}; \mathbb{R}^3)$. Under a cylindrical symmetry assumption corresponding to a photonic fiber geometry on the functions V and Γ the variational problem can be posed in a symmetric subspace of $H(\text{curl}; \mathbb{R}^3)$. For a defocusing case $\sup \Gamma < 0$ with large negative values of Γ at infinity we obtain ground states by the direct minimization method. For the focusing case $\inf \Gamma > 0$ the concentration compactness principle produces ground states under the assumption that zero lies outside the spectrum of the linear operator $\nabla \times \nabla \times +V(x)$. Examples of cylindrically symmetric functions V are provided for which this holds.

Mathematics Subject Classification. Primary 35Q60, 58E15; Secondary 47J30, 78A25.

Keywords. Curl-curl problem, Maxwell's equation, Ground state, Variational methods, Symmetric subspace, Concentration compactness.

1. Introduction

For given real-valued functions V, Γ we consider the nonlinear curl-curl problem

$$\nabla \times \nabla \times U + V(x)U = \Gamma(x)|U|^{p-1}U \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $p > 1$, and look for real (weak) solutions. As we show below in Sect. 1.3, Eq. (1.1) arises in the description of three-dimensional monochromatic waves by Kerr nonlinear Maxwell's equations. Solutions $U \in H^1(\mathbb{R}^3)$ then correspond to fully localized standing electromagnetic waves. The problem of localizing light in all three dimensions attracts strong interest in the physics community.

This is partly due to the potential applications of such “light bullets” as information carriers in future optical logic and optical computing devices. Standing bullets, in particular, can be used in optical memory [10,20,36].

Compared to the $-\Delta$ operator, the curl-curl operator exhibits major mathematical difficulties. For example, the kernel of $\nabla\times\nabla\times$ is an infinite dimensional space which substantially spoils the coercivity of the variational functional associated to (1.1), cf. (1.2). Moreover, the natural solution space $H(\mathrm{curl};\mathbb{R}^3)$ does not embed into any $L^q(\mathbb{R}^3)$-space for $q>2$. To overcome these difficulties we use variational methods in suitable function spaces based on underlying symmetries. To be specific, we look for weak solutions

$$U \in X := H(\mathrm{curl};\mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3),$$

where $L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ denotes the space of L^{p+1}-functions with respect to the measure $|\Gamma|\,dx$ and $H(\mathrm{curl};\mathbb{R}^3)$ is the space of functions $U\in L^2(\mathbb{R}^3)$ for which $\mathrm{curl}\,U$, defined in the sense of distributions, belongs to $L^2(\mathbb{R}^3)$; cf. Sect. 2 for more details on these spaces. The solutions $U\in X$ of (1.1) arise as critical points of the functional

$$J[U] = \int_{\mathbb{R}^3} \frac{1}{2}(|\nabla\times U|^2 + V(x)|U|^2) - \frac{\Gamma(x)}{p+1}|U|^{p+1}\,dx, \quad U\in X. \tag{1.2}$$

We find ground state solutions, i.e. minimizers of J within a certain subspace of X (defocusing case) or under a natural constraint (focusing case). Note that although we limit our attention to real solutions, the methods are in principle applicable in the complex case $U(x)\in\mathbb{C}^3$ as well.

1.1. Variational aspects of the curl-curl problem

In the literature there are only few results on the nonlinear curl-curl problem. In [9] Benci, Fortunato opened the discussion about ground states for the problem

$$\nabla\times\nabla\times U = W'(|U|^2)U. \tag{1.3}$$

The problem was solved by Azzollini, Benci, D’Aprile, Fortunato in [4] using variational and symmetry-based methods. Using a different class of symmetries D’Aprile, Siciliano also obtained in [12] solutions of (1.3). Recently, Bartsch and Mederski [6,7] considered ground states as well as bound states of (1.3) on a bounded domain Ω with the boundary condition $\nu\times U=0$ on $\partial\Omega$. In [21] Mederski considered (1.1) where, e.g., the right hand side is of the form $\Gamma(x)f(u)$ with $f(u)\sim|u|^{p-1}u$ if $|u|\gg 1$ and $f(u)\sim|u|^{q-1}u$ if $|u|\ll 1$ for $1<p<5<q$ and where $\Gamma>0$ is periodic and bounded, $V\le 0$, $V\in L^{\frac{p+1}{p-1}}(\mathbb{R}^3)\cap L^{\frac{q+1}{q-1}}(\mathbb{R}^3)$.

Let us point out that on top of the common obstacle of J being unbounded from below in the case $\Gamma>0$, the variational formulation has the following additional difficulties:

- For $p>1$ the space $H(\mathrm{curl};\mathbb{R}^3)$ does not embed into $L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ even when Γ is bounded. In the so-called focusing case $\Gamma>0$ it is therefore hard to control the X-norm of any Palais–Smale sequence $(U_k)_{k\in\mathbb{N}}$, i.e., any sequence with $(J[U_k])_{k\in\mathbb{N}}$ bounded and $J'[U_k]\to 0$ as $k\to\infty$.

- Note that $\|\nabla U\|_2^2 = \|\nabla \times U\|_2^2 + \|\nabla \cdot U\|_2^2$. Hence restriction of J to the space $X_0 = \{U \in X : \nabla \cdot U = 0\}$ on one hand allows at least for $\Gamma \in L^\infty(\mathbb{R}^3)$ the embedding $X_0 \to L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ but on the other hand it generates an additional gradient term in the Euler-Lagrange equation.

Therefore, finding critical points of J directly in the whole space X is out of the scope of the current paper. Instead we will look for critical points on a suitable subspace by exploiting symmetries of (1.1). As proposed in [4] and earlier in [31], one such subspace is given by functions U of the form

$$U(x) = \frac{u(r, x_3)}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad r^2 = x_1^2 + x_2^2, \tag{1.4}$$

where $u : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a real valued, scalar function. Note that such functions U satisfy $\nabla \cdot U = 0$. Assuming cylindrical symmetry also for the potentials V and Γ, i.e., $V = V(r, x_3), \Gamma = \Gamma(r, x_3)$, this ansatz leads to the equation

$$\left(-\partial_r^2 - \partial_{x_3}^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} + V(r, x_3)\right) u = \Gamma(r, x_3)|u|^{p-1}u.$$

We also define the linear operator in the vector valued Eq. (1.1) as

$$\mathcal{L} := (\nabla \times \nabla \times) + V(x) \tag{1.5}$$

and study its spectrum $\sigma(\mathcal{L})$ when restricted to a suitable subspace of functions which exhibit symmetries like the functions given in (1.4). Under the above symmetry assumptions, we will study (1.1) in the following three scenarios:

- Fully radially symmetric case: $V = V(\rho)$, $\Gamma = \Gamma(\rho)$ with $\rho^2 = r^2 + x_3^2$.
- Strongly defocusing case: $\sup_{\mathbb{R}^3} V < 0$ and $\Gamma(x) \le -C(1 + |x|)^\alpha$ for some constants $C > 0$ and $\alpha > \frac{3(p-1)}{2}$.
- Focusing case: $\inf_{\mathbb{R}^3} \Gamma > 0, 0 \notin \sigma(\mathcal{L})$. Examples of such potentials $V(r, x_3)$ are given. They are periodic in x_3, satisfy $\lim_{r\to\infty} V(r, x_3) = V_\infty(x_3)$ and $\sup_{\mathbb{R}} V_\infty(x_3) > 0 > \inf_{\mathbb{R}^3} V$. Hence the potential has non-vanishing negative and, as $r \to \infty$, also non-vanishing positive part.

From a physical point of view the latter two scenarios may be criticized. Because Γ corresponds to the electric susceptibility of the considered medium, see Sect. 1.3, the strongly defocusing case implies unrealistically high defocusing nature of the material. And since $V(x)$ is proportional to $-n^2(x)$, where n is the refractive index, the condition of the non-vanishing positive part of V at infinity in the focusing case implies an imaginary refractive index. Hence it will be desirable to overcome these limitations in future work.

1.2. Main results

Now we state our main results. The first result is concerned with those solutions of (1.1) that are fully radially symmetric.

Theorem 1.1. (Fully radially symmetric case) *Let $p > 1$ and assume that $V, \Gamma \in L^\infty_{loc}(\mathbb{R}^3)$ and $0 \le V\Gamma^{-1} \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^3)$. Additionally suppose the full radial* *symmetry of V and Γ in $\mathbb{R}^3$, i.e. $V(x) = \tilde{V}(|x|)$ and $\Gamma(x) = \tilde{\Gamma}(|x|)$ for almost all $x \in \mathbb{R}^3$ and $\tilde{V}, \tilde{\Gamma} \in L^\infty_{loc}([0,\infty))$. Under the full radial symmetry condition $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in O(3)$, all distributional solutions $U \in L^p_{loc}(\mathbb{R}^3)$ of (1.1) satisfy $\nabla \times U = 0$ and have the form*

$$U(x) = s(|x|) \left(\frac{V(x)}{\Gamma(x)} \right)^{\frac{1}{p-1}} \frac{x}{|x|}, \tag{1.6}$$

where $s : (0,\infty) \to \{-1, 1\}$ is an arbitrary measurable function. If additionally $(V\Gamma^{-1})^{\frac{2}{p-1}}$, $(V\Gamma^{-1})^{\frac{p+1}{p-1}}\Gamma \in L^1(\mathbb{R}^3)$ then $U \in X$ and hence it is a critical point of J.

Thus, the assumption of full radial symmetry does not lead to interesting solutions of (1.1). We therefore relax the fully radial symmetry and look for solutions having only cylindrical symmetry. For this purpose we use in Theorems 1.2 and 1.3 the space X_{G_1} which will be defined in Sect. 2 and may be thought of as the subspace of $X = H(\text{curl}; \mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ consisting of vector fields of the form (1.4).

Theorem 1.2. (Strongly defocusing case) *Let $p > 1$ and assume that $V = V(r, x_3)$ and $\Gamma = \Gamma(r, x_3)$ have cylindrical symmetry and satisfy*

(i) *$\Gamma(x) \le -C(1 + |x|)^\alpha$ in $\mathbb{R}^3$ with $\alpha > \frac{3}{2}(p-1)$ and $C > 0$,*
(ii) *$V \in L^\infty(\mathbb{R}^3)$ and $\sup_{\mathbb{R}^3} V < 0$.*

Then (1.1) has a ground state on X_{G_1}.

Theorem 1.3. (Focusing case) *Let $1 < p < 5$ and assume that $V = V(r, x_3)$ and $\Gamma = \Gamma(r, x_3)$ have cylindrical symmetry and satisfy*

(i) *$\inf_{\mathbb{R}^3} \Gamma > 0$,*
(ii) *$V, \Gamma \in L^\infty(\mathbb{R}^3)$ are 1-periodic in x_3, i.e., $V(r, x_3) = V(r, x_3 + 1)$, $\Gamma(r, x_3) = \Gamma(r, x_3 + 1)$ for a.a. $r > 0, x_3 \in \mathbb{R}$,*
(iii) *$0 \notin \sigma(\mathcal{L})$.*

Then (1.1) has a ground state on X_{G_1}, which is moreover a minimizer of J restricted to a natural constraint set (the so-called Nehari–Pankov manifold, cf. Sect. 5).

Examples of potentials $V(r, x_3)$ with $0 \notin \sigma(\mathcal{L})$ are given in Section 4. They have non-vanishing positive and negative parts.

1.3. Physical context of the problem

Here we show how (1.1) arises in the study of fully localized monochromatic standing waves in Kerr nonlinear Maxwell's equations when higher harmonics are neglected. As mentioned before, such standing "light bullets" could be of potential use for optical data storage, cf. [10, 20, 36].

So far, to our knowledge, standing light bullets have not been observed in experiments: neither in homogeneous or periodic media nor in radial or cylindrical geometries, corresponding to the choice in this paper. Nevertheless, at least one theoretical prediction of such waves exists. In the Kerr nonlinear fiber Bragg grating (a cylindrical geometry with periodicity in the longitudinal direction) an asymptotic model for broad wavepackets and a small periodicity contrast supports localized waves, so called gap solitons [3]. The model is the system of one dimensional coupled mode equations and the gap solitons come in a family including standing solutions. Gap solitons have been experimentally observed with velocities as low as $0.23\frac{c}{n}$, where c is the speed of light in vacuum and n the average refractive index of the fiber core [23], but not with velocity zero. On the other hand, moving localized pulses have been demonstrated in numerous other nonlinear geometries including standard optical fibers [24] and arrays of waveguides arranged in the plane [22]. In fact, moving bullets exist even in homogenous linear materials, see the recent work on Airy bullets [11]. Note also that standing bullets in the form of defect states at localized defects of a linear medium have been studied e.g. in photonic crystals [18]. Clearly, our cylindrical geometry does not correspond to a defect localized in all three dimensions. In most physics articles theoretical predictions of light bullets are made based on the nonlinear Schrödinger equation (NLS). For instance in homogeneous materials the NLS is known to have radially symmetric localized solutions, so called Townes solitons, in all dimensions [33]. In periodic media [27] and at interfaces of two periodic structures [13] standing ground state H^1-solutions exist. The NLS is, however, only an asymptotic approximation of Maxwell's equations. Moreover, for inhomogeneous media in two and three dimensions the approximation has not been rigorously justified. This paper, in contrast, deals with the full three-dimensional Maxwell problem.

The three-dimensional Maxwell equations in the absence of charges and currents read

$$\nabla \times \mathcal{E} + \partial_t \mathcal{B} = 0, \quad \nabla \cdot \mathcal{D} = 0,$$
$$\nabla \times \mathcal{H} - \partial_t \mathcal{D} = 0, \quad \nabla \cdot \mathcal{B} = 0.$$

Here $\mathcal{E}, \mathcal{H} : \mathbb{R}^4 \to \mathbb{R}^3$ denote the electric and magnetic field, respectively, and $\mathcal{D}, \mathcal{B} : \mathbb{R}^4 \to \mathbb{R}^3$ denote the displacement field and the magnetic induction, respectively. For the relation between the magnetic field $\mathcal{H}$ and the magnetic induction $\mathcal{B}$ we assume $\mathcal{B} = \mu_0 \mathcal{H}$ with μ_0 constant. By taking the curl of the first equation one finds

$$\nabla \times \nabla \times \mathcal{E} + \mu_0 \partial_t^2 \mathcal{D} = 0, \quad \nabla \cdot \mathcal{D} = 0. \tag{1.7}$$

For a Kerr-type nonlinear medium the material law between the electric field $\mathcal{E}$ and the displacement field $\mathcal{D}$ is given by

$$\mathcal{D} = \epsilon_0 \left(n^2(x)\mathcal{E} + \mathcal{P}_{\mathrm{NL}}(x, \mathcal{E}) \right) \quad \text{with} \quad \mathcal{P}_{\mathrm{NL}}(x, \mathcal{E}) = \chi^{(3)}(x)(\mathcal{E} \cdot \mathcal{E})\mathcal{E}, \tag{1.8}$$

where $n^2(x) = 1 + \chi^{(1)}(x)$ is the square of the linear refractive index and where $\mathcal{P}_{\mathrm{NL}}$ denotes the nonlinear part of the polarization. Note that in this section we use the notation $w \cdot z = w_1 z_1 + w_2 z_2 + w_3 z_3$ both for real and complex valued vectors $w, z \in \mathbb{C}^3$. The functions $\chi^{(1)}$ and $\chi^{(3)}$ denote the linear and cubic susceptibilities of the medium respectively. Although $\chi^{(3)}$ is generally a tensor, symmetries in the atomic structure of the material allow a reduction to a scalar, see [25, Sec. 2d]. The resulting second order equation for the electric field $\mathcal{E}$ is then given by the quasilinear wave equation

$$\nabla\times\nabla\times\mathcal{E}+\frac{1}{c^2}\partial_t^2\big(n(x)^2\mathcal{E}+\chi^{(3)}(x)(\mathcal{E}\cdot\mathcal{E})\mathcal{E}\big)=0,\quad (x,t)\in\mathbb{R}^4 \tag{1.9}$$

together with $\nabla\cdot\mathcal{D}=0$. Here $c=(\epsilon_0\mu_0)^{-1/2}$ is the speed of light in vacuum. If $\mathcal{E}$ solves (1.9), then $\mathcal{D}$ is known from (1.8) and $\mathcal{B}$ can be obtained from $\nabla\times\mathcal{E}$ by a time integration and thus also $\mathcal{H}$ is known. Moreover, the fields $\mathcal{D},\mathcal{B}$ will be divergence free provided they are divergence free at some fixed time, e.g., $t=0$.

The question of light bullets is that of the existence of solutions of Maxwell's equations in nonlinear dispersive media which are localized in space, i.e., which at all times t are decaying to 0 as $|x|\to\infty$. In this paper we cannot give a complete answer to this question. Instead we will solve a related problem. Motivated by Fourier-expansion in time, one might look for a solution of (1.9) of the form $\mathcal{E}(x,t)=\sum_{k=0}^\infty\big(e^{-\mathrm{i}(2k+1)\omega t}E_k(x)+\text{c.c.}\big)$ with $E_k(x)\in\mathbb{C}^3$. If such solution existed under the additional assumption of localization, i.e., $\mathcal{E}(x,t)$ decaying to 0 as $|x|\to\infty$ for all t, then it would be a standing light bullet. Here we consider the simpler monochromatic ansatz

$$\mathcal{E}(x,t)=e^{-\mathrm{i}\omega t}E(x)+\text{c.c.}\quad\text{with}\quad E(x)\in\mathbb{C}^3. \tag{1.10}$$

If we insert these monochromatic fields into the constitutive relation (1.8) and neglect the generation of higher harmonics, i.e., we cancel all terms with factors $e^{\pm3\mathrm{i}\omega t}$, then we obtain the new simplified constitutive relation $\mathcal{D}=\epsilon_0\left(n^2(x)\mathcal{E}+\mathcal{P}_{\mathrm{NL}}^{(\mathrm{a})}(x,\mathcal{E})\right)$ with

$$\mathcal{P}_{\mathrm{NL}}^{(\mathrm{a})}(x,e^{-\mathrm{i}\omega t}E+\text{c.c.})=\chi^{(3)}(x)e^{-\mathrm{i}\omega t}\left(2|E|^2E+(E\cdot E)\bar{E}\right)+\text{c.c.} \tag{1.11}$$

Note that here $E\cdot E=E_1^2+E_2^2+E_3^2\in\mathbb{C}$ whereas $|E|^2=|E_1|^2+|E_2|^2+|E_3|^2$ denotes the Hermitian inner product of E with itself. The second order elliptic equation for the E-field resulting from (1.7) is

$$\nabla\times\nabla\times E-\frac{\omega^2}{c^2}\big(n^2(x)E+\chi^{(3)}(x)\big(2|E|^2E+(E\cdot E)\,\overline{E}\big)\big)=0,\quad x\in\mathbb{R}^3. \tag{1.12}$$

Note that the divergence conditions $\nabla\cdot\mathcal{D}=0$ is automatically satisfied due to the monochromatic ansatz and the curl-curl structure of the equation.

Another model of the nonlinear polarization which effectively removes higher harmonics and results in equation (1.12) is given by time-averaging $\mathcal{E}\cdot\mathcal{E}$. In detail, for a $T-$periodic $\mathcal{E}(x,t)\cdot\mathcal{E}(x,t)$ one defines

$$\mathcal{P}_{\mathrm{NL}}^{(\mathrm{b})}(x,\mathcal{E})=\chi^{(3)}(x)\frac{1}{T}\int_0^T\mathcal{E}(x,t)\cdot\mathcal{E}(x,t)dt\;\mathcal{E}(x,t), \tag{1.13}$$

see, e.g., [32,34]. For $\mathcal{E}$ as in (1.10), where $T=\pi/\omega$, we get the same as in (1.11), i.e.

$$\mathcal{P}_{\mathrm{NL}}^{(\mathrm{a})}(x,e^{-\mathrm{i}\omega t}E+\text{c.c.})=\mathcal{P}_{\mathrm{NL}}^{(\mathrm{b})}(x,e^{-\mathrm{i}\omega t}E+\text{c.c.}).$$

To sum up, we may say that a solution $E:\mathbb{R}^3\to\mathbb{C}^3$ of (1.12) gives via (1.10) rise to a complete solution of the Maxwell system provided we consider the constitutive relation (1.11) or (1.13) instead of (1.8).

With the notation

$$V(x) := -\frac{\omega^2}{c^2}n^2(x), \quad \Gamma(x) := 3\frac{\omega^2}{c^2}\chi^{(3)}(x)$$

Eq. (1.12) reads

$$\nabla \times \nabla \times E + V(x)E = \frac{1}{3}\Gamma(x)\big(2|E|^2\, E + (E \cdot E)\,\overline{E}\big) \text{ in } \mathbb{R}^3. \tag{1.14}$$

Restricting to real valued solutions $E \in H^1(\mathbb{R}^3)$, Eq. (1.14) is equivalent to (1.1) with $p = 3$.

1.4. Structure of the paper

The rest of the paper is structured as follows. In Sect. 2 Theorem 1.1 is first proved. Next, for the case of cylindrical symmetry of V and Γ a subspace of X is chosen in which minimization of J is possible. In Sect. 3 the strongly defocusing case (i.e. Theorem 1.2) is handled by the direct minimization method. Sections 4 and 5 treat the more delicate focusing case (i.e. Theorem 1.3). In Sect. 4 we study the spectrum of the linear operator in (1.1) and find examples of V for which zero lies outside the spectrum. This is a necessary condition for our minimization approach. Finally, in Sect. 5 J is minimized on the so called Nehari–Pankov manifold within the symmetric subspace using the concentration-compactness principle.

2. Variational formulation of (1.1)

We begin with the definition of some spaces of vector valued functions $U : \mathbb{R}^3 \to \mathbb{R}^3$. For a measurable weight-function $\sigma : \mathbb{R}^3 \to (0, \infty)$ the corresponding weighted L^q-space for $1 \le q < \infty$ is defined by

$$L^q_\sigma(\mathbb{R}^3) = \left\{U : \mathbb{R}^3 \to \mathbb{R}^3 : \int_{\mathbb{R}^3} \sigma(x)\,|U|^q\,dx < \infty\right\}$$

with the norm

$$\|U\|_{\sigma,q} = \left(\int_{\mathbb{R}^3} \sigma(x)\,|U|^q\,dx\right)^{\frac{1}{q}}.$$

The space $H^1(\mathbb{R}^3)$ is defined by

$$H^1(\mathbb{R}^3) = \left\{U : \mathbb{R}^3 \to \mathbb{R}^3 : U^i, \frac{\partial U^i}{\partial x_j} \in L^2(\mathbb{R}^3) \text{ for } i, j = 1, 2, 3\right\},$$

with the norm

$$\|U\|^2_{H^1} = \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \left(\frac{\partial U^i}{\partial x_j}\right)^2 + |U|^2\,dx.$$

The space $H(\text{curl}; \mathbb{R}^3)$ is defined by

$$H(\text{curl}; \mathbb{R}^3) = \{U : \mathbb{R}^3 \to \mathbb{R}^3 : U, \nabla \times U \in L^2(\mathbb{R}^3)\},$$

with the norm

$$\|U\|_{H(\mathrm{curl})}^2 = \int_{\mathbb{R}^3} |\nabla \times U|^2 + |U|^2 \, dx$$

and where $\nabla \times U$ is understood in the distributional sense, i.e., it satisfies $\int_{\mathbb{R}^3} (\nabla \times U) \cdot \phi \, dx = \int_{\mathbb{R}^3} U \cdot (\nabla \times \phi) \, dx$ for all C^∞-functions $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ with compact support.

Notice that for $U \in H^1(\mathbb{R}^3)$ we have the identity

$$\int_{\mathbb{R}^3} |\nabla \times U|^2 + (\nabla \cdot U)^2 \, dx = \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \left(\frac{\partial U^i}{\partial x_j} \right)^2 \, dx$$

with $\nabla \cdot U$ denoting the distributional divergence of U. Therefore

$$H(\mathrm{curl}; \mathbb{R}^3) \cap \{U : \nabla \cdot U = 0\} = H^1(\mathbb{R}^3) \cap \{U : \nabla \cdot U = 0\}. \tag{2.1}$$

This property will be used when we single out a suitable subspace of $H^1(\mathbb{R}^3)$, one in which we can solve (1.1). For this purpose we first study the symmetries of (1.1).

Lemma 2.1. *Assume that the locally bounded measurable functions $V, \Gamma : \mathbb{R}^3 \to \mathbb{R}^3$ are radially symmetric.*

(a) *If $U \in L^p_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ is a distributional solution of (1.1) and $M \in O(3)$ then $\tilde{U}(x) := M^T U(Mx)$ also solves (1.1) in the sense of distributions.*
(b) *Suppose $U : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in O(3)$. Then $U(x) = f(|x|)\frac{x}{|x|}$ for some $f : [0, \infty) \to \mathbb{R}$. If additionally $U \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ then $\nabla \times U = 0$ in the sense of distributions.*

Proof. (a) Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞-function with compact support, let $M \in O(3)$ and define

$$\psi(y) := M\phi(M^T y), \qquad y \in \mathbb{R}^3.$$

Then a direct computation yields

$$\nabla \times \nabla \times \psi(y) = M(\nabla \times \nabla \times \phi)(M^T y)$$

and thus

$$(\mathcal{L}\psi)(y) = M(\mathcal{L}\phi)(M^T y).$$

Therefore, if $U \in L^p_{loc}(\mathbb{R}^3; \mathbb{R}^3)$ satisfies

$$\int_{\mathbb{R}^3} U(y) \cdot (\mathcal{L}\psi)(y) - \Gamma(y)|U(y)|^{p-1} U(y) \cdot \psi(y) \, dy = 0 \quad \text{for all} \quad \psi \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)$$

then

$$\begin{aligned}
&\int_{\mathbb{R}^3} \tilde{U}(x)\cdot(\mathcal{L}\phi)(x) - \Gamma(x)|\tilde{U}(x)|^{p-1}\tilde{U}(x)\cdot\phi(x)\,dx \\
&= \int_{\mathbb{R}^3} M^T U(y)\cdot(\mathcal{L}\phi)(M^T y) - \Gamma(y)|U(y)|^{p-1}M^T U(y)\cdot\phi(M^T y)\,dy \\
&= \int_{\mathbb{R}^3} M^T U(y)\cdot M^T\mathcal{L}\psi(y) - \Gamma(y)|U(y)|^{p-1}U(y)\cdot\psi(y)\,dy \\
&= \int_{\mathbb{R}^3} U(y)\cdot\mathcal{L}\psi(y) - \Gamma(y)|U(y)|^{p-1}U(y)\cdot\psi(y)\,dy \\
&= 0.
\end{aligned}$$

(b) Let $x \in \mathbb{R}^3$ be such that $U(x) = M^T U(Mx)$ for all $M \in O(3)$. Then $U(x) = MU(x)$ for all those rotations M which leave x fixed, i.e., for all rotations around the axis $\mathbb{R}x$. Hence $U(x) \in \mathbb{R}x$ and we may write $U(x) = f(|x|)\frac{x}{|x|}$. Under the assumption $U \in L^1_{loc}(\mathbb{R}^3)$ we see that $f \in L^1(I)$ for any compact interval $I \subset (0,\infty)$. Therefore we may define the function $F(r) := \int_1^r f(s)\,ds$ for $r > 0$ which is absolutely continuous in $\mathbb{R}^+$. Moreover, for any $R > 1$ using polar coordinates and Fubini's theorem we see that

$$\begin{aligned}
\int_{B_R(0)} |F(|x|)|\,dx &= 4\pi\int_0^R \left|\int_1^r f(t)\,dt\right| r^2\,dr \\
&\le 4\pi\int_0^1\int_r^1 |f(t)|\,dt\,r^2\,dr + 4\pi\int_1^R\int_1^r |f(t)|\,dt\,r^2\,dr \\
&\le \frac{4\pi}{3}\int_0^1 |f(t)|t^3\,dt + \frac{4\pi R^3}{3}\int_1^R |f(t)|\,dt \\
&\le \frac{4\pi R^3}{3}\int_0^R |f(t)|t^2\,dt \\
&= \frac{R^3}{3}\int_{B_R(0)} |U(x)|\,dx < \infty
\end{aligned}$$

since $U \in L^1_{loc}(\mathbb{R}^3)$. Hence the function $F(|\cdot|)$ belongs to $L^1_{loc}(\mathbb{R}^3)$ and due to the absolute continuity of F it has the strong derivative $U(x)$ almost everywhere. Since both $F(|\cdot|)$ and U are $L^1_{loc}(\mathbb{R}^3)$, one can see that $U(x) = \nabla(F(|x|))$ in $\mathbb{R}^3$ in the weak sense. This implies $\nabla\times U = 0$ in the distributional sense. □

Proof of Theorem 1.1: Suppose $U \in L^p_{loc}(\mathbb{R}^3)$ is a distributional solution of (1.1). Lemma 2.1 shows that the requirement of full radial symmetry of V and Γ and the solution symmetry $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in O(3)$ reduces (1.1) to the algebraic equation

$$V(x)U = \Gamma(x)|U|^{p-1}U \text{ in } \mathbb{R}^3.$$

Provided $0 \le V\Gamma^{-1}$ the function U has the form (1.6). Now let us reversely assume that U has the form (1.6). Since $0 \le V\Gamma^{-1} \in L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^3)$ we see that

$U \in L^p_{loc}(\mathbb{R}^3)$ and in particular $U \in L^1_{loc}(\mathbb{R}^3)$. Moreover, with an absolutely continuous function $F : (0,\infty) \to \mathbb{R}$ given by

$$F(t) = \int_1^t s(\tau) \left(\frac{\tilde{V}(\tau)}{\tilde{\Gamma}(\tau)} \right)^{1/(p-1)} d\tau, \quad t > 0$$

we have $U(x) = \nabla (F(|x|))$ in the distributional sense. As in Lemma 2.1 we find $F(|\cdot|) \in L^1_{loc}(\mathbb{R}^3)$ and, moreover, $\nabla \times U = 0$. Hence U solves (1.1). Finally, the assumption $(V\Gamma^{-1})^{\frac{2}{p-1}}$, $(V\Gamma^{-1})^{\frac{p+1}{p-1}}\Gamma \in L^1(\mathbb{R}^3)$ implies that any U defined by (1.6) belongs to the space X and thus is a critical point of J. □

Although the $H(\mathrm{curl};\mathbb{R}^3)$ solutions in Theorem 1.1 are valid localized solutions of (1.1), in the rest of the paper we consider solutions that are not gradient fields.

Since the requirement of full radial symmetry does not lead to interesting solutions of (1.1), we look for solutions which are invariant only under a subgroup of $O(3)$ (this idea is due to Azzollini et al. [4]). For this we define the following copy of $SO(2)$ as a subset of $O(3)$

$$G_0 := \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}.$$

Assume that the measurable weight $\sigma : \mathbb{R}^3 \to (0,\infty)$ satisfies $\sigma(Mx) = \sigma(x)$ for all $x \in \mathbb{R}^3$ and all $M \in G_0$. Then the group G_0 operates isometrically on $L^q_\sigma(\mathbb{R}^3)$, on $H(\mathrm{curl};\mathbb{R}^3)$ and on $H^1(\mathbb{R}^3)$ by the group action $U \mapsto M^T U(M\cdot)$. Due to this result we can now define the corresponding G_0-fixed point subspaces of $L^q_\sigma(\mathbb{R}^3)$, $H(\mathrm{curl};\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$, $k \in \mathbb{N}$ by

$$\begin{aligned} L^q_{\sigma,G_0}(\mathbb{R}^3) &= \{U \in L^q_\sigma(\mathbb{R}^3) : U(x) = M^T U(Mx)\ \forall x \in \mathbb{R}^3, \forall M \in G_0\}, \\ H_{G_0}(\mathrm{curl};\mathbb{R}^3) &= \{U \in H(\mathrm{curl};\mathbb{R}^3) : U(x) = M^T U(Mx)\ \forall x \in \mathbb{R}^3, \forall M \in G_0\}, \\ H^k_{G_0}(\mathbb{R}^3) &= \{U \in H^k(\mathbb{R}^3) : U(x) = M^T U(Mx)\ \forall x \in \mathbb{R}^3, \forall M \in G_0\}, \\ X_{G_0} &= H_{G_0}(\mathrm{curl};\mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3). \end{aligned}$$

Observe that the functional J is invariant under the action of G_0. Thus, by Palais' principle of symmetric criticality [26,35], every critical point of $J|_{X_{G_0}}$ is also a critical point of J on X. Next we want to restrict the spaces $L^q_{\sigma,G_0}(\mathbb{R}^3)$, $H_{G_0}(\mathrm{curl};\mathbb{R}^3)$ and $H^k_{G_0}(\mathbb{R}^3)$ even further. In order to do so we need two lemmas—the first one being analogous to Lemma 2.1. We omit the proofs because they are contained in Lemma 1 and Proposition 1 in [4].

Lemma 2.2. *Suppose a measurable function $U : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $U(x) = M^T U(Mx)$ for a.a. $x \in \mathbb{R}^3$ and all $M \in G_0$. Then there are unique measurable functions $Q, S, T : \mathbb{R}^3 \to \mathbb{R}^3$ such that*

$$U(x) = Q(x) + S(x) + T(x)$$

with

$$Q(x)=\frac{q(r,x_3)}{r}\begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix},\quad S(x)=\frac{s(r,x_3)}{r}\begin{pmatrix}x_1\\ x_2\\ 0\end{pmatrix},\quad T(x)=\begin{pmatrix}0\\ 0\\ t(r,x_3)\end{pmatrix}. \tag{2.2}$$

where $q,s,t:(0,\infty)\times\mathbb{R}\to\mathbb{R}$ are measurable functions. If $U\in L^q_\sigma(\mathbb{R}^3)$ or $H^1(\mathbb{R}^3)$, then $Q,S,T\in L^q_\sigma(\mathbb{R}^3)$ or $H^1(\mathbb{R}^3)$, respectively.

Lemma 2.3. *Let Y be one of the spaces $L^q_{\sigma,G_0}(\mathbb{R}^3)$, $H_{G_0}(\mathrm{curl};\mathbb{R}^3)$ or $H^k_{G_0}(\mathbb{R}^3)$, $k\in\mathbb{N}$ and define the map*

$$g_1:\begin{cases}Y\to Y,\\ U=Q+S+T\mapsto Q-S-T.\end{cases}$$

The map g_1 is a linear isometry and satisfies $g_1\circ g_1=Id$. Hence $G_1=\{\mathrm{Id},g_1\}$ is a group of order 2. Moreover, the functional $J|_{X_{G_0}}$ is invariant under the action of G_1.

Remark. The proof of the isometry and invariance statement relies on the fact that pointwise $|U|^2=|Q|^2+|S|^2+|T|^2$ and $|\nabla U|^2=|\nabla Q|^2+|\nabla S|^2+|\nabla T|^2$, cf. [4]. However, for the curl only $|\nabla\times U|^2=|\nabla\times Q|^2+|\nabla\times(S+T)|^2$ holds. But this is sufficient for our claim.

This result allows to define the spaces

$$\begin{aligned}L^q_{\sigma,G_1}(\mathbb{R}^3)&=\{U\in L^q_{\sigma,G_0}(\mathbb{R}^3):g_1U=U\},\\ H_{G_1}(\mathrm{curl};\mathbb{R}^3)&=\{U\in H_{G_0}(\mathrm{curl};\mathbb{R}^3):g_1U=U\},\\ H^k_{G_1}(\mathbb{R}^3)&=\{U\in H^k_{G_0}(\mathbb{R}^3):g_1U=U\},\quad k\in\mathbb{N},\\ X_{G_1}&=H_{G_1}(\mathrm{curl};\mathbb{R}^3)\cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3).\end{aligned}$$

All the spaces with the suffix G_1 may be thought of as the subspaces of $L^q_\sigma(\mathbb{R}^3)$, $H(\mathrm{curl};\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$ consisting of vector fields of the form (1.4). Again, Palais' principle of symmetric criticality ensures that every critical point of $J|_{X_{G_1}}$ is also a critical point of J on X. Finally, note that

$$H_{G_1}(\mathrm{curl};\mathbb{R}^3)=H^1_{G_1}(\mathbb{R}^3) \tag{2.3}$$

because the members of both spaces have vanishing divergence, cf. (2.1).

To summarize the results of this section recall that the energy functional related to (1.1) is

$$J[U]=\int_{\mathbb{R}^3}\frac{1}{2}(|\nabla\times U|^2+V(x)|U|^2)-\frac{\Gamma(x)}{p+1}|U|^{p+1}\,dx,$$

which is well defined on $X=H(\mathrm{curl};\mathbb{R}^3)\cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$. Due to Lemma 2.2, Palais' principle of symmetric criticality (cf. Palais [26], Willem [35]) and (2.3) one can seek critical points of the functional J restricted to the subspace $X_{G_1}=H^1_{G_1}(\mathbb{R}^3)\cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ and these critical points will be solutions of (1.1). The elements of the subspace $H^1_{G_1}(\mathbb{R}^3)$ have the favorable property of vanishing divergence.

3. Ground states in the defocusing case

We assume $p > 1$ and

$$\text{(H-defoc)} \quad \Gamma(x) \le -C(1+|x|)^{\alpha} \text{ in } \mathbb{R}^3 \text{ with } \alpha > \frac{3}{2}(p-1) \text{ and } C > 0.$$

We work in the following reflexive Banach space

$$X_{G_1} := H^1_{G_1}(\mathbb{R}^3) \cap L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$$

where the norm on X_{G_1} is given by

$$\|U\|_X := \|U\|_{H^1} + \|U\|_{|\Gamma|,p+1}.$$

The basic tool for proving existence of ground states in the defocusing case is the following embedding result, which is due to Benci, Fortunato [8].

Lemma 3.1. *Assume $p > 1$ and* (H-defoc)*. Then the space $L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ embeds continuously into $L^2(\mathbb{R}^3)$ and the space X_{G_1} embeds compactly into $L^2(\mathbb{R}^3)$.*

Proof. Let $\beta = \frac{p+1}{p-1}$ and $\beta' = \frac{p+1}{2}$. Then

$$\begin{aligned}\int_{\mathbb{R}^3} |U|^2\,dx &= \int_{\mathbb{R}^3} |\Gamma(x)|^{-\frac{1}{\beta'}} |\Gamma(x)|^{\frac{1}{\beta'}} |U|^2\,dx \\ &\le \left(\int_{\mathbb{R}^3} |\Gamma(x)|^{-\frac{2}{p-1}}\,dx\right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^3} |\Gamma(x)||U|^{p+1}\,dx\right)^{\frac{2}{p+1}},\end{aligned}$$

and the first integral is finite since by assumption (H-defoc) $|\Gamma(x)|^{-\frac{2}{p-1}} \le C(1+|x|)^{-\frac{2\alpha}{p-1}}$ and $-\frac{2\alpha}{p-1} < -3$. This proves the first part of the claim. For the second part, let us define the positive and continuous radially symmetric function $\rho : \mathbb{R}^3 \to (0,\infty)$ by setting $\rho(x) = 1$ for $|x| \le 1$ and $\rho(x) = |x|^{\gamma}$ with $\gamma > 0$ so small that $\gamma\beta - \frac{2\alpha}{p-1} < -3$. Then $\rho(x) \to \infty$ as $|x| \to \infty$ and $\int_{\mathbb{R}^3} \rho(x)^{\beta}(1+|x|)^{-\frac{2\alpha}{p-1}}\,dx < \infty$. We obtain

$$\begin{aligned}\|U\|^2_{\rho,2} &= \int_{\mathbb{R}^3} \rho(x)|\Gamma(x)|^{-\frac{1}{\beta'}} |\Gamma(x)|^{\frac{1}{\beta'}} |U|^2\,dx \\ &\le \left(\int_{\mathbb{R}^3} \rho(x)^{\beta}|\Gamma(x)|^{-\frac{2}{p-1}}\,dx\right)^{\frac{p-1}{p+1}} \left(\int_{\mathbb{R}^3} |\Gamma(x)||U|^{p+1}\,dx\right)^{\frac{2}{p+1}} \\ &\le C\|U\|^2_{|\Gamma|,p+1}.\end{aligned}$$

This shows that $L^{p+1}_{|\Gamma|}(\mathbb{R}^3)$ embeds continuously into $L^2_{\rho}(\mathbb{R}^3)$. Finally, by Theorem 3.1 of Benci, Fortunato [8] we have that $H^1(\mathbb{R}^3) \cap L^2_{\rho}(\mathbb{R}^3)$ embeds compactly into $L^2(\mathbb{R}^3)$. Both facts together imply the second claim of the lemma. ☐

Lemma 3.2. *Assume $p > 1$,* (H-defoc) *and $V \in L^{\infty}(\mathbb{R}^3)$. Then the functional J is a weakly lower-semicontinuous, coercive C^1-functional on X_{G_1} and hence has a minimizer.*

Proof. Since

$$J_1[U] = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla \times U|^2 - \frac{\Gamma(x)}{p+1}|U|^{p+1}\,dx$$

is convex on X_{G_1} and

$$J_2[U] = \int_{\mathbb{R}^3} \frac{V(x)}{2}|U|^2\,dx$$

is weakly continuous on X_{G_1} by Lemma 3.1, we find that the functional $J = J_1 + J_2$ is weakly lower-semicontinuous on X_{G_1}. Moreover, there exist constants $K_1, \ldots, K_5 > 0$ such that the following estimates hold for $U \in X_{G_1}$:

$$\begin{aligned} J[U] &\geq \frac{1}{2}\|\nabla \times U\|_2^2 + \frac{1}{p+1}\|U\|_{|\Gamma|,p+1}^{p+1} - \frac{\|V\|_\infty}{2}\|U\|_2^2 \\ &\geq \frac{1}{2}\|\nabla \times U\|_2^2 + \frac{1}{p+1}\|U\|_{|\Gamma|,p+1}^{p+1} - K_1\|U\|_{|\Gamma|,p+1}^2 \\ &\geq \frac{1}{2}\|\nabla \times U\|_2^2 + K_2\|U\|_{|\Gamma|,p+1}^2 - K_3 \\ &\geq \frac{1}{2}\|\nabla \times U\|_2^2 + K_4\|U\|_2^2 + \frac{K_2}{2}\|U\|_{|\Gamma|,p+1}^2 - K_3 \\ &\geq K_5\|U\|_X^2 - K_3, \end{aligned}$$

which shows the coercivity of J. It is clear that the quadratic parts of the functional J are C^1 and it is standard (cf. Struwe [30]) to verify that the functional $\int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1}$ has a Gâteaux derivative which depends continuously on $U \in X_{G_1}$. Hence J is a C^1-functional on X_{G_1} and the minimizer of J is a weak solution of (1.1). □

Proof of Theorem 1.2: We set $U_0(x) = sW(tx)$ for some vector-valued function $W \in C_0^\infty(\mathbb{R}^3)$ and take $s, t > 0$. Since $\sup_{\mathbb{R}^3} V < 0$ we obtain

$$\begin{aligned} J[U_0] &= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla \times U_0(x)|^2 - \frac{\Gamma(x)}{p+1}|U_0(x)|^{p+1} + \frac{V(x)}{2}|U_0(x)|^2\,dx \\ &\leq \frac{s^2}{t^3}\int_{\mathbb{R}^3} \frac{t^2}{2}|\nabla \times W(y)|^2 - \frac{s^{p-1}\Gamma(y/t)}{p+1}|W(y)|^{p+1} + \frac{\sup_{\mathbb{R}^3} V}{2}|W(y)|^2\,dy \\ &< 0 \end{aligned}$$

provided $t > 0$ is chosen so small that $\int_{\mathbb{R}^3} t^2|\nabla \times W|^2 + \sup_{\mathbb{R}^3} V|W|^2\,dy < 0$ and then $s > 0$ is chosen sufficiently small. Thus the minimizer of J over X_{G_1} is non-trivial and therefore a ground state of (1.1) within X_{G_1}.

4. Spectrum of the linear operator $\mathcal{L}$

In the focusing case we can only show the existence of ground states (cf. Sect. 5) when zero does not lie in the spectrum of the linear operator

$$\mathcal{L} := (\nabla \times \nabla \times) + V(r, x_3).$$

Of course an easy example is given by the class of potentials $V = V(r, x_3)$ with $\inf V_{\mathbb{R}^3} > 0$. However, since $V(x)$ is proportional to $-n^2(x)$ with $n(x)$

being the refractive index, the physically interesting case consists of functions V which are negative (or at least have non-vanishing negative part). In this section we construct potentials V with non-vanishing negative part and where 0 lies in a spectral gap of the operator $\mathcal{L}$, cf. Lemmas 4.7, 4.8, and 4.9.

The construction of such examples needs various preparations. We consider $\mathcal{L}$ as an operator defined on

$$D(\mathcal{L}) = H^2_{G_1}(\mathbb{R}^3) \subset L^2_{G_1}(\mathbb{R}^3)$$

and we will show in Lemma 4.4 that $\mathcal{L}$ is a selfadjoint operator, whose spectrum has a particular additive structure whenever the potential is separable, i.e., $V(r, x_3) = W(r) + P(x_3)$, cf. Lemma 4.6. The key to these results is the following observation: if

$$U(x) = u(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \quad \text{with} \quad r = \sqrt{x_1^2 + x_2^2}$$

then

$$\mathcal{L}U(x) = \Big((Lu)(r, x_3)\Big)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \tag{4.1}$$

with

$$L = -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right) - \frac{\partial^2}{\partial x_3^2} + V(r, x_3) \tag{4.2}$$

where the first two terms correspond to a five-dimensional Laplacian with cylindrical symmetry. Let us now start with the detailed analysis of the operators and their spectra.

Define the maps

$$\Psi_{\text{rad}} : \begin{cases} \mathbb{R}^4 \to \mathbb{R}, \\ (y_1, \dots, y_4) \mapsto \sqrt{y_1^2 + \dots + y_4^2} \end{cases}$$

$$\Psi : \begin{cases} \mathbb{R}^5 \to \mathbb{R}^2, \\ (y_1, \dots, y_5) \mapsto (\Psi_{\text{rad}}(y_1, \dots, y_4), y_5) \end{cases}$$

In the following we use the index *rad* for spaces of functions $u : (0, \infty) \to \mathbb{R}$ of the single radial variable r; the index *cyl* refers to spaces of functions $u : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ of two variables (r, x_3). For the following Hilbert spaces we also denote in brackets the measure with respect to which integration is performed.

$$\begin{aligned}
L^2_{\text{rad}}(r^3 dr) &= \big\{u : (0,\infty) \to \mathbb{R} : u \circ \Psi_{\text{rad}} \in L^2(\mathbb{R}^4)\big\} \\
&= \big\{u : (0,\infty) \to \mathbb{R} : u \in L^2_{r^3}(0,\infty)\big\}, \\
L^2_{\text{cyl}}(r^3 dr dx_3) &= \big\{u : (0,\infty) \times \mathbb{R} \to \mathbb{R} : u \circ \Psi \in L^2(\mathbb{R}^5)\big\} \\
&= \big\{u : (0,\infty) \to \mathbb{R} : u \in L^2_{r^3}((0,\infty) \times \mathbb{R})\big\}, \\
H^1_{\text{rad}}(r^3 dr) &= \big\{u : (0,\infty) \to \mathbb{R} : u \circ \Psi_{\text{rad}} \in H^1(\mathbb{R}^4)\big\} \\
&= \big\{u : (0,\infty) \to \mathbb{R} : u, u' \in L^2_{r^3}(0,\infty)\big\},
\end{aligned}$$

$$\begin{aligned}
H^1_{\mathrm{cyl}}(r^3 dr dx_3) &= \left\{u:(0,\infty)\times\mathbb{R}\to\mathbb{R} : u\circ\Psi\in H^1(\mathbb{R}^5)\right\}\\
&= \left\{u:(0,\infty)\times\mathbb{R}\to\mathbb{R} : u, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial x_3}\in L^2_{r^3}((0,\infty)\times\mathbb{R})\right\},\\
H^2_{\mathrm{rad}}(r^3 dr) &= \left\{u:(0,\infty)\to\mathbb{R} : u\circ\Psi_{\mathrm{rad}}\in H^2(\mathbb{R}^4)\right\}\\
&= \left\{u\in H^1_{\mathrm{rad}}(r^3 dr): \frac{u'}{r}, u''\in L^2_{r^3}(0,\infty)\right\},\\
H^2_{\mathrm{cyl}}(r^3 dr dx_3) &= \left\{u:(0,\infty)\times\mathbb{R}\to\mathbb{R} : u\circ\Psi\in H^2(\mathbb{R}^5)\right\}\\
&= \left\{u\in H^1_{\mathrm{cyl}}(r^3 dr dx_3): \frac{1}{r}\frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial x_3^2}\in L^2_{r^3}((0,\infty)\times\mathbb{R})\right\}.
\end{aligned}$$

These identities may be well known. For the sake of clarity we explain the last one for $H^2_{\mathrm{cyl}}(r^3 dr dx_3)$ on the level of the derivatives of highest order: $u\circ\Psi$ has second order derivatives in $L^2(\mathbb{R}^5)$ if and only if for all $i,j\in\{1,2,3,4\}$ we have

$$\left(\frac{\partial^2 u}{\partial r^2}-\frac{1}{r}\frac{\partial u}{\partial r}\right)\frac{y_i y_j}{r^2}+\frac{\partial u}{\partial r}\frac{\delta_{ij}}{r}, \frac{\partial^2 u}{\partial x_3^2}, \frac{\partial^2 u}{\partial r\partial x_3}\frac{y_i}{r}\in L^2(\mathbb{R}^5). \tag{4.3}$$

In view of the fact that $\int_{\mathbb{R}^3}\sum_{i,j=1}^3\left(\frac{\partial^2 v(x)}{\partial x_i\partial x_j}\right)^2 dx = \int_{\mathbb{R}^3}(\Delta v(x))^2\,dx$ for $v\in C_0^\infty(\mathbb{R}^3)$ we see that

$$u\circ\Psi, \Delta(u\circ\Psi)\in L^2(\mathbb{R}^5)\Longleftrightarrow u\circ\Psi\in H^2(\mathbb{R}^5)$$

Hence, for (4.3) it is sufficient to have

$$u,\ \frac{1}{r}\frac{\partial u}{\partial r},\ \frac{\partial^2 u}{\partial r^2},\ \frac{\partial^2 u}{\partial x_3^2}\in L^2_{r^3}((0,\infty)\times\mathbb{R}) \tag{4.4}$$

since the L^2-norm of $\frac{\partial^2 u}{\partial r\partial x_3}$ may be estimated by the L^2-norm of $\Delta(u\circ\Psi)$, i.e., by sums of $L^2_{r^3}((0,\infty)\times\mathbb{R})$-norms of $\frac{1}{r}\frac{\partial u}{\partial r}$, $\frac{\partial^2 u}{\partial r^2}$, $\frac{\partial^2 u}{\partial x_3^2}$. Let us also show that (4.4) is necessary. If we square all the entries in (4.3) and add them up then we see that

$$\int_0^\infty\int_{-\infty}^\infty\left(\left(\frac{\partial^2 u}{\partial r^2}\right)^2+\frac{3}{r^2}\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial^2 u}{\partial x_3^2}\right)^2+\left(\frac{\partial^2 u}{\partial r\partial x_3}\right)^2\right)r^3\,dr dx_3<\infty,$$

which implies that (4.4) is also necessary.

The spaces carry natural inner products. Instead of listing all of them we just write out the ones for $H^2_{\mathrm{rad}}(r^3 dr)$ and $H^2_{\mathrm{cyl}}(r^3 dr dx_3)$:

$$\begin{aligned}
\langle u,v\rangle_{H^2_{\mathrm{rad}}} &= \int_0^\infty\left(uv+u'v'+\frac{1}{r^2}u'v'+u''v''\right)r^3 dr\\
\langle u,v\rangle_{H^2_{\mathrm{cyl}}} &= \int_0^\infty\int_{-\infty}^\infty\left(uv+\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{\partial u}{\partial x_3}\frac{\partial v}{\partial x_3}+\frac{1}{r^2}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\right)r^3 dr dx_3\\
&\quad+\int_0^\infty\int_{-\infty}^\infty\left(\frac{\partial^2 u}{\partial r^2}\frac{\partial^2 v}{\partial r^2}+\frac{\partial^2 u}{\partial x_3^2}\frac{\partial^2 v}{\partial x_3^2}\right)r^3 dr dx_3.
\end{aligned}$$

The above identities of spaces have the following implication.

Lemma 4.1. (Hardy's inequality)

(i) *There exists a constant* $C > 0$ *such that for all* $u \in H^1_{\mathrm{cyl}}(r^3 dr dx_3)$

$$\int_0^\infty \int_{-\infty}^\infty \frac{u^2}{r^2} r^3 dr dx_3 \le C \int_0^\infty \int_{-\infty}^\infty \left(\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \right) r^3 dr dx_3.$$

(ii) *There exists a constant* $C > 0$ *such that for all* $u \in H^2_{\mathrm{cyl}}(r^3 dr dx_3)$

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{r^2} \left(\frac{\partial u}{\partial x_3}\right)^2 r^3 dr dx_3 \le C \int_0^\infty \int_{-\infty}^\infty \left(\left(\frac{\partial^2 u}{\partial r^2}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2}\right)^2 \right) r^3 dr dx_3.$$

Proof. We use the identities of spaces as explained above. Part (i) can be found as Theorem C in [5] set up in $\mathbb{R}^5$ where $r = \mathrm{dist}(y, K)$ and $K = \{y \in \mathbb{R}^5 : y_1 = y_2 = y_3 = y_4 = 0\}$. Note that $H^1_0(\mathbb{R}^5 \backslash K) = H^1_0(\mathbb{R}^5)$, cf. Theorem 2.43 in [17], because as a subset of $\mathbb{R}^5$ the set K has zero 2-capacity, cf. Section 4.7.2 in [15]. Part (ii) is a consequence of (i) when applied to $\partial U / \partial y_5 \in H^1_0(\mathbb{R}^5)$. □

Lemma 4.2. *The following identity holds between the group invariant spaces* (*denoted with suffix* G_1) *and the spaces of scalar functions with cylindrical symmetry* (*denoted with suffix* cyl):

$$L^2_{G_1}(\mathbb{R}^3) = \left\{ u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} : u \in L^2_{\mathrm{cyl}}(r^3 dr dx_3) \right\}$$

$$H^k_{G_1}(\mathbb{R}^3) = \left\{ u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} : u \in H^k_{\mathrm{cyl}}(r^3 dr dx_3) \right\}, \quad k = 1, 2.$$

Proof. Let

$$U(x_1, x_2, x_3) = u(r, x_3) \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

The identity of the L^2-spaces is obvious since

$$\int_{\mathbb{R}^3} |U(x)|^2 \, dx = 2\pi \int_0^\infty \int_{-\infty}^\infty |u(r, x_3)|^2 r^3 dr dx_3.$$

Next we prove the identity of the H^1-spaces. Clearly $U \in H^1(\mathbb{R}^3)$ if and only if $x_i u \in H^1(\mathbb{R}^3)$ for $i = 1, 2$, i.e., if and only if

$$u\delta_{ij} + \frac{x_i x_j}{r} \frac{\partial u}{\partial r}, \ x_i \frac{\partial u}{\partial x_3}, \ x_i u \in L^2(\mathbb{R}^3).$$

By squaring, summing over $i, j = 1, 2$ and rearranging terms this in turn is equivalent to

$$\int_0^\infty \int_{-\infty}^\infty \left(u^2 + \left(u + r\frac{\partial u}{\partial r}\right)^2 + r^2 \left(\frac{\partial u}{\partial x_3}\right)^2 + r^2 u^2 \right) r \, dr dx_3 < \infty.$$

The above is equivalent to $\frac{u}{r}, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial x_3}, u \in L^2_{r^3}((0,\infty)\times\mathbb{R})$. Hardy's inequality of Lemma 4.5 tells us that the $L^2_{r^3}((0,\infty)\times\mathbb{R})$-norm of the first term is bounded by the norm of the remaining terms, and hence $U \in H^1(\mathbb{R}^3)$ if and only if $u \in H^1_{\text{cyl}}(r^3 drdx_3)$. Finally, let us prove the identity for the H^2-spaces. The second derivatives of U lie in $H^2(\mathbb{R}^3)$ if and only if

$$\delta_{ij}\frac{x_k}{r}\frac{\partial u}{\partial r} + \delta_{ik}\frac{x_j}{r}\frac{\partial u}{\partial r} + \delta_{jk}\frac{x_i}{r}\frac{\partial u}{\partial r} - \frac{x_i x_j x_k}{r^3}\frac{\partial u}{\partial r} + \frac{x_i x_j x_k}{r^2}\frac{\partial^2 u}{\partial r^2} \in L^2(\mathbb{R}^3)$$

and

$$\delta_{ij}\frac{\partial u}{\partial x_3} + \frac{x_i x_j}{r}\frac{\partial^2 u}{\partial r\partial x_3},\ x_i\frac{\partial^2 u}{\partial x_3^2} \in L^2(\mathbb{R}^3)$$

for $i,j,k = 1,2$. By squaring, summing over $i,j,k = 1,2$ and rearranging this becomes

$$\int_0^\infty\int_{-\infty}^\infty \left(3\left(\frac{\partial u}{\partial r}\right)^2 + \left(2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right)^2\right) r\, drdx_3 < \infty$$

$$\int_0^\infty\int_{-\infty}^\infty \left(\left(\frac{\partial u}{\partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_3} + r\frac{\partial^2 u}{\partial r\partial x_3}\right)^2\right) r\, drdx_3 < \infty$$

$$\int_0^\infty\int_{-\infty}^\infty \left(\frac{\partial^2 u}{\partial x_3^2}\right)^2 r^3\, drdx_3 < \infty.$$

Therefore a necessary and sufficient condition for $U \in H^2(\mathbb{R}^3)$ is given by

$$\int_0^\infty\int_{-\infty}^\infty \left(\frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial^2 u}{\partial r^2}\right)^2\right) r^3\, drdx_3$$
$$+\int_0^\infty\int_{-\infty}^\infty \left(\frac{1}{r^2}\left(\frac{\partial u}{\partial x_3}\right)^2 + \left(\frac{\partial^2 u}{\partial r\partial x_3}\right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2}\right)^2\right) r^3\, drdx_3 < \infty.$$

By the relation $\|D^2u\|_{L^2(\mathbb{R}^3)} = \|\Delta u\|_{L^2(\mathbb{R}^3)}$ and by Hardy's inequality of Lemma 4.5 the above is equivalent to

$$\int_0^\infty\int_{-\infty}^\infty \left(\frac{1}{r^2}\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial^2 u}{\partial r^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_3^2}\right)^2\right) r^3\, drdx_3 < \infty.$$

Hence U has second derivatives in $L^2(\mathbb{R}^3)$ if and only if $\frac{1}{r}\frac{\partial u}{\partial r}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial x_3^2} \in L^2_{r^3}((0,\infty)\times\mathbb{R})$. In view of the definition of $H^2_{\text{cyl}}(r^3 drdx_3)$ this establishes the claim. □

Lemma 4.3. *Let $V \in L^\infty(\mathbb{R}^3)$ and suppose $V = V(r, x_3)$ has cylindrical symmetry. Then the operator $L : D(L) := H^2_{\text{cyl}}(r^3 drdx_3) \subset L^2_{\text{cyl}}(r^3 drdx_3) \to L^2_{\text{cyl}}(r^3 drdx_3)$ given by (4.2) is selfadjoint.*

Proof. For $u \in D(L)$ we have

$$(Lu)\circ\Psi = -\Delta(u\circ\Psi) + (Vu)\circ\Psi,$$

i.e., L coincides with the five-dimensional Schrödinger operator $-\Delta + V$ in the space of functions with cylindrical symmetry. □

Lemma 4.4. *Let $V \in L^\infty(\mathbb{R}^3)$ and suppose $V = V(r, x_3)$ has cylindrical symmetry. The operator $\mathcal{L} := (\nabla\times\nabla\times) + V(r, x_3)$ defined on $D(\mathcal{L}) = H^2_{G_1}(\mathbb{R}^3) \subset L^2_{G_1}(\mathbb{R}^3) \to L^2_{G_1}(\mathbb{R}^3)$ is selfadjoint and $\sigma(\mathcal{L}) = \sigma(L)$.*

Proof. First we check the symmetry of $\mathcal{L}$. Let $U, \tilde{U} \in D(\mathcal{L})$, i.e., by Lemma 4.2

$$U(x) = u(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad \tilde{U}(x) = \tilde{u}(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

for some $u, \tilde{u} \in D(L)$. Thus

$$\begin{aligned} \langle \mathcal{L}U, \tilde{U}\rangle_{L^2(\mathbb{R}^3)} &= \left\langle \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}(Lu)(r, x_3), \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}\tilde{u}(r, x_3)\right\rangle_{L^2(\mathbb{R}^3)} \\ &= 2\pi\langle Lu, \tilde{u}\rangle_{L^2_{\text{cyl}}} \\ &= 2\pi\langle u, L\tilde{u}\rangle_{L^2_{\text{cyl}}} \text{ since } L \text{ is selfadjoint} \\ &= \langle U, \mathcal{L}\tilde{U}\rangle_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{4.5}$$

To show that $\mathcal{L}$ is selfadjoint it suffices to show that for some $\mu \in \mathbb{R}$ the operator

$$\mathcal{L} - \mu \,\mathrm{Id} : D(\mathcal{L}) \to L^2_{G_1}(\mathbb{R}^3)$$

is onto, cf. [14, Theorem 4.2]. We choose any μ in the resolvent set of L, e.g. $\mu = -\|V\|_\infty - 1$. Let $F \in L^2_{G_1}(\mathbb{R}^3)$, i.e., there exists $f \in L^2_{\text{cyl}}(r^3 dr dx_3)$ with

$$F(x) = f(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}.$$

Since μ lies in the resolvent set of L we can find $u \in D(L) = H^2_{\text{cyl}}(r^3 dr dx_3)$ such that $Lu - \mu u = f$. Defining

$$U(x) = u(r, x_3)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$$

and using (4.1) we get $\mathcal{L}U - \mu U = F$. This finishes the proof of the selfadjointness of $\mathcal{L}$ and also of the identity of the resolvent sets of $\mathcal{L}$ and of L and hence $\sigma(\mathcal{L}) = \sigma(L)$ follows. □

We assume now that our cylindrical waveguide geometry is 1-periodic along the x_3-direction, i.e., $V(r, x_3) = V(r, x_3 + 1)$ for a.a. $r > 0$ and $x_3 \in \mathbb{R}$. Besides the cylindrical symmetry and the periodicity in x_3-direction, the existence of ground states in the focusing case relies on the assumption that $0 \notin \sigma(\mathcal{L})$. In the following we will construct an example of a potential $V = V(r, x_3)$ which is 1-periodic w.r.t. x_3, with $0 \notin \sigma(\mathcal{L})$ and with $\sigma(\mathcal{L}) \cap (-\infty, 0) \neq \emptyset$. Recall that the physical significance of the sign of V has been explained at the beginning of this section.

Let us assume that the linear potential V is separable, i.e.

$$V(r,x_3)=W(r)+P(x_3). \tag{4.6}$$

with $W\in L^\infty(0,\infty)$, $P\in L^\infty(\mathbb{R})$ and $P(x_3+1)=P(x_3)$ for all $x_3\in\mathbb{R}$ [later we will assume that P is piecewise continuous in order to have (4.7)]. The splitting of the potential implies a splitting of the operator L as follows (so far we consider this only on a formal level): if $u(r,x_3)=v(r)w(x_3)$ then

$$(Lu)(r,x_3)=w(x_3)(L_rv)(r)+v(r)(L_pw)(x_3)$$

where L_r, L_p are given by the following differential expressions

$$L_r=-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right)+W(r),\quad L_p=-\frac{\partial^2}{\partial x_3^2}+P(x_3).$$

We can give these differential expressions the meaning of proper selfadjoint operators by specifying their domains of definition properly as in the following lemma.

Lemma 4.5. *Let $W\in L^\infty(0,\infty), P\in L^\infty(\mathbb{R})$. The operator L_p defined on $D(L_p)=H^2(\mathbb{R})\subset L^2(\mathbb{R})\to L^2(\mathbb{R})$ is selfadjoint. The operator L_r defined on $D(L_r)=H^2_{\mathrm{rad}}(r^3dr)\subset L^2_{\mathrm{rad}}(r^3dr)\to L^2_{\mathrm{rad}}(r^3dr)$ is selfadjoint.*

Proof. The statement for L_p is clear. The statement for L_r follows from the observation that for $v\in H^2_{\mathrm{rad}}(r^3dr)$

$$(L_rv)\circ\Psi_{\mathrm{rad}}=-\Delta(v\circ\Psi_{\mathrm{rad}})+(Wv)\circ\Psi_{\mathrm{rad}},$$

i.e., L_r coincides with the four-dimensional Schrödinger operator $-\Delta+W(r)$ with radial symmetry. □

Now we are in a position to state that for $V(r,x_3)=W(r)+P(x_3)$ the spectrum of $\mathcal{L}$ can be computed by separation of variables.

Lemma 4.6. *Let $W\in L^\infty(0,\infty), P\in L^\infty(\mathbb{R})$ and $V(r,x_3)=W(r)+P(x_3)$. Then $\sigma(\mathcal{L})=\sigma(L)=\sigma(L_r)+\sigma(L_p)$.*

Proof. Let us define the subspace

$$D_0=\left\{\sum_{k=1}^N v_k(r)w_k(x_3): N\in\mathbb{N}, v_k\in D(L_r), w_k\in D(L_p) \text{ for } k=1,\dots,N\right\}$$

and the operator L_r+L_p on D_0 by

$$(L_r+L_p)\left(\sum_{k=1}^N v_kw_k\right):=\sum_{k=1}^N\big((L_rv_k)w_k+v_k(L_pw_k)\big).$$

The subspace D_0 is dense in $L^2_{\mathrm{cyl}}(r^3drdx_3)$ because $D(L_r)\subset L^2_{\mathrm{rad}}(r^3dr)$ and $D(L_p)\subset L^2(\mathbb{R})$ are dense. Since L_r+L_p is symmetric, it is therefore closable. Let us recall the definition of the closure of an operator and its domain:

$$
\begin{aligned}
&D(\overline{L_r+L_p})\\
&\quad=\Big\{u\in L^2_{\mathrm{cyl}}(r^3drdx_3):\exists(u_n)_{n\in\mathbb{N}}\text{ in }D_0, z\in L^2_{\mathrm{cyl}}(r^3drdx_3)\text{ s.t.}\\
&\qquad\quad u_n\to u\text{ in }L^2_{\mathrm{cyl}}(r^3drdx_3)\text{ and }(L_r+L_p)u_n\to z\text{ in }L^2_{\mathrm{cyl}}(r^3drdx_3)\Big\}.
\end{aligned}
$$

For $u\in D(\overline{L_r+L_p})$ one defines $\overline{L_r+L_p}(u):=z$. By Theorem VIII.33 and its Corollary from [29], the selfadjointness of L_r, L_p is passed on and yields selfadjointness of $\overline{L_r+L_p}$. Note also that $L_r+L_p=L|_{D_0}$ and hence $\overline{L_r+L_p}=\overline{L|_{D_0}}$. Since by Lemma 4.3 the operator L defined on $H^2_{\mathrm{cyl}}(r^3drdx_3)$ is a selfadjoint extension of $L|_{D_0}$ and hence also of the operator $\overline{L|_{D_0}}=\overline{L_r+L_p}$ which is already selfadjoint, we find that $\overline{L_r+L_p}=L$. Again by Theorem VIII.33 and its Corollary from [29] we find the claim $\sigma(L)=\sigma(\overline{L_r+L_p})=\overline{\sigma(L_r)+\sigma(L_p)}=\sigma(L_r)+\sigma(L_p)$, where the last equality holds since the two spectra $\sigma(L_r),\sigma(L_p)$ are closed and bounded from below. □

Next to the periodicity and boundedness of P let us now sharpen the assumption by requiring additionally that P is piecewise continuous. Then the spectrum of L_p is purely continuous and consists of the union of countably many intervals

$$\sigma(L_p)=\bigcup_{k=1}^{\infty}[\nu_{2k-1},\nu_{2k}]\quad\text{with}\quad \nu_{2k-1}<\nu_{2k}\le\nu_{2k+1}, \tag{4.7}$$

see Theorem XIII.90 in [28]. We assume that the first gap is open, i.e.,

$$\nu_2<\nu_3.$$

Next we describe the radial part of the spectrum of L under some special assumptions on the potential W. We start with some properties of Bessel functions.

Lemma 4.7. *Let J_1 denote the order one Bessel function which is regular at 0 and K_1 the order one modified Bessel function which decreases exponentially at infinity. Let $0<j_1<j_2<\cdots$ be the positive zeroes of J_1 and $0<j_1'<j_2'<\cdots$ be the positive zeroes of J_1'. Let*

$$\eta_*=\sqrt{(j_1)^2-(j_1')^2},\quad \eta^*:=\sqrt{(j_2')^2-(j_1)^2}.$$

Then $\eta_*<\eta^*$ and for every $\eta\in[\eta_*,\eta^*]$ there exists a unique value $\xi=\xi(\eta)\in(j_1',j_1)$ with the properties

$$\frac{J_1(\xi)}{\xi J_1'(\xi)}=\frac{K_1(\eta)}{\eta K_1'(\eta)}\quad\text{and}\quad (j_1)^2<\xi^2+\eta^2<(j_2')^2. \tag{4.8}$$

Proof. Define $\tilde g:(0,j_2')\backslash\{j_1'\}\to\mathbb{R}$ and $\tilde h:(0,\infty)\to\mathbb{R}$ by

$$\tilde g(\xi):=\frac{J_1(\xi)}{\xi J_1'(\xi)},\quad \tilde h(\eta):=\frac{K_1(\eta)}{\eta K_1'(\eta)}.$$

Let us mention that the properties of $\tilde g,\tilde h$ used in this proof are proved in Lemma 6.1 in the Appendix. Since $\tilde g(\xi)\to 1$ as $\xi\to 0+$ and $\tilde h(\eta)\to -1$ as $\eta\to$

0+, the two functions can be extended continuously to 0. Moreover, on $[0, j_1')$ the function $\tilde{g}$ is strictly increasing from 1 to $+\infty$, on (j_1', j_2') it increases strictly from $-\infty$ to $+\infty$ with a zero at j_1. The function $\tilde{h}$ is negative and strictly increases on $[0,\infty)$ from -1 to 0. Suppose a value $\eta > 0$ is given. By the strict monotonicity of $\tilde{g}$ we can find a unique solution $\xi = \xi(\eta)$ of $\tilde{g}(\xi) = \tilde{h}(\eta)$ within the interval (j_1', j_1). Now we want to ensure that the pair $(\xi(\eta), \eta)$ satisfies the constraint in (4.8). Since $\xi(\eta) \in (j_1', j_1)$ the constraint $(j_1)^2 < \xi^2 + \eta^2 < (j_2')^2$ is certainly satisfied if we impose the following restriction on η:

$$\eta^2 \in \big((j_1)^2 - (j_1')^2, (j_2')^2 - (j_1)^2\big) = (\eta_*^2, \eta^{*2}).$$

Note that $\eta^* > \eta_*$ is equivalent to $2(j_1)^2 < (j_1')^2 + (j_2')^2$. This inequality can be checked using the numerical values in Table 9.5 of [2]. Up to an error in omitted digits the values are $j_1 = 3.83171$, $j_1' = 1.84118$, and $j_2' = 5.33144$ so that $2(j_1)^2 < (j_1')^2 + (j_2')^2$ holds. □

Lemma 4.8. *Assume $\nu_2 < \nu_3$ and choose values μ_0, W_∞ such that*

$$-\nu_3 < \mu_0 < -\nu_2 < -\nu_1 < W_\infty. \tag{4.9}$$

Let $\eta \in [\eta_*, \eta^*]$ and let $\xi(\eta)$ be as in Lemma 4.7. If we define

$$\delta := \frac{\eta}{\sqrt{W_\infty - \mu_0}} \quad \text{and} \quad W_0 := \mu_0 - \left(\frac{\xi(\eta)}{\delta}\right)^2$$

as well as

$$W(r) = \begin{cases} W_0, & 0 \le r < \delta, \\ W_\infty, & r \ge \delta, \end{cases} \tag{4.10}$$

then μ_0 is an eigenvalue of L_r. There are no other eigenvalues below the essential spectrum $[W_\infty, \infty)$ and hence

$$\sigma(L_r) = \{\mu_0\} \cup [W_\infty, \infty).$$

Proof. Due to the form of W as in (4.10) we have $\sigma(L_r) \subset [W_0, \infty)$ and $\sigma_{\text{ess}}(L_r) = [W_\infty, \infty)$. Now consider the eigenvalue equation

$$-u''(r) - \frac{3}{r}u'(r) = (-W(r) + \mu)\, u. \tag{4.11}$$

Let us check that neither W_0 nor W_∞ are eigenvalues of L_r. First suppose $\mu = W_0$ is an eigenvalue. Note that $-W(r) + W_0 \le 0$ on $[0, \infty)$ and < 0 on (δ, ∞). Multiplication of (4.11) with a corresponding eigenfunction u and integration with respect to the measure $r^3\,dr$ on $(0, \infty)$ yields a positive left hand side and a negative right hand side. Hence $\mu = W_0$ is not an eigenvalue. Now suppose $\mu = W_\infty$ is an eigenvalue and u is a corresponding eigenfunction. Then $-W(r) + W_0 = 0$ for $r \ge \delta$ so that $u(r) = \text{const.}\, r^{-2}$ on $[\delta, \infty)$. But no matter how r^{-2} extends to $[0, \delta)$ the function u does not belong to $L^2_{\text{rad}}(r^3 dr)$ because $\int_\delta^\infty r^{-4} \cdot r^3 dr = \infty$. So $\mu = W_\infty$ is also not an eigenvalue of L_r.

Since $\min \sigma(L_r) \ge W_0$ and since neither W_0 nor W_∞ are eigenvalues of L_r, we are looking for solutions of (4.11) with $W_0 < \mu < W_\infty$.

As $W(r)$ only takes the values W_0, W_∞, equation (4.11) is transformed via $u(r) = r^{-1}v(\sqrt{\mu - W_0}r), s := \sqrt{\mu - W_0}r$ into

$$s^2v'' + sv' + (s^2 - 1)v = 0 \quad \text{for} \quad 0 \le s \le \delta\sqrt{\mu - W_0}$$

and via $u(r) = r^{-1}w(\sqrt{W_\infty - \mu}r), s := \sqrt{W_\infty - \mu}r$ into

$$s^2w'' + sw' - (s^2 + 1)w = 0 \quad \text{for} \quad \delta\sqrt{W_\infty - \mu} \le s < \infty.$$

Thus, $v(s) = \alpha J_1(s)$ is a multiple of the order one Bessel function which is regular at 0 and $w(s) = \beta K_1(s)$ is a multiple of the order one modified Bessel function which is exponentially decaying at infinity. Altogether we obtain

$$u(r) = \begin{cases} \alpha r^{-1}J_1(\sqrt{\mu - W_0}r) & \text{for } 0 \le r \le \delta, \\ \beta r^{-1}K_1(\sqrt{W_\infty - \mu}r) & \text{for } \delta \le r < \infty. \end{cases}$$

We need to choose α, β, μ in order to obtain a C^1-function at $r = \delta$. This leads to the equation

$$g(\mu) := \frac{J_1(\sqrt{\mu - W_0}\delta)}{\sqrt{\mu - W_0}\delta J_1'(\sqrt{\mu - W_0}\delta)} = \frac{K_1(\sqrt{W_\infty - \mu}\delta)}{\sqrt{W_\infty - \mu}\delta K_1'(\sqrt{W_\infty - \mu}\delta)} =: h(\mu)$$

and our choice of W_0 and δ such that $\sqrt{\mu_0 - W_0}\delta = \xi$ and $\sqrt{W_\infty - \mu_0}\delta = \eta(\xi)$ guarantees the C^1-matching at $\mu = \mu_0$. We have therefore verified that μ_0 is indeed an eigenvalue of L_r. It remains to show that there is no other eigenvalue.

We analyze the two sides of the equation $g(\mu) = h(\mu)$ independently. We have already mentioned in the proof of the preceding lemma that $\frac{K_1(x)}{xK_1'(x)}$ is a negative and increasing function of x with $\lim_{x\to 0+} \frac{K_1(x)}{xK_1'(x)} = -1$. Likewise the function $\frac{J_1(x)}{xJ_1'(x)}$ satisfies $\lim_{x\to 0+} \frac{J_1(x)}{xJ_1'(x)} = 1$, has its zeroes at $j_1, j_2, \ldots$ and its poles at $j_1', j_2', \ldots$ and is increasing between 0 and j_1' and between two consecutive poles. The proof of these statements is given in the Appendix. Thus, as μ runs through $[W_0, W_\infty]$, the function $g(\mu)$ starts from the value 1 and increases up to its first pole at $W_0 + \left(\frac{j_1'}{\delta}\right)^2$. On the interval $(W_0 + \left(\frac{j_1'}{\delta}\right)^2, W_0 + \left(\frac{j_1}{\delta}\right)^2]$ it increases from $-\infty$ to 0 and on $[W_0 + \left(\frac{j_1}{\delta}\right)^2, W_\infty]$ it increases from 0 to a positive value. The function $h(\mu)$ stays negative and strictly decreases to the value -1 as μ ranges through the interval $[W_0, W_\infty]$, cf. Fig. 1 for a plot of an example of the two functions. Therefore, on $[W_0, W_\infty]$ the two functions g, h intersect exactly once provided W_∞ lies between the first zero and the second pole of g, i.e., provided

$$\left(\frac{j_1}{\delta}\right)^2 < W_\infty - W_0 < \left(\frac{j_2'}{\delta}\right)^2. \tag{4.12}$$

Equation (4.12) is equivalent to $j_1^2 < \xi^2 + \eta^2 < (j_2')^2$. The latter is guaranteed by our choice of ξ, η and from Lemma 4.7. □

As we have seen from Lemma 4.8, the piecewise constant function W given in (4.10) together with the choice of μ_0, W_∞, the restriction on η and the definition of δ and W_0 ensure the existence of exactly one eigenvalue of

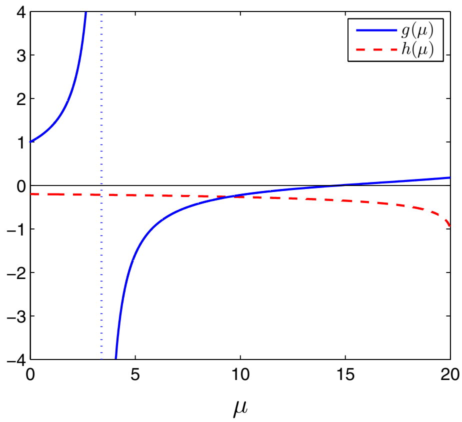

FIGURE 1. Functions g and h for $\delta = 1, W_0 = 0, W_\infty = 20$

L_r below the essential spectrum. Therefore these conditions can be considered as assumptions that ensure guiding of a single linear mode in the cylindrical waveguide.

Lemma 4.9. *Assume that P is piecewise continuous, 1-periodic such that $\sigma(L_p)$ has a bounded first open gap. Assume that W is as in (4.10) and that μ_0, W_∞, δ and W_0 are chosen as in Lemma 4.8. Then 0 is not in the spectrum of L.*

Proof. Because of Lemmas 4.6, 4.8 and (4.7) we see that

$$\sigma(L) = \bigcup_{k=1}^{\infty} [\mu_0 + \nu_{2k-1}, \mu_0 + \nu_{2k}] \cup [\nu_1 + W_\infty, \infty).$$

Thus, the inequalities $-\nu_3 < \mu_0 < -\nu_2 < -\nu_1 < W_\infty$ from Lemma 4.8 guarantee that 0 lies in the resolvent of L. □

Remark. Let us verify that the constructed potential V takes positive values on sets of positive measure for $r > \delta$. Since $V(x) = -\frac{\omega^2}{c^2} n^2(x)$, this implies the unphysical situation of an imaginary refractive index. It is an open problem to construct a negative $V = V(r, x_3)$, which satisfies the condition $0 \notin \sigma(\mathcal{L})$. First note that

$$\begin{aligned} \nu_1 = \min \sigma(L_p) &= \inf_{\psi \in H^1(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \psi'^2 + P(x)\psi^2 \, dx}{\int_{\mathbb{R}} \psi^2 \, dx} \\ &< \inf_{\psi \in H^1(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \psi'^2 + (\sup P)\psi^2 \, dx}{\int_{\mathbb{R}} \psi^2 \, dx} \\ &= \sup P, \end{aligned}$$

where the strict inequality comes from the fact that P is a periodic potential which generates an operator with a true first open gap and hence must be non-constant. Together with $W_\infty > -\nu_1$, cf. (4.9) we obtain

$$\sup V = W_\infty + \sup P > 0.$$

5. Ground states in the focusing case

We assume

$$\text{(H-foc)} \quad 1 < p < 5, \quad V, \Gamma \in L^\infty(\mathbb{R}^3), \quad \inf_{\mathbb{R}^3} \Gamma > 0, \quad \text{and} \quad 0 \notin \sigma(\mathcal{L}).$$

For the existence of a Palais–Smale sequence our only assumption is (H-foc). In the previous section we found examples of cylindrical potentials V of the form $V(r, x_3) = W(r) + P(x_3)$ with periodicity of P which produce $0 \notin \sigma(\mathcal{L})$. Nevertheless, also other potentials V may satisfy this condition. Note that for the potential V from the previous section the resulting operator $\mathcal{L}$ has negative spectrum, positive spectrum and 0 is in the resolvent set. The analysis of this chapter will, however, work also in the case where $0 < \min \sigma(\mathcal{L})$.

For the final step of the proof, i.e. the non-triviality of the limit of a suitable Palais–Smale sequence and the existence of a nontrivial ground state, we use the assumption of cylindrical symmetry and periodicity in x_3 of both V and Γ.

Since $\mathcal{L}$ is a selfadjoint operator on

$$D(\mathcal{L}) = H^2_{G_1}(\mathbb{R}^3) \subset L^2_{G_1}(\mathbb{R}^3)$$

the spectral theorem yields the existence of a spectral resolution $(P_\lambda)_{\lambda \in \mathbb{R}}$ and hence of projections

$$P^+ := \int_0^\infty 1 \, dP_\lambda, \quad P^- := \int_{-\infty}^0 1 \, dP_\lambda.$$

Therefore we obtain the spectral decomposition $H^1_{G_1}(\mathbb{R}^3) = H^+ \oplus H^-$ with $H^\pm := P^\pm(H^1_{G_1}(\mathbb{R}^3))$. We define $U^\pm := P^\pm U$ for all $U \in H^1_{G_1}(\mathbb{R}^3)$. Notice that $\mathcal{L}$ defines the bilinear form $b(U, V) := \int_{\mathbb{R}^3} (\nabla \times U) \cdot (\nabla \times V) + V(x) U \cdot V \, dx$ which is positive/negative definite on the spaces $H^\pm$ by construction. Therefore, we may define the scalar product

$$\langle U, V \rangle := b(U^+, V^+) - b(U^-, V^-) \quad \text{on} \quad H^1_{G_1}(\mathbb{R}^3) \times H^1_{G_1}(\mathbb{R}^3)$$

which has the property that H^+ is orthogonal to H^-, i.e., that $P^\pm$ are orthogonal projections. We denote the norm on $H^1_{G_1}(\mathbb{R}^3)$ corresponding to $\langle \cdot, \cdot \rangle$ by $||| \cdot |||$. It is equivalent to the H^1-norm $\| \cdot \|_{H^1}$ and has the property

$$|||U|||^2 = \pm \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2 \, dx \quad \text{for all} \quad U \in H^\pm$$

such that

$$|||U|||^2 = |||U^+|||^2 + |||U^-|||^2 \quad \text{for all} \quad U \in H^1_{G_1}(\mathbb{R}^3).$$

If the potential V has the unphysical property that $\inf V > 0$ (imaginary refractive index in all of $\mathbb{R}^3$) then $H^- = \{0\}$. But in general $H^- \neq \{0\}$ and therefore the functional J has linking geometry. In any case J is unbounded from below on $H^1_{G_1}(\mathbb{R}^3)$ so that a direct minimization is impossible. Therefore we choose to minimize over the Nehari–Pankov manifold

$$N := \{U \in H^1_{G_1}(\mathbb{R}^3) \backslash \{0\} : J'[U]\Phi = 0 \ \forall \Phi \in [U] \oplus H^-\}.$$

This approach is analogous to that in [27]. Later, in Lemma 5.4, we will see that the constraint set N does not produce Lagrange multipliers. Note also that for $U \in N$

$$J[U] = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1}\, dx = \frac{p-1}{2(p+1)} \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2\, dx.$$

Remark. (a) Another common approach to obtain a critical point of J is minimization under the constraint $\|U\|_{L^2} = 1$. This however produces a Lagrange multiplier κ, which is generally nonzero, so that the minimizer does not solve (1.1) but the equation $\mathcal{L}U + \kappa U = \Gamma(r, x_3)|U|^{p-1}U$. (b) If $H^- = \{0\}$ then another common approach consists in minimizing $K(U) = \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2\, dx$ under the constraint $\|U\|_{\Gamma,p+1} = 1$. This produces a Lagrange multiplier which however can be scaled out. If H^- is not trivial then one may still find a critical point of K under this constraint.

Lemma 5.1. *Under the assumption* (H-foc) *there exist values* $\epsilon_1, \epsilon_2, C > 0$ *such that*

$$\|U\|_{H^1} \geq \epsilon_1, \quad J[U] \geq \epsilon_2, \quad \|U\|_{H^1} \leq CJ[U]^{\frac{p}{p+1}}$$

for all $U \in N$.

Proof. For $U \in N$ we have

$$\begin{aligned} |||U^+|||^2 &= \int_{\mathbb{R}^3} |\nabla \times U^+|^2 + V(x)|U^+|^2\, dx \\ &= \langle U, U^+ \rangle \\ &= J'[U]U^+ + \int_{\mathbb{R}^3} \Gamma(x)|U|^{p-1}U \cdot U^+\, dx \\ &= \underbrace{J'[U](U - U^-)}_{=0} + \int_{\mathbb{R}^3} \Gamma(x)|U|^{p-1}U \cdot U^+\, dx, \end{aligned}$$

where the first term in the last equation vanishes due to the definition of the manifold N. Hence, by using Hölder's and Sobolev's inequality and again the definition of N we obtain

$$\begin{aligned} |||U^+|||^2 &\leq \left(\int_{\mathbb{R}^3} (\Gamma(x)|U|^p)^{\frac{p+1}{p}}\, dx \right)^{\frac{p}{p+1}} \left(\int_{\mathbb{R}^3} |U^+|^{p+1}\, dx \right)^{\frac{1}{p+1}} \\ &\leq C_1 \|\Gamma\|_\infty^{\frac{1}{p+1}} \left(\int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1}\, dx \right)^{\frac{p}{p+1}} \|U^+\|_{H^1} \qquad (5.1) \\ &\leq C_2 \left(\frac{2(p+1)}{p-1} J[U] \right)^{\frac{p}{p+1}} |||\, U^+ \,||| \,. \end{aligned}$$

We may repeat the above argument with U^- and find

$$\begin{aligned} |||U^-|||^2 &= -\int_{\mathbb{R}^3} |\nabla \times U^-|^2 + V(x)|U^-|^2\, dx \\ &= \langle U, U^- \rangle \end{aligned}$$

$$= -\underbrace{J'[U]U^-}_{=0} - \int_{\mathbb{R}^3} \Gamma(x)|U|^{p-1}U \cdot U^- \, dx,$$

and from there the same Hölder- and Sobolev-estimates as before lead to

$$|||U^-||| \le C_2 \left(\frac{2(p+1)}{p-1} J[U] \right)^{\frac{p}{p+1}}.$$

Together with (5.1) this establishes the third of the three claims.

To see the first of the three claims, we use (5.1) and the corresponding estimate for U^- to get

$$|||U^+|||^2 \le C_3 \|U\|_{L^{p+1}}^p \, ||| \, U^+|||, \quad |||U^-|||^2 \le C_3 \|U\|_{L^{p+1}}^p \, ||| \, U^-|||$$

from which we obtain $|||U||| \le \sqrt{2} C_3 \|U\|_{L^{p+1}}^p$. Due to the Sobolev inequality and since $||| \cdot |||$ and $\| \cdot \|_{H^1}$ are equivalent we also have $\|U\|_{L^{p+1}}^p \le C_4 \, ||| \, U|||^p$. Hence $|||U||| \le \sqrt{2} C_3 C_4 \, ||| \, U|||^p$ and thus $\|U\|_{H^1} \le C_5 \|U\|^p$. Since $U \ne 0$ by the definition of N we obtain the first of the three estimates. Finally, the second estimate follows from the first and the third. □

Lemma 5.2. *Assume* (H-foc). *The map*

$$G : \begin{cases} H^1_{G_1}(\mathbb{R}^3) \backslash H^- \to H^1_{G_1}(\mathbb{R}^3), \\ U \mapsto \langle \nabla J[U], U^+ \rangle \frac{U^+}{|||U^+|||^2} + P^- \nabla J[U]. \end{cases}$$

is a C^1-map. If $U \in N$ and $X_U := [U] + H^-$ then the map

$$\partial_{X_U} G(U) : X_U \to X_U$$

is negative definite uniformly with respect to bounded subsets of N, i.e., if $N_0 \subset N$ is bounded, then there exists a value $\delta > 0$ such that

$$\langle \partial_{X_U} G(U) v, v \rangle \le -\delta \, ||| \, v \, |||^2 \qquad \text{for all} \qquad v \in X_U \text{ and all } U \in N_0.$$

In particular $\partial_{X_U} G(U) : X_U \to X_U$ has a bounded inverse.

Proof. For $U \in N$ and $v \in H^1_{G_1}(\mathbb{R}^3)$ we have

$$\begin{aligned} G'(U)v = & \langle \nabla^2 J[U]v, U^+ \rangle \frac{U^+}{|||U^+|||^2} + \langle \nabla J[U], v^+ \rangle \frac{U^+}{|||U^+|||^2} \\ & + \langle \nabla J[U], U^+ \rangle \frac{v^+}{|||U^+|||^2} - 2 \langle U^+, v \rangle \langle \nabla J[U], U^+ \rangle \frac{U^+}{|||U^+|||^4} \\ & + (\nabla^2 J[U]v)^-. \end{aligned}$$

If we take $v \in X_U = [U] + H^- = [U^+] + H^-$, i.e., $v = tU + w$ with $t \in \mathbb{R}$, $w \in H^-$ then the above formula simplifies to

$$\partial_{X_U} G(U)v = \langle \nabla^2 J[U]v, U^+ \rangle \frac{U^+}{|||U^+|||^2} + (\nabla^2 J[U]v)^- = P_{X_U} \nabla^2 J[U]v,$$

where $P_{X_U} : H \to X_U$ denotes the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$. Therefore

$$\begin{aligned}\langle \partial_{X_U} G(U)v, v\rangle &= \langle \nabla^2 J[U]v, v\rangle \\ &= t^2 \int_{\mathbb{R}^3} |\nabla \times U|^2 + V(x)|U|^2 - p\Gamma(x)|U|^{p+1}\, dx \\ &\quad + \int_{\mathbb{R}^3} |\nabla \times w|^2 + V(x)|w|^2 - p\Gamma(x)|U|^{p-1}|w|^2\, dx \\ &\quad +2t \int_{R^3} \nabla \times U \cdot \nabla \times w + V(x)U \cdot w - p\Gamma(x)|U|^{p-1}U \cdot w\, dx\end{aligned}$$

and by using $U \in N$ we obtain

$$\begin{aligned}&\langle \partial_{X_U} G(U)v, v\rangle \\ &\quad = - \||\, w \,\||^2 - \int_{\mathbb{R}^3} \Gamma(x)|U|^{p-1} \left(t^2(p-1)|U|^2 + p|w|^2 + 2t(p-1)U \cdot w\right)\, dx.\end{aligned}$$

Now we use the identity

$$\begin{aligned}t^2(p-1)|U|^2 + p|w|^2 + 2t(p-1)U \cdot w &= t^2 \frac{p-1}{p}|U|^2 + \left|\sqrt{p}w + \frac{p-1}{\sqrt{p}} tU\right|^2 \\ &\geq t^2 \frac{p-1}{p}|U|^2\end{aligned}$$

to deduce

$$\langle \partial_{X_U} G(U)v, v\rangle \leq - \||\, w \,\||^2 - t^2 \frac{p-1}{p} \int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1}\, dx.$$

Next we obtain from Lemma 5.1 that $J[U] = (\frac{1}{2} - \frac{1}{p+1}) \int_{\mathbb{R}^3} \Gamma(x)|U|^{p+1}\, dx \geq \epsilon_2$ and hence we find

$$\langle \partial_{X_U} G(U)v, v\rangle \leq - \||\, w \,\||^2 - 2t^2 \frac{p+1}{p} \epsilon_2 \leq -C \||\, w + tU \||^2$$

by using the boundedness of N_0. This finishes the proof. □

Lemma 5.3. *Assume* (H-foc). *The set N is a C^1-manifold such that*

$$N = G^{-1}\{0\} \quad \text{and} \quad T_U N = \operatorname{Ker} G'(U)$$

for every $U \in N$.

Proof. As before, for $U \in N$ let $X_U = [U] + H^-$ and denote by $P_{X_U} : H^1_{G_1}(\mathbb{R}^3) \to X_U$ the orthogonal projection w.r.t. $\langle \cdot, \cdot \rangle$. Recall that the map $G : H^1_{G_1}(\mathbb{R}^3) \backslash H^- \to H^1_{G_1}(\mathbb{R}^3)$ satisfies

$$G(X_U \backslash H^-) \subset X_U \quad \text{for all} \quad U \in H^1_{G_1}(\mathbb{R}^3) \backslash H^-. \tag{5.2}$$

Now fix $U \in N = G^{-1}(\{0\})$. There exists an open neighbourhood $\mathcal{O}$ of U in $H^1_{G_1}(\mathbb{R}^3) \backslash H^-$ such that

$$X_V \cap X_U^\perp = \{0\} \quad \text{for all} \quad V \in \mathcal{O}. \tag{5.3}$$

For $V \in \mathcal{O}$ we have by (5.2), (5.3) the equivalence

$$G(V) = 0 \Leftrightarrow (P_{X_U} \circ G)(V) = 0.$$

This shows that $N \cap \mathcal{O} = (P_{X_U} \circ G\,|_{\mathcal{O}})^{-1}(\{0\})$. But the map $P_{X_U} \circ G\,|_{\mathcal{O}}$: $\mathcal{O} \to X_U$ is a submersion, i.e., its derivative is surjective at every point of $\mathcal{O}$, because

$$(P_{X_U} \circ G)'(V) = P_{X_U} \circ G'(V) : H^1_{G_1}(\mathbb{R}^3) \overset{G'(V)}{\to} X_V \overset{P_{X_U}}{\to} X_U$$

and the first map $G'(V) : H^1_{G_1}(\mathbb{R}^3) \to X_V$ is surjective by Lemma 5.2 and the second map $P_{X_U} : X_V \to X_U$ is an isomorphism by (5.3). Therefore, the submersion theorem of [1, Theorem 3.5.4], applies and states that $N \cap \mathcal{O}$ is a submanifold of $H^1_{G_1}(\mathbb{R}^3)\backslash H^-$ and $T_U N = \operatorname{Ker} G'(U)$. □

Notice that any nontrivial solution $U \in H^1_G(\mathbb{R}^3)$ of (1.1) belongs to the Nehari–Pankov manifold N. As a consequence, one can show that the constraint N produces a zero Lagrange multiplier. The following is a much stronger statement.

Lemma 5.4. *Assume* (H-foc). *Let N_0 be a bounded subset of N. Then there exists a constant $C_0 > 0$ such that the following holds: if $U \in N_0$ and $\nabla J[U] = \tau + \sigma$ where $\tau \in T_U N$ is the tangential component of $\nabla J[U]$ and $\sigma \perp \tau$ is the transversal component of $\nabla J[U]$ then*

$$\|\nabla J[U]\|_{H^1} \le C_0 \|\tau\|_{H^1}.$$

Proof. By Lemma 5.2 the map $\partial_{X_U} G(U) : X_U \to X_U$ has a bounded inverse and hence a closed range. Moreover, $\partial_{X_U} G(U)|_{X_U} = P_{X_U} \nabla^2 J[U]$ is symmetric as a composition of a second derivative and an orthogonal projection. Therefore

$$\operatorname{Rg} \partial_{X_U} G(U)|_{X_U} = (\operatorname{Ker} \partial_{X_U} G(U)|_{X_U})^{\perp} = (T_U N)^{\perp}.$$

If we consider

$$\nabla J[U] = \tau + \sigma \quad \text{with} \quad \tau \in T_U N \text{ and } \sigma \in (T_U N)^{\perp}$$

then there exists $h \in X_U$ such that

$$\nabla J[U] = \tau + \partial_{X_U} G(U) h. \tag{5.4}$$

Hence, using $h \in X_U = [U] + H^-$ and thus $\langle \nabla J[U], h \rangle = 0$ we get from Lemma 5.2

$$\langle \partial_{X_U} G(U) h, h \rangle = \langle -\tau, h \rangle \le -\delta \|h\|^2_{H^1}.$$

Using the Cauchy–Schwarz inequality and the equivalence of the norms $\|\cdot\|_{H^1}$ and $|||\cdot|||$ we get

$$\|h\|_{H^1} \le C_0 \|\tau\|_{H^1}.$$

By (5.4) and the boundedness of $G'(U)$ on bounded subsets of N this implies the claim. □

By the previous lemma the tangential component of the gradient of J at a point in N controls the entire gradient. As a consequence there exists special minimizing sequences of $J|_N$, where the tangential part of the gradient converges to zero (a consequence of Ekeland's variational principle) and hence the entire gradient converges to 0.

Lemma 5.5. *Assume* (H-foc). *There exists a bounded Palais–Smale sequence* $(U_k)_{k\in\mathbb{N}}$ *in* N *such that*

$$J[U_k] \to c := \inf_N J, \quad J'[U_k] \to 0 \text{ as } k \to \infty.$$

Proof. As a consequence of Ekeland's variational principle, cf. Struwe [30], there exists a minimizing sequence $(U_k)_{k\in\mathbb{N}}$ of $J|_N$ such that $(J|_N)'(U_k) \to 0$, hence $J'(U_k) \to 0$ by Lemma 5.4. □

Proof of Theorem 1.3: Consider the Palais–Smale sequence $(U_k)_{k\in\mathbb{N}}$ from Lemma 5.5. If any subsequence of $(U_k)_{k\in\mathbb{N}}$ converges in $L^{p+1}(\mathbb{R}^3)$ to zero, then by the definition of the Nehari–Pankov manifold this sequence also converges to zero in $H^1(\mathbb{R}^3)$. This is impossible by Lemma 5.1. Therefore, the concentration compactness principle (cf. Lions [19] or Lemma 1.21 in Willem [35] suitably adapted to the vectorial case) implies that for every radius $R > 0$

$$\liminf_{k\in\mathbb{N}} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |U_k|^2\,dx > 0.$$

Fix a value $R > 0$. Then there exists a subsequence again denoted $(U_k)_{k\in\mathbb{N}}$, centers $y_k \in \mathbb{R}^3$ and $\eta > 0$ such that

$$\int_{B_R(y_k)} |U_k|^2\,dx \geq \eta \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.5}$$

By possibly adding the value 1 to R we may assume that $y_k^3 \in \mathbb{Z}$ and (5.5) still holds. Now we claim that $\rho_k^2 := (y_k^1)^2 + (y_k^2)^2$ is bounded. Assume the contrary. Due to the symmetries in $H^1_{G_1}(\mathbb{R}^3)$ we have that

$$\int_{B_R(y_k)} |U_k|^2\,dx = \int_{B_R(\tilde{y}_k)} |U_k|^2\,dx$$

whenever the point $\tilde{y}_k$ is such that

$$y_k^3 = \tilde{y}_k^3 \text{ and } (y_k^1)^2 + (y_k^2)^2 = (\tilde{y}_k^1)^2 + (\tilde{y}_k^2)^2. \tag{P}$$

Notice that the number of disjoint balls $B_R(\tilde{y}_k)$ with centers $\tilde{y}_k$ satisfying (P) tends to infinity if $\rho_k \to \infty$ as $k \to \infty$. But this is impossible since the L^2-norm of $(U_k)_{k\in\mathbb{N}}$ is bounded. Thus, if we define

$$\rho := R + \sup_{k\in\mathbb{N}} \rho_k$$

then

$$\int_{B_\rho(0,0,y_k^3)} |U_k|^2\,dx \geq \eta \quad \text{for all} \quad k \in \mathbb{N}.$$

Set

$$\bar{U}_k(x_1, x_2, x_3) := U_k(x_1, x_2, x_3 + y_k^3).$$

Then, due to the periodicity of V, Γ in the x_3-variable we have

$$\bar{U}_k \in N, \quad J[\bar{U}_k] = J[U_k] \to c, \quad \|J'[\bar{U}_k]\|_* = \|J'[U_k]\|_* \to 0 \text{ as } k \to \infty$$

and

$$\int_{B_\rho(0)} |\bar{U}_k|^2 \, dx \geq \eta \quad \text{for all} \quad k \in \mathbb{N}.$$

Now we may take a weakly converging subsequence (which is again denoted by $(\bar{U}_k)_{k\in\mathbb{N}}$) with $\bar{U}_k \rightharpoonup \bar{U} \neq 0$ in $H^1_{G_1}(\mathbb{R}^3)$ for some $\bar{U} \in H^1_{G_1}(\mathbb{R}^3)$. Moreover, for every $\phi \in H^1_{G_1}(\mathbb{R}^3) \cap C_0^\infty(\mathbb{R}^3)$ we have $J'[\bar{U}]\phi = \lim_{k\to\infty} J'[\bar{U}_k]\phi = 0$ due to weak convergence and the compact embedding $H^1_{G_1}(\mathbb{R}^3) \to L^q(K)$, $K = \operatorname{supp}\phi$ and $q \in [1,6)$. Hence $\bar{U}$ is a critical point of J so that $\bar{U} \in N$. Hence

$$\begin{aligned} J[\bar{U}] &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} \Gamma(x)|\bar{U}|^{p+1} \, dx \\ &\leq \liminf_{k\in\mathbb{N}} \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} \Gamma(x)|\bar{U}_k|^{p+1} \, dx \\ &= \liminf_{k\in\mathbb{N}} J[\bar{U}_k] \\ &= c \end{aligned}$$

and therefore "=" holds and $\bar{U}$ is a minimizer of J restricted to N. □

Appendix

Lemma 6.1. *Let J_1 denote the order one Bessel function which is regular at* 0. *Then the function*

$$\alpha(x) := \frac{J_1(x)}{xJ_1'(x)}$$

satisfies $\lim_{x\to 0+} \alpha(x) = 1$ *and is strictly increasing on the intervals* $(0, j_1')$, (j_k', j_{k+1}') *for* $k = 1, 2, 3, \ldots$. *Let K_1 denote the order one modified Bessel function which decreases exponentially at infinity. Then the function*

$$\beta(x) := \frac{K_1(x)}{xK_1'(x)}$$

satisfies $\lim_{x\to 0+} \beta(x) = -1$ *and is negative and strictly increasing on* $(0, \infty)$.

Proof. Since J_1 is analytic and since $J_1(0) = 0, J_1'(0) = 1/2$ the relation $\alpha(x) \to 1$ as $x \to 0$ follows immediately. Differentiating $\alpha(x)$, we need to show

$$\frac{x(J_1'(x))^2 - J_1(x)(J_1'(x) + xJ_1''(x))}{x^2(J_1'(x))^2} > 0$$

and by using the differential equation for J_1 this amounts to

$$x\left((J_1'(x))^2 + \left(1 - \frac{1}{x^2}\right) J_1(x)^2\right) > 0,$$

i.e.,

$$x^2(J_1'(x))^2 > (1 - x^2)J_1(x)^2 \text{ for } x \in (0, \infty).$$

If we multiply the differential equation

$$x^2 J_1''(x) + x J_1'(x) = (1 - x^2) J_1(x)$$

by $J_1'(x)$ and integrate from 0 to x we obtain

$$\int_0^x s^2 \frac{d}{ds}(J_1'(s))^2 + 2s(J_1'(s))^2 \, ds = \int_0^x (1 - s^2) \frac{d}{ds}(J_1(s)^2) \, ds.$$

Integration by parts and using $J_1(0) = 0$ leads to

$$x^2 (J_1'(x))^2 = (1 - x^2) J_1(x)^2 + \int_0^x 2s J_1(s)^2 \, ds > (1 - x^2) J_1(x)^2 \quad \text{for} \quad x \in (0, \infty)$$

and hence the result is proved.

Now we turn to the statement for β. First we recall that $x K_0(x) \to 0$ (cf. [16, 8.447]), $K_1(x) \to \infty$ as $x \to 0$ (cf. [16, 8.451(6.)]) and that $x K_1'(x) + K_1(x) = -x K_0(x)$ (cf. [16, 8.486(12.)]). This implies $\beta(x) \to -1$ as $x \to 0$. For the monotoniticy of $\beta(x)$ it suffices by differentiation to prove that

$$x(K_1'(x))^2 - K_1(x) K_1'(x) - x K_1(x) K_1''(x) > 0$$

and using the differential equation this amounts to showing that

$$x^2 (K_1'(x))^2 > (1 + x^2) K_1(x)^2 \quad \text{for all} \quad x \in (0, \infty).$$

If we multiply the differential equation

$$x^2 K_1''(x) + x K_1'(x) = (1 + x^2) K_1(x)$$

by $K_1'(x)$ and integrate from x to ∞ we obtain

$$\int_x^\infty s^2 \frac{d}{ds}(K_1'(s))^2 + 2s(K_1'(s))^2 \, ds = \int_x^\infty (1 + s^2) \frac{d}{ds}(K_1(s)^2) \, ds.$$

Integration by parts and using the exponential decay of K_1, K_1', K_1'' at infinity leads to

$$\begin{aligned} -x^2 (K_1'(x))^2 &= -(1 + x^2) K_1(x)^2 - \int_x^\infty 2s K_1(s)^2 \, ds \\ &< -(1 + x^2) K_1(x)^2 \text{ for } x \in (0, \infty). \end{aligned}$$

This proves the result. □

Acknowledgements

The authors thank Alejandro Aceves for fruitful discussions on the physics background. M. Plum and W. Reichel gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173. Likewise, T. Dohnal acknowledges the support through the DFG grant DO1467/3-1.

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Received: 22 January 2016.
Accepted: 26 July 2016.