Ground-Water Hydraulics

GEOLOGICAL SURVEY PROFESSIONAL PAPER 708

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By S. W. LOHMAN

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UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1972

UNITED STATES DEPARTMENT OF THE INTERIOR

ROGERS C. B. MORTON, *Secretary*

GEOLOGICAL SURVEY

V. E. McKelvey, *Director*

First printing 1972

Second printing 1975

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington, D.C. 20402 – Price $4.15 (paper cover)
Stock Number 024–001–01194

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SYMBOLS AND DIMENSIONS

[Number in parentheses refers to the equation, page, or illustration where the symbol first appears or where additional clarification may be obtained. Symbols are also defined in the text. There is some duplication of symbols because of the desire to preserve the notation used in the original papers]

Symbol	Dimensions	Description
A	L^2	Area (8); area of influence (145).
B	----------	$1/\sqrt{T/(K'/b')}$ (95).
B	L^{-1}	$\sqrt{T/\alpha S_l}$ (107).
B	L^3T^{-1}	Constant (140).
BE	----------	Barometric efficiency of artesian well (22).
C	----------	Constant (7).
C, C'	L^2	Constant of proportionality (144).
D	LT^{-1}	Discharge rate per unit area (151).
$D(u)_q$	----------	D (drain) function of u for constant discharge (112).
$D(u)_h$	----------	D (drain) function of u for constant drawdown (118).
E_s	$ML^{-1}T^{-2}$	Bulk modulus of elasticity of solid skeleton of aquifer (20).
E_w	$ML^{-1}T^{-2}$	Bulk modulus of elasticity of water (20).
$F(\theta_0, \theta/\theta_0, r/a, Tt/r^2S)$	----------	F function of θ_0, θ/θ_0, r/a, Tt/r^2S (p. 49).
$G(\alpha)$	----------	G function of α (68).
$G(\alpha, r_w/B)$	----------	G function of α, r_w/B (96).
H	L	Head inside well at time t after injection or removal of "slug" (75).
H_0	L	Head inside well at instant of injection or removal of "slug" (75).
$H(u,\beta)$	----------	H function of u, β (91).
J	ML^2T^{-2}	Joule (9).
J_0, $J_0(x)$	----------	Bessel function of zero order, first kind (68), (100).
$J_1(x)$	----------	Bessel function of first order, first kind (75).
K	LT^{-1}	Hydraulic conductivity (13), (92).
$\bar{K}$	LT^{-1}	Average hydraulic conductivity (p. 11).
K	----------	Constant r_i/r_p (150).
K', K''	LT^{-1}	Vertical hydraulic conductivity of confining beds (86), (92).
K_r	LT^{-1}	Radial hydraulic conductivity (102).
K_z	LT^{-1}	Vertical hydraulic conductivity (102).
K_0	----------	Modified Bessel function of second kind, zero order (96).
K_1	----------	Modified Bessel function of second kind, first order (96).
L_1, L_2	L	Lengths of two concentric closed contours (137).
$L(u,v)$	----------	L (leakance) function of u, v (87).
M	----------	Moisture equivalent (18).
N_r	----------	Ratio: specific retention/moisture equivalent (18).
Q	L^3T^{-1}	Flow rate (8); constant discharge rate (19); total flow (132).
Q_b	L^2T^{-1}	Constant discharge rate of drain per unit length of drain (112).
Q_b	L^2T^{-1}	Discharge of aquifer to drain per unit length of drain (120).
P	$ML^{-1}T^{-2}$	Pressure (fig. 3).
R	LT^{-1}	Recharge rate per unit area (151).
S	----------	Storage coefficient (19).
S'	----------	Corrected value of storage coefficient (64).
S', S''	----------	Storage coefficients of aquifer and semipervious confining layers (92).
S_e	----------	Early time storage coefficient (107).
S_l	----------	Later time specific yield (107).
S_r	----------	Specific retention (17).
S_s, S_s', S_s''	L^{-1}	Specific storage of aquifer and confining beds, respectively (92).
S_y	----------	Specific yield (16).
T	MT^{-2}	Surface tension of fluid (1).
T	L^2T^{-1}	Transmissivity (26), (19).
TE	----------	Tidal efficiency of artesian well (22).
V	L^3	Total volume (4).
$V(\psi,\tau)$	----------	V function of ψ and τ (103).
W	LT^{-1}	Rate of accretion (126).
$W(u)$	----------	W (well) function of u (46).
$Y_0(x)$	----------	Bessel function of zero order, second kind (68).
$Y_1(x)$	----------	Bessel function of first order, second kind (77).
a	L	Finite length (128).
b	L	Thickness of aquifer (p. 6).
b'	L	Thickness of confining bed (86).
cm	L	Centimeter (p. 3), (11).
cgs	LMT	Centimeter-gram-second (11).
d	L	Mean grain diameter (7).
d	----------	Derivative (8).
e	----------	Base of Naperian logarithms, 2.71828 (p. 19).
ft	L	Foot (13).
g	LT^{-2}	Standard acceleration due to gravity (8).
gal	L^3	U.S. gallon (36).
g	M	Gram.
gpd	L^3T^{-1}	Gallons per day (p. 6).
gpm	L^3T^{-1}	Gallons per minute (p. 8).
h	L	Head (8), (116), (130).
h_0	L	Head at node (well) 0 (128), (129).
h_1 to h_4	----------	Head at nodes (wells) 1 to 4 (fig. 26), (128).
h_c	L	Height of capillary rise (1).
in.	L	Inch (p. 9).
k	L^2	Intrinsic permeability (7), (8).
kg	M	Kilogram (9).

Symbol	Dimensions	Description
l	L	Length of flow (8).
lb	MLT^{-2}	Pound (p. 9).
m	L	Meter (9).
mm	L	Millimeter (p. 3).
min	T	Minute (36).
n	----------	S_s+S_l/S_s (108).
n_d	----------	Number of potential drops (133).
n_f	----------	Number of flow channels (132).
p	$ML^{-1}T^{-2}$	Pressure (11).
q	LT^{-1}	Rate of flow per unit area (specific discharge) (8), (151).
r	L	Radius or radial distance (1), (19).
r_c	L	Radius of casing in interval over which water level fluctuates (76).
r_i	L	Radial distance from observation well to image well (148).
r_p	L	Radial distance from observation well to pumped well (148).
r_s	L	Radius of well screen or open hole (76).
r_w	L	Radius of discharging well (67), (139).
s	L	Drawdown (19), (114).
s'	L	Residual drawdown (81).
s_i	L	Drawdown in image well (146).
s_0	L	Abrupt change in drain level at $t=0$ (118); abrupt change in water level (fig. 32).
s_o	L	Algebraic sum of s_p and s_i (146).
s_p	L	Drawdown in pumped well (146).
s_w	L	Drawdown in discharging well (67), (139).
sec	T	Second (9).
t	T	Time since discharge began or stopped (19).
u	----------	Variable of integration (19); $r^2S/4Tt$ (45), (85), (91); $x\sqrt{S/4Tt}$ (113).
u_i	----------	$(r_i/r_p)^2u_p$ (149).
u_p	----------	$u_i/(r_i/r_p)^2$ (149).
v	L^3T^{-1}	Volume of water per unit time (37).
v	----------	$r/_2\sqrt{K'/b'T}$ (85), (86).
$\bar{v}$	LT^{-1}	Average velocity (28).
v_g	L^3	Volume of water drained by gravity (16).
v_i	L^3	Volume of interstices (4).
v_m	L^3	Volume of mineral particles (4).
v_r	L^3	Volume of water retained against gravity (17).
v_w	L^3	Volume of water (4).
w	L	Length (130).
x	----------	Variable of integration (68), (108).
x	L	Distance from drain to point of observation (112).
x	L	Coordinate in x direction (126).
y	L	Coordinate in y direction (126).
y	----------	Variable of integration (85), (91).
z	L	Elevation head (99).
z	L	Coordinate in z direction (fig. 21).
∞	----------	Infinity (19).
Σ	----------	Summation (128).
α	----------	Angle (1); Tt/Sr_w^2 (67), (94); r_s^2S/r_c^2 (76); $(r/B)^2/r^2S_l$ (107).
α	$M^{-1}LT^2$	$1/E_s$ (21).
β	$M^{-1}LT^2$	$1/E_w$ (21).
β	----------	Tt/r_c^2 (77); $\frac{r}{4b}\left(\sqrt{\frac{K'S_s'}{KS_s}}+\sqrt{\frac{K''S_s''}{KS_s}}\right)$ (92).
γ	$ML^{-2}T^{-2}$	Specific weight per unit area (20).
Δ	----------	Finite difference, change in (24).
∂	----------	Partial derivative (37).
ϵ	----------	$\frac{x^2}{x^2+1}\exp\{-\alpha nt(x^2+1)\}$ (108).
η	$ML^{-1}T^{-1}$	Dynamic viscosity (10).
θ	----------	Porosity (4); angle (138).
θ_0	----------	Angle (138).
μ	----------	Micro (10^{-6}) (p. 5).
λ	----------	Variable of integration (100).
ν	L^2T^{-1}	Kinematic viscosity (8), (10).
π	----------	3.1416.
ρ	ML^{-3}	Density of fluid (1).
ρ_d	ML^{-3}	Density of dry sample (bulk density) (5).
ρ_m	ML^{-3}	Mean density of mineral particles (grain density) (5).
ρ_w	ML^{-3}	Density of water (18).
τ	----------	Kt/Sb (100), (101); variable of integration (107).
φ	L^2T^{-2}	Potential (8).
ψ	----------	r/b (100), (101).

GROUND-WATER HYDRAULICS

By S. W. LOHMAN

INTRODUCTION

The science of ground-water hydrology is concerned with evaluating the occurrence, availability, and quality of ground water. Although many ground-water investigations are qualitative in nature, quantitative studies are necessarily an integral part of the complete evaluation of occurrence and availability. The worth of an aquifer as a source of water depends largely upon two inherent characteristics—its ability to store and to transmit water.

Thorough knowledge of the geologic framework is essential to understand the operation of the natural plumbing system within it. Ground-water hydraulics is concerned with the natural or induced movement of water through permeable rock formations. The principal method of analysis in ground-water hydraulics is the application, generally by field tests of discharging wells, of equations derived for particular boundary conditions. Prior to 1935, such equations were known only for the relatively simple steady flow condition, which incidentally generally does not occur in nature. The development by Theis (1935) of an equation for the nonsteady flow of ground water was a milestone in ground-water hydraulics. Since 1935 the number of equations and methods has grown rapidly and steadily. These are described in a wide assortment of publications, some of which are not conveniently available to many engaged in ground-water studies. The essence of many of these will be presented and briefly discussed, but frequent recourse should be made to the more exhaustive treatments given in the references cited.

The material presented herein was adapted from the lecture notes which I prepared for a series of five lectures on ground-water hydraulics presented in May 1967 to the students of the 1967 Ground Water School of the Australian Water Resources Council at Adelaide, South Australia. Problems given in the lecture notes have been changed to examples in this report, and the solutions of these examples are complete with tabulated data and data plots. Nine plates and three figures of type curves are reproduced at scales to fit readily available logarithmic or semilogarithmic translucent graph paper, and most of the data plots also are reproduced at scales to fit the proper type curves. Thus, all the type curves may be used in the solution of actual field problems.

I am indebted to the following colleagues of the Geological Survey for their critical reviews of the lecture notes, or the present version, or both: R. R. Bennett, R. H. Brown, H. H. Cooper, Jr., W. J. Drescher, J. M. Dumeyer, P. A. Emery, J. G. Ferris, C. L. McGuinness, E. A. Moulder, E. A. Sammel, R. W. Stallman, C. V. Theis, and E. P. Weeks.

Before getting into ground-water hydraulics, let us review briefly the divisions of subsurface water and some of the fundamental properties of aquifers.

DIVISIONS OF SUBSURFACE WATER IN UNCONFINED AQUIFERS

Unconfined aquifers composed of granular materials, such as mixtures of clay, silt, sand, and gravel, may contain all or part of the divisions of subsurface water shown in figure 1. All divisions generally are present in areas of relatively deep water table after rather prolonged dry spells. In other areas, the divisions may be present only in part, in order from bottom to top. Thus, beneath lakes, streams, and some swamps, surface water is underlain directly by unconfined ground water and the capillary fringe is absent. In some swamps the saturated part of the capillary fringe reaches the surface, but the unsaturated zone is absent.

SATURATED ZONE

WATER TABLE

The unconfined ground water below the water table (fig. 1) is under pressure greater than atmospheric.

When a well is sunk a few feet into an unconfined aquifer, the water level remains, for a time, at the same altitude at which it was first reached in drilling (fig. 1), but of course this level may fluctuate later in response to many factors. This level is one point on the **water table,** which may be defined as that imaginary surface within an unconfined aquifer at which the pressure is atmospheric. (See Hubbert, 1940, p. 897, 898; Lohman, 1965, p. 92.) The water level in wells sunk to greater depths in unconfined aquifers may stand at, above, or below the

Pressure	Zone	Divisions	Well
Gas phase, equals atmospheric	Unsaturated zone		
Liquid phase, less than atmospheric		Capillary fringe	
Less than atmospheric	Saturated zone[1]		
Atmospheric		Water table	
Greater than atmospheric		Unconfined ground water	

[1] As redefined by Hubbert (1940, p. 897, 898). See also Lohman (1965, p. 92).

FIGURE 1.—Divisions of subsurface water in unconfined aquifers.

water table, depending upon whether the well is in the discharge or recharge area of the aquifer. (See "Flowing Wells—Unconfined Aquifers" and fig. 7.)

Unconfined aquifers containing bodies of perched ground water above the regional water table may have repetitions of all or part of the divisions shown in figure 1.

CAPILLARY FRINGE

The capillary fringe ranges in thickness from a small fraction of an inch in coarse gravel to more than 5 ft in silt. Its lower part is completely saturated, like the material below the water table, but it contains water under less than atmospheric pressure, and hence the water in it normally does not enter a well. The capillary fringe rises and declines with fluctuations of the water table, and may change in thickness as it moves through materials of different grain sizes. Some capillary water may be drawn into wells by way of the saturated zone if the body of capillary water declines into coarser material and moves below the water table within the cone of depression of a discharging well. The saturated part of the capillary fringe was termed the "zone of complete capillary saturation" by Terzaghi (1942) and the "capillary stage" by Versluys (1917).

UNSATURATED ZONE

The **unsaturated zone** contains water in the gas phase under atmospheric pressure, water temporarily or permanently under less than atmospheric pressure, and air or other gases. The fine-grained materials may be temporarily or permanently saturated with water under less than atmospheric pressure, but the coarse-grained materials are unsaturated and generally contain liquid water only in rings surrounding the contacts between grains, as shown in figure 2. The soil may be temporarily saturated with soil water during or after periods of precipitation or flooding. The unsaturated zone may be absent beneath swamps, streams, or lakes. For a more sophisticated account of the unsaturated zone, see Stallman (1964).

CAPILLARITY

The rise of water or other fluids in tubes or in the interstices in rocks or soil may be considered to be caused by (1) the molecular attraction (adhesion) between the solid material and the fluid, and (2) the surface tension of the fluid, an expression of the attraction (cohesion) between the molecules of the fluid.

The molecular attraction between the solid material and the fluid depends in part upon the composition of the fluid and upon the composition and cleanliness of the material, and, as will be shown below, the height of capillary rise is governed by the size of the tube or opening. Water will wet and adhere to a clean floor, whereas it will remain in drops without wetting a floor covered with dust.

The surface of water resists considerable tension without losing its continuity. Thus, a carefully placed greased needle floats on water, as do certain insects having greasy pads on their feet.

In figure 3, the water has risen a height h_c in a tube of radius r immersed in a vessel of water. The relations shown in figure 3 may be expressed

$$\pi r^2 \rho g h_c = 2\pi r T \cos\alpha \qquad [MLT^{-2}], \qquad (1)$$

where

r = radius of capillary tube,
ρ = density of fluid,
g = acceleration due to gravity,
h_c = height of capillary rise,
T = surface tension of fluid, and
α = angle between meniscus and tube.

Note that, according to equation 1, weight equals lift by surface tension. Solving equation 1 for h_c,

$$h_c = \frac{2T}{r\rho g}\cos\alpha \qquad [L]. \qquad (2)$$

For pure water in clean glass, $\alpha = 0$, and $\cos\alpha = 1$. At 20°C, $T = 72.8$ dyne cm^{-1}, ρ may be taken as 1 g cm^{-3}, and

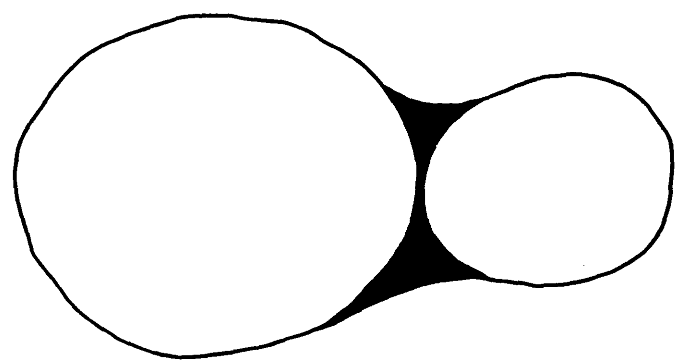

FIGURE 2.—Water in the unsaturated zone.

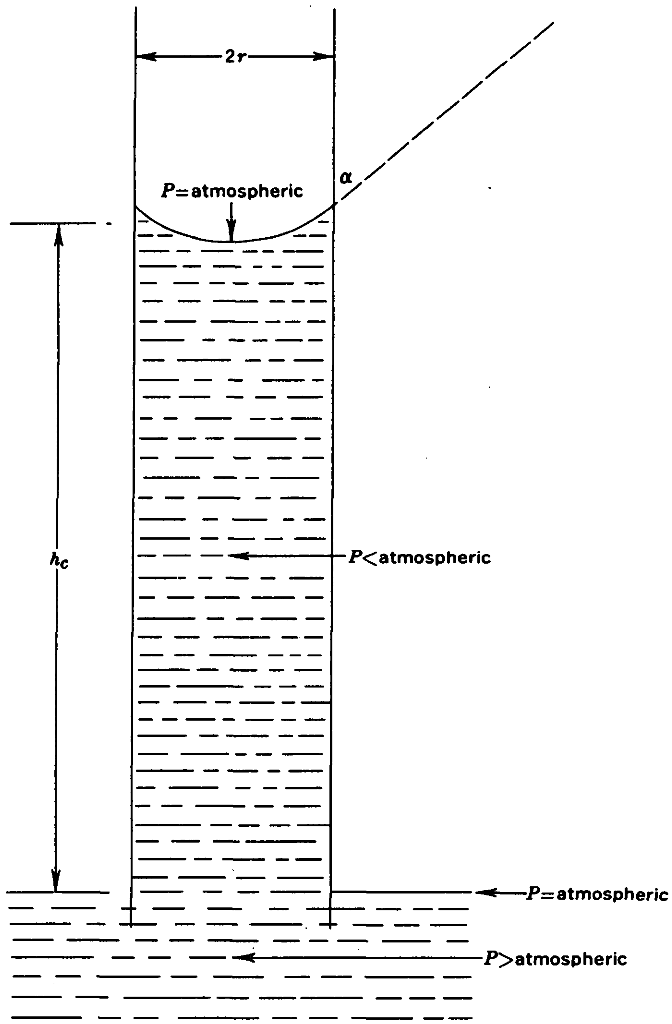

FIGURE 3.—Capillary rise of water in a tube (diameter greatly exaggerated).

$g = 980.665$ cm sec^{-2}, whence

$$h_c = \frac{0.15}{r} \qquad [L]. \qquad (3)$$

Surface tension is sometimes given in grams per centimeter and for pure water in contact with air, at 20°C, its value is

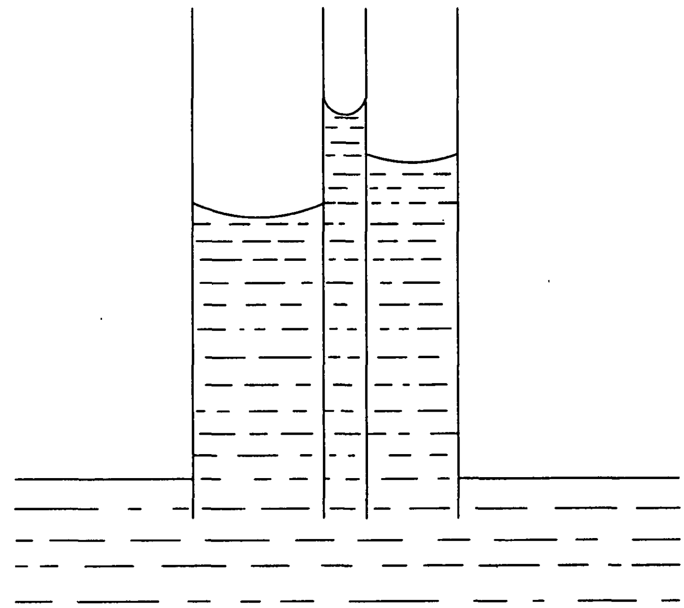

FIGURE 4.—Rise of water in capillary tubes of different diameters (diameters greatly exaggerated).

0.074 g cm^{-1}. In order to express it in grams per centimeter, we must divide 72.8 by g, the standard acceleration of gravity; thus 72.8 dyne cm^{-1}/980.665 cm sec^{-2} = 0.074 g cm^{-1}.

From equation 3 it is seen that the *height of capillary rise* in tubes is inversely proportional to the radius of the tube. The rise of water in interstices of various sizes in the capillary fringe (fig. 1) may be likened to the rise of water in a bundle of capillary tubes of various diameters, as shown in figure 4. In table 1, note that the capillary rise is nearly inversely proportional to the grain size.

TABLE 1.—*Capillary rise in samples having virtually the same porosity, 41 percent, after 72 days*

[From A. Atterberg, cited in Terzaghi (1942)]

Material	Grain size (mm)	Capillary rise (cm)
Fine gravel	5–2	2.5
Very coarse sand	2–1	6.5
Coarse sand	1–0.5	13.5
Medium sand	0.5–0.2	24.6
Fine sand	.2–0.1	42.8
Silt	.1–0.05	105.5
Silt	.05–0.02	[1]200

[1] Still rising after 72 days.

HYDROLOGIC PROPERTIES OF WATER-BEARING MATERIALS

POROSITY

The porosity of a rock or soil is simply its property of containing interstices. It can be expressed quantitatively as the ratio of the volume of the interstices to the total volume, and may be expressed as a decimal fraction or as a percentage. Thus

$$\theta = \frac{v_i}{V} = \frac{v_w}{V} = \frac{V - v_m}{V} = 1 - \frac{v_m}{V} \qquad [\text{dimensionless}] \quad (4)$$

where

θ = porosity, as a decimal fraction,
v_i = volume of interstices,
V = total volume,
v_w = volume of water (in a saturated sample), and
v_m = volume of mineral particles.

Porosity may be expressed also as

$$\theta = \frac{\rho_m - \rho_d}{\rho_m} = 1 - \frac{\rho_d}{\rho_m} \quad \text{[dimensionless]} \qquad (5)$$

where

ρ_m = mean density of mineral particles (grain density) and

ρ_d = density of dry sample (bulk density).

Multiplying the right-hand sides of equations 4 and 5 by 100 gives the porosity as a percentage.

PRIMARY

Primary porosity comprises the original interstices created when a rock or soil was formed in its present state. In soil and sedimentary rocks the primary interstices are the spaces between grains or pebbles. In intrusive igneous rocks the few primary interstices result from cooling and crystallization. Extrusive igneous rocks may have large openings and high porosity resulting from the expansion of gas, but the openings may or may not be connected. Metamorphism of igneous or sedimentary rocks generally reduces the primary porosity and may virtually obliterate it.

SECONDARY

Fractures such as joints, faults, and openings along planes of bedding or schistosity in consolidated rocks having low primary porosity and permeability may afford appreciable *secondary porosity*. In some rocks such secondary porosity affords the only means for the storage and movement of ground water. Solution of carbonate rocks such as limestone or dolomite by water containing dissolved carbon dioxide takes place mainly along joints and bedding planes and may greatly increase the secondary porosity. Similarly, solution of gypsum or anhydrite by water alone may greatly increase the secondary porosity.

CONDITIONS CONTROLLING POROSITY OF GRANULAR MATERIALS

ARRANGEMENT OF GRAINS (ASSUMED SPHERICAL AND OF EQUAL SIZE)

If a hypothetical granular material were composed of spherical particles of equal size, the porosity would be independent of particle size (whether the particles were the size of silt or the size of the earth) but would vary with the packing arrangement of the particles. As shown by Slichter (1899, p. 305–328), the lowest porosity of 25.95 (about 26) percent would result from the most compact rhombohedral arrangement (fig. 5A) and the highest porosity of 47.64 (about 48) percent would result from the least compact cubical arrangement (fig. 5C). The porosity of the other arrangements, such as that shown in figure 5B, would be between these limits.

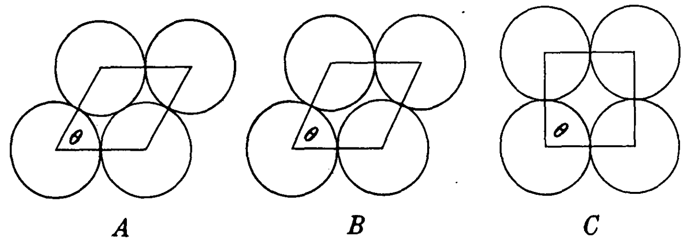

FIGURE 5.—Sections of four contiguous spheres of equal size. *A*, most compact arrangement, lowest porosity; *B*, less compact arrangement, higher porosity; *C*, least compact arrangement, highest porosity. Sketches from Slichter (1899, pl. 1).

SHAPE OF GRAINS

Angularity of particles causes wide variations in porosity and may increase or decrease it, according to whether the particles tend to bridge openings or pack together like pieces of a mosaic.

DEGREE OF ASSORTMENT

The greater the range in particle size the lower the porosity, as the small particles occupy the voids between the larger ones.

VOID RATIO

The **void ratio** of a rock or soil is the ratio of the volume of its interstices to the volume of its mineral particles. It may be expressed:

$$\text{Void ratio} = \frac{v_i}{v_m} = \frac{v_w}{v_m} = \frac{\theta}{1-\theta} \quad \text{[dimensionless]}, \qquad (6)$$

where the symbols are as defined for equation 4.

PERMEABILITY

The **permeability** of a rock or soil is a measure of its ability to transmit fluid, such as water, under a hydropotential gradient. Many earlier workers found that the permeability is approximately proportional to the square of the mean grain diameter,

$$k \approx Cd^2 \quad [L^2], \qquad (7)$$

where

k = intrinsic permeability,

C = a dimensionless constant depending upon porosity, range and distribution of particle size, shape of grains, and other factors, and

d = the mean grain diameter of some workers and the effective grain diameter of others.

INTRINSIC PERMEABILITY

Inasmuch as permeability is a property of the medium alone and is independent of the nature or properties of the fluid, the U.S. Geological Survey is adopting the term "intrinsic permeability," which is not to be confused with hydraulic conductivity as the latter includes the properties of natural ground water. Intrinsic permeability may be expressed

$$k = -\frac{q\nu}{g(dh/dl)} = -\frac{q\nu}{(d\varphi/dl)} \qquad [L^2] \qquad (8)$$

where

k = intrinsic permeability,
q = rate of flow per unit area = Q/A,
ν = kinematic viscosity,
g = acceleration of gravity,
dh/dl = gradient, or unit change in head per unit length of flow, and
$d\varphi/dl$ = potential gradient, or unit change in potential per unit length of flow.

From equation 8 it may be stated that a porous medium has an **intrinsic permeability** of one unit of length squared if it will transmit in unit time a unit volume of fluid of unit kinematic viscosity through a cross section of unit area measured at right angles to the flow direction under a unit potential gradient.

If q is measured in meters per second, ν in square meters per second, φ in joules per kilogram, and l in meters, the unit for k is in square meters. Thus, equation 8 may be written

$$k = -\frac{(\text{m}^3)(\text{m}^2\,\text{sec}^{-1})}{(\text{m}^2)(\text{sec})(-\text{J kg}^{-1}\,\text{m}^{-1})}$$

$$= -\frac{(\text{m}^3)(\text{m}^2\,\text{sec}^{-1})}{(\text{m}^2)(\text{sec})(-\text{kg m sec}^{-2}\,\text{m kg}^{-1}\,\text{m}^{-1})} = \text{m}^2 \qquad [L^2]. \qquad (9)$$

The Geological Survey will express k in square micrometers, $(\mu\text{m})^2 = 10^{-12}\ \text{m}^2 = 10^{-8}\ \text{cm}^2$, which is 10^{-12} times the value in equation 9.

The kinematic viscosity (ν) is related to the dynamic viscosity (η) thus

$$\eta = \nu\rho \qquad [ML^{-1}T^{-1}], \qquad (10)$$

where ρ = density.

Other expressions for intrinsic permeability referred to in the literature (table 2) involve pressure gradients rather than head or potential gradients and were intended mainly for laboratory use where gas (generally nitrogen) permeameters rather than water permeameters are used. Although, as pointed out by Hubbert (1940, p. 921) "* * * this equation is physically erroneous as an expression of Darcy's law, owing to the use of pressure as a potential function * * *," at least one of the expressions has been widely used, so they will be taken up briefly.

TABLE 2.—*Relation of units of hydraulic conductivity, permeability, and transmissivity*

[Equivalent values shown in same horizontal lines. † indicates abandoned term]

A. Hydraulic conductivity

Hydraulic conductivity (K)		†Field coefficient of permeability (P_f)
Feet per day (ft day^{-1})	Meters per day (m day^{-1})	†Gallons per day per square foot †(gal day^{-1} ft^{-2})
One	0.305	7.48
3.28	**One**	24.5
.134	.041	**One**

B. Transmissivity (T)

Square feet per day (ft^2 day^{-1})	Square meters per day (m^2 day^{-1})	†Gallons per day per foot †(gal day^{-1} ft^{-1})
One	0.0929	7.48
10.76	**One**	80.5
.134	.0124	**One**

C. Permeability

Intrinsic permeability $k = -\frac{q\nu}{d\varphi/dl}$ [$(\mu\text{m})^2 = 10^{-8}$ cm^2]	Darcy = $-\frac{q\mu}{dp/dl + \rho g dz/dl}$ [0.987×10^{-8} cm^2]	†Coefficient of permeability P or $P_m = -\frac{q(\text{at } 60°\text{F.})}{dh/dl}$ †[gal day^{-1} ft^{-2} at 60°F.]
One	1.01	18.4
0.987	**One**	18.2
.054	.055	**One**

In 1930, Nutting (1930, p. 1348) defined a "rational cgs measure of permeability" then in use in his U.S. Geological Survey laboratory, and intended for general use by the petroleum industry, as "the flow in cubic centimeters per second through each square centimeter, of a fluid of 0.01 [poise] viscosity under a pressure of 1 megadyne per centimeter * * *." Nutting doubtless meant a pressure gradient of 1 megabarye per centimeter, or 1 megadyne per square centimeter per centimeter. Thus corrected, Nutting's definition may be expressed, in centimeter-gram-second units,

$$k = -\frac{q\eta}{dp/dl}$$

$$= -\frac{(\text{cm}^3)(10^{-2}\ \text{dyne-sec cm}^{-3})}{(\text{cm}^2)(\text{sec})(-10^6\ \text{dyne cm}^{-2}\,\text{cm}^{-1})}$$

$$= 10^{-8}\ \text{cm}^2 = (\mu\text{m})^2 \qquad [L^2]. \qquad (11)$$

Four years later Wyckoff, Botset, Muskat, and Reed (1934, p. 166) seemingly ignored the Nutting definition of permeability in consistent units and proposed the darcy in inconsistent units, wherein the atmosphere was used in place of the megabarye. Thus

$$\text{darcy} = -\frac{(\text{cm}^3)(10^{-2}\text{ dyne-sec cm}^{-2})}{(\text{cm}^2)(\text{sec})(-1.0132\times 10^6\text{ dyne cm}^{-2}\text{ cm}^{-1})}$$

$$=0.987\times 10^{-8}\text{ cm}^2 = 0.987(\mu\text{m})^2 \quad [L^2]. \qquad (12)$$

To make matters still worse, in 1935 the American Petroleum Institute (1942, p. 4) redefined the darcy for adoption by the petroleum industry by changing the volume in equation 12 from cubic centimeters to milliliters. Inasmuch as 1 milliliter is 27 parts per million greater than 1 cubic centimeter, at ordinary temperatures, the darcy, as redefined, embodies the doubly inconsistent units milliliter, centimeter, and atmosphere.

HYDRAULIC CONDUCTIVITY

The Water Resources Division, U.S. Geological Survey, is adopting hydraulic conductivity (K) in consistent units to replace (P) the "coefficient of permeability" in the inconsistent units gpd ft^{-2} (gallons a day per square foot). K may be defined thus: A medium has a **hydraulic conductivity** of unit length per unit time if it will transmit in unit time a unit volume of ground water at the prevailing viscosity through a cross section of unit area, measured at right angles to the direction of flow, under a hydraulic gradient of unit change in head through unit length of flow. The suggested units are:

$$K = -\frac{q}{dh/dl}$$

$$= -\frac{\text{ft}^3}{\text{ft}^2\text{ day}(-\text{ft ft}^{-1})} = \text{ft day}^{-1} \quad [LT^{-1}], \qquad (13)$$

or

$$K = -\frac{\text{m}^3}{\text{m}^2\text{ day}(-\text{m m}^{-1})} = \text{m day}^{-1} \quad [LT^{-1}], \qquad (14)$$

where the symbols are as defined for equation 12. The minus signs in equations 13 and 14 result from the fact that the water moves in the direction of decreasing head. The relation of the new and old units is given in table 2.

TRANSMISSIVITY

The **transmissivity** (T) is the rate at which water of the prevailing kinematic viscosity is transmitted through a unit width of the aquifer under a unit hydraulic gradient. It replaces the term "coefficient of transmissibility" because it is considered by convention a property of the aquifer, which is transmissive, whereas the contained liquid is transmissible. Hence, though spoken of as a property of the aquifer, it is a property of the confined liquid also. It is equal to Kb, where b is the thickness of the aquifer. In the units of equations 13 and 14, T becomes

$$T = \text{ft}^2\text{ day}^{-1}, \text{ or m}^2\text{ day}^{-1} \quad [L^2T^{-1}]. \qquad (15)$$

The relation of the new and old units is given in table 2.

WATER YIELDING AND RETAINING CAPACITY OF UNCONFINED AQUIFERS

SPECIFIC YIELD[1]

In general terms, the specific yield is the water yielded from water-bearing material by gravity drainage, as occurs when the water table declines. More exactly, the **specific yield** of a rock or soil has been defined (Meinzer, 1923, p. 28) as the ratio of (1) the volume of water which, after being saturated, it will yield by gravity to (2) its own volume. This may be expressed

$$S_y = \frac{v_g}{V} \quad [\text{dimensionless}], \qquad (16)$$

where

S_y = specific yield, as a decimal fraction,
v_g = volume of water drained by gravity, and
V = total volume.

Note that the duration of the drainage has not been specified; I suggest that it should be stated when known. Multiplying the right-hand side of equation 16 by 100 gives the result in percent.

SPECIFIC RETENTION

The **specific retention** of a rock or soil with respect to water has been defined (Meinzer, 1923, p. 28, 29) as the ratio of (1) the volume of water which, after being saturated, it will retain against the pull of gravity to (2) its own volume. It may be expressed

$$S_r = \frac{v_r}{V} = \theta - S_y \quad [\text{dimensionless}], \qquad (17)$$

where

S_r = specific retention, as a decimal fraction, and
v_r = volume of water retained against gravity, mostly by molecular attraction.

From equation 17, it may be noted also that $S_y = \theta - S_r$.

MOISTURE EQUIVALENT

As used in the Hydrologic Laboratory of the U.S. Geological Survey, the **moisture equivalent** of water-bearing materials is the ratio of (1) the weight of water which the material, after saturation, will retain against

[1] See also "Storage Coefficient."

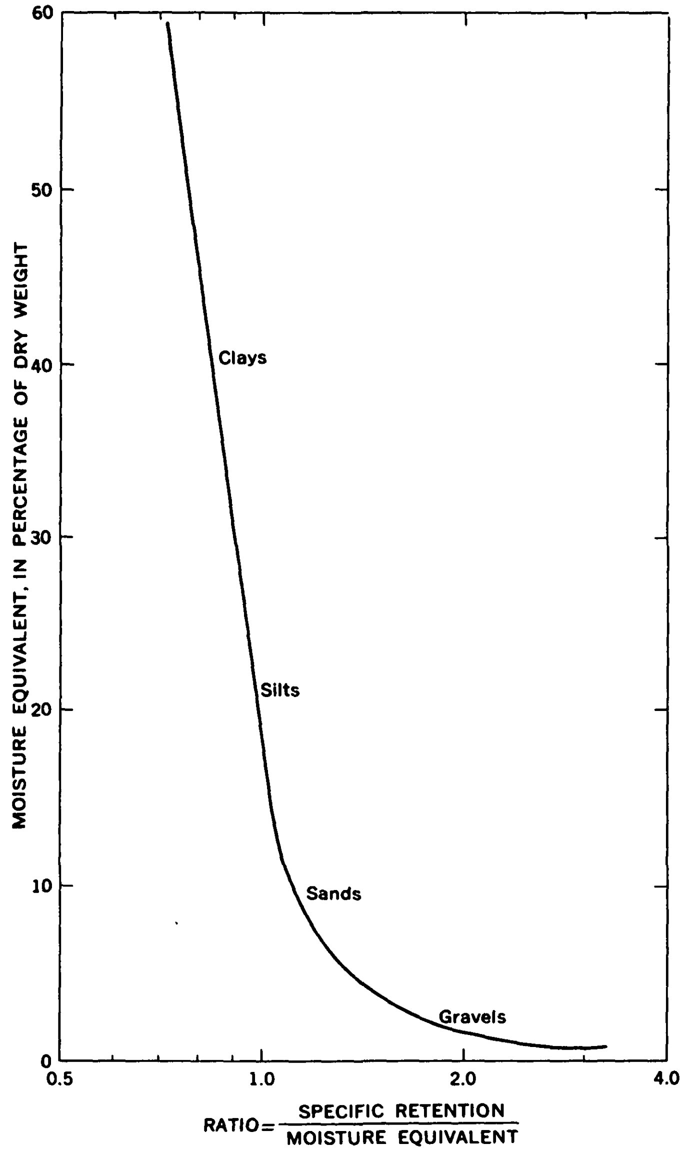

FIGURE 6.—Relation between moisture equivalent and specific retention from Piper (1933, p. 485). Modified by A. I. Johnson.

a centrifugal force 1,000 times the force of gravity, for 2 hours at 20°C., and under 100 percent humidity, to (2) the weight of the material when dry. Note that this ratio is by weight, whereas specific retention is a ratio by volume. The relation between the two concepts may be expressed

$$S_r = MN_r \frac{\rho_d}{\rho_w} \quad \text{[dimensionless]}, \qquad (18)$$

where

S_r = specific retention, in percent by volume,
M = moisture equivalent, in percent by weight,
N_r = ratio: specific retention/moisture equivalent,
ρ_d = dry density of sample, and
ρ_w = density of water.

Piper (1933) found that the relations shown in figure 6 prevail between specific retention and moisture equivalent. Note that the ratio is essentially 1 for moisture equivalents between 34 and 12 percent but ranges from 1 to 2.5 for values between 12 and 2 percent.

ARTESIAN WELLS—CONFINED AQUIFERS

Confined aquifers, as the name suggests, contain ground water that is confined under pressure between relatively impermeable or significantly less permeable material and that will rise above the top of the aquifer. If the water rises above the land surface it will flow naturally. A well drilled into such a confined aquifer is an artesian well, and if the water rises above the land surface, it may be termed a "flowing artesian well." As will be shown in the next section, however, flowing wells also may be constructed in unconfined aquifers.

FLOWING WELLS—UNCONFINED AQUIFERS

Consider a hilly area underlain by uniformly permeable material that receives recharge from precipitation in interstream areas and from which water discharges into streams. The approximate flow pattern is illustrated by the solid lines with arrows in an idealized cross section (fig. 7);

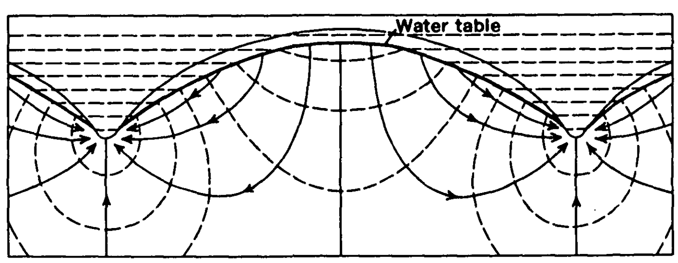

FIGURE 7.—Approximate flow pattern in uniformly permeable material which receives recharge in interstream areas and from which water discharges into streams. From Hubbert (1940, p. 930, fig. 45).

the dashed lines at right angles to the flow lines are lines of equipotential. There is an infinity of flow and equipotential lines, only a few of which are shown. Cased wells at or near the streams reach water under greater head as the depth increases and, as may be inferred from the horizontal dashed lines, wells open at moderate depth will flow at the surface. Note also in figure 7 that cased wells on the hill reach water at progressively lower heads as the depth increases.

CONFINED AQUIFERS

POTENTIOMETRIC SURFACE

The **potentiometric surface** is an imaginary surface connecting points to which water would rise in tightly cased wells from a given point in an aquifer. It may be above or below the land surface. The water table (p. 1) is a particular potentiometric surface. Potentiometric is preferable to the term "piezometric," which was used by many in the past.

Confined and unconfined aquifers are compared in figure 8. The well tapping the confined aquifer in figure 8 is a flowing well; if the potentiometric surface were at or below the ground surface, however, this well would have to be pumped.

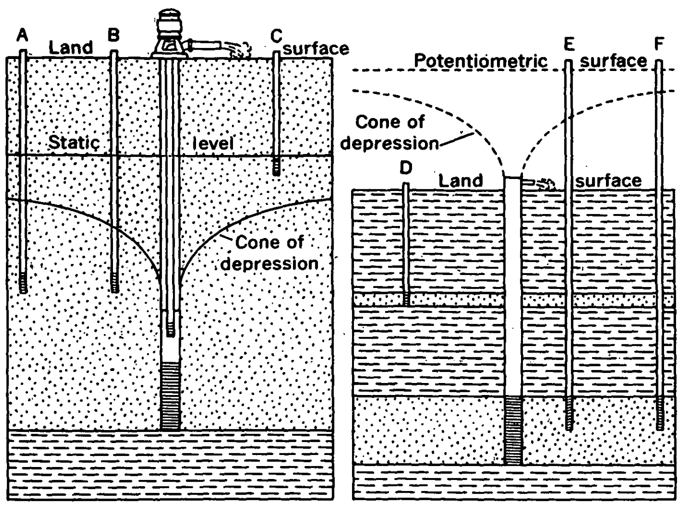

FIGURE 8.—Discharging wells in a confined aquifer (right) and an unconfined aquifer (left). Although the water levels in wells A and B have declined because of pumping from the nearby well, wells A and B remain usable; however, shallower well C has been "dried up" by the pumping. The water levels in wells E and F have declined because of flow from the nearby well, but in well D, which taps a shallower aquifer, the water level is not affected by flow from the deeper aquifer.

STORAGE PROPERTIES

Prior to 1925, confined, or artesian, aquifers were considered mainly as conduits for delivering water from recharge areas to distant wells or springs. They were not thought of as having storage properties except, of course, for volume times porosity. Confining beds generally were thought to be wholly or relatively impermeable, whereas they are now known to range from nearly impermeable to moderately permeable.

In 1925 Meinzer and Hard (1925, p. 92), from studies of flowing artesian wells tapping the Dakota Sandstone in the Ellendale area, South Dakota, postulated that, although in the preceding 38 years the average rate of discharge of a selected group of wells was 3,000 gpm, only about 500 gpm could have been transmitted from the recharge area and that the remaining 2,500 gpm was released from storage by elastic compression of the aquifer as the pressure supporting the load gradually declined. This led to Meinzer's classic theory of the compressibility and elasticity of artesian aquifers (Meinzer, 1928). It is now known also that part of the water released from storage comes from expansion of the water. (See also, Swenson, 1968.) It is also known that much of the water released by lowering of head in some aquifers comes from inelastic compression of silty or clayey lenses or beds within or adjacent to the aquifers. (See "Nonelastic Confined Aquifers and Oil-Bearing Strata.")

STORAGE COEFFICIENT

The storage property of confined aquifers was given quantitative significance for the first time by Theis (1935), who introduced the storage coefficient (S) in his classic equation:

$$s=\frac{Q}{4\pi T}\int_{r^2S/4Tt}^{\infty}\left(\frac{e^{-u}}{u}\right)du \qquad [L], \qquad (19)$$

where

s = drawdown,
Q = constant discharge rate from well,
T = transmissivity,
r = distance from discharging well to point of observation of s,
S = storage coefficient,
t = time since discharge began, and
u = variable of integration.

The current version of Theis' definition (1938) of the **storage coefficient** is: The volume of water an aquifer releases from or takes into storage per unit surface area of the aquifer per unit change in head. Note from the definition that the storage coefficient is dimensionless.

The storage coefficient of unconfined aquifers is virtually equal to the specific yield, as most of the water is released from storage by gravity drainage and only a very small part comes from compression of the aquifer and expansion of the water.

The storage coefficient of most confined aquifers ranges from about 10^{-5} to 10^{-3} and is about 10^{-6} per foot of thickness. In contrast, the specific yield of most unconfined aquifers ranges from about 0.1 to about 0.3 and averages about 0.2. Inasmuch as the storage coefficient of confined aquifers generally is so small, the question might be raised as to how much water can be released from storage. To illustrate that large quantities of water can be so released, assume that, in a confined aquifer having a storage coefficient of 2×10^{-4}, the head declines 400 ft throughout 1 square mile; then $(2\times10^{-4})(4\times10^2\ \text{ft})(2.8\times10^7\ \text{ft}^2)=2.24\times10^6\ \text{ft}^3$—a large volume of water. In the Denver artesian basin, Colorado, the head has declined much more than this in an area of perhaps 100 square miles.

COMPONENTS

Jacob (1940, p. 576) showed that for an elastic confined aquifer, neglecting any release of water from the confining beds,

$$S=\theta\gamma b\left(\frac{1}{E_w}+\frac{C}{\theta E_s}\right)\qquad [\text{dimensionless}],\qquad (20)$$

where

- θ = porosity, as a decimal fraction;
- γ = specific weight per unit area, 62.4 lb ft^{-3}/144 in^2 ft^{-2} = 0.434 lb in^{-2} ft^{-1};
- b = thickness, in feet;
- E_w = bulk modulus of elasticity of water, 3×10^5 lb in^{-2}, at ordinary temperatures; and
- C = a dimensionless ratio, which may be considered unity in an uncemented granular material. In a solid aquifer, as a limestone having tubular solution channels, C is apparently equal to the porosity. The value for a sandstone doubtless ranges between these limits, depending upon the degree of cementation.
- E_s = bulk modulus of elasticity of the solid skeleton of the aquifer, as confined in situ, in pounds per square inch.

An alternate expression of equation 20 for elastic confined aquifers in which C may be considered unity is

$$S=\theta\gamma b\left(\beta+\frac{\alpha}{\theta}\right)\qquad [\text{dimensionless}],\qquad (21)$$

where

- $\beta=1/E_w=3.3\times10^{-6}$ in^2 lb^{-1}, and
- $\alpha=1/E_s$, in square inches per pound.

Let us consider the part of the storage coefficient that results only from the expansion of water in a confined aquifer having $\theta=0.2$ and $b=100$ ft. From equation 21, $S=\theta\gamma b\beta=(0.2)(0.434\ \text{lb in}^{-2}\ \text{ft}^{-1})(100\ \text{ft})(3.3\times10^{-6}\ \text{in}^2\ \text{lb}^{-1})=2.9\times10^{-5}$. Although this value obviously is too small inasmuch as it does not include the compression of the aquifer, it is of value for comparison with the storage coefficient determined by testing an aquifer of this porosity and thickness. If the determined value is comparable to or less than this computed value for water alone, obviously the determined value is in error.

If S and other terms are known, α, the reciprocal of the modulus of elasticity of the aquifer, can be determined from equation 21. Jacob (1940, p. 583) showed also that

$$S=\theta\gamma b\beta\left(\frac{1}{\text{BE}}\right)=\theta\gamma b\beta\left(\frac{1}{1-\text{TE}}\right)\qquad [\text{dimensionless}],\qquad (22)$$

where

- BE = barometric efficiency of artesian well, and
- TE = tidal efficiency of artesian well near seacoast.

Other terms are defined for equations 20 and 21.

LAND SUBSIDENCE

ELASTIC CONFINED AQUIFERS

I (Lohman, 1961) showed that for elastic confined aquifers for which C may be assumed to equal 1, equations 20 and 21 may be rewritten

$$\frac{b}{E_s}=\frac{S}{\gamma}-\theta b\beta\qquad [L^2M^{-1}T^2]\qquad (23)$$

and that Hooke's Law (strain is proportional to stress, within the elastic limit) may be expressed

$$\Delta b=\frac{b}{E_s}\,\Delta p\qquad [L],\qquad (24)$$

where

- Δb = change in b, in feet, and
- Δp = change (generally decline) in artesian pressure, in pounds per square inch.

Combining equations 23 and 24,

$$\Delta b=\Delta p\left(\frac{S}{\gamma}-\theta b\beta\right)\qquad [L].\qquad (25)$$

Equation 25 gives the amount of land subsidence, Δb, for an elastic confined aquifer of known S, θ, and β, for a given decline in artesian pressure, Δp. For example, assume $S=2\times10^{-4}$, $\theta=0.3$, $b=100$ ft, $\Delta p=100$ lb in^{-2}, and note that it is convenient to use $1/\gamma$, which equals 2.31 ft lb^{-1} in^2. Then, from equation 25,

$$\Delta b=10^2\ \text{lb in}^{-2}(2\times10^{-4}\times2.31\ \text{ft lb}^{-1}\ \text{in}^2-0.3\times10^2\ \text{ft}\times3.3\times10^{-6}\ \text{in}^2\ \text{lb}^{-1})$$
$$=0.04\ \text{ft (rounded)}.$$

Similarly, for $b=1{,}000$ ft, $\Delta p=1{,}000$ lb in^{-2}, $S=10^{-3}$, $\theta=0.3$, $\Delta b=1.3$ ft.

NONELASTIC CONFINED AQUIFERS AND OIL-BEARING STRATA

Clay or silty clay beds or lenses in confined aquifers or oil-bearing strata, and in associated confining beds, are much

TABLE 3.—*Land subsidence in California oil and water fields*

Well field	Subsidence (ft)	Through year
Wilmington oil field (see Gilluly and Grant, 1949)	[1] 29	1966
Water fields:		
Santa Clara Valley (San Jose)	13	1967
San Joaquin Valley:		
Los Banos-Kettleman Hills area	26	1966
Tulare-Wasco area	12	1962
Arvin-Maricopa area	8	1965

[1] Stabilized at this amount by repressuring. This figure includes some recovery due to repressuring.

more porous than associated sands or gravels; hence, they contain more fluid per unit volume at a given fluid pressure. When the pressure is gradually reduced, as by discharge of fluids from wells, such beds slowly release fluids and undergo nonelastic (plastic), generally irreversible, compaction. (See Athy, 1930; Hedberg, 1936; and Poland and Evenson, 1966.) Compaction of this type is much greater than purely elastic compression, and it has caused appreciable subsidence of the land surface in both oil and water fields in California, Texas, and elsewhere. Latest available data for several California oil and water fields (J. F. Poland, U.S. Geol. Survey, written commun., Oct. 27, 1967) are given in table 3.

MOVEMENT OF GROUND WATER–STEADY-STATE FLOW

In steady-state flow, hereinafter referred to simply as steady flow, as of ground water through permeable material, there is no change in head with time. Mathematically, this statement is symbolized by $dh/dt=0$, which says that the change in head, dh, with respect to the change in time, dt, equals zero. Steady flow generally does not occur in nature, but it is a very useful concept in that steady flow can be closely approached in nature and in aquifer tests, and this condition may be symbolized by $dh/dt \to 0$.

Figure 9 shows a hypothetical example of true steady radial flow. Here steady radial flow will be reached and maintained when all the recoverable ground water in the cone of depression has been drained by gravity into the well discharging at constant rate Q.

DARCY'S LAW

Although Hagen (1839) and Poiseuille (1846) found that the rate of flow through capillary tubes is proportional to the hydraulic gradient, Darcy (1856) seemingly was the first to experiment with the flow of water through sand, and he found that the rate of laminar (viscous) flow of water through sand also is proportional to the hydraulic gradient. This is known as *Darcy's law* and it is generally

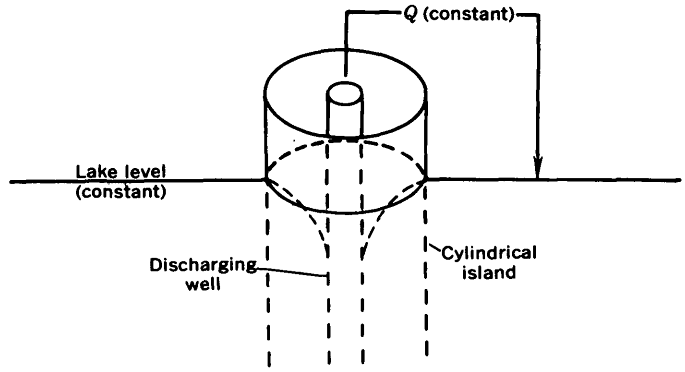

FIGURE 9.—Hypothetical example of steady flow (well discharging at constant rate Q from a cylindrical island in a lake of constant level).

expressed, by rewriting equation 13,

$$q=\frac{Q}{A}=-\frac{Kdh}{dl} \qquad [LT^{-1}]. \qquad (26)$$

It will be noted that K, the constant of proportionality in Darcy's law, is the hydraulic conductivity.

To illustrate the use of equation 26 (Darcy's law), assume that we wish to compute the total rate of ground-water movement in a valley where A, the cross-sectional area, is 100 ft deep times 1 mile wide, where $K=500$ ft day^{-1}, and $dh/dl=5$ ft per mile. Then

$$Q=-(100\text{ ft})(5{,}280\text{ ft})(500\text{ ft day}^{-1})\left(\frac{-5\text{ ft}}{5{,}280\text{ ft}}\right)$$
$$=250{,}000\text{ ft}^3\text{ day}^{-1}. \qquad (27)$$

VELOCITY

Because the hydraulic conductivity, K, has the dimensions of velocity, LT^{-1}, some might mistake this for the particle velocity of the water, whereas, as may be seen from equations 13 and 14, K is actually a measure of the volume rate of flow through unit cross-sectional area. For the average particle velocity, $\bar{v}$, we must also know the porosity of the material. Thus

$$Q=\bar{v}A\theta=-KA\frac{dh}{dl},$$

where

$\bar{v}$ = average velocity, in feet per day, and
θ = porosity, as a decimal fraction.

Other terms are defined for equation 13. Rewriting the above equation,

$$\bar{v}=-\frac{Kdh/dl}{\theta} \qquad [LT^{-1}]. \qquad (28)$$

For example, using the values of K and dh/dl given in equation 27, and assuming $\theta = 0.2$,

$$\bar{v} = -\frac{(500 \text{ ft day}^{-1})(-5 \text{ ft}/5{,}280 \text{ ft})}{0.2}$$

$$= 2.4 \text{ ft day}^{-1} \text{ (rounded)}. \qquad (29)$$

It should be stressed that the solution of equation 29 is the *average velocity* and does not necessarily equal the actual velocity between any two points in the aquifer, which may range from less than to more than this value, depending upon the flow path followed. Thus, equation 28 should not be used for predicting the velocity and distance of movement of, say, a contaminant introduced into the ground.

AQUIFER TESTS BY WELL METHODS—POINT SINK OR POINT SOURCE

STEADY RADIAL FLOW WITHOUT VERTICAL MOVEMENT

The first mathematical analysis of steady flow, using a discharging well, was made by Dupuit (1848), who made the important assumption that within the cone of depression of a discharging well the head is constant throughout any vertical line through the water body and therefore is represented by the elevation of the water table. Actually, this is true only in confined aquifers having uniform hydraulic conductivity and having a fully penetrating discharging well, or in confined aquifers remote from the discharging well. Nevertheless, methods based upon this assumption can be applied satisfactorily when certain precautions are taken.

Much of the mathematical analysis of Dupuit was repeated by Adolph Thiem (1887), but it remained for his son Günther Thiem (1906) to develop a readily usable solution, by determining that the Dupuit-Thiem methods could be applied to any two intermediate points on the cone of depression of a discharging well to determine the hydraulic conductivity of the aquifer. As will be shown later, it is now known that many more points may be used.

In order to derive the Thiem equation and show its relation to Darcy's law, let figure 10 represent half the cross section of the cone of depression in an unconfined aquifer around well A that has been pumped at constant rate Q long enough that steady flow is being closely approached, and the quantity of water still draining from storage is negligible compared with the quantity of water moving toward well A. Although figure 10 depicts an unconfined aquifer, the method is applicable also to confined aquifers. If the material is reasonably homogenous, and if the base of the aquifer and the undisturbed water table are assumed to be parallel and horizontal; then, by the law of continuity, and provided that changes in storage are negligible compared to Q, virtually equal quantities of water are discharged from well A (Q) and flow radially toward well A through any two concentric cylinders within the cone of depression, as at observation well B (Q_1) at radius r_1 or at observation well C (Q_2) at radius r_2. Thus $Q \approx Q_1 \approx Q_2$. Under these assumed conditions Darcy's law may be expressed as a first-order ordinary differential equation in cylindrical coordinates

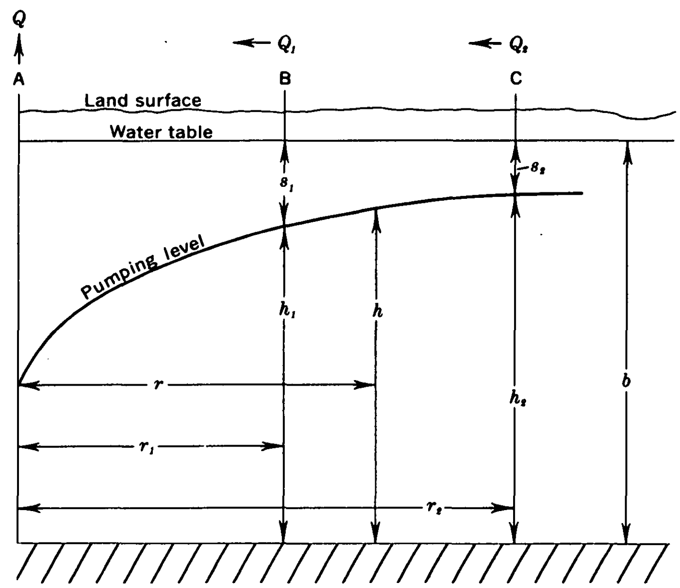

FIGURE 10.—Half the cross section of the cone of depression around a discharging well (A) in an unconfined aquifer.

$$Q = -K2\pi rh \frac{dh}{dr} \qquad [L^3T^{-1}]. \qquad (30)$$

Separating variables,

$$\frac{dr}{r} = -\frac{2\pi K}{Q} hdh.$$

Integrating between r_1 and r_2, h_1 and h_2,

$$\int_{r_1}^{r_2} \frac{dr}{r} = -\frac{2\pi K}{Q} \int_{h_1}^{h_2} hdh,$$

hence

$$\log_e \frac{r_2}{r_1} = -\frac{2\pi K}{Q} \left[\frac{h_2^2 - h_1^2}{2}\right].$$

Converting to common logarithms and solving for K,

$$K = -\frac{2.30Q \log_{10} r_2/r_1}{\pi(h_2^2 - h_1^2)} \qquad [LT^{-1}]. \qquad (31)$$

In confined aquifers (where there is no unwatering) or in thick unconfined aquifers (where s is negligible compared to b), $h_2 + h_1$ may be assumed equal to $2b$. Then, as $h_2^2 - h_1^2 = (h_2 + h_1)(h_2 - h_1)$, $h_2 - h_1 = s_1 - s_2$, and $T = \bar{K}b$, equation 31 may be rewritten

$$T=-\frac{2.30Q\log_{10}r_2/r_1}{2\pi(s_1-s_2)}\quad[L^2T^{-1}].\tag{32}$$

Equations 31 and 32 are forms of the Thiem equation, and later it will be shown how equation 32 can be derived also as a special solution of the nonsteady flow equation.

In thin unconfined aquifers, in which s is an appreciable proportion of b, Jacob (1963a) showed how to correct the drawdowns (s_1 and s_2) to the values that would have been observed had there been no diminution in saturated thickness (as in a confined aquifer of thickness b). Note from figure 10 that $h_2=b-s_2$, and $h_1=b-s_1$. Substituting these values in equation 31 and, for convenience, multiplying both sides of the equation by $2b$,

$$K=-\frac{2.30Q\log_{10}r_2/r_1}{2\pi b\left[\frac{(b^2-2bs_2+s_2^2)-(b^2-2bs_1+s_1^2)}{2b}\right]}$$

and

$$T=-\frac{2.30Q\log_{10}r_2/r_1}{2\pi[(s_1-s_1^2/2b)-(s_2-s_2^2/2b)]}\quad[L^2T^{-1}].\tag{33}$$

In equations 32 and 33 note that a straight line should result when values of $\log_{10}r$ are plotted at logarithmic scale against corresponding values of s or $s-s^2/2b$ at arithmetic scale. Thus, when using semilogarithmic paper, equations 32 and 33 may be written, respectively,

$$T=-\frac{2.30Q}{2\pi\Delta s/\Delta\log_{10}r}\quad[L^2T^{-1}]\tag{34}$$

and

$$T=-\frac{2.30Q}{2\pi\Delta(s-s^2/2b)/\Delta\log_{10}r}\quad[L^2T^{-1}].\tag{35}$$

Δs or $\Delta(s-s^2/2b)$ is taken over one $\log_{10}$ cycle of r, for which $\Delta\log_{10}r$ is 1. Equation 34 is used for tests in confined aquifers.

To my knowledge the first pumping test by the Thiem method made by a member of the U.S. Geological Survey was in 1929 by R. M. Leggette (1936, p. 117–119) at Meadville, Pa., and this may well have been the first one made in the United States. The method was thoroughly investigated and validated in 1931 by Wenzel (1936), who ran two elaborate Thiem tests near Grand Island, Nebr., and by Theis (1932, p. 137–140), who made a test by this method at Portales, N. Mex., in 1931 and two more tests in the same area in 1932 (Theis, 1934, p. 91–95). The fourth locality tested by this method was near Elizabeth City, N.C., by me (Lohman, 1936, p. 42–44) in the spring of 1933. Three additional Thiem tests were made in Nebraska in the fall of 1933 by Wenzel (1942), and I made an 18-day test, the longest known to me, by this method near Wichita, Kans., in 1937. (See Wenzel, 1942, p. 142–146; Williams and Lohman, 1949, p. 104–108; Jacob, 1963a, p. 249–254.)

Example

Use of the Thiem method as modified by Jacob for thin unconfined aquifers may be demonstrated from data (table 4) obtained in the 18-day pumping test mentioned above (given in Wenzel, 1942, p. 142; Williams and Lohman, 1949, p. 104–108; and Jacob, 1963a, p. 249–254). The discharge, Q, was held virtually constant at $1{,}000\pm7$ gpm for nearly 19 days, when lightening tripped the circuit breaker and stopped the test. The well tapped unconfined alluvial sand and gravel, and the initial saturated thickness, b, was 26.8 ft. Table 4 gives the data for six observation wells out of a total of 22 at the end of 18 days; three of the wells were on a line extending north from the pumped well, and three were on a line to the south. Wells N–3 and S–3 were planned to be 200 ft from the pumped well, but a property line made it necessary to reduce the distance to 190 ft. Inasmuch as the drawdown correction ($s^2/2b$) ranges from about 0.2 to 0.65 ft, it is necessary to use the corrected drawdowns given in the last column. In figure 11, the corrected drawdowns in the six observation wells are plotted on the linear scale against the radial distance on the logarithmic scale, then a straight line is drawn through the three graphic average points (x). Graphic rather than arithmetic averages should be used, because if one of the six drawdown values is spurious for some geologic or other reason, this point may be ignored in drawing the straight line. Note that, although two points determine a straight line, it is much more conforting to have three or more points that fall on or close to a straight line.

Using the slope of the straight line and other data given, T is computed from equation 35:

$$T=-\frac{(2.30)(1{,}000\text{ gal min}^{-1})(1{,}440\text{ min day}^{-1})}{(2\pi)(7.48\text{ gal ft}^{-3})[-(4.05\text{ ft}-0.65\text{ ft})]}$$
$$=20{,}700\text{ ft}^2\text{ day}^{-1}.\tag{36}$$

TABLE 4.—*Data for pumping test near Wichita, Kans.*

Line	Well	r (ft)	r^2 (ft²)	s (ft)	$s^2/2b$ (ft)	$s-s^2/2b$ (ft)
North	1	49.2	2,420	5.91	0.65	5.26
	2	100.7	10,140	4.58	.39	4.19
	3	189.4	35,900	3.42	.22	3.20
South	1	49.0	2,400	5.48	.56	4.92
	2	100.4	10,080	4.31	.35	3.96
	3	190.0	36,100	3.19	.19	3.00

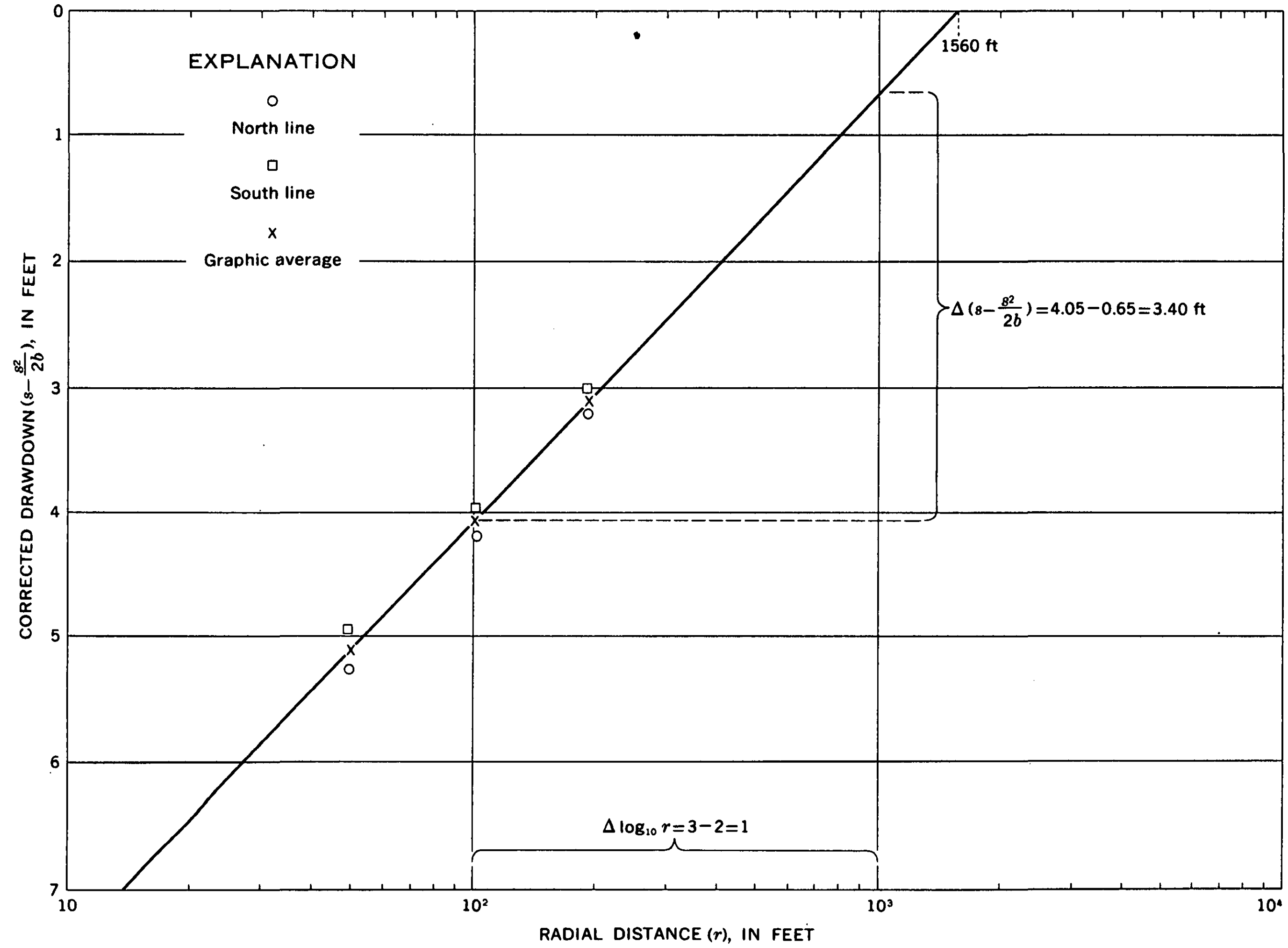

FIGURE 11.—Semilogarithmic plot of corrected drawdowns versus radial distance for aquifer test near Wichita, Kans.

Later (see "Storage Coefficient"), it will be shown how figure 11 may be used to determine the storage coefficient. Column 4 (r^2) is included in table 4 so that the data may be used also in the Theis equation, which gives the same value for T as equation 36, thus indicating that steady flow had been closely approached as far away as 190 ft after 18 days of pumping.

A good arrangement of observation wells for aquifer tests by the Thiem method, particularly for thin unconfined aquifers, was suggested to me by the late C. E. Jacob (written commun., Jan. 28, 1946) and was used successfully in 39 tests in the San Luis Valley, Colo. (Powell, 1958, table 6, p. 130–133). Three pairs of observation wells are put down along a straight line on one side of the pumped well extending in any convenient direction from the pumped well and spaced at distances of $1b$, $2b$, and $4b$ from the well (where b is the initial saturated thickness of the unconfined aquifer). One observation well of each pair is cased to the bottom of the aquifer; the other extends just below the cone of depression created by the pumped well. The drawdowns or corrected drawdowns in the six observation wells are plotted on semilogarithmic paper, and graphic averages are used to determine the position and slope of the straight line, as shown in figure 11. This arrangement is an effective means of correcting for partial penetration (see p. 35) of the aquifer by the pumped well and for local inhomogeneities along this line in the aquifer. (See also Jacob, 1936.)

PARTIAL DIFFERENTIAL EQUATIONS FOR RADIAL FLOW

Figure 12 represents two cylindrical sections of a confined aquifer of thickness b and radii r and $r+dr$, respectively, from which a central well is discharging at constant rate Q. Let the gradient across the annular cylindrical section of infinitesimal thickness dr, between points h_2 and h_1 on the potentiometric surface, be $\partial h/\partial r$. Then, according to R. W. Stallman (written commun., Feb. 1967),

$$\frac{\partial V}{\partial t} = -\frac{\partial Q}{\partial r}dr = 2\pi r S\frac{\partial h}{\partial t}dr \quad [L^3T^{-1}], \quad (37)$$

in which

$\frac{\partial V}{\partial t}$ = change in volume of water between h_2 and h_1, with time,

$\frac{\partial Q}{\partial r}$ = change in rate of flow between h_2 and h_1, with distance,

$\frac{\partial h}{\partial t}$ = change in head between h_2 and h_1, with time, and

S = storage coefficient.

The expression of Darcy's law in equation 26 may be altered to the form

$$Q = -2\pi Tr\frac{\partial h}{\partial r} \quad [L^3T^{-1}], \quad (38)$$

in which $T=\bar{K}b$, b replaces h, and $\partial h/\partial r$, the partial derivative, replaces dh/dr. Differentiating equation 38 with respect to ∂r,

$$\frac{\partial Q}{\partial r} = -2\pi T\left[\frac{\partial\left(r\frac{\partial h}{\partial r}\right)}{\partial r}\right]$$

$$= -2\pi T\left(\frac{\partial r}{\partial r}\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right) \quad [L^2T^{-1}]. \quad (39)$$

Combining equations 37 and 39, we obtain

$$2\pi T\left(\frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right) = 2\pi r S\frac{\partial h}{\partial t}.$$

Dividing both sides of this equation by $2\pi Tr$, we obtain

$$\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{T}\frac{\partial h}{\partial t} \quad [L^{-1}], \quad (40)$$

which is the partial differential equation for nonsteady radial flow. For steady radial flow, $\partial h/\partial t=0$, and equation 40 becomes

$$\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = 0 \quad [L^{-1}]. \quad (41)$$

Note that when $\partial h/\partial t=0$, the entire right-hand member of equation 40 is zero; this indicates that there are no changes in storage in the aquifer. Equation 41 may be expressed also in ordinary differentials.

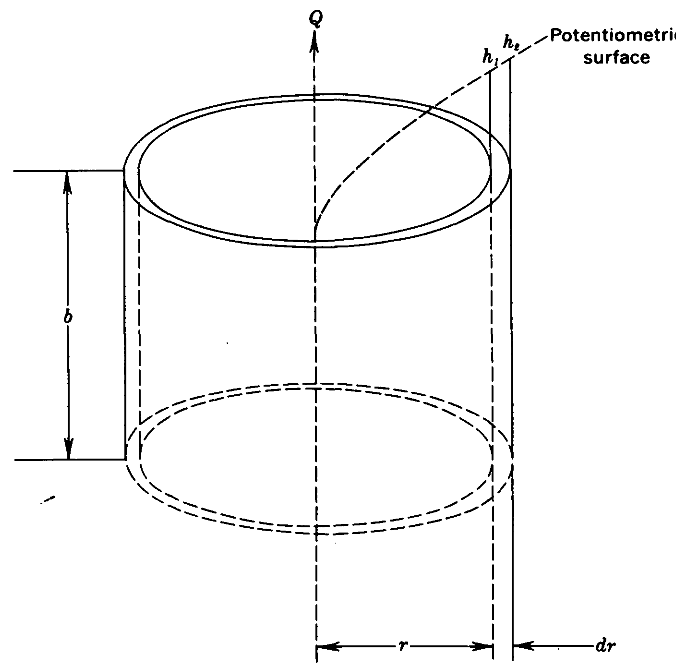

FIGURE 12.—Cylindrical sections of a confined aquifer.

For the benefit of those who have difficulty in visualizing the meaning of the differential terms in equation 41, let us multiply both sides of this equation by r to reduce it to the dimensionless form

$$\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}r = 0. \quad (42)$$

In figure 13, the curve represents a part of the cross section

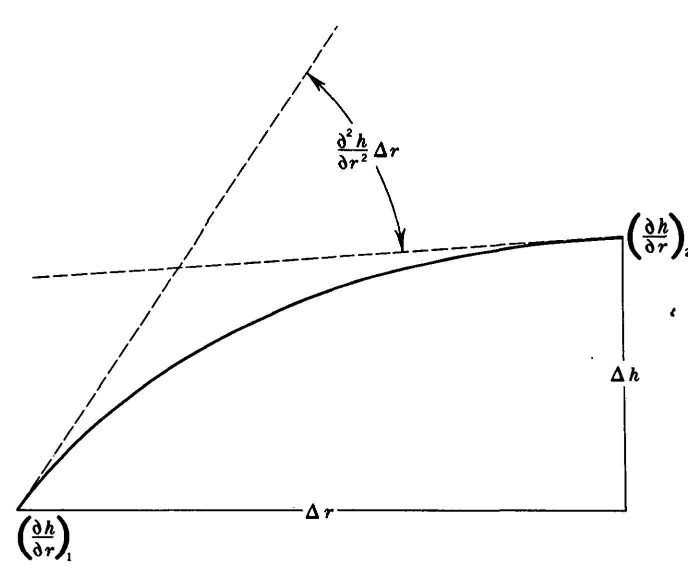

FIGURE 13.—Sketch to illustrate partial differential equation for steady radial flow.

of a cone of depression in which steady radial flow has been reached. Let

$$\left(\frac{\partial h}{\partial r}\right)_1 \text{ and } \left(\frac{\partial h}{\partial r}\right)_2$$

be the slopes of the curve at each end of radial distance Δr; then note that the difference in the two slopes, represented by dashed tangent lines, is graphically equal to the change in slope over distance Δr, represented by the arc labeled

$$\frac{\partial^2 h}{\partial r^2}\,\Delta r.$$

Stated mathematically,

$$\left(\frac{\partial h}{\partial r}\right)_1 - \left(\frac{\partial h}{\partial r}\right)_2 = \frac{\partial^2 h}{\partial r^2}\,\Delta r,$$

or

$$\left(\frac{\partial h}{\partial r}\right)_2 - \left(\frac{\partial h}{\partial r}\right)_1 + \frac{\partial^2 h}{\partial r^2}\,\Delta r = 0. \qquad (43)$$

Equation 43 is equivalent in form to equation 42 when applied over distance r.

I have not found a practicable way to portray graphically the meaning of equation 40, but it may help to note that the

$$S\,\frac{\partial h}{\partial t}$$

in the right-hand member represents the change in storage per unit area of the aquifer, as the head changes with time.

NONSTEADY RADIAL FLOW WITHOUT VERTICAL MOVEMENT

CONSTANT DISCHARGE

In 1935 C. V. Theis introduced equation 19 with the assistance of C. I. Lubin, who developed the equation for a continuous point source for the heat conduction problem. Equation 19 is a solution of equation 40 for constant discharge that involves the following assumptions, stated by Theis (1935): (1) the aquifer is homogeneous and isotropic, (2) the water body has infinite areal extent (practically its boundaries are beyond the effects of the well in the time considered), (3) the discharging well penetrates the entire thickness of the aquifer, (4) the well has an infinitesimal diameter (of no practical significance for periods of pumping longer than a few minutes) and (5) the water removed from storage is discharged instantaneously with decline in head. Thus, the assumption of a constant coefficient of storage has been added to the assumptions of homogeneity, isotropy, and complete well penetration which characterize the steady state equations that have been given so far. The assumption of a constant coefficient of storage, which is used in all the transient flow equations that have been developed (there are a few exceptions where modifications of the assumption are explicitly stated), is of doubtful validity, especially when applied to unconfined water bodies. The justification for this assumption is entirely empirical; it has been applied with some success for some decades, and deviations from it involve generally complex numerical computations. The student should be wary of many solutions for Darcian flow that do not explicitly state the tacit assumptions made.

Equation 19 cannot be integrated directly, but its value is given by the infinite series in the following equation:

$$s = \frac{Q}{4\pi T}\left[-0.577216 - \log_e u + u - \frac{u^2}{2\cdot 2!} + \frac{u^3}{3\cdot 3!} - \cdots\right] \qquad [L], \quad (44)$$

where

$$u = \frac{r^2 S}{4Tt} \qquad [\text{dimensionless}], \qquad (45)$$

which is the lower limit of integration in equation 19; the value of the series is commonly expressed as $W(u)$—the well function of u. Values of $W(u)$ for values of u from 10^{-15} to 9.9 are tabulated in Wenzel (1942, p. 89), in Ferris, Knowles, Brown, and Stallman (1962, p. 96, 97), and are given in table 5. For given values of u and $W(u)$, T may be determined from

$$T = \frac{Q}{4\pi s}\,W(u) \qquad [L^2T^{-1}], \qquad (46)$$

and S may be determined by rewriting equation 45,

$$S = \frac{4Ttu}{r^2} \quad \text{or} \quad \frac{4Tu}{r^2/t} \qquad [\text{dimensionless}]. \qquad (47)$$

C. V. Theis (Wenzel, 1942, p. 88, 89) devised a simple graphical method of superposition that makes it possible to obtain solutions of equations 46 and 47. Selected values of $W(u)$ versus u from table 5 were plotted on logarithmic graph paper to form the type curve shown in figure 14. Equations 46 and 47 may be rearranged to obtain

$$s = \left[\frac{Q}{4\pi T}\right] W(u)$$

or

$$\log_{10} s = \left[\log_{10}\frac{Q}{4\pi T}\right] + \log_{10} W(u) \qquad [L], \qquad (48)$$

TABLE 5.—*Values of* W(u) *for values of* u *between* 10^{-15} *and 9.9*

[From Ferris, Knowles, Brown, and Stallman (1962, p. 96)]

N \ u	$N\times10^{-15}$	$N\times10^{-14}$	$N\times10^{-13}$	$N\times10^{-12}$	$N\times10^{-11}$	$N\times10^{-10}$	$N\times10^{-9}$	$N\times10^{-8}$	$N\times10^{-7}$	$N\times10^{-6}$	$N\times10^{-5}$	$N\times10^{-4}$	$N\times10^{-3}$	$N\times10^{-2}$	$N\times10^{-1}$	N
1.0	33.9616	31.6590	29.3564	27.0538	24.7512	22.4486	20.1460	17.8435	15.5409	13.2383	10.9357	8.6332	6.3315	4.0379	1.8229	0.2194
1.1	33.8662	31.5637	29.2611	26.9585	24.6559	22.3533	20.0507	17.7482	15.4456	13.1430	10.8404	8.5379	6.2363	3.9436	1.7371	.1860
1.2	33.7792	31.4767	29.1741	26.8715	24.5689	22.2663	19.9637	17.6611	15.3586	13.0560	10.7534	8.4509	6.1494	3.8576	1.6595	.1584
1.3	33.6992	31.3966	29.0940	26.7914	24.4889	22.1863	19.8837	17.5811	15.2785	12.9759	10.6734	8.3709	6.0695	3.7785	1.5889	.1355
1.4	33.6251	31.3225	29.0199	26.7173	24.4147	22.1122	19.8096	17.5070	15.2044	12.9018	10.5993	8.2968	5.9955	3.7054	1.5241	.1162
1.5	33.5561	31.2535	28.9509	26.6483	24.3458	22.0432	19.7406	17.4380	15.1354	12.8328	10.5303	8.2278	5.9266	3.6374	1.4645	.1000
1.6	33.4916	31.1890	28.8864	26.5838	24.2812	21.9786	19.6760	17.3735	15.0709	12.7683	10.4657	8.1634	5.8621	3.5739	1.4092	.08631
1.7	33.4309	31.1283	28.8258	26.5232	24.2206	21.9180	19.6154	17.3128	15.0103	12.7077	10.4051	8.1027	5.8016	3.5143	1.3578	.07465
1.8	33.3738	31.0712	28.7686	26.4660	24.1634	21.8608	19.5583	17.2557	14.9531	12.6505	10.3479	8.0455	5.7446	3.4581	1.3089	.06471
1.9	33.3197	31.0171	28.7145	26.4119	24.1094	21.8068	19.5042	17.2016	14.8990	12.5964	10.2939	7.9915	5.6906	3.4050	1.2649	.05620
2.0	33.2684	30.9658	28.6632	26.3607	24.0581	21.7555	19.4529	17.1503	14.8477	12.5451	10.2426	7.9402	5.6394	3.3547	1.2227	.04890
2.1	33.2196	30.9170	28.6145	26.3119	24.0093	21.7067	19.4041	17.1015	14.7989	12.4964	10.1938	7.8914	5.5907	3.3069	1.1829	.04261
2.2	33.1731	30.8705	28.5679	26.2653	23.9628	21.6602	19.3576	17.0550	14.7524	12.4498	10.1473	7.8449	5.5443	3.2614	1.1454	.03719
2.3	33.1286	30.8261	28.5235	26.2209	23.9183	21.6157	19.3131	17.0106	14.7080	12.4054	10.1028	7.8004	5.4999	3.2179	1.1099	.03250
2.4	33.0861	30.7835	28.4809	26.1783	23.8758	21.5732	19.2706	16.9680	14.6654	12.3628	10.0603	7.7579	5.4575	3.1763	1.0762	.02844
2.5	33.0453	30.7427	28.4401	26.1375	23.8349	21.5323	19.2298	16.9272	14.6246	12.3220	10.0194	7.7172	5.4167	3.1365	1.0443	.02491
2.6	33.0060	30.7035	28.4009	26.0983	23.7957	21.4931	19.1905	16.8880	14.5854	12.2828	9.9802	7.6779	5.3776	3.0983	1.0139	.02185
2.7	32.9683	30.6657	28.3631	26.0606	23.7580	21.4554	19.1528	16.8502	14.5476	12.2450	9.9425	7.6401	5.3400	3.0615	.9849	.01918
2.8	32.9319	30.6294	28.3268	26.0242	23.7216	21.4190	19.1164	16.8138	14.5113	12.2087	9.9061	7.6038	5.3037	3.0261	.9573	.01686
2.9	32.8968	30.5943	28.2917	25.9891	23.6865	21.3839	19.0813	16.7788	14.4762	12.1736	9.8710	7.5687	5.2687	2.9920	.9309	.01482
3.0	32.8629	30.5604	28.2578	25.9552	23.6526	21.3500	19.0474	16.7449	14.4423	12.1397	9.8371	7.5348	5.2349	2.9591	.9057	.01305
3.1	32.8302	30.5276	28.2250	25.9224	23.6198	21.3172	19.0146	16.7121	14.4095	12.1069	9.8043	7.5020	5.2022	2.9273	.8815	.01149
3.2	32.7984	30.4958	28.1932	25.8907	23.5880	21.2855	18.9829	16.6803	14.3777	12.0751	9.7726	7.4703	5.1706	2.8965	.8583	.01013
3.3	32.7676	30.4651	28.1625	25.8599	23.5573	21.2547	18.9521	16.6495	14.3470	12.0444	9.7418	7.4395	5.1399	2.8668	.8361	.008939
3.4	32.7378	30.4352	28.1326	25.8300	23.5274	21.2249	18.9223	16.6197	14.3171	12.0145	9.7120	7.4097	5.1102	2.8379	.8147	.007891
3.5	32.7088	30.4062	28.1036	25.8010	23.4985	21.1959	18.8933	16.5907	14.2881	11.9855	9.6830	7.3807	5.0813	2.8099	.7942	.006970
3.6	32.6806	30.3780	28.0755	25.7729	23.4703	21.1677	18.8651	16.5625	14.2599	11.9574	9.6548	7.3526	5.0532	2.7827	.7745	.006160
3.7	32.6532	30.3506	28.0481	25.7455	23.4429	21.1403	18.8377	16.5351	14.2325	11.9300	9.6274	7.3252	5.0259	2.7563	.7554	.005448
3.8	32.6266	30.3240	28.0214	25.7188	23.4162	21.1136	18.8110	16.5085	14.2059	11.9033	9.6007	7.2985	4.9993	2.7306	.7371	.004820
3.9	32.6006	30.2980	27.9954	25.6928	23.3902	21.0877	18.7851	16.4825	14.1799	11.8773	9.5748	7.2725	4.9735	2.7056	.7194	.004267
4.0	32.5753	30.2727	27.9701	25.6675	23.3649	21.0623	18.7598	16.4572	14.1546	11.8520	9.5495	7.2472	4.9482	2.6813	.7024	.003779
4.1	32.5506	30.2480	27.9454	25.6428	23.3402	21.0376	18.7351	16.4325	14.1299	11.8273	9.5248	7.2225	4.9236	2.6576	.6859	.003349
4.2	32.5265	30.2239	27.9213	25.6187	23.3161	21.0136	18.7110	16.4084	14.1058	11.8032	9.5007	7.1985	4.8997	2.6344	.6700	.002969
4.3	32.5029	30.2004	27.8978	25.5952	23.2926	20.9900	18.6874	16.3884	14.0823	11.7797	9.4771	7.1749	4.8762	2.6119	.6546	.002633
4.4	32.4800	30.1774	27.8748	25.5722	23.2696	20.9670	18.6644	16.3619	14.0593	11.7567	9.4541	7.1520	4.8533	2.5899	.6397	.002336
4.5	32.4575	30.1549	27.8523	25.5497	23.2471	20.9446	18.6420	16.3394	14.0368	11.7342	9.4317	7.1295	4.8310	2.5684	.6253	.002073
4.6	32.4355	30.1329	27.8303	25.5277	23.2252	20.9226	18.6200	16.3174	14.0148	11.7122	9.4097	7.1075	4.8091	2.5474	.6114	.001841
4.7	32.4140	30.1114	27.8088	25.5062	23.2037	20.9011	18.5985	16.2959	13.9933	11.6907	9.3882	7.0860	4.7877	2.5268	.5979	.001635
4.8	32.3929	30.0904	27.7878	25.4852	23.1826	20.8800	18.5774	16.2748	13.9723	11.6697	9.3671	7.0650	4.7667	2.5068	.5848	.001453
4.9	32.3723	30.0697	27.7672	25.4646	23.1620	20.8594	18.5568	16.2542	13.9516	11.6491	9.3465	7.0444	4.7462	2.4871	.5721	.001291
5.0	32.3521	30.0495	27.7470	25.4444	23.1418	20.8392	18.5366	16.2340	13.9314	11.6289	9.3263	7.0242	4.7261	2.4679	.5598	.001148
5.1	32.3323	30.0297	27.7271	25.4246	23.1220	20.8194	18.5168	16.2142	13.9116	11.6091	9.3065	7.0044	4.7064	2.4491	.5478	.001021
5.2	32.3129	30.0103	27.7077	25.4051	23.1026	20.8000	18.4974	16.1948	13.8922	11.5896	9.2871	6.9850	4.6871	2.4306	.5362	.0009086
5.3	32.2939	29.9913	27.6887	25.3861	23.0835	20.7809	18.4783	16.1758	13.8732	11.5706	9.2681	6.9659	4.6681	2.4126	.5250	.0008086
5.4	32.2752	29.9726	27.6700	25.3674	23.0648	20.7622	18.4596	16.1571	13.8545	11.5519	9.2494	6.9473	4.6495	2.3948	.5140	.0007198
5.5	32.2568	29.9542	27.6516	25.3491	23.0465	20.7439	18.4413	16.1387	13.8361	11.5336	9.2310	6.9289	4.6313	2.3775	.5034	.0006409
5.6	32.2388	29.9362	27.6336	25.3310	23.0285	20.7259	18.4233	16.1207	13.8181	11.5155	9.2130	6.9109	4.6134	2.3604	.4930	.0005708
5.7	32.2211	29.9185	27.6159	25.3133	23.0108	20.7082	18.4056	16.1030	13.8004	11.4978	9.1953	6.8932	4.5958	2.3437	.4830	.0005085
5.8	32.2037	29.9011	27.5985	25.2959	22.9934	20.6908	18.3882	16.0856	13.7830	11.4804	9.1779	6.8758	4.5785	2.3273	.4732	.0004532
5.9	32.1866	29.8840	27.5814	25.2789	22.9763	20.6737	18.3711	16.0685	13.7659	11.4633	9.1608	6.8588	4.5615	2.3111	.4637	.0004039
6.0	32.1698	29.8672	27.5646	25.2620	22.9595	20.6569	18.3543	16.0517	13.7491	11.4465	9.1440	6.8420	4.5448	2.2953	.4544	.0003601
6.1	32.1533	29.8507	27.5481	25.2455	22.9429	20.6403	18.3378	16.0352	13.7326	11.4300	9.1275	6.8254	4.5283	2.2797	.4454	.0003211
6.2	32.1370	29.8344	27.5318	25.2293	22.9267	20.6241	18.3215	16.0189	13.7163	11.4138	9.1112	6.8092	4.5122	2.2645	.4366	.0002864
6.3	32.1210	29.8184	27.5158	25.2133	22.9107	20.6081	18.3055	16.0029	13.7003	11.3978	9.0952	6.7932	4.4963	2.2494	.4280	.0002555
6.4	32.1053	29.8027	27.5001	25.1975	22.8949	20.5923	18.2898	15.9872	13.6846	11.3820	9.0795	6.7775	4.4806	2.2346	.4197	.0002279
6.5	32.0898	29.7872	27.4846	25.1820	22.8794	20.5768	18.2742	15.9717	13.6691	11.3665	9.0640	6.7620	4.4652	2.2201	.4115	.0002034
6.6	32.0745	29.7719	27.4693	25.1667	22.8641	20.5616	18.2590	15.9564	13.6538	11.3512	9.0487	6.7467	4.4501	2.2058	.4036	.0001816
6.7	32.0595	29.7569	27.4543	25.1517	22.8491	20.5465	18.2439	15.9414	13.6388	11.3362	9.0337	6.7317	4.4351	2.1917	.3959	.0001621
6.8	32.0446	29.7421	27.4395	25.1369	22.8343	20.5317	18.2291	15.9265	13.6240	11.3214	9.0189	6.7169	4.4204	2.1779	.3883	.0001448
6.9	32.0300	29.7275	27.4249	25.1223	22.8197	20.5171	18.2145	15.9119	13.6094	11.3608	9.0043	6.7023	4.4059	2.1643	.3810	.0001293
7.0	32.0156	29.7131	27.4105	25.1079	22.8053	20.5027	18.2001	15.8976	13.5950	11.2924	8.9899	6.6879	4.3916	2.1508	.3738	.0001155
7.1	32.0015	29.6989	27.3963	25.0937	22.7911	20.4885	18.1860	15.8834	13.5808	11.2782	8.9757	6.6737	4.3775	2.1376	.3668	.0001032
7.2	31.9875	29.6849	27.3823	25.0797	22.7771	20.4746	18.1720	15.8694	13.5668	11.2642	8.9617	6.6598	4.3636	2.1246	.3599	.00009219
7.3	31.9737	29.6711	27.3685	25.0659	22.7633	20.4608	18.1582	15.8556	13.5530	11.2504	8.9479	6.6460	4.3500	2.1118	.3532	.00008239
7.4	31.9601	29.6575	27.3549	25.0523	22.7497	20.4472	18.1446	15.8420	13.5394	11.2368	8.9343	6.6324	4.3364	2.0991	.3467	.00007364
7.5	31.9467	29.6441	27.3415	25.0389	22.7363	20.4337	18.1311	15.8286	13.5260	11.2234	8.9209	6.6190	4.3231	2.0867	.3403	.00006583
7.6	31.9334	29.6308	27.3282	25.0257	22.7231	20.4205	18.1179	15.8153	13.5127	11.2102	8.9076	6.6057	4.3100	2.0744	.3341	.00005886
7.7	31.9203	29.6178	27.3152	25.0126	22.7100	20.4074	18.1048	15.8022	13.4997	11.1971	8.8946	6.5927	4.2970	2.0623	.3280	.00005263
7.8	31.9074	29.6048	27.3023	24.9997	22.6971	20.3945	18.0919	15.7893	13.4868	11.1842	8.8817	6.5798	4.2842	2.0503	.3221	.00004707
7.9	31.8947	29.5921	27.2895	24.9869	22.6844	20.3818	18.0792	15.7766	13.4740	11.1714	8.8689	6.5671	4.2716	2.0386	.3163	.00004210
8.0	31.8821	29.5795	27.2769	24.9744	22.6718	20.3692	18.0666	15.7640	13.4614	11.1589	8.8563	6.5545	4.2591	2.0269	.3106	.00003767
8.1	31.8697	29.5671	27.2645	24.9619	22.6594	20.3568	18.0542	15.7516	13.4490	11.1464	8.8439	6.5421	4.2468	2.0155	.3050	.00003370
8.2	31.8574	29.5548	27.2523	24.9497	22.6471	20.3445	18.0419	15.7393	13.4367	11.1342	8.8317	6.5298	4.2346	2.0042	.2996	.00003015
8.3	31.8453	29.5427	27.2401	24.9375	22.6350	20.3324	18.0298	15.7272	13.4246	11.1220	8.8195	6.5177	4.2226	1.9930	.2943	.00002699
8.4	31.8333	29.5307	27.2282	24.9256	22.6230	20.3204	18.0178	15.7152	13.4126	11.1101	8.8076	6.5057	4.2107	1.9820	.2891	.00002415
8.5	31.8215	29.5189	27.2163	24.9137	22.6112	20.3086	18.0060	15.7034	13.4008	11.0982	8.7957	6.4939	4.1990	1.9711	.2840	.00002162
8.6	31.8098	29.5072	27.2046	24.9020	22.5995	20.2969	17.9943	15.6917	13.3891	11.0865	8.7840	6.4822	4.1874	1.9604	.2790	.00001936
8.7	31.7982	29.4957	27.1931	24.8905	22.5879	20.2853	17.9827	15.6801	13.3776	11.0750	8.7725	6.4707	4.1759	1.9498	.2742	.00001733
8.8	31.7868	29.4842	27.1816	24.8790	22.5765	20.2739	17.9713	15.6687	13.3661	11.0635	8.7610	6.4592	4.1646	1.9393	.2694	.00001552
8.9	31.7755	29.4729	27.1703	24.8678	22.5652	20.2626	17.9600	15.6574	13.3548	11.0523	8.7497	6.4480	4.1534	1.9290	.2647	.00001390
9.0	31.7643	29.4618	27.1592	24.8566	22.5540	20.2514	17.9488	15.6462	13.3437	11.0411	8.7386	6.4368	4.1423	1.9187	.2602	.00001245
9.1	31.7533	29.4507	27.1481	24.8455	22.5429	20.2404	17.9378	15.6352	13.3326	11.0300	8.7275	6.4258	4.1313	1.9087	.2557	.00001115
9.2	31.7424	29.4398	27.1372	24.8346	22.5320	20.2294	17.9268	15.6243	13.3217	11.0191	8.7166	6.4148	4.1205	1.8987	.2513	.000009988
9.3	31.7315	29.4290	27.1264	24.8238	22.5212	20.2186	17.9160	15.6135	13.3109	11.0083	8.7058	6.4040	4.1098	1.8888	.2470	.000008948
9.4	31.7208	29.4183	27.1157	24.8131	22.5105	20.2079	17.9053	15.6028	13.3002	10.9976	8.6951	6.3934	4.0992	1.8791	.2429	.000008018
9.5	31.7103	29.4077	27.1051	24.8025	22.4999	20.1973	17.8948	15.5922	13.2896	10.9870	8.6845	6.3828	4.0887	1.8695	.2387	.000007185
9.6	31.6998	29.3972	27.0946	24.7920	22.4895	20.1869	17.8843	15.5817	13.2791	10.9765	8.6740	6.3723	4.0784	1.8599	.2347	.000006439
9.7	31.6894	29.3868	27.0843	24.7817	22.4791	20.1765	17.8739	15.5713	13.2688	10.9662	8.6637	6.3620	4.0681	1.8505	.2308	.000005771
9.8	31.6792	29.3766	27.0740	24.7714	22.4688	20.1663	17.8637	15.5611	13.2585	10.9559	8.6534	6.3517	4.0579	1.8412	.2269	.000005173
9.9	31.6690	29.3664	27.0639	24.7613	22.4587	20.1561	17.8535	15.5509	13.2483	10.9458	8.6433	6.3416	4.0479	1.8320	.2231	.000004637

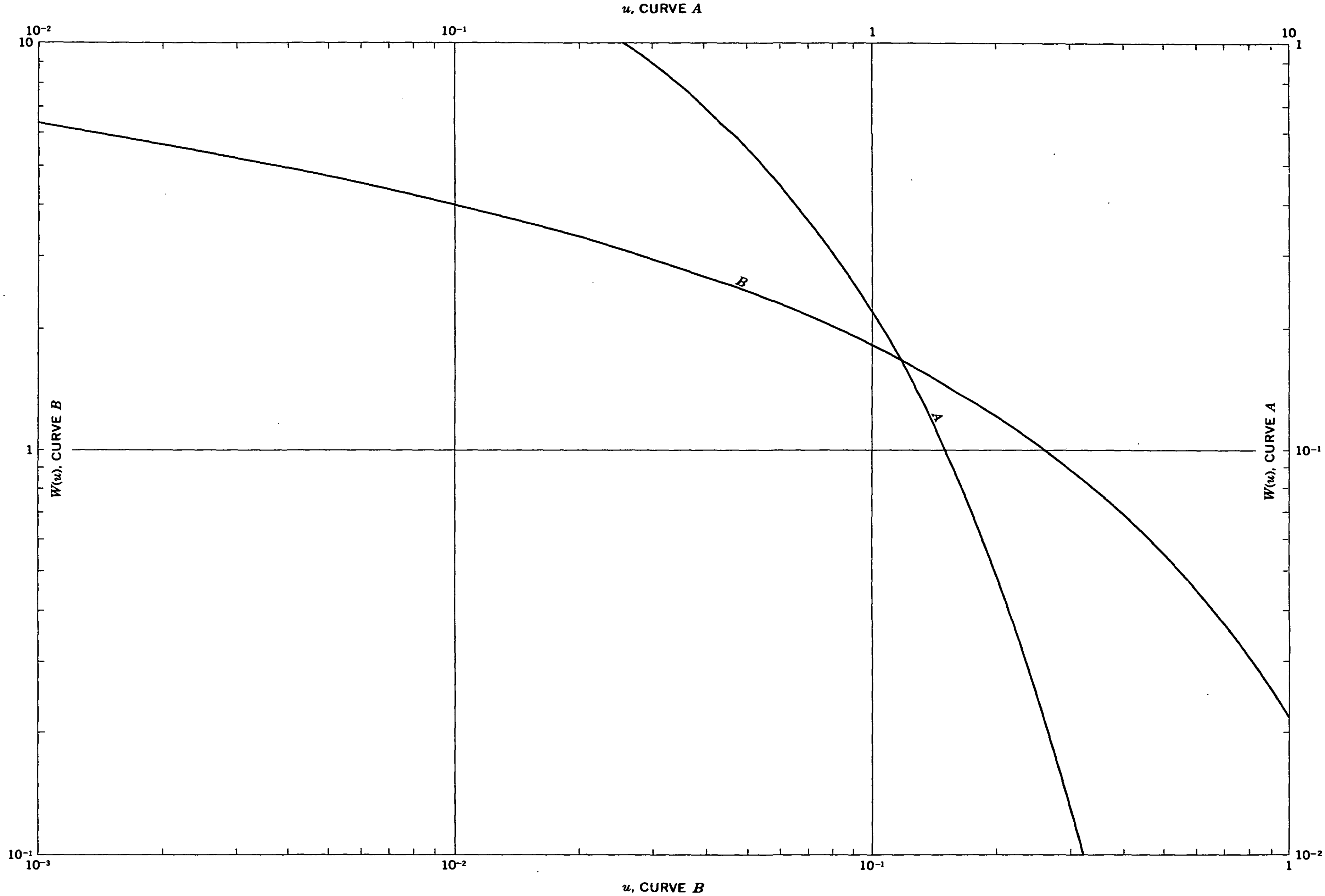

FIGURE 14.—Logarithmic graph of $W(u)$ versus u.

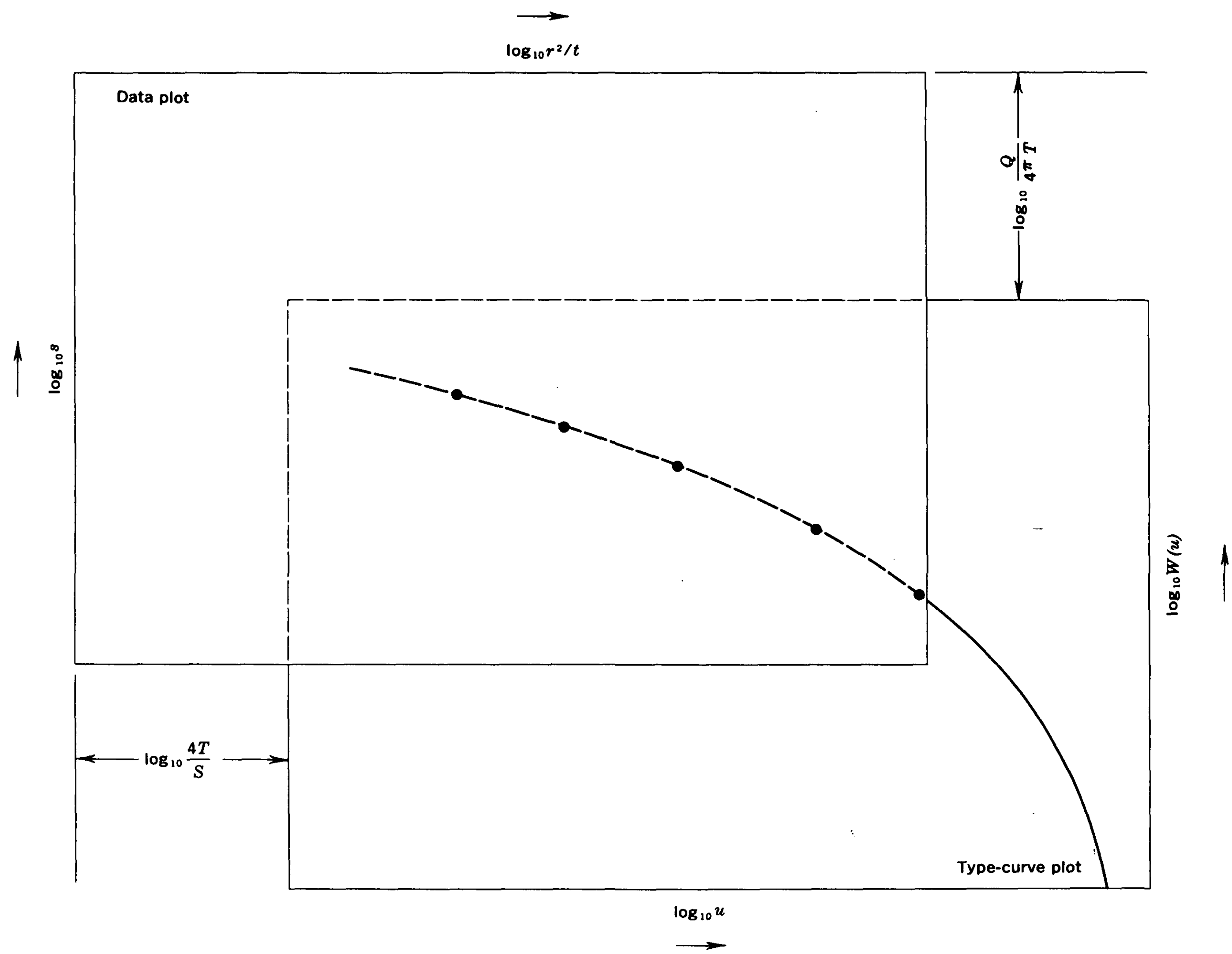

FIGURE 15.—Relation of $W(u)$ and u to s and r^2/t, and displacements of graph scales by amounts of constants shown.

and

$$\frac{r^2}{t} = \left[\frac{4T}{S}\right] u$$

or

$$\log_{10} \frac{r^2}{t} = \left[\log_{10} \frac{4T}{S}\right] + \log_{10} u \qquad [L^2T^{-1}]. \qquad (49)$$

If the discharge, Q, is held constant, the bracketed parts of equations 48 and 49 are constant for a given pumping test, and $W(u)$ is related to u in the manner that s is related to r^2/t, as shown graphically in figure 15. Therefore, if values of s are plotted against r^2/t (or $1/t$ if only one observation well is used) on logarithmic tracing paper to the same scale as the type curve, the data curve will be similar to the type curve except that the two curves will be displaced both vertically and horizontally by the amounts of the bracketed constants in equations 48 and 49. The data curve is superimposed on the type curve, and a fit, or near fit, is obtained, keeping the coordinate axes of the two curves parallel. An arbitrary match point is selected anywhere on the overlapping parts of the two sheets, the four values of which (two for each sheet) are then used in solving equations 46 and 47. It is convenient to choose a point whose coordinates on the type curve are both unity—that is, where $W(u) = 1.0$ and $u = 1.0$. In some plots it may be desirable to use a power of 10 for one coordinate. (See fig. 16.)

A convenient alternative method is to plot $W(u)$ versus $1/u$ as the type curve; then for the data curve, s may be plotted against t/r^2 (or t, if only one observation well is used). This procedure is illustrated on plate 9, which also may be used for solutions of the Theis equation by superposing plots of t/r^2 or t versus s on the heavy parent type curve.

TABLE 6.—*Drawdown of water level in observation wells N-1, N-2, and N-3 at distance r from well being pumped at constant rate of 96,000 ft³ day⁻¹*

[Logarithmic plot of data, except values preceded by an asterisk, shown in figure 16. Data from J. G. Ferris]

Time since pumping started, t_m (min)	N-1 (r=200 ft) Observed drawdown, s (ft)	r^2/t (ft² day⁻¹)	N-2 (r=400 ft) Observed drawdown, s (ft)	r^2/t (ft² day⁻¹)	N-3 (r=800 ft) Observed drawdown, s (ft)	r^2/t (ft² day⁻¹)
1.0	0.66	5.76×10^7	0.16	2.3×10^8	0.0046	9.23×10^8
1.5	.87	3.84×10^7	.27	1.53×10^8	.02	6.15×10^8
2.0	.99	2.88×10^7	.38	1.15×10^8	.04	4.6×10^8
2.5	1.11	2.30×10^7	.46	9.2×10^7	.07	3.7×10^8
3.0	1.21	1.92×10^7	.53	7.65×10^7	.09	3.1×10^8
4	1.36	1.44×10^7	.67	*5.75×10^7	.16	2.3×10^8
5	1.49	1.15×10^7	.77	4.6×10^7	.22	1.85×10^8
6	1.59	9.6×10^6	.87	*3.82×10^7	.27	1.54×10^8
8	1.75	7.2×10^6	.99	*2.87×10^7	.37	1.15×10^8
10	1.86	5.76×10^6	1.12	*2.3×10^7	.46	*9.23×10^7
12	1.97	4.80×10^6	1.21	1.92×10^7	.53	*7.7×10^7
14	2.08	4.1×10^6	1.26	1.75×10^7	.59	6.6×10^7
18	2.20	3.2×10^6	1.43	1.28×10^7	.72	5.1×10^7
24	2.36	2.4×10^6	1.58	*9.6×10^6	.87	*3.84×10^7
30	2.49	1.92×10^6	1.70	7.65×10^6	.95	3.1×10^7
40	2.65	1.44×10^6	1.88	*5.75×10^6	1.12	*2.3×10^7
50	2.78	1.15×10^6	2.00	4.6×10^6	1.23	*1.85×10^7
60	2.88	9.6×10^5	2.11	3.82×10^6	1.32	1.54×10^7
80	3.04	7.2×10^5	2.24	2.87×10^6	1.49	*1.15×10^7
100	3.16	5.76×10^5	2.38	2.3×10^6	1.62	*9.23×10^6
120	3.28	4.8×10^5	2.49	*1.92×10^6	1.70	*7.7×10^6
150	3.42	3.84×10^5	2.62	1.53×10^6	1.83	6.15×10^6
180	3.51	3.2×10^5	2.72	1.28×10^6	1.94	5.1×10^6
210	3.61	2.74×10^5	2.81	1.1×10^6	2.03	4.4×10^6
240	3.67	2.5×10^5	2.88	*9.6×10^5	2.11	*3.84×10^6

EXAMPLE

Use of equations 46 and 47 for determining T and S by the curve-matching procedure may be demonstrated from the data given in table 6, which gives the drawdowns in water levels in a theoretical confined aquifer at distances of 200, 400, and 800 ft from a well being pumped at the constant rate of 96,000 ft³ day⁻¹. Most of these data are plotted in figure 16 except for values preceded by an asterisk, which would plot too close to adjacent points, and except for values of r^2/t of 10^8 or larger, which would have required 2×4 cycle paper. Superposition of figure 16 on curve B of figure 14 gave the match point shown, whose values are $W(u)=1.0$, $u=10^{-1}$, $s=0.56$ ft, and $r^2/t=2.75\times10^7$ ft² day⁻¹. Using equation 46,

$$T=\frac{(96{,}000\text{ ft}^3\text{ day}^{-1})(1.0)}{(4\pi)(0.56\text{ ft})}$$

$$=13{,}700\text{ ft}^2\text{ day}^{-1}=14{,}000\text{ ft}^2\text{ day}^{-1}\text{ (rounded)}.$$

Using equation 47,

$$S=\frac{(4)(13{,}700\text{ ft}^2\text{ day}^{-1})(10^{-1})}{2.75\times10^7\text{ ft}^2\text{ day}^{-1}}=2\times10^{-4}.$$

STRAIGHT-LINE SOLUTIONS

TRANSMISSIVITY

Cooper and Jacob (1946) showed that for values of $u=r^2S/4Tt\leq$ about 0.01, all but the first two terms between brackets in equation 44 may be neglected. Under these conditions, equation 44 may be closely approximated by

$$s=\frac{Q}{4\pi T}\left[-0.577216-\log_e\frac{r^2S}{4Tt}\right]\quad [L]. \qquad (50)$$

This may be rewritten and simplified;

$$T=\frac{Q}{4\pi s}\left[\log_e 0.562+\log_e\frac{4Tt}{r^2S}\right]$$

$$=\frac{2.30Q}{4\pi s}\log_{10}\frac{2.25Tt}{r^2S}\quad [L^2T^{-1}]. \qquad (51)$$

Note that in equation 51 Q, T, and S are constants, and that either t or $1/r^2$ may be considered a constant. Thus equation 51 may be written

$$T=\frac{2.30Q}{4\pi s}\left[\log_{10}\frac{2.25T}{S}+\log_{10}\frac{t}{r^2}\right]\quad [L^2T^{-1}].$$

By differential calculus, remembering that the derivative of a constant is zero, we may obtain from the above equation

$$T=\frac{2.30Q}{4\pi ds/d\log_{10} t/r^2}\quad [L^2T^{-1}]. \qquad (52)$$

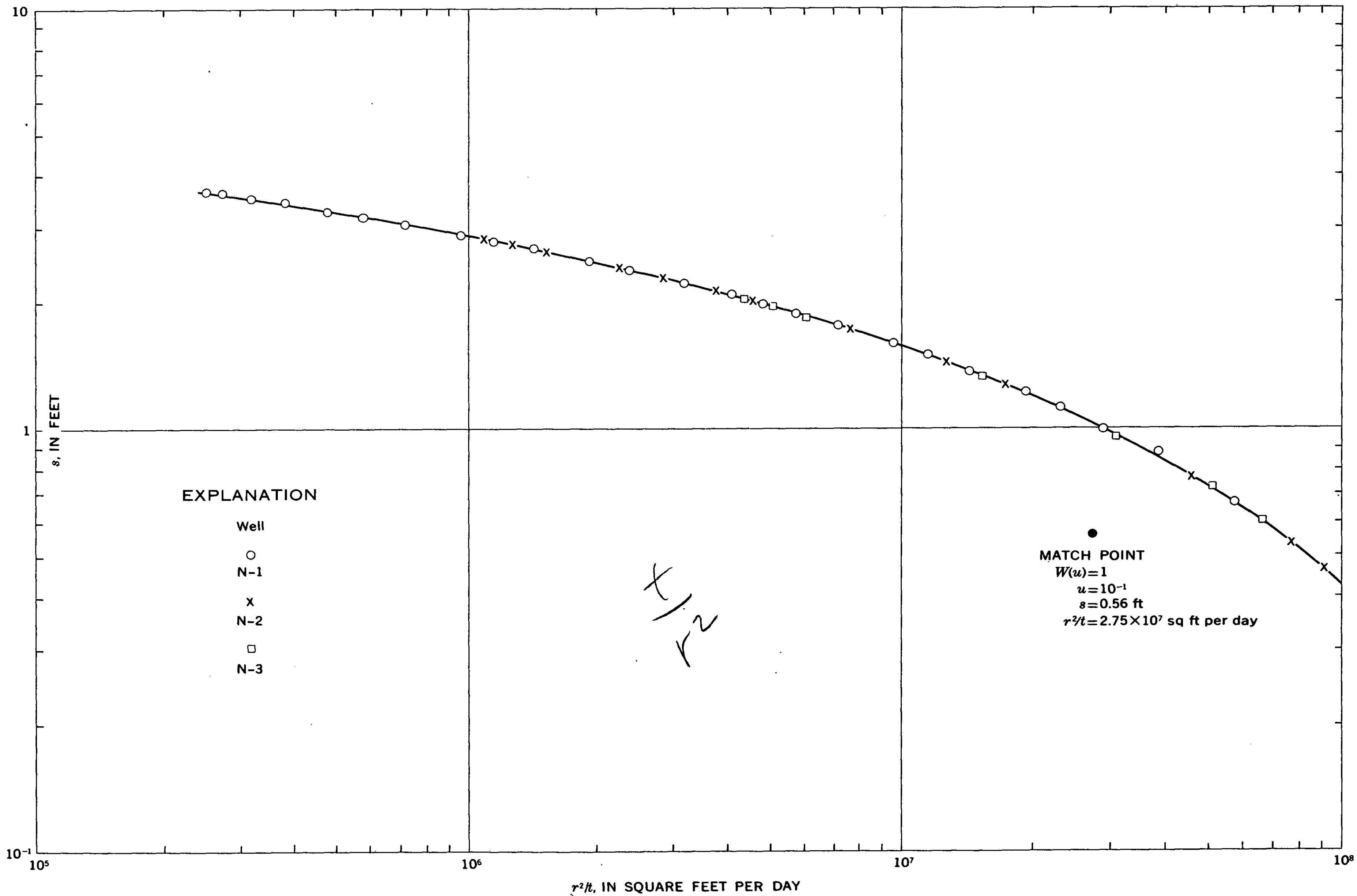

FIGURE 16.—Logarithmic plot of s versus r^2/t from table 6.

Similarly, considering also r as a constant,

$$T=\frac{2.30Q}{4\pi ds/d\log_{10} t} \quad [L^2T^{-1}]. \qquad (53)$$

Considering t as a constant,

$$T=\frac{2.30Q}{4\pi ds/d\log_{10} 1/r^2}=-\frac{2.30Q}{2\pi ds/d\log_{10} r} \quad [L^2T^{-1}]. \qquad (54)$$

As the relation of s to the log terms is linear on semilogarithmic plots, we may change from infinitesimals (derivatives) to finite values, whence equations 52 though 54 become, respectively,

$$T=\frac{2.30Q}{4\pi \Delta s/\Delta\log_{10} t/r^2} \quad [L^2T^{-1}], \qquad (55)$$

$$T=\frac{2.30Q}{4\pi \Delta s/\Delta\log_{10} t} \quad [L^2T^{-1}], \qquad (56)$$

and

$$T=-\frac{2.30Q}{2\pi \Delta s/\Delta\log_{10} r} \quad [L^2T^{-1}]. \qquad (57)$$

Note that equation 57, obtained by simplification and differentiation of equation 51 for nonsteady flow, is identical to equation 34—the steady flow, or Thiem, equation for confined aquifers or for relatively thick unconfined aquifers. In testing relatively thin unconfined aquifers, equation 35, which contains Jacob's drawdown correction, should be used rather than equations 34 or 57, for reasons given earlier.

Equations 55 through 57 are readily solved using semilogarithmic paper, by plotting values of s on the linear scale against corresponding values of the log terms on the logarithmic scale and determining the value of Δs for one log cycle. Figure 11 illustrates the method.

For preparing a composite graph for several observation wells and several wells being pumped at constant but different rates, Q_1, Q_2, Q_3, and so on, the following equation may be used:

$$T=\frac{2.30}{4\pi \Delta(s/Q)/\Delta\log_{10} t/r^2} \quad [L^2T^{-1}]. \qquad (58)$$

In using equation 58 for, say, three pumping wells within an area of overlapping cones of depression, values of s_1/Q_1, t_1/r_1^2; s_2/Q_2, t_2/r_2^2; and s_3/Q_3, t_3/r_3^2 are plotted on the same semilogarithmic graph, and, if u is less than or equal to 0.01 for all values, all points should fall on or near a straight line and may be used in determining a single set of values of T and S for the entire well field.

Cooper and Jacob (1946) used r^2/t rather than t/r^2 in equations 55 and 58, whence a minus sign preceded the right-hand sides of their equations and the slope of the straight line was reversed.

STORAGE COEFFICIENT

In plotting data for the straight-line solutions of equations 52 through 54, it is convenient to consider the top of the arithmetic scale as the line of zero drawdown, as shown on figure 11. The drawdown then increases downward, as in the well. Cooper and Jacob (1946) showed that by extending the straight line until it intersects the line of zero drawdown, and noting the value of t, t/r^2, r^2/t, or r at the point of intersection, the storage coefficient can be determined as follows. At zero drawdown, from equation 50,

$$s=\frac{Q}{4\pi T}\left[\log_e \frac{4Tt}{r^2S}-0.577216\right]=0,$$

$$\log_e \frac{4Tt}{r^2S}=0.577216=\log_e 1.781,$$

$$\frac{4Tt}{r^2S}=1.781,$$

and

$$S=2.25T\left(\frac{t}{r^2}\right)_0 \quad \text{(at point of zero drawdown)}. \qquad (59)$$

Equation 59 gives the desired result satisfactorily if the straight line has sufficient slope so that its zero arithmetic coordinate can be found within the confines of the plot, as in figure 11. However, if the slope is nearly flat, so that the zero arithmetic coordinate occurs outside the confines of the plot, error may result from graphical extrapolation onto an adjoining sheet of graph paper. In order to avoid such errors, I described the following method (Lohman, 1963) for determining S within the data region of straight-line plots without the need for mechanical extrapolation. Equation 51 may be rewritten

$$S=\frac{2.25Tt/r^2}{\log_{10}^{-1}[4\pi Ts/2.30Q]}. \qquad (60)$$

Combining equations 55 and 60 for the value of Δs for one $\log_{10}$ cycle of t/r^2 (unity), we obtain

$$S=\frac{2.25Tt/r^2}{\log_{10}^{-1}[s/\Delta s]}. \qquad (61)$$

Note that in equation 61 the values of T, s, and Δs are the same as those used in equations 55 and 56 for determining T. For equations 57 and 58, the comparable equations for

S are, respectively,

$$S=\frac{2.25Tt/r^2}{\log_{10}^{-1}[-2s/\Delta s]} \qquad (62)$$

for plots of s versus r and

$$S=\frac{2.25Tt/r^2}{\log_{10}^{-1}[(s/Q)/\Delta(s/Q)]} \qquad (63)$$

for plots of s/Q versus t/r^2.

EXAMPLE

Note in figure 11 that the extrapolated straight line reaches the line of zero drawdown at $r=1{,}560$ ft. Using equation 59 and the results of the Thiem test accompanying figure 11,

$$S=\frac{(2.25)\,(20{,}700\text{ ft}^2\text{ day}^{-1})\,(18\text{ days})}{(1{,}560\text{ ft})^2}=0.35\text{ (rounded)}.$$

Similarly, using equation 62,

$$S=\frac{(2.25)\,(20{,}700\text{ ft}^2\text{ day}^{-1})\,(18\text{ days})/(100\text{ ft})^2}{\log_{10}^{-1}[-2(-4.05/3.40)]}$$

$$=\frac{83.9}{\log_{10}^{-1}[2.38235]}=\frac{83.9}{241.19}=0.35\text{ (rounded)}.$$

Note that in the final bracketed denominator .38235 is the mantissa, which determines the digits 24119, and that the 2. is the characteristic, which determines the three digits to the left of the decimal point, giving the final antilog as 241.19.

Jacob (1963a, p. 247) showed that in thin unconfined aquifers, such as the one tested near Wichita, Kans., a drawdown correction must be applied in solving for both T and S. The correction to be applied to S (after the drawdowns have been corrected) is

$$S'=\left(\frac{b-s}{b}\right)S, \qquad (64)$$

where

S' is the corrected value;
b, for the Wichita test, was 26.8 ft; and
s, for the Wichita test at the geometric mean distance of 100 ft, was 4.5 ft.

Hence,

$$S'=\left(\frac{26.8-4.5}{26.8}\right)0.35=0.29=0.3\text{ (rounded)}.$$

PRECAUTIONS

Although the straight-line solutions described above give very satisfactory results when properly used, it is unfortunately true that they have been improperly applied by many workers. It was stated in their derivation that they can be used only when u is less than or equal to about 0.01, but the importance of this seemingly has been forgotten all too often. Let us consider the several parameters involved in this criterion by reexamining equation 45:

$$u=\frac{r^2S}{4Tt},$$

and letting u be less than or equal to about 0.01.

First note that because S appears in the numerator (other things being equal including time, t), the value of u is considerably greater for an unconfined aquifer of specific yield from 0.1 to 0.3 than for a confined aquifer whose storage coefficient may range from only say 10^{-5} to 10^{-3}. To compensate for this, t must be greater by several orders of magnitude in testing an unconfined aquifer than in testing a confined aquifer. Equally important is the fact that pumping time in an unconfined aquifer must be long enough to allow reasonably complete drainage of material within that part of the cone of depression being observed. On the contrary, release of water from a reasonably elastic confined aquifer not unwatered during testing is virtually instantaneous.

Note that r appears as r^2 in the numerator. Thus, other things being equal, the greater the distance r, the longer t must be to allow u to reduce to the required maximum value.

In the denominator, 4 and T are constants, but again, other things being equal, the smaller the value of T, the more time is required for u to reduce to 0.01 or less.

For a given array of wells in a particular aquifer, assuming that sufficient pumping time is allowed to adequately drain the cone of depression in an unconfined aquifer, all parameters except t are constants; therefore pumping time generally is the only control for reducing u to or below its maximum permissible value, and this is the parameter generally neglected by many.

In a drawdown-time test (s versus $\log_{10} t$ or $\log_{10} t/r^2$), data points for any particular distance r will begin to fall on a straight line only *after* the time, t, is sufficiently long to satisfy the above criteria. In a drawdown-distance test (s versus $\log_{10} r$, Thiem test), the well must be pumped long enough that the data for the most distant observation will satisfy the requirements; then only the drawdowns at or *after* this value of t may be analyzed on a semilogarithmic plot for one particular value of t.

Equation 56 may be used for a time-drawdown or time-recovery test on a pumped well in a confined aquifer or in an unconfined aquifer that drains rapidly and reasonably completely during the test. When properly used, equation 56 gives the same results as the more complicated recovery equation of Theis (1935). In an unconfined aquifer that drains very slowly or incompletely or both, however, the results obtained by use of equation 56 may be badly in error. In the absence of any observation wells, it is realized that drawdown or recovery of a pumped well may be the only means available for obtaining at least an *estimate* of T. Similarly, considerable *judgment* should be exercised in using equations 55 and 58 for tests in unconfined aquifers.

Let us now apply the above criteria to the Thiem test near Wichita, Kans. (table 4, fig. 11), for the most distant observation well at $r=190$ ft:

$$u=\frac{(190\text{ ft})^2(0.29)}{(4)(20{,}700\text{ ft}^2\text{ day}^{-1})(18\text{ day})}=0.007.$$

Thus, even at the most distant observation well, u was less than 0.01, and the data at 18 days are valid in this respect.

If S and T can be estimated in advance, as from the results of other tests in the same aquifer, then, for arrays having the farthest observation well(s) at distance r, the minimum permissible pumping time, t, may be estimated from

$$t=\frac{r^2S}{4Tu}\qquad [T]. \tag{65}$$

Assume that $S=0.2$, $T=20{,}000$ ft^2 day^{-1}, $r=200$ ft (farthest observation well), and $u=0.01$. Then

$$t=\frac{(200\text{ ft})^2(0.2)}{(4)(20{,}000\text{ ft}^2\text{ day}^{-1})(0.01)}=10\text{ days}.$$

Thus, under the assumed conditions, all observations made 10 days or *more* after the beginning of pumping may be used in straight-line plots.

CONSTANT DRAWDOWN

Jacob and Lohman (1952) derived equations for determining T and S from tests in which the drawdown is constant and the discharge varies with time. These conditions are met when a naturally flowing well in a confined aquifer is shut in long enough for the head to recover, then the well is opened and allowed to flow for a period of 2 to 4 hours, during which period timed measurements are made of the declining rate of flow. The equation, based upon the assumptions that the aquifer is homogeneous, isotropic, and extensive laterally and that T and S are constant at all times and places, was developed from analogous thermal conditions in an equivalent thermal system. The equation, which is another solution of equation 40, is

$$T=\frac{Q}{2\pi G(\alpha)s_w}\qquad [L^2T^{-1}], \tag{66}$$

where

$$\alpha=\frac{Tt}{Sr_w^2}\qquad [\text{dimensionless}] \tag{67}$$

and

s_w = constant drawdown in discharging well,
r_w = radius of discharging well, and
$G(\alpha)$ = the G function of α, given by

$$G(\alpha)=\frac{4\alpha}{\pi}\int_0^\infty x\,e^{-\alpha x^2}\left[\frac{\pi}{2}+\tan^{-1}\left(\frac{Y_0(x)}{J_0(x)}\right)\right]dx, \tag{68}$$

where $J_0(x)$ and $Y_0(x)$ are Bessel functions of zero order of the first and second kinds, respectively.

Equation 68 is not tractable by integration, but the integral was replaced by a summation and solved by numerical methods. The resulting values of $G(\alpha)$ for corresponding values of α are given in table 7.

Plate 1 is a logarithmic plot of the type curve from data given in table 7. On translucent logarithmic paper, to the same scale, values of Q/s_w are plotted against values of t/r_w^2, or values of Q may be plotted against values of t. Then, by placing the experimental curve over the type curve, the solutions are obtained in the same manner as described for the Theis equation (eqs. 46–49). T is obtained from equation 66, and S is determined by rewriting equation 67,

$$S=\frac{Tt}{\alpha r_w^2}. \tag{69}$$

This curve-matching method, though laborious, was very useful before the straight-line solutions described below were derived. It is recommended that the much simpler straight-line solutions be used, for in such a test on a single well in a confined aquifer, u reaches a value of <0.01 very quickly because of the small values of r and S. (See p. 27.)

STRAIGHT-LINE SOLUTIONS

Jacob and Lohman (1952) showed that for all but extremely small values of t, the function $G(\alpha)$ can be approximated very closely by $2/W(u)$. It was shown in equation 51 that for sufficiently small values of u, $W(u)$ can be closely approximated by 2.30 $\log_{10} 2.25Tt/r^2S$. Making these substitutions in equation 66, and adding the subscript w to r^2, we obtain

$$T=\frac{2.30Q}{4\pi s_w}\log_{10}\frac{2.25Tt}{r_w^2S}\qquad [L^2T^{-1}], \tag{70}$$

which is identical to equation 51 except for the subscripts. Differentiating equation 70 and changing from infinitesimals to finite values, as was done for equations 52 through 57, gives

$$T=\frac{2.30}{4\pi\Delta(s_w/Q)/\Delta\log_{10} t/r_w^2}\qquad [L^2T^{-1}] \tag{71}$$

TABLE 7.—*Values of* $G(\alpha)$ *for values of* α *between* 10^{-4} *and* 10^{15}
[Modified from Jacob and Lohman (1952, p. 561)]

	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^{2}	10^{3}	10^{4}	10^{5}
1	56.9	18.34	6.13	2.249	0.985	0.534	0.346	0.251	0.1964	0.1608
2	40.4	13.11	4.47	1.716	.803	.461	.311	.232	.1841	.1524
3	33.1	10.79	3.74	1.477	.719	.427	.294	.222	.1777	.1479
4	28.7	9.41	3.30	1.333	.667	.405	.283	.215	.1733	.1449
5	25.7	8.47	3.00	1.234	.630	.389	.274	.210	.1701	.1426
6	23.5	7.77	2.78	1.160	.602	.377	.268	.206	.1675	.1408
7	21.8	7.23	2.60	1.103	.580	.367	.263	.203	.1654	.1393
8	20.4	6.79	2.46	1.057	.562	.359	.258	.200	.1636	.1380
9	19.3	6.43	2.35	1.018	.547	.352	.254	.198	.1621	.1369
10	18.3	6.13	2.25	.985	.534	.346	.251	.196	.1608	.1360

	10^{6}	10^{7}	10^{8}	10^{9}	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}	10^{15}
1	0.1360	0.1177	0.1037	0.0927	0.0838	0.0764	0.0704	0.0651	0.0605	0.057
2	.1299	.1131	.1002	.0899	.0814	.0744				
3	.1266	.1106	.0982	.0883	.0801	.0733				
4	.1244	.1089	.0968	.0872	.0792	.0726				
5	.1227	.1076	.0958	.0864	.0785	.0720				
6	.1213	.1066	.0950	.0857	.0779	.0716				
7	.1202	.1057	.0943	.0851	.0774	.0712				
8	.1192	.1049	.0937	.0846	.0770	.0709				
9	.1184	.1043	.0932	.0842	.0767	.0706				
10	.1177	.1037	.0927	.0838	.0764	.0704				

or

$$T=\frac{2.30}{4\pi s_w(\Delta 1/Q)/\Delta\log_{10}t}\qquad [L^2T^{-1}]. \tag{72}$$

By extrapolating the straight line to $s_w/Q=0$, S may be determined from

$$S=2.25T\left(\frac{t}{r_w^2}\right)_0 \tag{73}$$

or, in the same manner as shown for equations 61 through 63, S may be determined within the data region of the straight-line plot from

$$S=\frac{2.25Tt/r_w^2}{\log_{10}^{-1}\left[\dfrac{(s_w/Q)}{\Delta(s_w/Q)}\right]} \tag{74}$$

Note that S may be determined when T is determined from equation 71, where r_w (radius of flowing well) is known, but S may not be determined when r_w is not known

TABLE 8.—*Field data for flow test on Artesia Heights well near Grand Junction, Colo., September 22, 1948*
[Valve opened at 10:29 a.m. s_w=92.33 ft; r_w=0.276 ft. Data from Lohman (1965, tables 6 and 7, well 28)]

Time of observation	Rate of flow (gpm)	Flow interval (min)	Total flow during interval (gal)	Time since flow started (min)	$\frac{s_w}{Q}$ (ft gal^{-1} min)	$\frac{t}{r_w^2}$ (min ft^{-2})
10:30	7.28	1	7.28	1	12.7	13.1
10:31	6.94	1	6.94	2	13.3	26.3
10:32	6.88	1	6.88	3	13.4	39.4
10:33	6.28	1	6.28	4	14.7	52.6
10:34	6.22	1	6.22	5	14.8	65.7
10:35	6.22	1	6.22	6	15.1	78.8
10:37	5.95	2	11.90	8	15.5	105
10:40	5.85	3	17.55	11	15.8	145
10:45	5.66	5	28.30	16	16.3	210
10:50	5.50	5	27.50	21	16.8	276
10:55	5.34	5	26.70	26	17.3	342
11:00	5.34	5	26.70	31	17.3	407
11:10½	5.22	10.5	54.81	41.5	17.7	345
11:20	5.14	9.5	48.83	51	18.0	670
11:30	5.11	10	51.10	61	18.1	802
11:45	5.05	15	75.75	76	18.3	999
12:00 (noon)	5.00	15	75.00	91	18.5	1,196
12:12	4.92	12	59.04	103	18.8	1,354
12:22	4.88	11	53.68	113	18.9	1,485
Total [1]		114	596.98			

[1] 596.98 gal per 114 min = 5.23 gal min^{-1}, weighted average discharge.

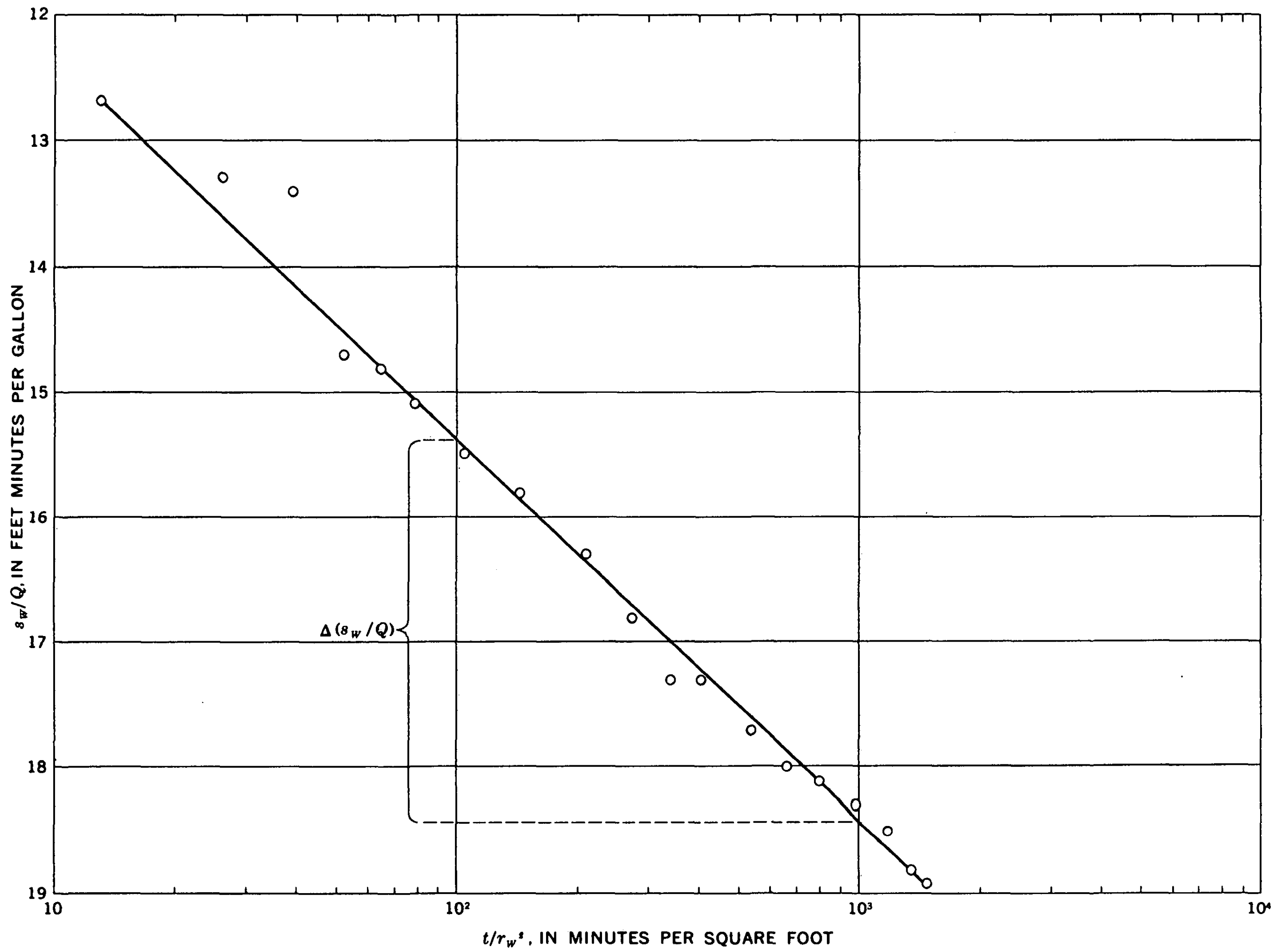

FIGURE 17.—Semilogarithmic plot of s_w/Q versus t/r_w^2.

as in equation 72. If the value of r_w is in doubt, owing to well construction, well development, or caving, do not try to determine S by these methods.

EXAMPLE

Table 8 gives the field data from a flow test on the Artesia Heights well near Grand Junction, Colo. (Lohman, 1965, tables 6 and 7, well 28), which I made with the help of Mahmood Hussain on September 22, 1948. After the well was shut in for a period of several days, the static head just prior to the test was 94.55 ft above the measuring point (92.33 ft above discharge point). Static and recovering heads were measured by an inkwell mercury gage which I designed and built and which reads directly in feet and tenths of feet of water. Hundredths of feet are readily interpolated. Discharge was measured by timing with a stopwatch the filling of a 4-gal container. Data in the last two columns of table 8 are plotted in figure 17. From the values of s_w/Q, $\Delta(s_w/Q)$, and t/r_w^2 obtained from this plot, we obtain, using equation 71,

$$T=\frac{(2.30)\,(1.44\times10^3\text{ min day}^{-1})}{(4\pi)[(18.4-15.38)\text{ ft gal}^{-1}\text{ min}](7.48\text{ gal ft}^{-3})}$$

$$=11.7\text{ ft}^2\text{ day}^{-1},$$

which rounds to 12 ft^2 day^{-1}. Note that it is not convenient to determine S using equation 73, because considerable extrapolation off the sheet would be required to reach the line where $s_w/Q=0$. However, S is readily determined from equation 74:

$$S=\frac{(2.25)\,(11.7\text{ ft}^2\text{ day}^{-1})\,(10^3\text{ min ft}^{-2})}{\log_{10}^{-1}[18.4/3.02](1{,}440\text{ min day}^{-1})}$$

$$=\frac{(2.25)\,(1.17\times10^1)\,(10^3)}{(1.23\times10^6)\,(1.44\times10^3)}=1.5\times10^{-5}.$$

Note that in table 8 and in figure 17 Q and t are in

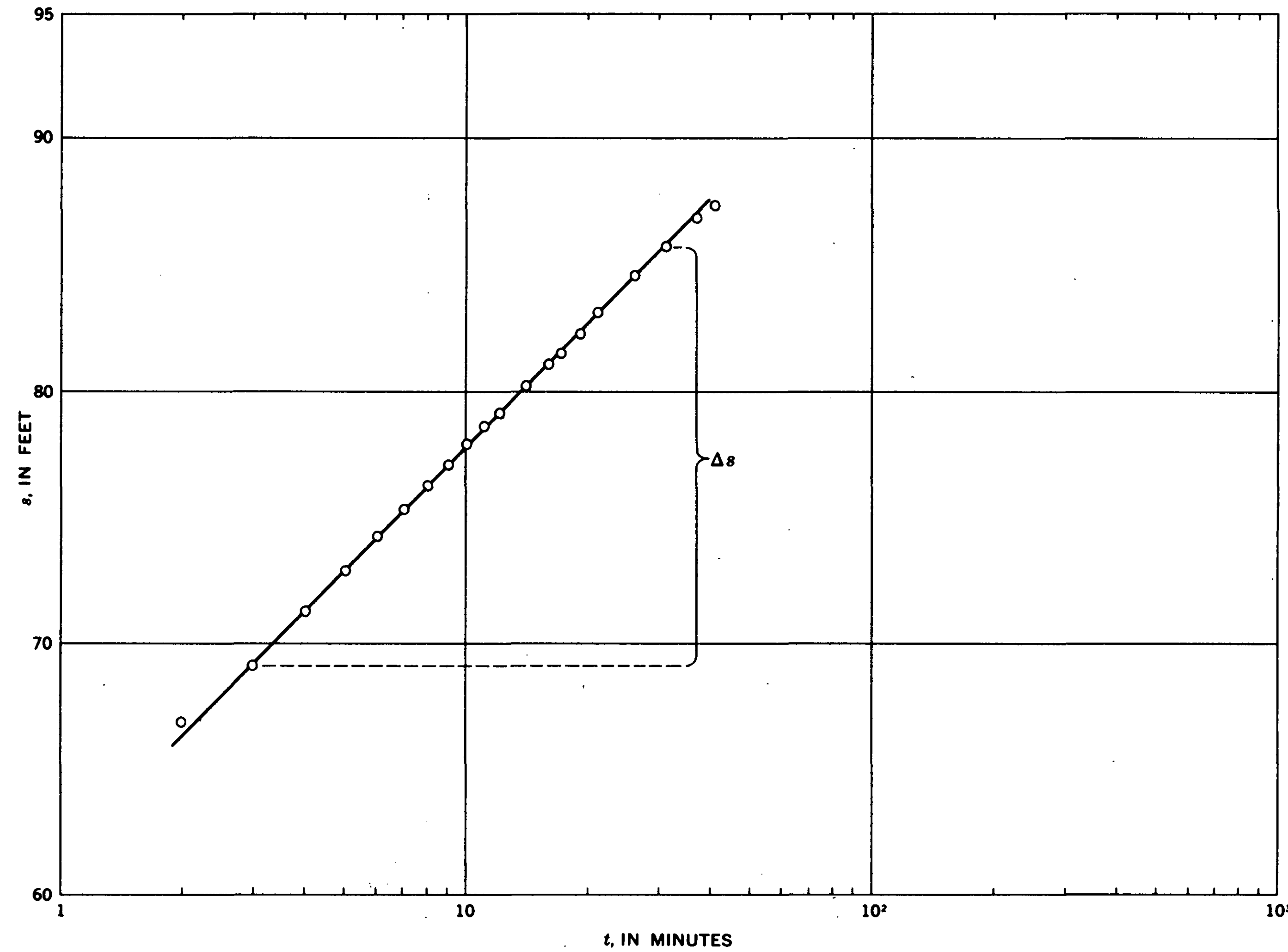

FIGURE 18.—Semilogarithmic plot of recovery (s_w) versus t.

gallons per minute and in minutes, respectively, whereas they are in cubic feet per day and in days in equations 71 and 74; hence, conversion factors were used in the solutions. The bracketed part of the denominator in equation 74 is a dimensionless ratio, so the units do not have to be converted. The slight deviations from the straight line of all data points except that for 10:32 a.m. are believed to have resulted from fluctuations in barometric pressure during the test. These were largely self-compensating and seem not to have materially affected the slope of the straight line; however, it is possible to correct the head and discharge with data from a barograph. The value at 10:32 a.m. may have resulted, at least in part, from an error in measuring or recording the discharge.

Columns 3 and 4 in table 8 are included to obtain a weighted average discharge during the 113-min flow period for use in applying the recovery method. The 2-in. gate valve on the well was closed at 12:23 p.m., and the slowly recovering head was measured as shown in table 9 and figure 18. The recovery method (eq. 56) is strictly

TABLE 9.—*Field data for recovery test on Artesia Heights well near Grand Junction, Colo., September 22, 1948*

[Valve closed at 12:23 p.m. Weighted average discharge, 5.23 gpm. Data from Lohman (1965, tables 6 and 7, well 28)]

Time of observation	Time since flow stopped, t (min)	Head (ft above land surface)
12:25	2	66.80
12:26	3	69.10
12:27	4	71.30
12:28	5	72.95
12:29	6	74.26
12:30	7	75.37
12:31	8	76.33
12:32	9	77.16
12:33	10	77.89
12:34	11	78.57
12:35	12	79.19
12:37	14	80.26
12:39	16	81.17
12:40	17	81.55
12:42	19	82.29
12:45	21	83.27
12:50	26	84.64
12:55	31	85.76
1:01	37	86.82
1:05	41	87.41

applicable only to tests of constant discharge and variable drawdown or recovery, whereas the flow test involves constant drawdown and gradually declining discharge. Nevertheless, recovery tests generally give values of T in close agreement with the flow tests and may be useful as corroborative checks. The results by the two methods generally do not agree precisely. For one thing, the weighted average discharge obtained by the method given in table 8, although close enough for the purpose, is not as accurate as would be obtained by use of an accurate water meter, which would integrate the entire discharge during the flow period.

Using the data from table 9 and figure 16 in equation 56,

$$T=\frac{(2.30)\,(5.23\text{ gal min}^{-1})\,(1{,}440\text{ min day}^{-1})}{(4\pi)[(85.75-69.25)\text{ ft}](7.48\text{ gal ft}^{-3})}$$

$$=11.2\text{ ft}^2\text{ day}^{-1},$$

which rounds to 11 ft² day⁻¹. This value is very close to the value of 12 ft² day⁻¹ obtained from the flow test. (Values before rounding were even closer: 11.7 and 11.2 ft² day⁻¹.)

Let us test the data using equation 65 to determine if even the earliest observations meet the requirement that u is less than or equal to about 0.01:

$$t=\frac{(0.076\text{ ft}^2)\,(1.5\times10^{-5})\,(1{,}440\text{ min day}^{-1})}{(4)\,(12\text{ ft}^2\text{ day}^{-1})\,(0.01)}$$

$$=3.4\times10^{-3}\text{ min}.$$

Thus, largely because of the low values of r^2 and S, all observations made 0.003 min or more after discharge or recovery began satisfy this requirement. The earliest observation was 1 min after discharge began (table 8).

INSTANTANEOUS DISCHARGE OR RECHARGE

"SLUG" METHOD

In areas lacking either flowing wells or wells equipped with pumps, it may be desirable to obtain at least an estimate of the transmissivity of the aquifer by use of the so-called "slug" method. In this method a known volume or "slug" of water is suddenly injected into or removed from a well and the decline or recovery of water level is measured at repeated closely spaced intervals during the ensuing minute or two. The method is strictly applicable only to fully penetrating or fully screened wells in confined aquifers of rather low transmissivity—say less than about 7,000 ft² day⁻¹. For partially penetrating wells, the value of transmissivity obtained generally would apply only to that part of the aquifer in which the well is screened or open. Application of the method to wells in unconfined aquifers would require considerable judgment, and the results should be regarded with skepticism.

Under the above conditions, and with the usual assumptions, Cooper, Bredehoeft, and Papadopulos (1967, p. 264, 265) derived the following equation, which is another solution of equation 40, for the response of a finite-diameter well to such an instantaneous "slug" of water:

$$H=8H_0\frac{\alpha}{\pi^2}\int_0^\infty \exp\left(\frac{-\beta u^2}{\alpha}\right)\frac{du}{u\Delta u}\qquad [L], \qquad (75)$$

where

H = head inside the well at time t after injection or removal of the "slug," above or below initial head $[L]$,

H_0 = head inside the well above or below initial head at instant of injection or removal of "slug" $[L]$,

and

$$\alpha=\frac{r_s^2 S}{r_c^2}\qquad [\text{dimensionless}], \qquad (76)$$

where

r_s = radius of well screen or open hole $[L]$, and

r_c = radius of casing in interval over which water level fluctuates $[L]$.

In equation 75

$$\beta=\frac{Tt}{r_c^2}\qquad [\text{dimensionless}] \qquad (77)$$

and

$$\Delta u=[uJ_0(u)-2\alpha J_1(u)]^2+[uY_0(u)-2\alpha Y_1(u)^2],$$

where

J_0 = Bessel function of first kind, zero order,
J_1 = Bessel function of first kind, first order,
Y_0 = Bessel function of second kind, zero order,
Y_1 = Bessel function of second kind, first order,

and

u = variable of integration.

Values of H/H_0 versus Tt/r_c^2 for five different values of α obtained by numerical solution of equation 75 given by Cooper, Bredehoeft, and Papadopulos (1967, table 1) are plotted as a family of semilogarithmic curves on plate 2, which is similar to their figure 3. V, the measured volume of water injected or removed from the well, obviously is equal to $H_0\pi r_c^2$, so the value of H_0 (at the instant of injection or removal) is obtained from

$$H_0=\frac{V}{\pi r_c^2}\qquad [L]. \qquad (78)$$

From measured values of H at repeated intervals, values

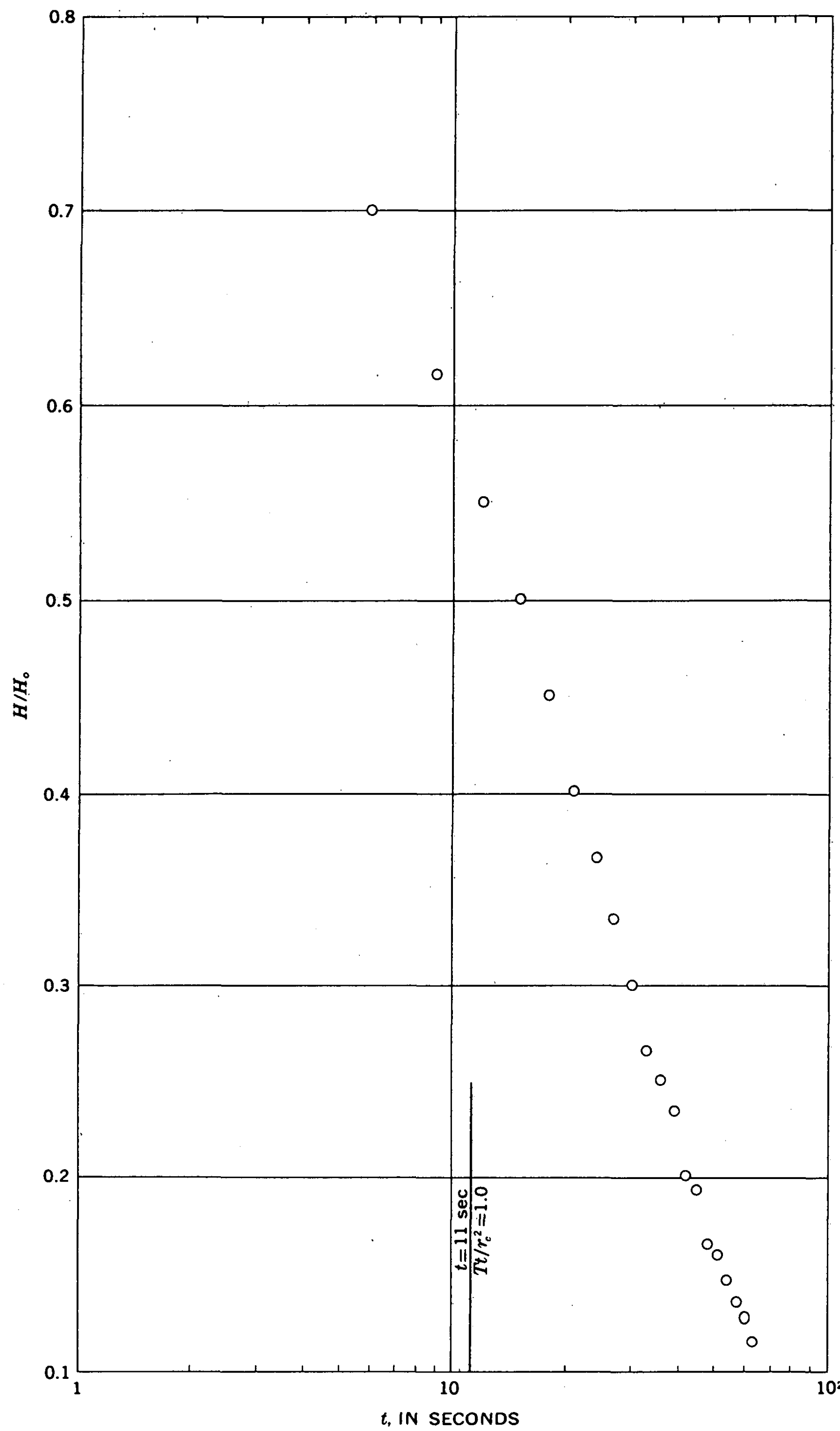

FIGURE 19.—Semilogarithmic plot of data from "slug" test on well at Dawsonville, Ga. From Cooper, Bredehoeft, and Papadopulos (1967, table 3).

of H/H_0 are computed and are plotted on the linear scale of semilogarithmic paper of the same scale as plate 2 against the time of measurement, t, in seconds, on the logarithmic scale. Note that H/H_0 is a dimensionless ratio, hence any convenient units of measurement may be used without affecting the final results in any way. The data curve is then superposed on plate 2 by the usual curve matching procedure, and a match line is selected for the value of t at $Tt/r_c^2 = 1.0$ (match point values of H/H_0 are not needed). The transmissivity is then determined from the following form of equation 77:

$$T=\frac{1.0r_c^2}{t} \qquad [L^2T^{-1}]. \qquad (79)$$

By rewriting equation 76, the storage coefficient may be determined from

$$S=\frac{r_c^2}{r_s^2}\alpha \qquad [\text{dimensionless}], \qquad (80)$$

but, as pointed out by Cooper, Bredehoeft, and Papadopulos (1967, p. 267): "However, because the matching of the data plot to the type curves depends upon the shapes of the type curves, which differ only slightly when α differs by an order of magnitude, a determination of S by this method has questionable reliability." They go on to say:

> The determination of T is not so sensitive to the choice of the curves to be matched. Whereas the determined value of S will change by an order of magnitude when the data plot is moved from one type curve to another, that of T will change much less. From a knowledge of the geologic conditions and other considerations one can ordinarily estimate S within an order of magnitude [see "Methods of Estimating Storage Coefficient" and "Methods of Estimating Specific Yield"] and thereby eliminate some of the doubt as to what value of α is to be used for matching the data plot.

In 1954 J. G. Ferris and D. B. Knowles (see Ferris and others, 1962, p. 104, 105) described a "slug" test based upon an instantaneous line source rather than a well of finite diameter. Their equation is identical to equation 81, except for algebraic sign. As shown by Cooper, Bredehoeft, and Papadopulos (1967, p. 265), however, the method of Ferris and Knowles is strictly applicable only for relatively large values of Tt/r_c^2, and hence of t, and should not be used for the values of t generally measured during a "slug" test.

Example

Cooper, Bredehoeft, and Papadopulos (1967, p. 265–268) illustrated the "slug" method using data obtained from a "slug" test on a well near Dawsonville, Ga., and described the well and procedure as follows. The well is cased to 24 m with 15.2-cm (6-in.) casing and drilled as a 15.2-cm open hole to a depth of 122 m. A nearly instantaneous decline in water level was obtained by the sudden withdrawal of a long weighted float whose total weight was 10.16 kg. From Archimedes' principle, they determined that the float had displaced a volume of 0.01016 m³ of water when floating in the well; hence, $V=0.01016$ m³. From equation 78, H_0 was found to be 0.560 m. Their recovery data, obtained from an electrically operated recorder actuated by a pressure transducer in the well, are given in table 10 and are shown in figure 19.

By superposition of figure 19 on plate 2, the data are found to fit the type curve for $\alpha=10^{-3}$. The value of t for the match line where $Tt/r_c^2=1.0$ is 11 sec. Therefore, from

TABLE 10.—*Recovery of water level in well near Dawsonville, Ga., after instantaneous withdrawal of weighted float*

[H_0=0.560 m. From Cooper, Bredehoeft, and Papadopulos (1967, table 3)]

t (sec)	Head above datum (m)	H (m)	H/H_0
−1	0.896		
0	.336	0.560	1.000
3	.439	.457	.816
6	.504	.392	.700
9	.551	.345	.616
12	.588	.308	.550
15	.616	.280	.500
18	.644	.252	.450
21	.672	.224	.400
24	.691	.205	.366
27	.709	.187	.334
30	.728	.168	.300
33	.747	.149	.266
36	.756	.140	.250
39	.765	.131	.234
42	.784	.112	.200
45	.788	.108	.193
48	.803	.093	.166
51	.807	.089	.159
54	.814	.082	.146
57	.821	.075	.134
60	.825	.071	.127
63	.831	.065	.116

equation 79,

$$T=\frac{(1.0)(7.6\text{ cm})^2}{11\text{ sec}}=5.3\text{ cm}^2\text{ sec}^{-1}$$

or

$$T=\frac{(1.0)(7.6\text{ cm})^2(8.64\times10^4\text{ sec day}^{-1})}{(11\text{ sec})(0.929\times10^3\text{ cm}^2\text{ ft}^{-2})}$$

$$=490\text{ ft}^2\text{ day}^{-1}.$$

BAILER METHOD

Skibitzke (1958) proposed a method for determining the transmissivity from the recovery of water level in a well that has been bailed. At any given point on the recovery curve the following equation applies:

$$T=\frac{V}{4\pi s't[e^{r_w^2S/4Tt}]} \qquad [L^2T^{-1}], \qquad (81)$$

where

s' = residual drawdown $[L]$,
V = volume of water removed in one bailing cycle $[L^3]$,
t = length of time since bailing stopped $[T]$, and
r_w = effective radius of the well $[L]$.

As r_w is small, the term in brackets in equation 81 approaches e^0, or unity, as t increases; therefore, for large values of t, equation 81 may be rewritten:

$$T=\frac{V}{4\pi s't} \qquad [L^2T^{-1}]. \qquad (82)$$

If the residual drawdown is observed at some time after the completion of n bailing cycles, the following equation applies:

$$T=\frac{1}{4\pi s'}\left[\frac{V_1}{t_1}+\frac{V_2}{t_2}+\frac{V_3}{t_3}+\cdots+\frac{V_n}{t_n}\right] \quad [L^2T^{-1}]. \quad (83)$$

If approximately the same volume of water is bailed during each cycle, equation 83 becomes

$$T=\frac{V}{4\pi s'}\left[\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\cdots+\frac{1}{t_n}\right] \quad [L^2T^{-1}]. \quad (84)$$

Equation 84 is applied to single values of V and s' and the summation of the reciprocal of the elapsed time between the time each bailer was removed from the well and the time of observation of s'. If T is to be expressed in square feet per day, then obviously V should be expressed in cubic feet, s' in feet, and t in days, or suitable conversions of units should be made.

The bailer method should give satisfactory *estimates* of T for wells in confined aquifers having sufficiently shallow water levels to permit short time intervals between bailing cycles. In wells in unconfined aquifers, or in wells having relatively deep water levels, the method should be used with considerable judgment or not at all. (See also "Precautions.")

Unfortunately, I have no data available with which to illustrate the bailer method.

LEAKY CONFINED AQUIFERS WITH VERTICAL MOVEMENT

The flow equations for confined aquifers under conditions of both constant discharge and constant drawdown discussed in earlier sections of this report all are based upon the assumptions that the confining beds are impermeable (or have very low permeability), that they release no water from storage, and that vertical flow components are negligible. It is well known that no rocks are wholly impermeable and that some confining beds have finite permeability. We will now take up the equations for both steady and nonsteady radial flow from infinite aquifers whose confining beds leak water either from or to the aquifer.

CONSTANT DISCHARGE

STEADY FLOW

Consider an aquifer overlain by a confining bed of low but finite permeability, which in turn is overlain by an unconfined aquifer. When discharge occurs from a well in a confined aquifer, the potentiometric surface is lowered throughout a large circular area (Cooper, 1963, p. 48). This lowering changes the relative head between the confined and unconfined aquifers and results in turn in a change in the rate of leakage through the confining bed. The change may be either a decrease in the rate of leakage out of the aquifer or an increase in the rate of leakage into the aquifer, but either way the change results in a net increase in the supply of water to the aquifer and, therefore, constitutes capture of water.

Jacob (1946) derived an equation of steady flow near a well discharging at a constant rate from such an infinite leaky confined aquifer and described a graphical method for determining the transmissivity of the aquifer and the "leakance" of the confining bed. The leakance is the ratio K'/b', in which K' and b' are the vertical hydraulic conductivity and the thickness, respectively, of the confining beds. Hantush and Jacob (1954) derived equations for steady flow in variously bounded leaky confined aquifers. Later, equations for the more generally encountered nonsteady flow in such aquifers were developed, and these will now be taken up.

NONSTEADY FLOW

HANTUSH-JACOB METHOD

Hantush and Jacob (1955) derived the following equation for nonsteady radial flow in an infinite leaky confined aquifer:

$$\frac{s}{Q/4\pi T}=2K_0(2v)-\int_{v^2/u}^{\infty}\frac{1}{y}\exp\left(\frac{-y-v^2}{y}\right)dy \quad [\text{dimensionless form}], \quad (85)$$

where

K_0 = the modified Bessel function of the second kind of zero order,

and

$$v=\frac{r}{2}\sqrt{\frac{K'}{b'T}} \quad [\text{dimensionless}], \quad (86)$$

where

K' = the vertical hydraulic conductivity of the confining bed $[LT^{-1}]$,
b' = the thickness of the confining bed $[L]$, and
T = the transmissivity of the aquifer $[L^2T^{-1}]$,

and

$u=r^2S/4Tt$ [dimensionless], and
y = the variable of integration.

The authors gave two series expressions for the formal solutions of equation 85—one for large values of t and one for small values—and gave a few examples in both tabular and graphic form. In January 1956, Hilton H. Cooper, Jr., computed many values and prepared two families of type curves which were later published (Cooper, 1963, pl. 4). Meanwhile, unknown to Cooper, Hantush (1955) also had computed many values. (See also Hantush, 1956.)

TABLE 11.—*Postulated water-level drawdowns in three observation wells during a hypothetical test of an infinite leaky confined aquifer*

[Pumped well began discharging 1,000 gal min^{-1} at t=0 min.] From Cooper (1963, p. 54)

Time since pumping began, t		Well 1 (r=100 ft)		Well 2 (r=500 ft)		Well 3 (r=1,000 ft)	
Min	Day	$\frac{t}{r^2}$ (day ft^{-2})	Drawdown, s (ft)	$\frac{t}{r^2}$ (day ft^{-2})	Drawdown, s (ft)	$\frac{t}{r^2}$ (day ft^{-2})	Drawdown, s (ft)
0.2	0.000139	1.39×10^{-8}	1.76	5.56×10^{-10}	0.01	1.39×10^{-10}	0.00
.5	.000347	3.47×10^{-8}	2.75	1.39×10^{-9}	.14	3.47×10^{-10}	.00
1	.000694	6.94×10^{-8}	3.59	2.78×10^{-9}	.45	6.94×10^{-10}	.02
2	.00139	1.39×10^{-7}	4.26	5.56×10^{-9}	.93	1.39×10^{-9}	.14
5	.00347	3.47×10^{-7}	5.28	1.39×10^{-8}	1.76	3.47×10^{-9}	.55
10	.00694	6.94×10^{-7}	5.90	2.78×10^{-8}	2.34	6.94×10^{-9}	.99
20	.0139	1.39×10^{-6}	6.47	5.56×10^{-8}	2.85	1.39×10^{-8}	1.46
50	.0347	3.47×10^{-6}	6.92	1.39×10^{-7}	3.31	3.47×10^{-8}	1.95
100	.0694	6.94×10^{-6}	7.11	2.78×10^{-7}	3.50	6.94×10^{-8}	2.10
200	.139	1.39×10^{-5}	7.20	5.56×10^{-7}	3.51	1.39×10^{-7}	2.11
500	.347	3.47×10^{-5}	7.21	1.39×10^{-6}	3.52	3.47×10^{-7}	2.11
1,000	.694	6.94×10^{-5}	7.21	2.78×10^{-6}	3.52	6.94×10^{-7}	2.11

As described by Cooper (1963), if the right-hand side of equation 85 is represented by $L(u, v)$, the L, or leakance, function of u and v, equation 85 may be written

$$s=\frac{Q}{4\pi T}L(u,v) \quad [L]. \tag{87}$$

S is determined by

$$S=4T\frac{t/r^2}{1/u} \quad [\text{dimensionless}] \tag{88}$$

and

$$\frac{K'}{b'}=4T\frac{v^2}{r^2}=\frac{S\left(\frac{v^2}{u}\right)}{t} \quad [T^{-1}]. \tag{89}$$

When K' and, hence, v *approach* zero, it can be shown that $L(u, v)$ approaches $W(u)$, and equation 87 becomes equation 46, the Theis equation. An enlargement of two families of type curves of $L(u, v)$ versus $1/u$ prepared by Cooper (1963, pl. 4) is shown on plate 3*A*. In one family of curves, v is the parameter; in the other, v^2/u is the parameter. The solid-line type curves (v) correspond to a plot of s (vertical) versus t at some constant r, plotted as t/r^2 (horizontal). The dashed-line curves (v^2/u) correspond to a plot of s versus t/r^2 at some constant t.

EXAMPLE

Table 11 from Cooper (1963, p. 54) gives postulated drawdowns in observation wells at distances of 100, 500, and 1,000 ft from a well discharging at the constant rate of 1,000 gpm for 1,000 min from a leaky confined aquifer. Values of s versus t/r^2 for the three wells are shown on plate 3*B* superposed on the type curves. Note that a match point was chosen where $\alpha(u, v)=1.0$, $1/u=1.0$, $s=1.15$ ft, and $t/r^2=1.87\times10^{-9}$ day ft^{-2}. Substituting appropriate values in equations 87 (solved for T) and 88 gives, respectively,

$$T=\frac{(1{,}000\text{ gal min}^{-1})(1{,}440\text{ min day}^{-1})(1.0)}{(4\pi)(1.15\text{ ft})(7.48\text{ gal ft}^{-3})}$$

$$=13{,}300\text{ ft}^2\text{ day}^{-1}\text{ (rounded)}$$

and

$$S=(4)(13{,}300\text{ ft}^2\text{ day}^{-1})\frac{(1.87\times10^{-9}\text{ day ft}^{-2})}{1.0}$$

$$=10^{-4}\text{ (rounded)}.$$

The plotted values for observation well 1 fall slightly below the solid-line curve for $v=0.02$, or at about 0.025. Substituting $r=100$ ft, $v=0.025$, and $T=13{,}300$ ft^2 day^{-1} in the first part of equation 89 gives

$$\frac{K'}{b'}=(4)(13{,}300\text{ ft}^2\text{ day}^{-1})\frac{(0.025)^2}{(100\text{ ft})^2}=0.0033\text{ day}^{-1}.$$

Assume $b'=100$ ft, then $K'=0.33$ ft day^{-1}.

As the data in table 11 represent idealized conditions, the same values for K'/b' would be obtained using the data for observation wells 2 and 3. Also, the same values of K'/b' would be obtained using the dashed-line curves by plotting the values of s for each observation well for some constant t, say 100 min (0.0694 day), and substituting the value of v^2/u, s, and t in the second part of equation 89.

Cooper (1963, p. 55) gives the following pertinent conclusions in regard to this method:

Because the adjustment of the hydraulic gradient through a confining bed generally lags considerably behind the decline in head, the water yielded by an artesian aquifer is derived largely, if not entirely, from storage in the confining bed. For this reason, most time-drawdown plots deviate from the Theis curve to a greater degree than if leakage alone were involved. The method for determining leakance is presented with reservation because, if applied under the mistaken assumption that the deviations are due to leakage, it yields erroneously large values. However, whenever the results of an aquifer test indicate that leakage occurs, the determination of T and S by use of the family of type curves described in this paper has advantages over that by use of the Theis type curve alone.

HANTUSH MODIFIED METHOD

Hantush (1960) presented an important modification of the theory of leaky confined aquifers in which the storage of water in the semipervious confining bed or beds is taken into account. His main equations are:

$$T=\frac{Q}{4\pi s}H(u,\beta)\qquad [L^2T^{-1}], \tag{90}$$

where

$$H(u,\beta)=\int_u^\infty \frac{e^{-y}}{y}\,\text{erfc}\left(\frac{\beta/\sqrt{u}}{\sqrt{y(y-u)}}\right)dy\quad [\text{dimensionless}], \tag{91}$$

$$u=\frac{r^2S}{4Tt}\qquad [\text{dimensionless}],$$

as in the Theis equation, and

$$\beta=\frac{r}{4b}\left(\sqrt{\frac{K'S_s'}{KS_s}}+\sqrt{\frac{K''S_s''}{KS_s}}\right)\qquad [\text{dimensionless}], \tag{92}$$

where

K = hydraulic conductivity of main aquifer,
K', K'' = hydraulic conductivities of semipervious confining layers,
$S=bS_s$, $S'=b'S_s'$, $S''=b''S_s''$ — Storage coefficients of the main aquifer and of the semipervious confining layers, respectively, and
S_s, S_s', S_s'' = specific storage (storage coefficient per vertical unit of thickness) of the main aquifer and confining layers (b, b', and b''), respectively.

The versatility of equations 90 through 92 lies in the fact that they are the general solutions for the drawdown distribution in all confined aquifers, whether they are leaky or nonleaky. Thus, if K' and K'' approach zero or are made equal to zero, β approaches or equals zero, and equation 90 becomes equation 46, the Theis equation for nonleaky confined aquifers. Hantush (1960, p. 3716–3718) gives general solutions for three different configurations of aquifers and sets of confining beds. If K'', S', and S'' approach zero or are made equal to zero, two of these solutions become equal to equation 85 of Hantush and Jacob (1955)—the equation for leaky confined aquifers for which release of stored water from the confining beds is considered negligible.

Plate 4 is a logarithmic plot of $1/u$ versus $H(u,\beta)$ for various indicated values of β, copied from a plot made by E. J. McClelland, U.S. Geological Survey, Sacramento, Calif., in 1961 from tabulated values by Hantush (1961). Time-drawdown or time-recovery data from tests in aquifers whose confining bed or beds are suspected of releasing water from storage are plotted (as s versus t) on 3×5-cycle logarithmic paper having the same scale as plate 4 (such as K & E 359–125G or 46–7522), and this is superposed on plate 4 until a fit is obtained on one of the type curves by the usual curve-matching procedure. From values of the four parameters at a convenient match point, T and S may be determined from equations 90 and 47, respectively.

Thorough knowledge of the geology, including the character of the confining beds, should indicate in advance which of the two leaky-aquifer type curves to use, or whether to use the Theis type curve for nonleaky aquifers.

EXAMPLE

Table 12 gives the time-drawdown measurements in an observation well at Pixley, Calif., 1,400 ft from a well pumping 750 gpm, supplied by Francis S. Riley (U.S. Geological Survey, Sacramento, Calif., written commun., March 5, 1968). The pumped well, which is 600 ft deep, obtains water from gravel, sand, sandy clay, and clay of the Tulare Formation in an area where considerable land subsidence has resulted from prolonged pumping from confined aquifers containing appreciable amounts of clay.

TABLE 12.—*Drawdown of water level in observation well 23S/25E-17Q2, 1,400 ft from a well pumping at constant rate of 750 gpm, at Pixley, Calif., March 13, 1963*

[Drawdown corrected for pretest trend. Data from Francis S. Riley (written commun., March 5, 1968)]

Time since pumping began, t (min)	Drawdown, s (ft)	Time since pumping began, t (min)	Drawdown, s (ft)
6.37	0.01	90	0.75
8.58	.02	100	.82
10.23	.03	137	1.04
11.90	.04	150	1.12
12.95	.05	160	1.17
14.42	.06	173	1.24
15.10	.07	184	1.27
16.88	.08	200	1.35
17.92	.10	210	1.40
21.35	.12	278	1.68
21.70	.13	300	1.76
22.70	.14	315	1.83
23.58	.15	335	1.87
24.65	.17	365	1.99
29	.21	390	2.10
30	.22	410	2.13
32	.24	430	2.20
34	.26	450	2.23
36	.28	470	2.29
38	.30	490	2.32
41	.33	510	2.39
44	.36	560	2.48
47	.38	740	2.92
50	.42	810	3.05
54	.46	890	3.19
60	.52	1,255	3.66
65	.56	1,400	3.81
70	.60	1,440	3.86
80	.65	1,485	3.90

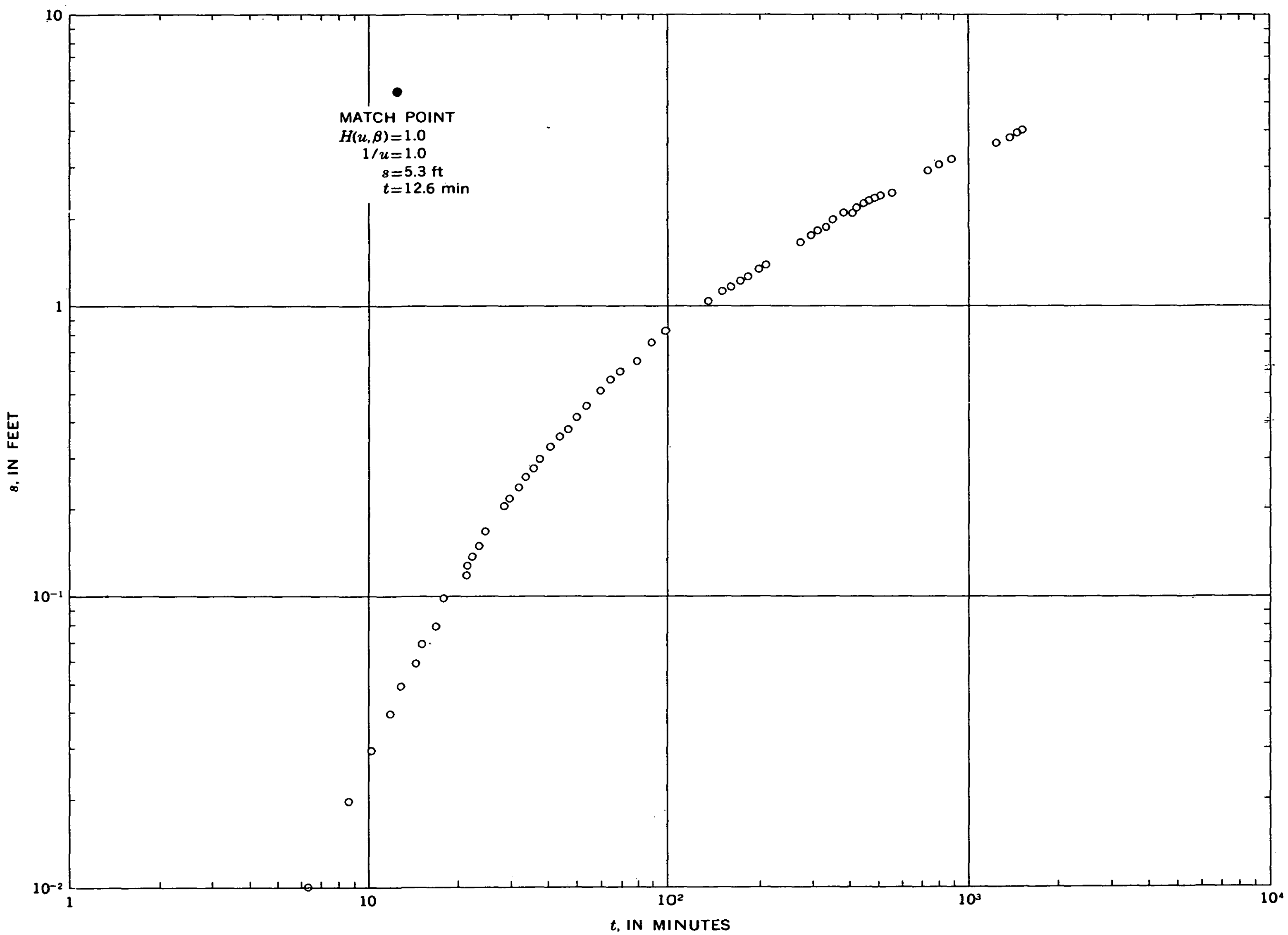

FIGURE 20.—Logarithmic plot of s versus t for observation well 23S/25E–17Q2 at Pixley, Calif.

(See table 3, "Tulare-Wasco area.") The aquifer is confined by the Corcoran Clay Member, about 6 ft thick, above which is an unconfined aquifer about 200 ft thick. A logarithmic plot of s versus t from table 12 is shown in figure 20, which shows also the match-point values of the four parameters obtained by superposition on plate 4. From these data, T and S are computed from equations 90 and 47, as follows:

$$T=\frac{Q}{4\pi s}H(u,\beta)$$

$$=\frac{(750\text{ gal min}^{-1})(1{,}440\text{ min day}^{-1})(1.0)}{(4\pi)(5.3\text{ ft})(7.48\text{ gal ft}^{-3})}$$

$$=2{,}170\text{ ft}^2\text{ day}^{-1},$$

rounded to 2,200 ft^2 day^{-1}, and

$$S=\frac{4Ttu}{r^2}=\frac{(4)(2{,}170\text{ ft}^2\text{ day}^{-1})(12.6\text{ min})}{(1{,}400\text{ ft})^2(1{,}440\text{ min day}^{-1})(1/1.0)}$$

$$=3.9\times10^{-5},$$

rounded to 4×10^{-5}.

Preliminary attempts to fit both early and late data from table 12, and similar drawdown and recovery data from two other observation wells at $r=650$ and 1,220 ft, to the Theis curve gave apparent values of T from 5 to 20 times the more realistic value computed above, and apparent values of S from 17 to 25 times the value computed above.

CONSTANT DRAWDOWN

Hantush (1959) derived an equation for determining T and S for a well of constant drawdown that is discharging by natural flow from an infinite leaky confined aquifer, and he also gave solutions for a circular leaky confined aquifer with zero drawdown on the outer boundary and for a closed circular aquifer. The equations for the infinite leaky confined aquifer follow:

$$T=\frac{Q}{2\pi s_w G(\alpha, r_w/B)}\quad [L^2T^{-1}], \tag{93}$$

where

$$\alpha=\frac{Tt}{Sr_w^2}\quad [\text{dimensionless}], \tag{94}$$

$$r_w/B=r_w\sqrt{T/(K'/b')}\quad [L^2], \tag{95}$$

and

$$G\left(\alpha,\frac{r_w}{B}\right)=\left(\frac{r_w}{B}\right)\frac{K_1(r_w/B)}{K_0(r_w/B)}+\frac{r}{\pi^2}\exp\left[-\alpha\left(\frac{r_w}{B}\right)^2\right]$$

$$\cdot\int_0^\infty \frac{u\exp(-\alpha u^2)}{J_0^2(u)+Y_0^2(u)}\cdot\frac{du}{u^2+(r_w/B)^2}\quad [\text{dimensionless}], \tag{96}$$

where

K_1 = Modified Bessel function of second kind, first order,
K_0 = Modified Bessel function of second kind, zero order,
J_0 = Bessel function of first kind, zero order,
Y_0 = Bessel function of second kind, zero order, and
u = variable of integration.

The integral in equation 96 cannot be integrated directly but was evaluated numerically, and values of the parameters are given by Hantush (1959, table 1) from which plate 5 was drawn after Walton (1962, pl. 4). When $B=\infty$, r_w/B and $K'b'$ (equal to T/B^2) $=0$, so that the parent-type curve on plate 5 is the same as on plate 1—the nonleaky-type curve of Jacob and Lohman (1952, fig. 5)—except, of course, that the values of the parameters differ.

On translucent logarithmic paper of the same scale as plate 5 (such as Codex 4123) values of Q are plotted on the vertical scale against values of t on the horizontal scale, and the data curve is superposed on plate 5. From the match point obtained by the usual curve matching procedure, preferably at $G(\alpha, r_w/B)$ and $\alpha=1.0$, values of the four parameters $G(\alpha, r_w/B)$, α, Q, and t are obtained. T is then determined using equation 93, and S is determined by rewriting equation 94:

$$S=\frac{Tt}{r_w^2\alpha}\quad [\text{dimensionless}]. \tag{97}$$

Unfortunately, I have no field data with which to illustrate this method.

UNCONFINED AQUIFERS WITH VERTICAL MOVEMENT

Boulton (1954a) derived an integral equation for the drawdown of the water table near a discharging well before the flow approaches steady state, which is founded partly on a consideration of vertical flow components, such as those that prevail near the well during the early stages of a pumping test in an unconfined aquifer. (See Stallman, 1961a.) In our notation, his partial differential equation describing the head (h) at the water table is

$$\frac{\partial h}{\partial t}=\frac{K}{S}\left[\left(\frac{\partial h}{\partial r}\right)^2+\left(\frac{\partial h}{\partial z}\right)^2-\frac{\partial h}{\partial z}\right]\quad [LT^{-1}]. \tag{98}$$

As equation 98 is nonlinear and cannot readily be solved, he assumes that the head gradients are small enough that their squares may be neglected, whence

$$\frac{\partial h}{\partial t}+\frac{K}{S}\frac{\partial h}{\partial z}=0\quad [LT^{-1}], \tag{99}$$

where h = pressure head ($p/g\rho$) plus elevation head (z).

Boulton's solution for an isotropic unconfined aquifer, in which the vertical and horizontal permeability are equal, is

$$s=\frac{Q}{2\pi Kb}\int_0^\infty \frac{J_0(\lambda\psi)}{\lambda}\left[1-\exp(\tau\lambda\tanh\lambda)\right]d\lambda \qquad [L], \qquad (100)$$

where

$$\left.\begin{aligned}\psi&=\frac{r}{b} &&[\text{dimensionless}],\\ \tau&=\frac{Kt}{Sb} &&[\text{dimensionless}],\end{aligned}\right\} \qquad (101)$$

and

J_0 = Bessel function of the first kind of zero order, and
λ = variable of integration.

For anisotropic aquifers, in which the vertical hydraulic conductivity, K_z, differs from the horizontal (radial) hydraulic conductivity, K_r, equation 101 becomes

$$\psi=\frac{r}{b}\sqrt{\frac{K_z}{K_r}}. \qquad (102)$$

Equation 100 may be written:

$$T=\frac{Q}{2\pi s}V(\psi,\tau) \qquad [L^2T^{-1}], \qquad (103)$$

where $V(\psi, \tau)$ = the V function of ψ and τ. When τ is sufficiently large, equation 103 reduces to the Theis equation (eq. 46). When τ is small, the Boulton equation 103 and the Theis equation (46) are related thus:

$$u=\frac{\psi^2}{4\tau}=\frac{r^2S}{4Tt} \qquad [\text{dimensionless}]. \qquad (104)$$

Boulton (1954a) gave a short table of values for $V(\psi, \tau)$ which was extended considerably by Stallman (1961b) with the aid of a digital computer. Stallman (1961a) also plotted values of $2V(\psi, \tau)$, or $W(u)$, versus $1/u$ for various values of ψ; values of $2V(\psi, \tau)$ versus ψ, for various values of τ; and s versus t/r^2, for values of ψ and τ, for pumping-test data for unconfined aquifers in Kansas and Nebraska.

From finite-difference expressions of partial differential equations similar to Boulton's, Stallman (1963a, 1965) designed electric-analog models simulating the assumed hydraulic model of an anisotropic aquifer, which he used to compute various values of the parameters for different penetrations of both pumping and observation wells. The principal results are given in his figures 10 and 12 (Stallman, 1965), which are here reproduced at larger scale on plates 6 and 7. These are nondimensional logarithmic plots of sT/Q versus Tt/r^2S, for observation wells at different values of ψ and for a pumping well for which ψ = 0.002. Plate 6 is for a fully penetrating pumping well, for five different penetrations of observation wells; plate 7 is for a pumping well open only for the bottom 0.3b and for the same five penetrations of observation wells.

For tests of aquifers whose values of K_z and K_r are suspected to differ appreciably, observed values of s versus t, t/r^2, or $1/r^2$ (for constant t) are plotted on translucent logarithmic graph paper of the same scale as plates 6 and 7 (such as K & E 359–125b or 46–7522, 3×5 cycle) and are fitted to the appropriate curve of plate 6 or 7 by the usual curve-matching procedure. From the four values of parameters at the match point, assuming that the match point is chosen so that both sT/Q and Tt/r^2S are equal to 1.0, T obviously is obtained from

$$T=1.0\frac{Q}{s} \qquad [L^2T^{-1}] \qquad (105)$$

and S is obtained from

$$S=\frac{Tt}{1.0r^2} \qquad [\text{dimensionless}]. \qquad (106)$$

Of course the values of any other match points, such as 10 or 10^{-1}, may be used in these equations, but the ones assumed are most convenient. Note that, in plotting his type curves, Stallman omitted the 4π and 4 from the parameters sT/Q and Tt/r^2S, respectively, thus omitting these pure numbers also in the computations using equations 105 and 106.

The relation of z to b in both the pumped and observation wells for curves on plates 6 and 7 is shown in figure 21. A well for which z = 0 would fully penetrate the aquifer but would be open only at the bottom. Dagan (1967) gave

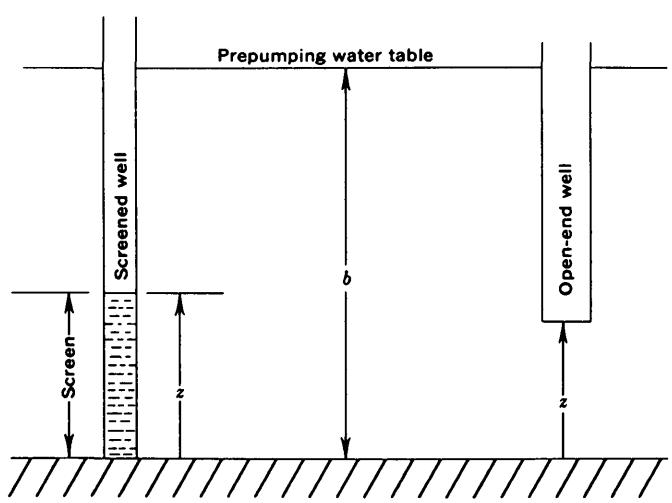

FIGURE 21.—Relation of z to b of pumped and observation wells on plates 6 and 7.

a digital computer solution for producing curves like those on plates 6 and 7 for any degree of penetration.

Boulton (1954b, 1963, 1964) also derived an equation to take account of the delayed yield from storage, which occurs in unconfined aquifers during the early part of the pumping. Boulton's (1963) differential equation is, in slightly modified notation,

$$T\left(\frac{\partial^2 s}{\partial r^2}+\frac{1}{r}\frac{\partial s}{\partial r}\right)=S_e\frac{\partial s}{\partial t}+\alpha S_l\int_0^t\frac{\partial s}{\partial t}e^{-\alpha(t-\tau)}d\tau \quad [LT^{-1}], \tag{107}$$

where

$$\alpha=\left(\frac{r}{B}\right)^2 T\Big/ r^2 S_l,$$

where

$$B=\sqrt{\frac{T}{\alpha S_l}},$$

and

S_e = early time apparent specific yield,
S_l = later time specific yield, and
τ = variable of integration.

When $n=\infty$, where

$$n=\frac{S_e+S_l}{S_e},$$

Boulton's solution of equation 107, for the drawdown at distance r from a pumped well that completely penetrates the aquifer, is

$$s=\frac{Q}{4\pi T}\int_0^\infty 2J_0\left(\frac{r}{B}x\right)\left\{1-\frac{1}{x^2+1}\exp\left(-\frac{\alpha t x^2}{x^2+1}\right)-\epsilon\right\}\frac{dx}{x} \quad [L], \tag{108}$$

where

J_0 = Bessel function of the first kind of zero order,
x = variable of integration,

and

$$\epsilon=\frac{x^2}{x^2+1}\exp\{-\alpha n t(x^2+1)\}.$$

For sufficiently small values of t, equation 108 becomes equal to equation 85, the leaky confined aquifer equation of Hantush and Jacob (1955).

Boulton (1963, p. 480, 481) gives tables of solutions of equation 108 for his W function $(4\pi Ts/Q)$ for various values of, in our notation, $1/u_e=4Tt/r^2S_e$, for his type A curves, for various values of $1/u_l=4Tt/r^2S_l$, for his type B curves, and for various values of r/B. Families of Boulton delayed-yield type curves based upon these tabulated values are shown on plate 8, which is similar to Boulton's (1963) figure 1. His type A curves $(1/u_e)$ are shown to the left of the break in the curves; his type B curves $(1/u_l)$ are shown to the right of the break. Note that the type A curves are essentially the same as those shown on plate 3A for leaky confined aquifers. Note also that the Theis type curve is asymptotic to the left of the type A family of curves and to the right of the type B family.

Logarithmic time-drawdown plots for tests of unconfined aquifers in which delayed yield from storage is suspected may be superposed on plate 8, and a match point may be obtained for a suitable value of r/B. From the four parameters s, t, $4\pi Ts/Q$, and $4Tt/r^2S_e$ or $4Tt/r^2S_l$ thus obtained, the desired values of T and S_e or S_l may be obtained as follows, assuming that the dimensionless parameters chosen on plate 8 are both equal to 1.0:

$$T=\frac{(1.0)Q}{4\pi s} \quad [L^2T^{-1}]. \tag{109}$$

For early values of t,

$$S_e=\frac{4Tt}{r^2(1.0)} \quad \text{[top scale, dimensionless]}; \tag{110}$$

for later values of t,

$$S_l=\frac{4Tt}{r^2(1.0)} \quad \text{[bottom scale, dimensionless]}. \tag{111}$$

Example for Anisotropic Aquifer

Table 13 gives the time-drawdown data for an observation well of $z=0.5b$ which was 63.0 ft from a fully penetrating, fully screened well $(z=b)$ pumped at an average rate of 1,170 gpm, near Ione, Colo. The wells are in unconfined alluvium having a prepumping saturated thickness (b) of 39.4 ft. The pumped well is 56.5 ft deep and the observation well is 25.8 ft $(0.52b)$ deep. Figure 22 is a logarithmic plot of the data given in table 13, and also it shows the values of the four parameters at the match point obtained by superposing figure 22 on plate 6D.

From equation 105,

$$T=\frac{(1.0)\,(1{,}170\text{ gal min}^{-1})\,(1{,}440\text{ min day}^{-1})}{(10.3\text{ ft})\,(7.48\text{ gal ft}^{-3})}$$

$$=2.2\times10^4\text{ ft}^2\text{ day}^{-1}.$$

From equation 106,

$$S=\frac{(2.2\times10^4\text{ ft}^2\text{ day}^{-1})\,(52\text{ min})}{(1.0)\,(63\text{ ft})^2(1{,}440\text{ min day}^{-1})}=0.2\text{ (rounded)}.$$

Using equation 102,

$$\psi=\frac{r}{b}\sqrt{\frac{K_z}{K_r}}=0.9,$$

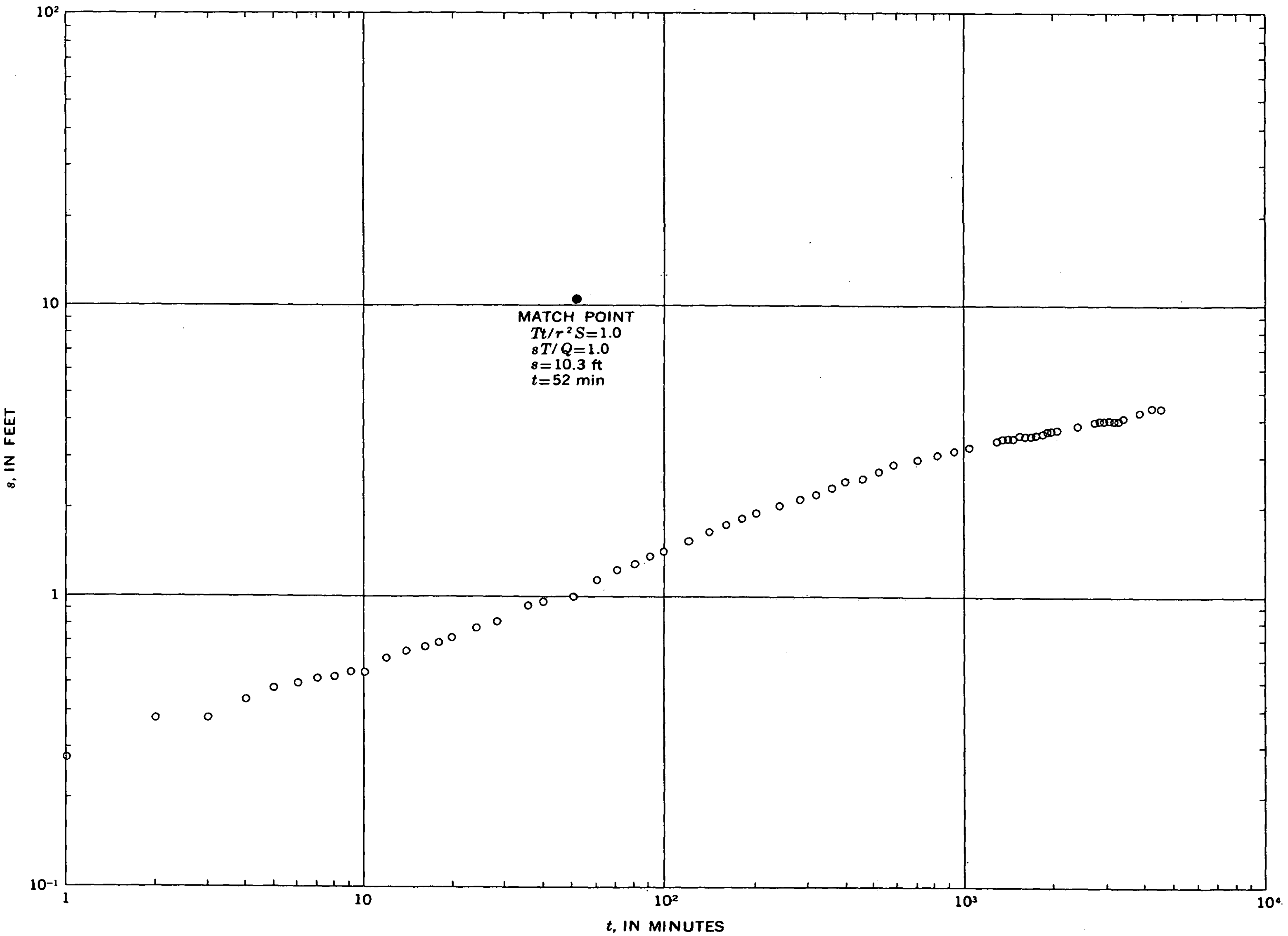

FIGURE 22.—Logarithmic plot of s versus t for observation well B2–66–7dda2, near Ione, Colo.

TABLE 13.—*Drawdown of water level in observation well B2-66-7dda2, 63.0 ft from a well pumping at average rate of 1,170 gpm, near Ione, Colo., August, 15-18 1967*

[Data from D. R. Albin, written commun., January 1968]

Time since pumping began, t (min)	Corrected drawdown, s (ft)	Time since pumping began, t (min)	Corrected drawdown, s (ft)
1	0.28	520	2.66
2	.38	580	2.74
3	.38	700	2.91
4	.44	820	3.02
5	.48	940	3.17
6	.50	1,060	3.22
7	.52	1,300	3.41
8	.53	1,360	3.44
9	.56	1,420	3.48
10	.56	1,480	3.48
12	.61	1,540	3.51
14	.65	1,600	3.56
16	.67	1,660	3.57
18	.70	1,720	3.59
20	.72	1,810	3.64
24	.79	1,900	3.67
28	.82	1,960	3.70
36	.92	2,020	3.73
40	.96	2,380	3.84
50	1.00	2,740	3.94
60	1.15	2,800	3.96
70	1.24	2,860	3.97
80	1.30	2,920	3.98
90	1.38	2,980	3.99
100	1.42	3,040	4.00
120	1.55	3,100	4.01
140	1.67	3,160	4.02
160	1.74	3,220	4.04
180	1.84	3,280	4.03
200	1.93	3,340	4.05
240	2.05	3,400	4.05
280	2.17	3,460	4.07
320	2.27	3,820	4.14
360	2.36	4,180	4.20
400	2.48	4,240	4.21
460	2.55	4,270	4.20

TABLE 14.—*Drawdown of water level in observation well 139, 73 ft from a well pumping at constant rate of 1,080 gpm, near Fairborn, Ohio, October 19-21, 1954*

[Data from S. E. Norris (written commun., Apr. 29, 1968)]

Time since pumping began, t (min)	Corrected drawdown, s (ft)	Time since pumping began, t (min)	Corrected drawdown, s (ft)
0.165	0.12	10	1.02
.25	.195	12	1.03
.34	.255	15	1.04
.42	.33	18	1.05
.50	.39	20	1.06
.58	.43	25	1.08
.66	.49	30	1.13
.75	.53	35	1.15
.83	.57	40	1.17
.92	.61	50	1.19
1.00	.64	60	1.22
1.08	.67	70	1.25
1.16	.70	80	1.28
1.24	.72	90	1.29
1.33	.74	100	1.31
1.42	.76	120	1.36
1.50	.78	150	1.45
1.68	.82	200	1.52
1.85	.84	250	1.59
2.00	.86	300	1.65
2.15	.87	350	1.70
2.35	.90	400	1.75
2.50	.91	500	1.85
2.65	.92	600	1.95
2.80	.93	700	2.01
3.0	.94	800	2.09
3.5	.95	900	3.15
4.0	.97	1,000	2.20
4.5	.975	1,200	2.27
5.0	.98	1,500	2.35
6.0	.99	2,000	2.49
7.0	1.00	2,500	2.59
8.0	1.01	3,000	2.66
9.0	1.015		

by interpolation,

$$\frac{K_z}{K_r}=\left[\frac{(0.9)\,(39.4\text{ ft})}{(63.0\text{ ft})}\right]^2=0.3,$$

$$K_z=0.3K_r\text{ ft day}^{-1},$$

and

$$K_r=\frac{T}{b}=\frac{2.2\times10^4\text{ ft}^2\text{ day}^{-1}}{39.4\text{ ft}}=560\text{ ft day}^{-1};$$

therefore,

$$K_z=(0.3)\,(560\text{ ft day}^{-1})=168\text{ ft day}^{-1}.$$

For additional examples of this method and evaluations of results, see Norris and Fidler (1966).

EXAMPLE FOR DELAYED YIELD FROM STORAGE

Table 14 gives the time-drawdown measurements in an observation well 73 ft from a well pumping at constant rate of 1,080 gpm near Fairborn (near Dayton), Ohio, supplied by S. E. Norris (U.S. Geological Survey, Columbus, Ohio, written commun., Apr. 29, 1968). The pumped well, which is 85 ft deep and is reportedly screened to full depth, obtains water from glacial sand and gravel. The observation well is 95 ft deep, but it penetrates only 75 ft of water-bearing material, the rest being 20 feet of clay in four beds. This is the same test as that for observation well 1 analyzed by Boulton (1963, fig. 2, p. 475–476) and by Walton (1960). The water-level measurements from 0 to 2.80 min were made using a technique described by Walton (1963). A logarithmic plot of s versus t from table 14 is shown in figure 23, which also shows the match-point values of the four parameters obtained by superposition on plates 6*C* and 8.

Using the parameters of the lower match point in figure 23 for Boulton's type B curves on plate 8, T is obtained from equation 109:

$$T=\frac{(1.0)\,(1.08\times10^3\text{ gal min}^{-1})\,(1.44\times10^3\text{ min day}^{-1})}{(1.257\times10^1)\,(4.22\times10^{-1}\text{ ft})\,(0.748\times10^1\text{ gal ft}^{-3})}$$

$$=4\times10^4\text{ ft}^2\text{ day}^{-1}\text{ (rounded)}.$$

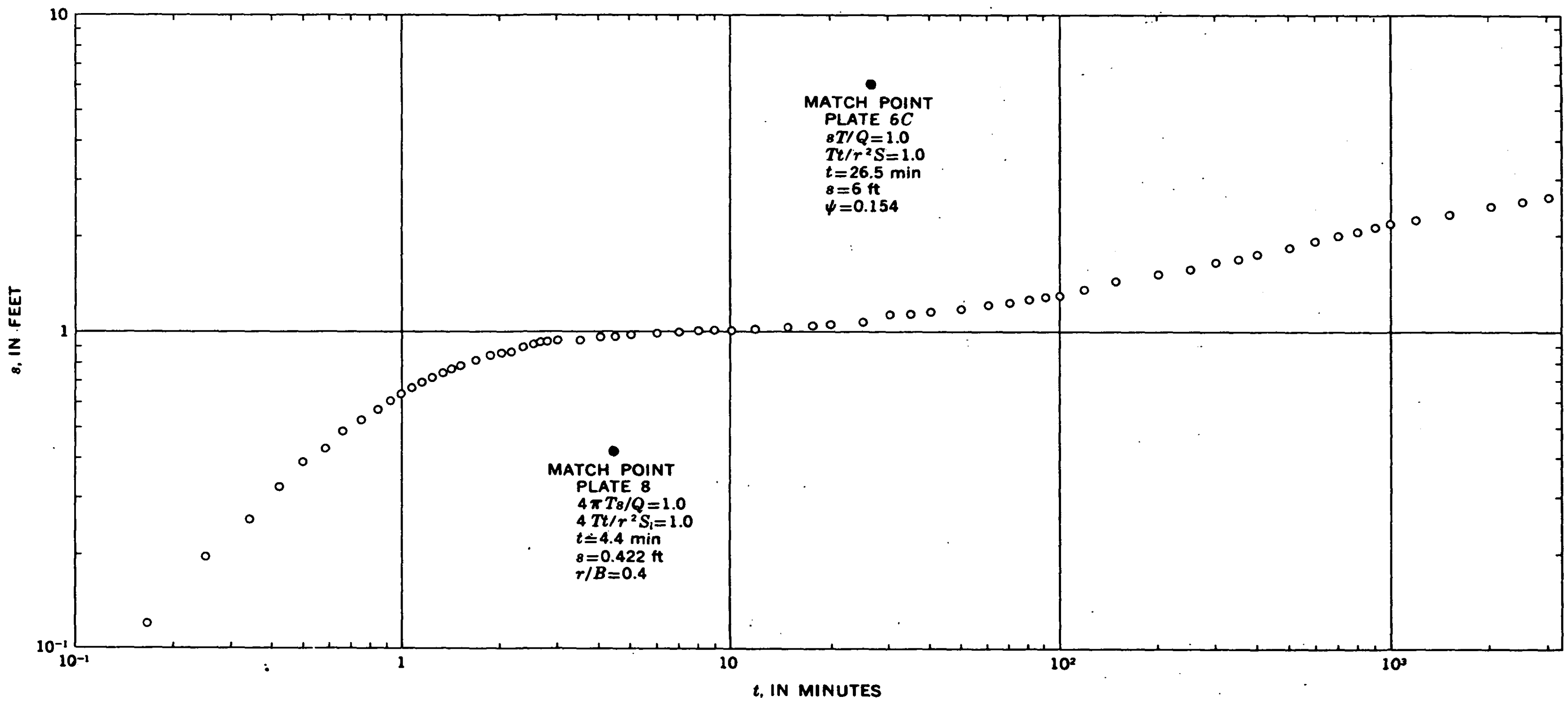

FIGURE 23.—Logarithmic plot of s versus t for observation well 139, near Fairborn, Ohio. From Walton (1960, fig. 4).

Similarly, using equation 111,

$$S_l = \frac{(4)\,(4\times 10^4 \text{ ft}^2 \text{ day}^{-1})\,(4.4 \text{ min})}{(5.33\times 10^3 \text{ ft}^2)\,(1.44\times 10^3 \text{ min day}^{-1})} = 0.09.$$

Matching the early data to Boulton's type A curves gave the same value for T, but a value of S_e of 3×10^{-3}. This value of S_e seems to be about one order of magnitude too large for a confined aquifer less than 100 ft thick; on the other hand, the value of S_e seems too small for the unconfined aquifer and suggests that it is only an apparent value observed before gravity drainage was completed.

Using the upper match point in figure 23 for Stallman's type curve in plate 6C, T is obtained from equation 105:

$$T = \frac{(1.0)\,(1.08\times 10^3 \text{ gal min}^{-1})\,(1.44\times 10^3 \text{ min day}^{-1})}{(6 \text{ ft})\,(0.748\times 10^1 \text{ gal ft}^{-3})}$$

$$= 3.5\times 10^4 \text{ ft}^2 \text{ day}^{-1},$$

which is of the same order of magnitude as that obtained from Boulton's curves (pl. 8).

Similarly, from equation 106,

$$S = \frac{(3.5\times 10^4 \text{ ft}^2 \text{ day}^{-1})\,(26.5 \text{ min})}{(5.33\times 10^3 \text{ ft}^2)\,(1.44\times 10^3 \text{ min day}^{-1})} = 0.1 \text{ (rounded)},$$

which is virtually identical to the 0.09 value. As the observation well is reported to be fully screened through the aquifer, figure 23 should have matched one of the type curves on plate 6A. The fact that it exactly matches the curve for $\psi = 0.154$ on plate 6C for $z = 0.75b$ suggests that the intercalated clay beds may have changed the shape of the response curves, but this is only speculation.

From equation 102,

$$\frac{73}{78}\sqrt{\frac{K_z}{K_r}} = 0.154,$$

$$\frac{K_z}{K_r} = \left[\frac{(0.154)\,(78)}{73}\right]^2 = 0.027,$$

$$K_r = \frac{3.5\times 10^4 \text{ ft}^2 \text{ day}^{-1}}{.78\times 10^2 \text{ ft}} = 4.5\times 10^2 \text{ ft day}^{-1},$$

and

$$K_z = 2.7\times 10^{-2}\times 4.5\times 10^2 \text{ ft day}^{-1} = 12 \text{ ft day}^{-1}.$$

The low vertical hyraulic conductivity compared to the radial value indicates that the aquifer is anisotropic and suggests a valid reason for the delayed drainage from storage, even after some 50 hours of pumping. This also suggests the desirability of trying both plates 6 or 7 and 8 for matching data curves similar to figure 23, knowledge of the local geology may help decide on which results to choose if they differ significantly.

AQUIFER TESTS BY CHANNEL METHODS—LINE SINK OR LINE SOURCE (NONSTEADY FLOW, NO RECHARGE)

CONSTANT DISCHARGE

In 1938 C. V. Theis (Wenzel and Sand, 1942, p. 45) developed an equation for determining the decline in head at any distance from a drain discharging water at a constant rate from a confined aquifer. The equation is based upon the following assumptions: The aquifer is homogeneous, isotropic, and of semi-infinite areal extent (bounded on one side only by the drain); the discharging drain completely penetrates the aquifer; the aquifer is bounded above and below by impermeable strata; the flow is laminar and unidimensional; the release of water from storage is instantaneous and in proportion to the decline in head; and the drain discharges at a constant rate. Theis derived his equation by analogy with heat flow in an analogous thermal system; later Ferris (1950) derived a similar equation from hydrologic concepts. In slightly modified form, Ferris' equation (Ferris and others, 1962, p. 123) may be written:

$$T = \frac{Q_b x}{2s}\left[\frac{e^{-u^2}}{u\sqrt{\pi}} - 1 + \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{Tt/S}} e^{-u^2} du\right] \quad [L^2T^{-1}], \tag{112}$$

where

$$u = x\sqrt{\frac{S}{4Tt}} \quad \text{[dimensionless]} \tag{113}$$

or

$$u^2 = \frac{x^2 S}{4Tt} \quad \text{[dimensionless]},$$

and

$$S = \frac{4Ttu^2}{x^2} \quad \text{[dimensionless]}, \tag{114}$$

s = drawdown at any point in the vicinity of the drain,
Q_b = Constant discharge rate (base flow) of the drain, per unit length of drain,
x = distance from drain to point of observation, and
t = time since drain began discharging.

The part of equation 112 in brackets may be written $D(u)_q$, the drain function of u; the subscript q identifies the constant discharge of the drain. Equation 112, therefore, may be written:

$$T = \frac{Q_b x}{2s} D(u)_q \quad [L^2T^{-1}]. \tag{115}$$

Values of $D(u)_q$ for corresponding values of u and u^2 are

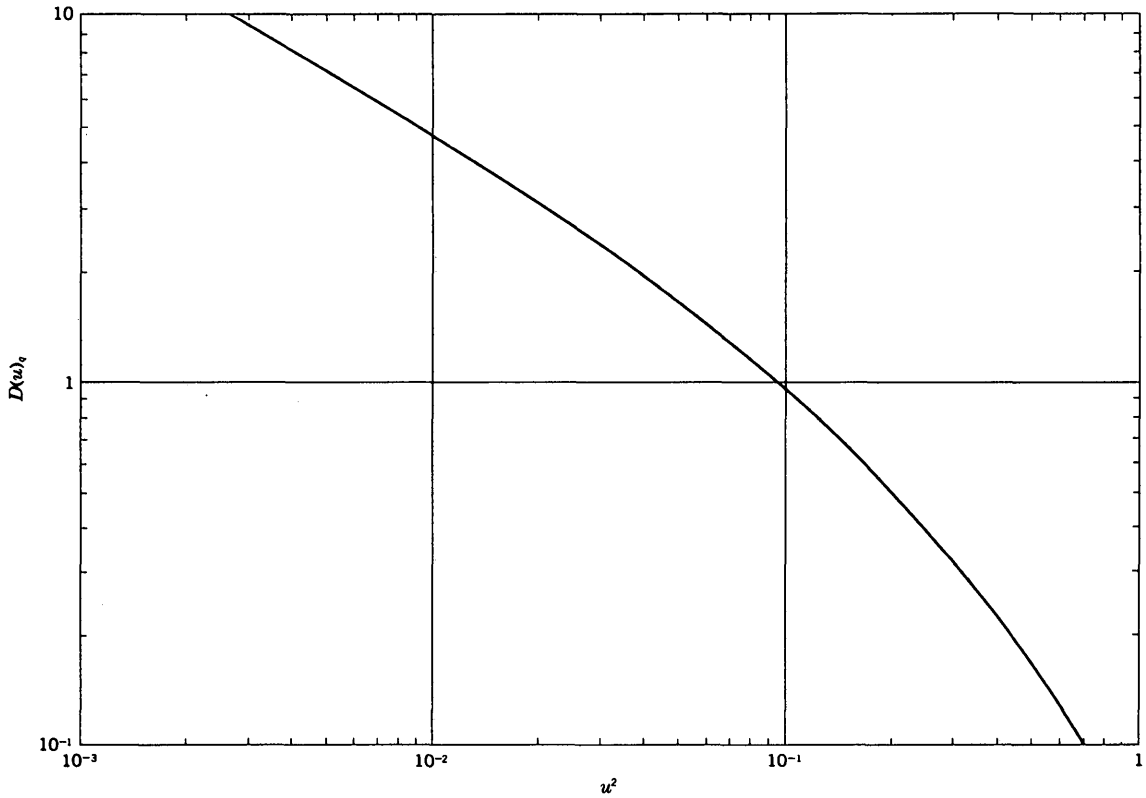

FIGURE 24.—Logarithmic plot of $D(u)_q$ versus u^2 for channel method—constant discharge.

given in table 15, and a logarithmic plot of $D(u)_q$ versus u^2 is shown in figure 24.

Observed values of s versus x^2/t are plotted on translucent logarithmic graph paper of the same scale as figure 24 (such as K&E 358–112) and are fitted to figure 24 by the usual curve-matching procedure. From the four values of parameters at the match point, assuming that the match point is chosen so that both $D(u)_q$ and u^2 are equal to 1.0, T is obtained from

$$T=\frac{Q_b x(1.0)}{2s} \quad [L^2T^{-1}] \qquad (116)$$

and S is obtained from

$$S=\frac{4T(1.0)}{x^2/t} \quad [\text{dimensionless}]. \qquad (117)$$

Unfortunately reliable field data to illustrate the method were not available.

TABLE 15.—*Values of* $D(u)_q$, u, *and* u^2 *for channel method—constant discharge*

[From Ferris, Knowles, Brown, and Stallman (1962, table 5)]

u	u^2	$D(u)_q$	u	u^2	$D(u)_q$
0.0510	0.0026	10.091	0.2646	0.070	1.280
.0600	.0036	8.437	.3000	.090	1.047
.0700	.0049	7.099	.3317	.110	.8847
.0800	.0064	6.097	.3605	.130	.7641
.0900	.0081	5.319	.4000	.160	.6303
.1000	.010	4.698	.4359	.190	.5327
.1140	.013	4.013	.4796	.230	.4370
.1265	.016	3.531	.5291	.280	.3516
.1414	.020	3.069	.5745	.330	.2895
.1581	.025	2.657	.6164	.380	.2426
.1732	.030	2.355	.6633	.440	.1996
.1871	.035	2.120	.7071	.500	.1666
.2000	.040	1.933	.7616	.580	.1333
.2236	.050	1.648	.8124	.660	.1084
.2449	.060	1.440	.8718	.760	.08503
			.9487	.900	.06207
			1.0000	1.000	.05026

CONSTANT DRAWDOWN

Stallman (in Ferris and others, 1962, p. 126–131) found a solution for a similar drain, in which the head abruptly changes by a constant amount and the discharge declines slowly, by borrowing the solution to an analogous heat-flow problem (Ingersol and others, 1954, p. 88). The basic assumptions are the same as those for equation 112 just described. Stallman's equation is

$$s=s_0\left[1-\frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{Tt/S}} e^{-u^2}du\right]=s_0D(u)_h \quad [L], \qquad (118)$$

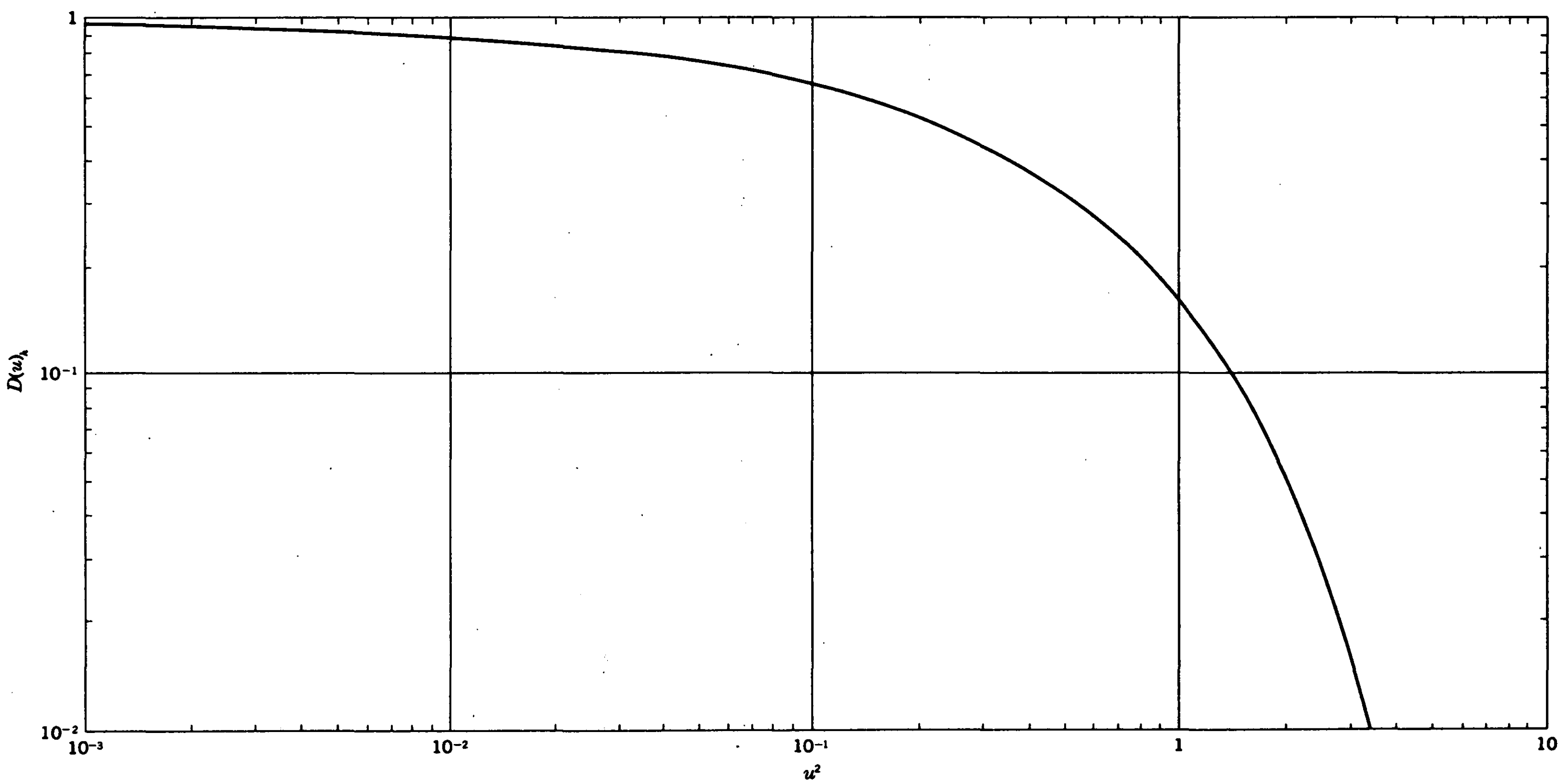

FIGURE 25.—Logarithmic plot of $D(u)_h$ versus u^2 for channel method—constant drawdown.

TABLE 16.—*Values of* $D(u)_h$, u, *and* u^2 *for channel method—constant drawdown*

[From Ferris, Knowles, Brown, and Stallman (1962, table 6)]

u	u^2	$D(u)_h$	u	u^2	$D(u)_h$
0.03162	0.0010	0.9643	0.6325	0.40	0.3711
.04000	.0016	.9549	.7746	.60	.2733
.05000	.0025	.9436	.8944	.80	.2059
.06325	.0040	.9287	1.000	1.00	.1573
.07746	.0060	.9128	1.140	1.30	.1069
.08944	.0080	.8994	1.265	1.60	.0736
.1000	.010	.8875	1.378	1.90	.0513
.1265	.016	.8580	1.483	2.20	.0359
.1581	.025	.8231	1.581	2.50	.0254
.2000	.040	.7730	1.643	2.70	.0202
.2449	.060	.7291	1.732	3.00	.0143
.2828	.080	.6892	1.789	3.20	.0114
.3162	.10	.6548			
.4000	.16	.5716			
.5000	.25	.4795			

where

s_0 = the abrupt change in drain level at $t=0$.

$D(u)_h$ represents the bracketed part of equation 118 and is the drain function of u for constant drawdown, and where

$$u^2=\frac{x^2S}{4Tt} \quad [\text{dimensionless}], \qquad (119)$$

the bracketed part of equation 118 is the complementary error function, cerf, solutions of which are available.

The discharge of the aquifer from both sides of the drain per unit length of drain, Q_b, resulting from the change in drain stage, s_0, is

$$Q_b=\frac{2s_0}{\sqrt{\pi t}}\sqrt{ST} \quad [L^2T^{-1}]. \qquad (120)$$

Solving equation 120 for ST, we obtain

$$ST=\frac{Q_b{}^2\pi t}{4s_0{}^2} \quad [L^2T^{-1}]. \qquad (121)$$

Dividing equation 121 by equation 119 to eliminate S, and replacing s_0 by $s/D(u)_h$,

$$T=\frac{Q_bxD(u)_h}{4su}\sqrt{\pi} \quad [L^2T^{-1}]. \qquad (122)$$

Solving equation 119 for S,

$$S=\frac{4Tu^2}{x^2/t} \quad [\text{dimensionless}]. \qquad (123)$$

Values of $D(u)_h$ for corresponding values of u and u^2 are given in table 16, and a logarithmic plot of $D(u)_h$ versus u^2 is shown in figure 25.

Observed values of s versus x^2/t are plotted on translucent logarithmic graph paper of the same scale as figure 25 (such as K&E 358–112) and are fitted to figure 25 by the usual curve-matching procedure. From the four values of the parameters at the match point, assuming that the match point is chosen so that both $D(u)_h$ and u^2 are equal to 1.0, whence u is also equal to 1.0, T is obtained by rewriting equation 122,

$$T=\frac{Q_bx(1.0)}{4s(1.0)}\sqrt{\pi} \quad [L^2T^{-1}], \qquad (124)$$

and S is obtained from equation 123 using the value of T determined from equation 124,

$$S=\frac{4T(1.0)}{x^2/t} \quad [\text{dimensionless}]. \qquad (125)$$

Unfortunately, field data to illustrate the method were not available to me, but the method was successfully used by Bedinger and Reed (1964). (See also Pinder and others, 1969.)

Jacob (1943) developed methods for an unconfined aquifer subject to a constant rate of recharge (W) and bounded by two parallel and assumedly fully penetrating streams. The base flow of streams or the average rate of ground-water recharge may be estimated from the shape of the water table, as determined from water-level measurements in wells, in such a bounded aquifer. Rorabaugh (1960) gave methods, equations, and charts for estimating the aquifer constant T/S (hydraulic diffusivity) from natural fluctuations of water levels in observation wells in finite aquifers having parallel boundaries. Examples of such aquifers are: a long island or peninsula, an aquifer bounded by parallel streams, and an aquifer bounded by a stream and a valley wall. For similar bounded aquifers, Rorabaugh (1964) also developed methods for estimating ground-water outflow into streams and for forecasting streamflow recession curves. The component of outflow related to bank storage is computed from river fluctuations; the component related to recharge from precipitation and irrigation is computed from water levels in a well. Rorabaugh's methods have widespread application in areas having the required boundary conditions.

AQUIFER TESTS BY AREAL METHODS

NUMERICAL ANALYSIS

The equations given above for the radial flow of ground water were derived from ordinary or partial differential equations by means of the calculus, for various assumed boundary conditions. Stallman (1956, 1962) showed that, after the manner of Southwell (1940, 1946), the partial differential equation for two-dimensional nonsteady flow in an unconfined homogeneous and isotropic aquifer subject to a steady rate of accretion, W, can be closely approximated by a finite-difference equation in which, for example, $\partial h/\partial t$ is replaced by $\Delta h/\Delta t$. He has since (written commun., 1965) developed a simplified application for use during winter periods when there is little or no transpiration from plants and no recharge from precipitation and, hence, when $W=0$. He (later he and C. T. Jenkins) developed comparable equations for nonhomogeneous isotropic aquifers (R. W. Stallman and C. T. Jenkins, written commun., January 1969).

For homogeneous isotropic aquifers, the equations with and without W are

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t}-\frac{W}{T} \quad [L^{-1}] \qquad (126)$$

and

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t} \quad [L^{-1}] \qquad (127)$$

where h is the head at any point whose coordinates are x and y. Let the infinitesimal lengths dx and dy be expanded so that each is equivalent to a finite length, a, and similarly, let dt be considered equivalent to Δt. A plan representation of the region of flow to be studied may then be subdivided by two systems of equally spaced parallel lines at right angles to each other. One system is oriented in the x direction and the other, in the y direction; the spacing of lines equals the distance a. A set of five gridline intersections, or nodes (observation wells), as shown in figure 26, is called an array.

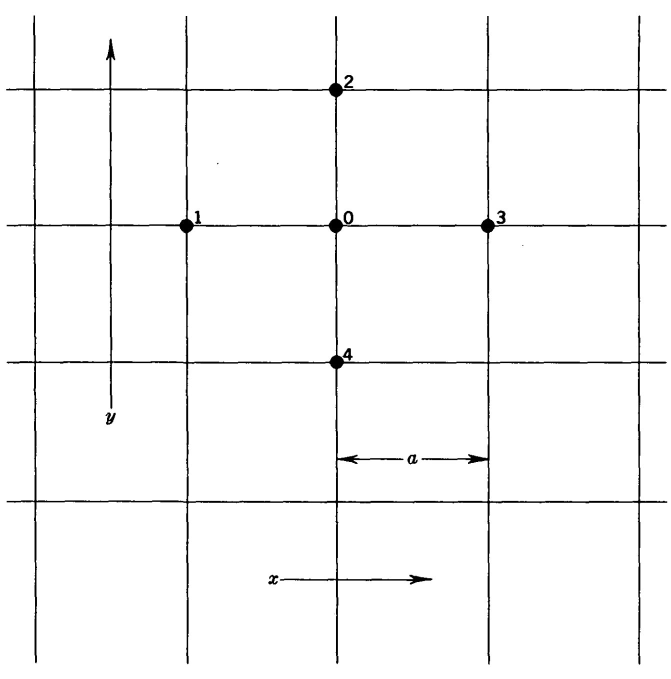

FIGURE 26.—Array of nodes used in finite-difference analysis.

The first two differentials in equations 126 and 127 can be expressed in terms of the head values at the nodes (wells) in the array, thus

$$\frac{\partial^2 h}{\partial x^2}\approx\frac{h_1+h_3-2h_0}{a^2} \quad [L^{-1}]$$

and

$$\frac{\partial^2 h}{\partial y^2}\approx\frac{h_2+h_4-2h_0}{a^2} \quad [L^{-1}]$$

where the subscripts refer to the numbered nodes in figure 26. Substituting these closely equivalent expressions in equations 126 and 127, and letting $\partial h/\partial t$ be considered equivalent to $\Delta h_0/\Delta t$, we obtain

$$h_1+h_2+h_3+h_4-4h_0=\sum h\approx\frac{a^2S}{T}\frac{\Delta h_0}{\Delta t}-\frac{a^2W}{T} \quad [L] \qquad (128)$$

and

$$h_1+h_2+h_3+h_4-4h_0=\sum h\approx\frac{a^2S}{T}\frac{\Delta h_0}{\Delta t} \quad [L] \qquad (129)$$

where Δh_0 is the change in head at node (well) 0 during the time interval Δt.

EXAMPLE

R. W. Stallman tried this method on several such arrays in the Arkansas River valley, Colorado, during the winter of 1965–66 and the summer of 1966. Wells 1–4 were spaced 1,000 ft apart so that $a=1{,}000$ ft$\sqrt{2}/2=707$ ft, and $a^2=5\times10^5$ ft^2. From estimated values of T and S, a normally is determined from the convenient empirical relation $a^2S/T=$about 10 days, but in the Arkansas River valley, nearby boundaries made it necessary to use $a^2S/T=$about 4 days. The elevations of the measuring points at each of the five wells were determined by precise leveling above a convenient arbitrary datum, and the water levels in feet above datum were obtained from automatic water-level sensors.

The winter data from a test near Lamar, Colo., are shown in figure 27. The slope of the straight line in figure 27 is $\Delta\sum h/(\Delta h_0/\Delta t)=4.25$ days, whence $a^2S/T=4.25$ days. S was obtained from neutron-moisture-probe tests (see Meyer, 1962), made during periods of both high and low water table, and was determined to be about 0.18. Then, using equation 129,

$$T \approx \frac{a^2S}{\Delta \sum h/(\Delta h_0/\Delta t)} \approx \frac{(5\times10^5\ \text{ft}^2)(0.18)}{4.25\ \text{days}}$$

$$\approx 2\times10^4\ \text{ft}^2\ \text{day}^{-1}\ \text{(rounded)}.$$

The straight line in figure 27 has been transferred to the plot of spring and summer data shown in figure 28. In figure 28, points to the right of the straight line indicating $W=0$ show recharge to the water table; those to the left show discharge from the water table by evapotranspiration. The average value of $\sum h$ above the line is about 0.1 ft. Using $T\approx 2\times10^4$ ft^2 day^{-1} and $a^2=5\times10^5$ ft^2, from equation 128,

$$0.1\ \text{ft} \approx -\frac{(5\times10^5\ \text{ft}^2)(W)}{2\times10^4\ \text{ft}^2\ \text{day}^{-1}}$$

and

$$W \approx -\frac{(2\times10^4\ \text{ft}^2\ \text{day}^{-1})(0.1\ \text{ft})}{5\times10^5\ \text{ft}^2}$$

$$\approx -4\times10^{-3}\ \text{ft day}^{-1}\ \text{(rounded)}.$$

FLOW-NET ANALYSIS

The following discussion of flow-net analysis has been adapted in part from Bennett (1962) and from Bennett and Meyer (1952, p. 54–58), to whose reports you are referred for further details.

In analyzing problems of steady ground-water flow, a graphical representation of the flow pattern may be of considerable assistance and may provide solutions to problems not readily amenable to mathematical solution. The first significant development in graphical analysis of flow patterns was made by Forchheimer (1930), but additional information was given by Casagrande (1937, p. 136, 137) and Taylor (1948).

A flow net, which is a graphical illustration of a flow pattern, is composed of two families of lines or curves. (See fig. 30.) One family of curves, called equipotential lines (solid lines on map), represents contours of equal head in the aquifer on the potentiometric surface or on the water table. Intersecting the equipotential lines at right angles (in isotropic aquifers) is another family of curves (dashed lines on map) representing the streamlines, or flow lines, where each curve indicates the path followed by a particle (molecule) of water as it moves through the aquifer in the direction of decreasing head.

Although the real flow pattern contains an infinity of possible flow and equipotential lines, it may be represented conveniently by constructing a net that uses only a few such lines, the spacing being conveniently determined by the contour interval of the equipotential lines. The contour interval indicates that the total drop in head in the system is evenly divided between adjacent pairs of equipotential lines; similarly the flow lines are selected so that the total flow is equally divided between adjacent pairs of flow lines. The movement of each particle of water between adjacent equipotential lines will be along flow paths involving the least work, hence it follows that, in isotropic aquifers, such flow paths will be normal to the equipotential lines, and the paths are drawn orthogonal to the latter.

The net is constructed so that the two sets of lines form a system of "squares." Note on the map that some of the lines are curvilinear, but that the "squares" are constructed so that the sum of the lengths of each line in one system is closely equal to the sum of the lengths in the other system. Figure 29 represents one idealized "square" of figure 30, whose dimensions are Δw and Δl. By rewriting Darcy's law (eq. 26) as a finite-difference equation for the flow, ΔQ, through this elemental "square" of thickness b, we obtain

$$\Delta Q = -Kb\Delta w\frac{\Delta h}{\Delta l} = -T\Delta w\frac{\Delta h}{\Delta l} \qquad [L^3T^{-1}]. \qquad (130)$$

But $\Delta w = \Delta l$, by construction, so

$$\Delta Q = -T\Delta h \qquad [L^3T^{-1}]. \qquad (131)$$

If n_f = number of flow channels, n_d = number of potential drops, and Q = total flow, then

$$Q = n_f\Delta Q, \quad \text{or} \quad \Delta Q = \frac{Q}{n_f} \qquad [L^3T^{-1}], \qquad (132)$$

and

$$h = n_d\Delta h, \quad \text{or} \quad \Delta h = \frac{h}{n_d} \qquad [L]. \qquad (133)$$

Substituting equations 132 and 133 in equation 131, we obtain

$$Q = -T\frac{n_f}{n_d}h \qquad [L^3T^{-1}], \qquad (134)$$

or

$$T = -\frac{Q}{(n_f/n_d)h} \qquad [L^2T^{-1}]. \qquad (135)$$

Example

According to Bennett and Meyer (1952, p. 55), the average discharge from the Patuxent Formation in the Sparrows Point district in 1945 was 1 million ft^3 day^{-1}. The map (fig. 30) shows 15 flow channels surrounding the

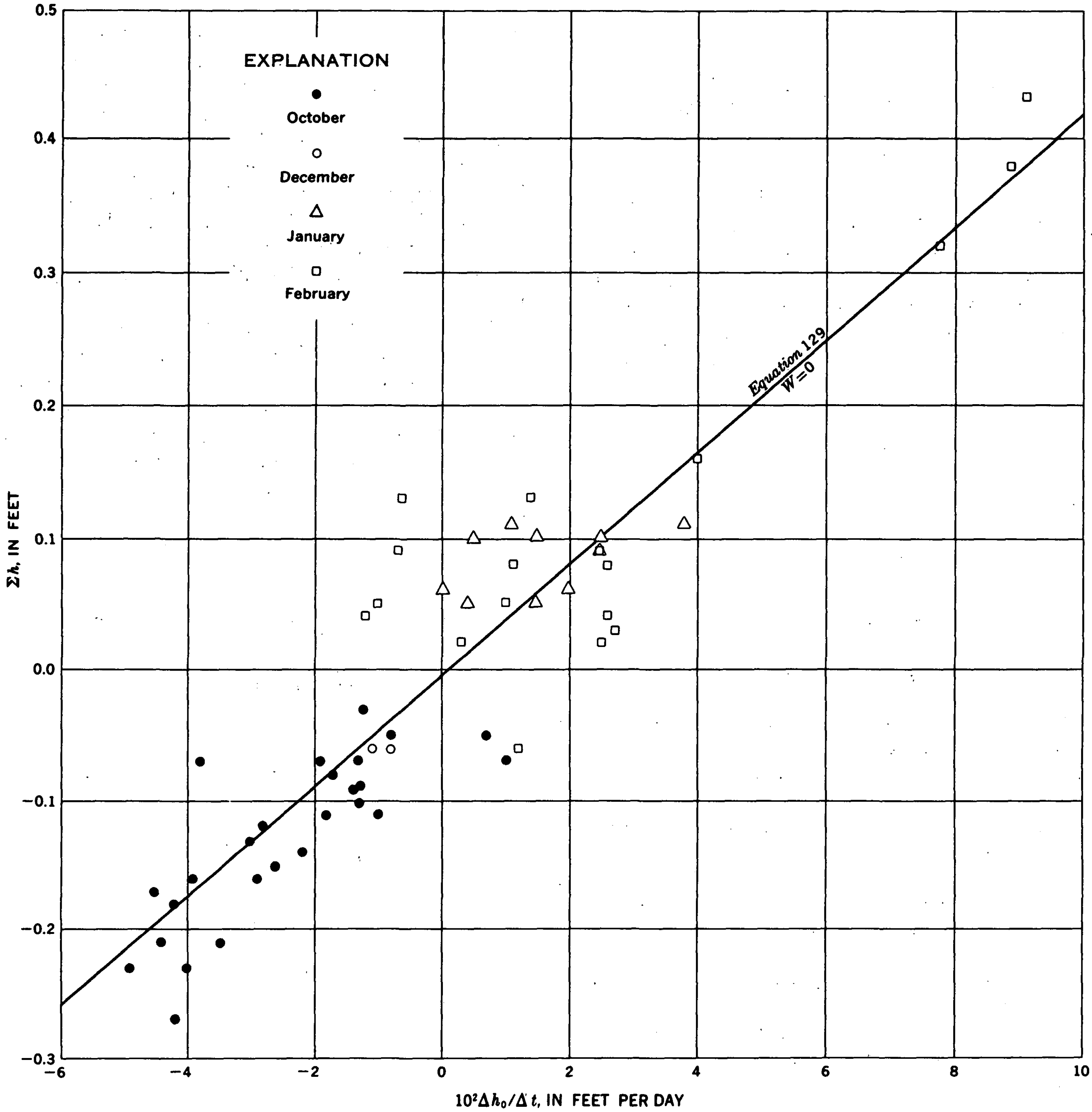

FIGURE 27.—Plot of Σh versus $\Delta h_0/\Delta t$ for winter of 1965–66, when $W=0$.

district, hence $n_f=15$. The number of equipotential drops between the 30- and 60-ft contours is three, so $n_d=3$. The total potential drop between the 30- and 60-ft contours is 30 ft, so $h=30$ ft. Then, from equation 135,

$$T=-\frac{10^6 \text{ ft}^3 \text{ day}^{-1}}{(15/3)(-30 \text{ ft})}=6{,}670 \text{ ft}^2 \text{ day}^{-1}=6{,}700 \text{ ft}^2 \text{ day}^{-1}.$$

Note that the value of T thus determined is for a much larger sample of the aquifer than that determined by a pumping test on a single well. This method has been largely neglected and is deserving of more widespread application.

CLOSED-CONTOUR METHOD

A water-level contour map containing closed contours around a well or group of wells of known discharge rate may be used to determine or estimate the transmissivity

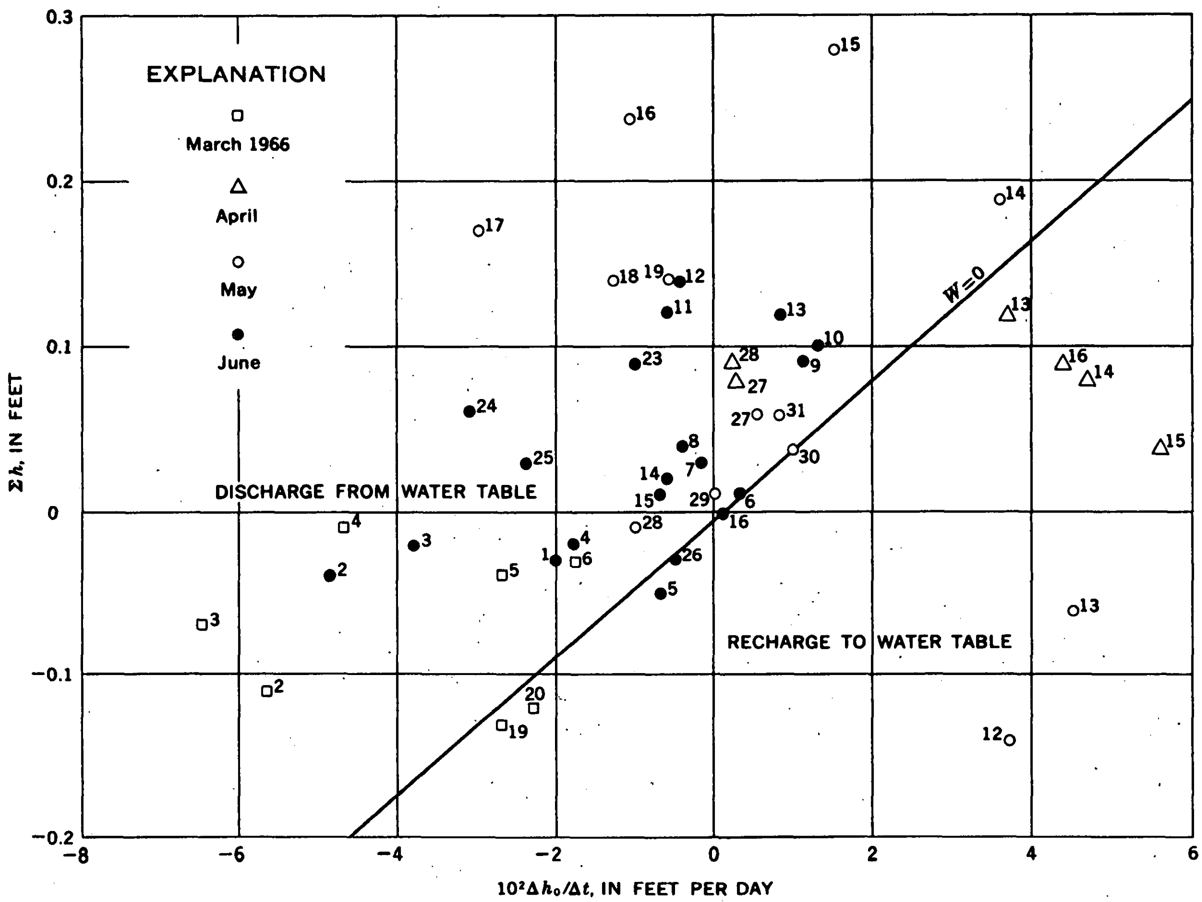

FIGURE 28.—Plot of Σh versus $\Delta h_0/\Delta t$ for spring of 1966.

of an aquifer under steady flow conditions. Equation 26 may be rewritten:

$$Q=-\frac{KA\Delta h}{\Delta r}=-\frac{TL\Delta h}{\Delta r}\qquad [L^3T^{-1}], \qquad (136)$$

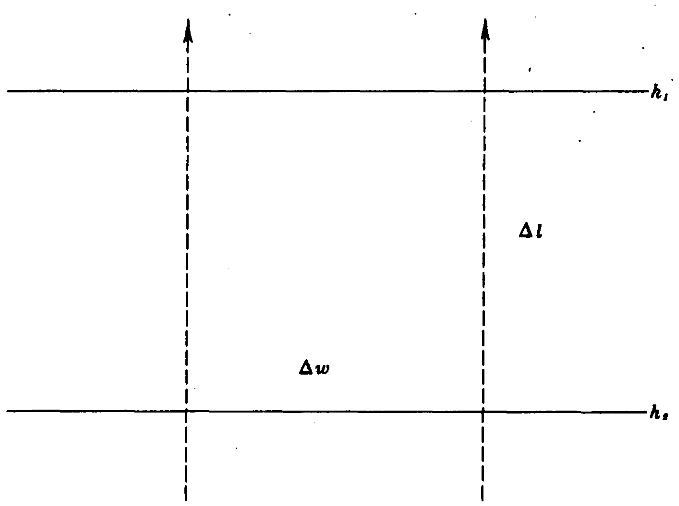

FIGURE 29.—Idealized square of flow net.

which, for any two concentric closed contours of length L_1 and L_2, may be written

$$T=-\frac{2Q}{(L_1+L_2)\,\Delta h/\Delta r}\qquad [L^2T^{-1}], \qquad (137)$$

where Δh is the contour interval and Δr is the average distance between the two closed contours. An example will illustrate the method.

Assume that two irregularly shaped closed contours have measured lengths (as by wheel-type map measure) of 27,600 and 44,000 ft, respectively, that the contour interval is 10 ft, that the average distance between the two contours is (1,800+2,200+2,100+1,700)/4=1,950 ft, and that the rate of withdrawal from a well field within the lowest closed contour is 1 million gal day^{-1}. Using equation 137,

$$T=-\frac{(2)\,(10^6\text{ gal day}^{-1})}{(7.16\times10^4\text{ ft})[-(10\text{ ft})/(1.95\times10^3\text{ ft})](7.48\text{ gal ft}^{-3})}$$

$=730$ ft^2 day^{-1} (rounded).

The regularity or irregularity of the shape and spacing of the contours, the density and accuracy of the water-

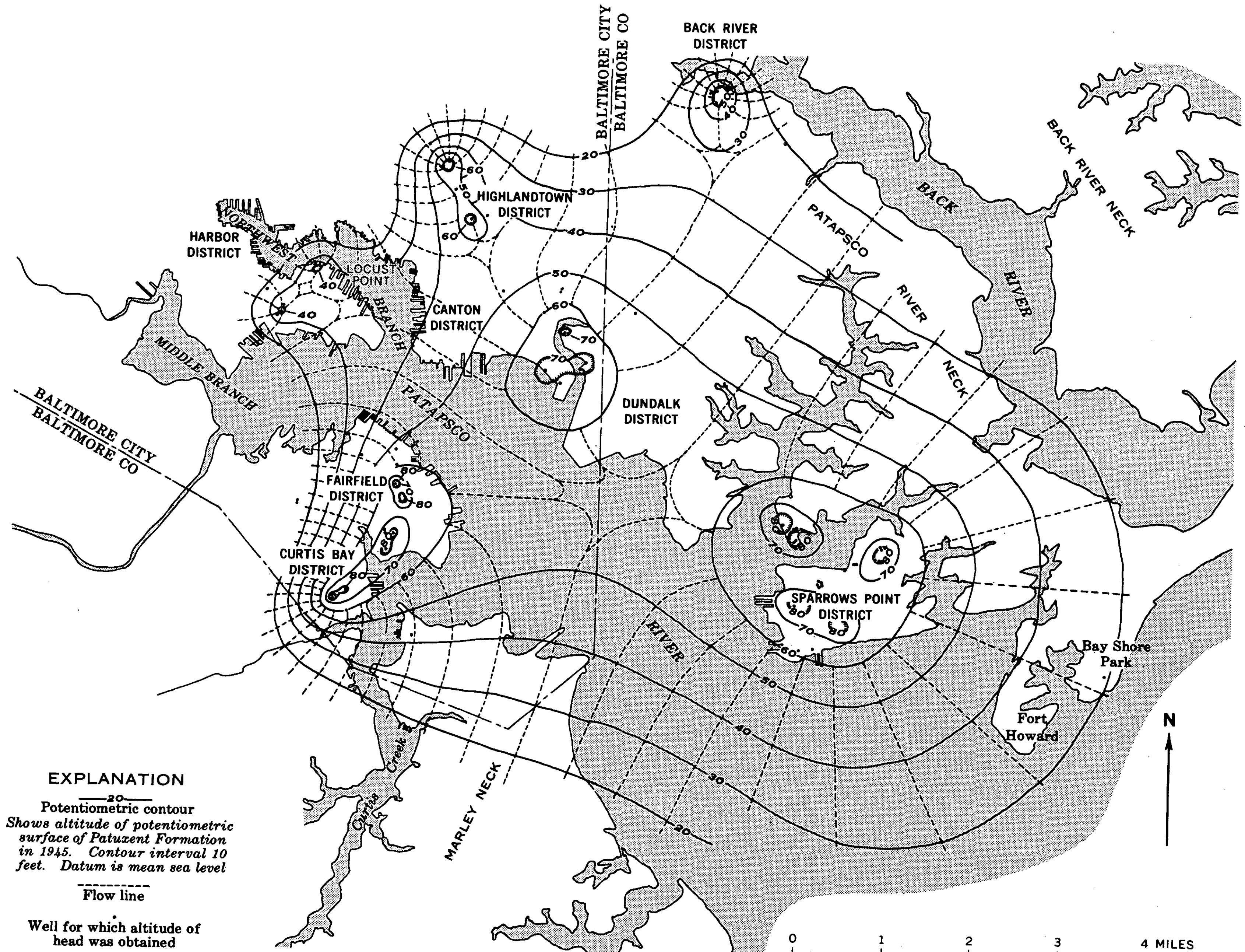

FIGURE 30.—Map of Baltimore industrial area, Maryland, showing potentiometric surface in 1945 and generalized flow lines in the Patuxent Formation. From Bennett and Meyer (1952, pl. 7).

level data, and the accuracy to which Q is known control the accuracy of T and should be carefully considered to guide the rounding of the final result. In the above hypothetical example, greater irregularity in the contours would necessitate rounding the result to 700 ft^2 day^{-1}. In the example, four measurements of Δr were averaged, but the number required would range from one, for concentric circles, to perhaps eight or 10 for more complicated patterns. Use of my method may save the trouble of drawing a flow net.

UNCONFINED WEDGE-SHAPED AQUIFER BOUNDED BY TWO STREAMS

Stallman and Papadopulos (1966) presented a method for determining T/S (hydraulic diffusivity) from water-level recession in an observation well caused by dissipation of recharge from an unconfined wedge-shaped aquifer between two perennial streams. The hydraulic system here is analogous to a nonsteady heat-flow problem solved by Jaeger (1942) by means of a complex integral equation, which may be evaluated only by very laborious numerical methods (Papadopulos, 1963). The close fit between observed and theoretical water-level recession curves computed from Jaeger's equation for three observation wells in Wisconsin (Weeks, 1964) led to the computation of many evaluations by a digital computer. The following four illustrations from Stallman and Papadopulos (1966) show the method.

A simplified form of Jaeger's equation is

$$\frac{s}{s_0}=F\left(\theta_0,\frac{\theta}{\theta_0},\frac{r}{a},\frac{Tt}{r^2S}\right) \quad \text{[dimensionless]}, \quad (138)$$

where F is simply a function of the four parameters in parentheses: θ_0, θ, r, and a are as shown in figure 31; s and s_0 are as shown in figure 32; the components of Tt/r^2S are as defined previously; and the solution for T/S is given in figure 33.

Note in the example in figure 31 that observation well A is near the confluence of two of several streams that drain an unconfined aquifer. The two tributaries form a wedge having an angle θ_0, of approximately 75°, and the angle θ, between the well and one side of the wedge, is 15°. Radius r, to the well, is about 5 miles. Radius a, the distance from the apex to the circumference along which water levels are presumed to be constant, was chosen to be 20 miles, so that $r/a=0.25$. Note in figure 34 (and on many of the plates in the report by Stallman and Papadopulos) that as r/a approaches zero (larger and larger values of a), the response curves form an envelope on the lower right, and that values larger than 20 miles would not affect the final result in the example given.

In the hypothetical hydrograph in figure 32, the water level was declining until about mid-May, when the aquifer received recharge during the spring thaw; this raised the water level by late May by the amount s_0 at $t=0$. For times after $t=0$, values of s were determined by subtracting altitudes of the projected water-level trend, had no recharge occurred, from the smoothed curve of actual water levels. Values of s/s_0 (linear scale) and t (log scale) were

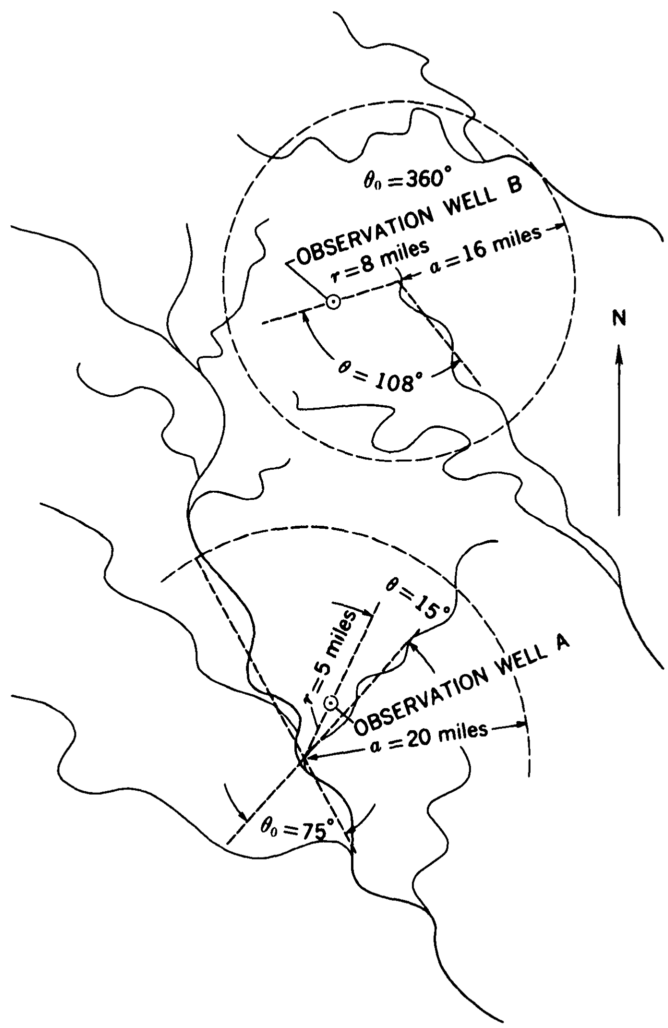

FIGURE 31.—Surface drainage pattern, showing location of observation wells that penetrate an unconfined aquifer.

then plotted on semilogarithmic tracing paper (such as Codex 31,227) to the same scale as figure 34, and the data curve was then matched to the type curves by a procedure slightly different from those described earlier. The s/s_0 axes are kept coincident, and the data curve is moved from side to side until the data curve fits the theoretical response curve for $r/a=0.25$. Any convenient match of Tt/r^2S in figure 34 and t on the data curve is then selected; the one

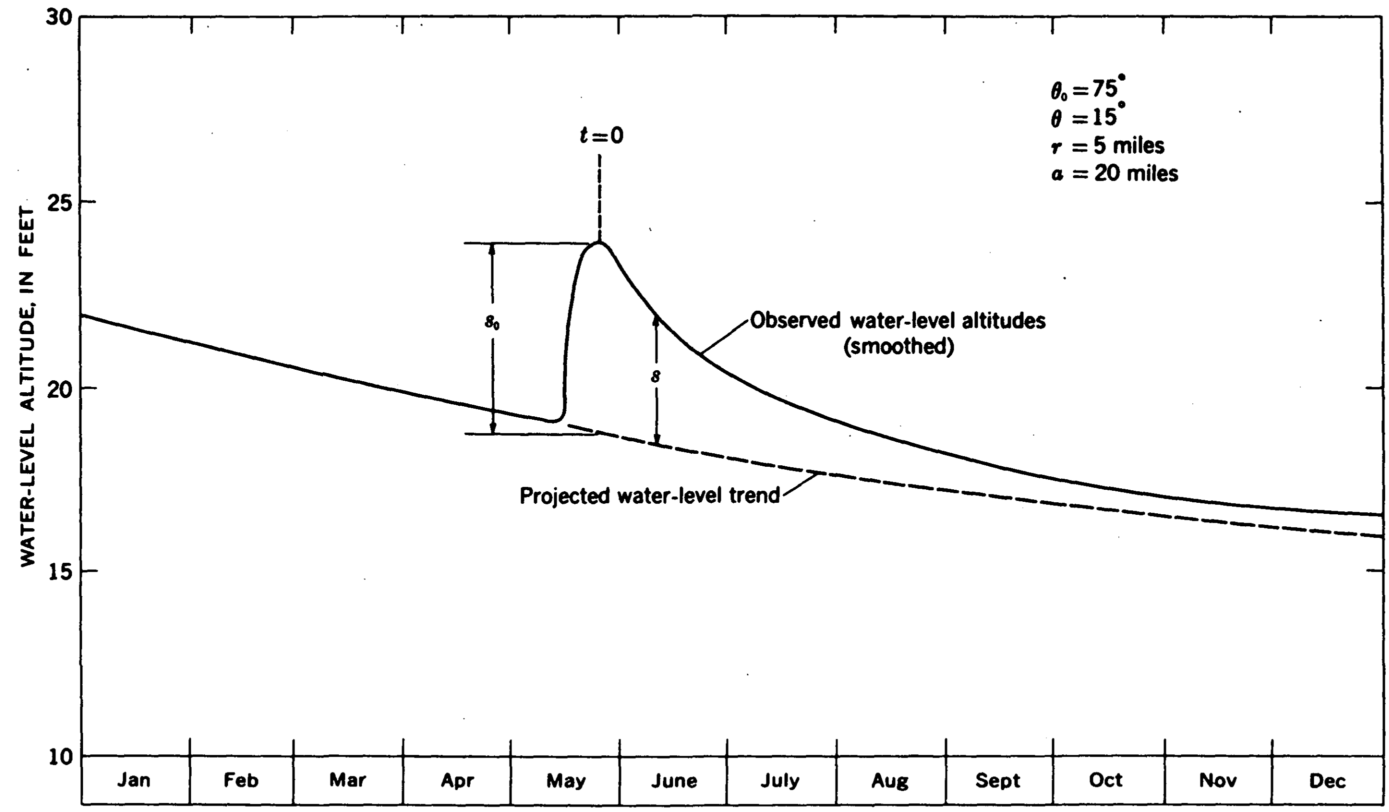

FIGURE 32.—Example hydrograph from well A of figure 31, showing observed and projected water-level altitudes.

chosen in the example in figure 33 was for $Tt/r^2S = 1.0$ and $t = 360$ days. From the value $Tt/r^2S = 1.0$, $T/S = 1.0 \times r^2/t = 1.0 \times 6.98 \times 10^8$ ft²/360 days $= 1.94 \times 10^6$ ft² day⁻¹. If S is known or estimated to be, say 0.2, then $T = 0.2 \times 1.94 \times 10^6$ ft² day⁻¹ = about 4×10^5 ft² day⁻¹. Note that in figure 34,

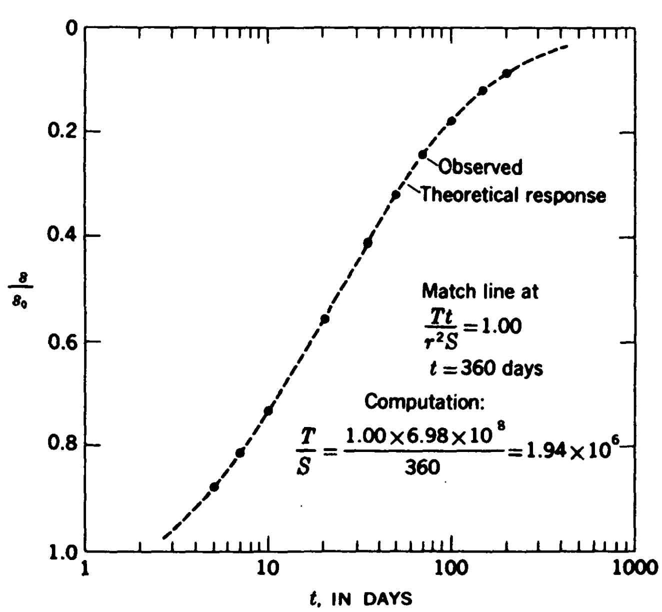

FIGURE 33.—s/s_0 versus t taken from hydrograph of well A (see fig. 32), showing computation of T/S.

which is a nondimensional plot, Stallman and Papadopulos omitted the pure number 4 from the numerator of Tt/r^2S, thus eliminating the necessity of using it in computations.

In the hypothetical data plotted in figures 32 and 33, values of s/s_0 were plotted for t from 5 to 200 days, but the authors warn that in actual practice it would be difficult to reliably project the water-level trend much beyond July and that, in general, values of s/s_0 for only about 50 days after cessation of recharge should be considered useful.

Note in figure 31 that observation well B is considered to be within a circular area of $\theta_0 = 360°$ and a radius of 16 miles surrounded by streams but that only the stream at $\theta = 108°$ was considered. R. W. Stallman (U.S. Geological Survey, oral commun., 1968) indicated that this rather extreme example might be improved by reducing radius a to about 12 miles, so that it just intersects the streams to the northwest, west, and south.

Figure 34 is but one of 120 sheets containing in all some 1,500 response curves for various values of θ_0, θ/θ_0, and r/a. This method should have widespread application in many places where unconfined aquifers are traversed by perennial streams, and where at least a few wells are available for observation of water levels preceding and following periods of recharge. In some studies this method might provide the only values of T/S and estimates of T; in others, it could conveniently supplement values obtained by other

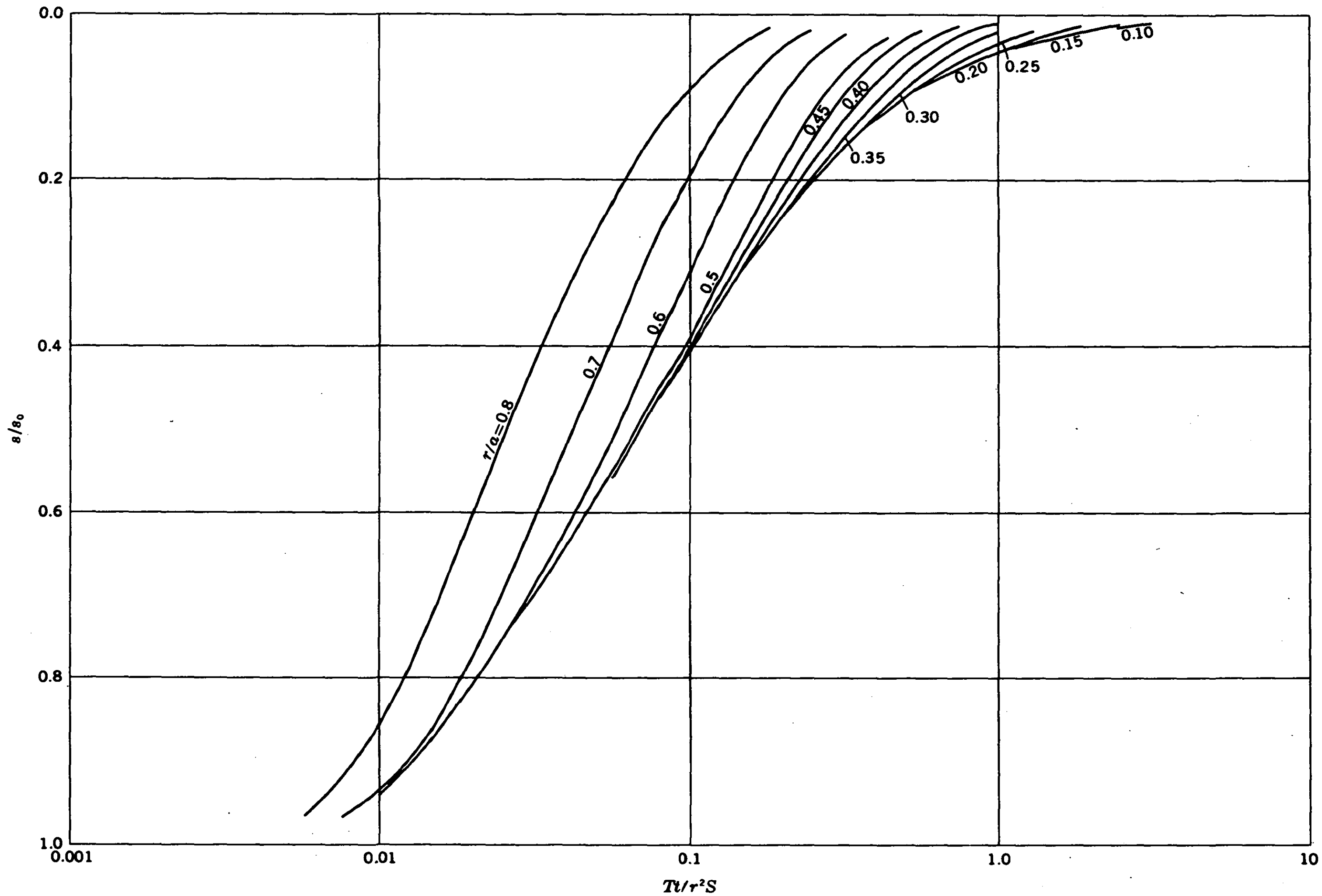

FIGURE 34.—s/s_0 versus Tt/r^2S for $\theta_0=75°$; $\theta/\theta_0=0.20$.

methods. In areas where T is known, this method also could be used to estimate S.

METHODS OF ESTIMATING TRANSMISSIVITY

In some ground-water investigations, such as those of a reconnaissance type, it may be necessary to estimate the transmissivity of an aquifer from the specific capacity (yield per unit of drawdown) of wells, as the determination of T by use of some of the equations discussed above may not be feasible. On the other hand, some of our modern quantitative studies, such as those for which electric-analog models or mathematical models are constructed, require a sufficiently large number of values of T that transmissivity-contour maps (T maps) may be constructed. In unconfined aquifers, such T maps generally require also the construction of water-level contour maps and bedrock-contour maps, from which may be obtained maps showing lines of equal saturated thickness, b, for we have seen that $T=\bar{K}b$. For example, a quantitative investigation of a 150-mile reach of the Arkansas River valley, in eastern Colorado, required a T map based upon about 750 values, or about 1⅔ values per square mile. About 25 of these values were obtained from pumping tests, selected as reliable tests from a greater number of tests conducted. About 200 values of T were estimated from the specific capacity of wells, by one of the methods to be described. About 525 values were estimated by geologists from studies of logs of wells and test holes, by methods to be described. Thus, only about 3 percent of the values were actually determined from pumping tests.

SPECIFIC CAPACITY OF WELLS

Several methods for estimating transmissivity from specific capacity have been published, some of which are cited below. If we solve equation 51 for Q/s_w (specific capacity), using s_w as the drawdown in the discharging well, and r_w as the radius of the well, and assuming that the well is 100 percent efficient, we obtain

$$\frac{Q}{s_w} \approx \frac{4\pi T}{2.30\log_{10} 2.25Tt/r_w{}^2S} \quad [L^2T^{-1}], \qquad (139)$$

which shows the manner in which Q/s_w is approximately related to the other constants (T, S) and variables (r_w, t). As r_w is constant for a particular well being pumped, we see that Q/s_w is nearly proportional to T at a given value of t, but gradually diminishes as t increases, by the amount $1/\log_{10} t$. Thus, for a given well, considered 100-percent efficient, and assuming that water is discharged instantaneously from storage with decline in head, we may symbolize the foregoing statements by the following equation:

$$\frac{Q}{s_w} \approx \frac{B}{\log_{10} t} \quad [L^2T^{-1}], \qquad (140)$$

where B = a constant for the well, including other terms as in equation 139.

No wells are 100-percent efficient, but, according to construction, age, and so forth, some wells are more efficient than others. Jacob (1947, p. 1048) has approximated the head loss resulting from the relatively high velocity of water entering a well or well screen as being proportional to some power of the velocity approaching the square of the velocity, which in turn is nearly proportional to Q^2; thus head loss is nearly equal to CQ^2, where C = a constant of proportionality. Adding this to equation 140,

$$\frac{Q}{s_w} \approx \frac{B}{\log_{10} t} + CQ^2 \quad [L^2T^{-1}]. \qquad (141)$$

Thus we see that Q/s_w diminishes not only with time but with pumping rate Q. In unconfined aquifers it may be necessary to adjust factor B further to account for delayed yield from storage.

In an uncased well in, say, sandstone, r_w may be assumed equal to the radius of the well, but in screened wells in unconsolidated material, in which the finer particles have been removed near the screen by well development, or in gravel-packed wells, the effective r_w generally is larger than the screen diameter. Jacob (1947) described a method for determining the effective r_w and the well loss (CQ^2) from a multiple-step drawdown test.

Most other investigators have neglected well loss in their equations, which are then equations for wells of assumed 100 percent efficiency, such as equation 140, but some have arbitrarily adjusted for this loss by selection of an arbitrary constant for wells of similar construction in a particular area or aquifer, which generally gives satisfactory results when used with caution.

Theis (1963a) gave equations and a chart, based upon the Theis equation, for estimating T from specific capacity for constant S and variable t, with allowance for variable well diameter but not well efficiency. Brown (1963) showed how Theis' results may be adapted to artesian aquifers. Meyer (1963) gave a chart for estimating T from the specific capacity at the end of 1 day of pumping, for different values of S and for well diameters of 0.5, 1.0, and 2.0 ft. Bedinger and Emmett (1963) gave equations and a chart for estimating T from specific capacity, based upon a combination of the Thiem and Theis equations and upon average values of T and S for a *specific* area, for well diameters of 0.5, 1.0, and 2.0 ft. Hurr (1966) gave equations and charts based upon the Theis and Boulton

(1954a) equations, which allow for delayed yield from storage, for determining T from specific capacity at different values of t for a well 1.0 ft in diameter. None of the methods just cited includes corrections for well efficiency, but this can be added in an approximate manner.

LOGS OF WELLS AND TEST HOLES

As noted above, about 525 values of T out of 750 total values in the Arkansas River valley of eastern Colorado were estimated by geologists from studies of logs of wells and test holes and from drill cuttings from test holes. Wherever possible, pumping tests were made on wells for which or near which logs were available; otherwise, test holes were drilled near the well tested. From several or many such pumping tests accompanied by logs, the values of T were carefully compared with the water-bearing bed or beds, and, as $T=\bar{K}b$, the total T was distributed by cut and try among the several beds, according to the following equation:

$$T=\sum_{1}^{n} K_m b_m = K_1b_1+K_2b_2+K_3b_3+\cdots+K_nb_n \quad [L^2T^{-1}]. \tag{142}$$

From this, table 17 was prepared, comparing *average* values of K for different alluvial materials in the valley. Equation 142 may be solved also by multiple regression using a digital computer or graphical method (Jenkins, 1963).

R. T. Hurr, who prepared table 17, then carefully examined the logs of other wells and test holes for which no pumping tests were available. He assigned values of K to each bed of known thickness on the basis of the descriptive words used by the person who prepared the log. The values of K that were assigned may have been (1) equal to, (2) more than, or (3) less than values given in the table (depending upon cleanliness, sorting, and so forth), and thus they necessarily involved subjective judgment. As experience was gained, however, the geologist who prepared the table generally could estimate K and T with fair to good accuracy. The T values from all sources also were compared carefully with the saturated-thickness map. This method for estimating T has been used successfully in the Arkansas River valley in Colorado, in the Arkansas Valley in Arkansas and Oklahoma (Bedinger and Emmett, 1963), in Nebraska, in California, and elsewhere.

Laboratory determinations for K of cores of consolidated rocks, such as partly to well cemented sandstone, may be used in place of estimates. Reconstitution of disturbed samples of unconsolidated material is *not possible*, however, so laboratory determinations for K generally do not give reliable values. However, they may be very useful in indicating relative values, as was done in Arkansas and Oklahoma.

The above methods may also be used by the geologist for estimating the hydraulic properties of exposed sections of rocks containing water-bearing beds.

TABLE 17.—*Average values of hydraulic conductivity of alluvial materials in the Arkansas River valley, Colorado*
[Courtesy of R. T. Hurr]

Material	Hydraulic conductivity[1] (ft day^{-1})
Gravel:	
Coarse	1,000
Medium	950
Fine	900
Sand:	
Gravel to very coarse	800
Very coarse	700
Very coarse to coarse	500
Coarse	250
Coarse to medium	100
Medium	50
Medium to fine	30
Fine	15
Fine to very fine	5
Very fine	3
Clay	1

[1] Values were converted from gallons per day per square foot and were rounded.

METHODS OF ESTIMATING STORAGE COEFFICIENT

In examining logs of wells or test holes in confined aquifers, or in measuring sections of exposed rocks that dip down beneath confining beds to become confined aquifers, the storage coefficient may be estimated from the following rule-of-thumb relationship:

b (ft)	S	$\frac{S}{b}$ (ft^{-1})
1	10^{-6}	10^{-6}
10	10^{-5}	
100	10^{-4}	
1,000	10^{-3}	

One may either multiply the thickness in feet times 10^{-6} ft^{-1} or interpolate between values in the first two columns; thus, for $b=300$ ft, $S\approx 3\times10^{-4}$, and so on. Values thus estimated are not absolutely correct, as no allowances have been made for porosity or for compressibility of the aquifer, but they are fairly reliable for most purposes. Such estimates may be improved upon by comparison with values obtained from reliable pumping or flow tests, then extrapolated to other parts of an aquifer with adjustments for thickness if needed.

METHODS OF ESTIMATING SPECIFIC YIELD

Earlier it was stated that the specific yield generally ranges between 0.1 and 0.3 (10–30 percent) and that long periods of pumping may be required to drain water-bearing material. Thus, in the absence of any determination, as in a rapid reconnaissance, we would not be very far off in assuming that, for supposedly long periods of draining, the specific yield of an unconfined aquifer is about 0.2—the average value between the general limits indicated.

Better estimates of specific yield—which might be slightly more or less than the average—could be obtained from (1) careful study of the grain sizes and degree ot sorting, if logs of wells or test holes are available, (2) data from a few reliable pumping tests, (3) values obtained from the use of neutron-moisture probes (Meyer, 1962), and (4) laboratory determinations of the specific yield of disturbed samples (values of laboratory determinations are likely to be larger than those obtained in the field). Data from the sources listed could also be extrapolated to similar types of material elsewhere in the aquifer.

TABLE 18.—*Computations of drawdowns produced at various distances from a well discharging at stated rates for 365 days from a confined aquifer for which* $T=20$ *ft² day⁻¹ and* $S=5\times10^{-5}$

$\frac{S}{4Tt}$ (ft⁻²)	r (ft)	r^2 (ft²)	u	$W(u)$	$\frac{1}{4\pi T}$ (ft⁻² day)	s (ft) for Q (ft³ day⁻¹)						
						10^3	2×10^3	3×10^3	4×10^3	5×10^3	6×10^3	7×10^3
1.71×10^{-9}	1	1	1.71×10^{-9}	19.61	3.98×10^{-3}	78.1	156	234	312	391	469	547
	10	10^2	1.71×10^{-7}	15.01		59.7	119	179	239	299	358	418
	10^2	10^4	1.71×10^{-5}	10.40		41.4	82.8	124	166	207	248	290
	10^3	10^6	1.71×10^{-3}	5.80		23.1	46.2	69.3	92.4	116	139	162
	2×10^3	4×10^6	6.84×10^{-3}	4.41		17.6	35.1	52.2	70.2	87.8	105	123
	4×10^3	1.6×10^7	2.74×10^{-2}	3.23		12.9	25.7	38.6	51.4	64.3	77.2	90.0
	6×10^3	3.6×10^7	6.16×10^{-2}	2.27		9.0	18.0	27.0	36.0	45.0	54.0	63.0
	8×10^3	6.4×10^7	1.09×10^{-1}	1.74		6.9	13.8	20.7	27.6	34.5	41.4	48.3
	10^4	10^8	1.71×10^{-1}	1.35		5.4	10.8	16.2	21.6	21.0	32.4	37.8
	1.5×10^4	2.25×10^8	3.85×10^{-1}	.73		2.9	5.8	8.7	11.6	14.5	17.4	20.3
	2×10^4	4×10^8	6.84×10^{-1}	.39		1.6	3.2	4.8	6.4	8.0	9.6	11.2
	3×10^4	9×10^8	1.54	.10		.15	.30	.45	.60	.75	.90	1.05
	4×10^4	1.6×10^9	2.74	.02		.08	.16	.24	.32	.40	.48	.56

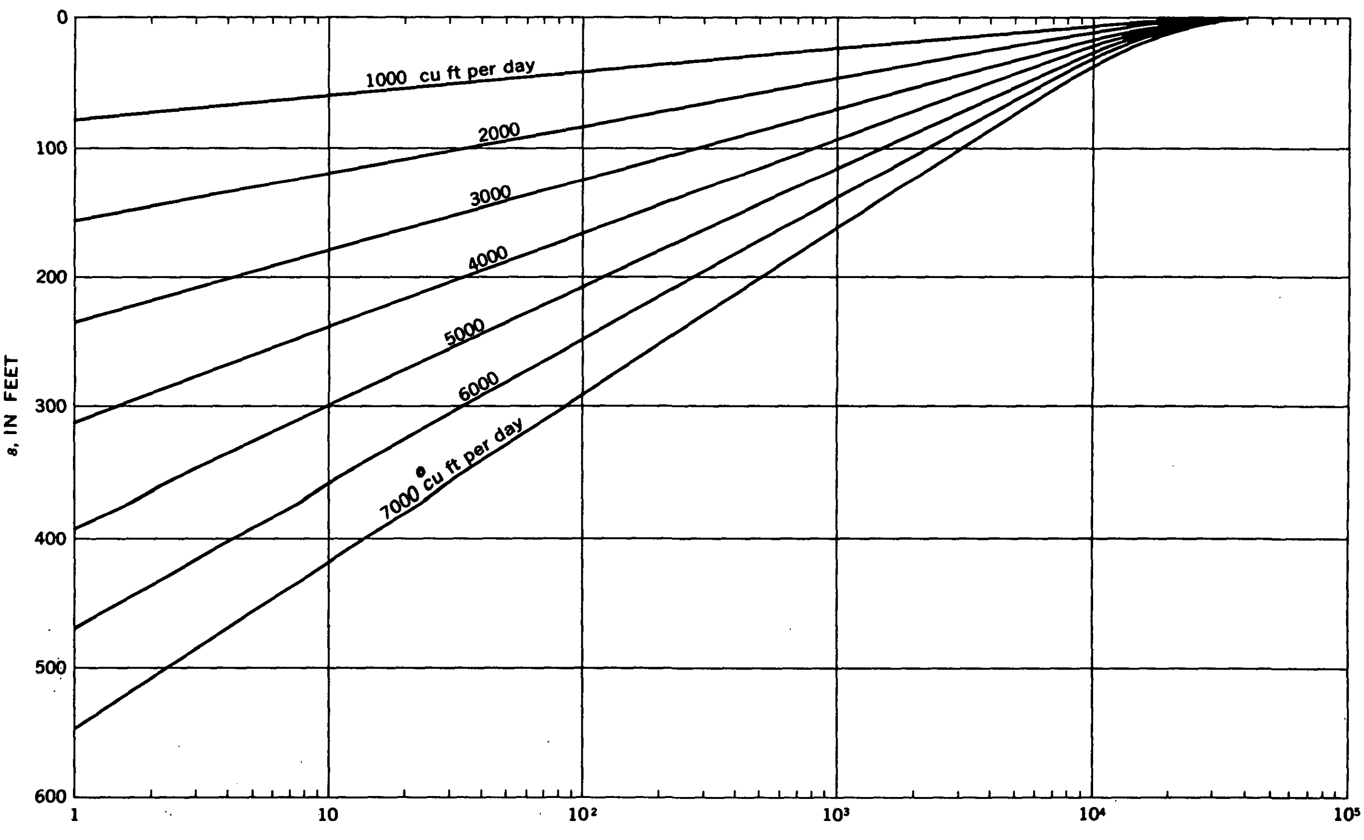

FIGURE 35.—Family of semilogarithmic curves showing the drawdown produced at various distances from a well discharging at stated rates for 365 days from a confined aquifer for which $T=20$ ft² day⁻¹ and $S=5\times10^{-5}$.

DRAWDOWN INTERFERENCE FROM DISCHARGING WELLS

If T and S are known, as from discharging well tests, the effect of one discharging well upon a nondischarging well, or other point, in an infinite, or at least extensive, homogeneous and isotropic aquifer is readily obtained by use of equation 47 solved for u, equation 46, and table 4, for any distance (r) for known or assumed values of t and constant Q, or for any time t for known or assumed values of r and Q. Tables 18 and 19 show the methods of computation of the drawdowns and figures 35 and 36 illustrate the results. These examples are modified from Lohman (1965, fig. 43). (See also Theis, 1963c).

Note on tables 18 and 19 that the computations are greatly facilitated by proper arrangement of headings.

TABLE 19.—*Computations of drawdowns produced after various times at a distance of 1,000 ft from a well discharging at stated rates from a confined aquifer for which* T = *20 ft² day⁻¹ and* S = *5×10⁻⁵*

$\frac{r^2S}{4T}$ (day)	t (day)	$\frac{1}{t}$ (day⁻¹)	u	$W(u)$	$\frac{1}{4\pi T}$ (ft⁻² day)	s (ft) for Q (ft³ day⁻¹): 10^3	2×10^3	3×10^3	4×10^3	5×10^3	6×10^3	7×10^3
6.25×10^{-1}	0.2	5.0	3.13	10^{-2}	3.98×10^{-3}	0.0398	0.0796	0.119	0.159	0.199	0.239	0.279
	.3	3.3	2.06	5×10^{-2}		.199	.398	.597	.796	.995	1.19	1.39
	.4	2.5	1.56	9×10^{-2}		.358	.716	1.07	1.43	1.79	2.15	2.51
	.6	1.66	1.04	2.1×10^{-1}		.836	1.67	2.51	3.34	4.18	5.02	5.85
	.8	1.25	7.8×10^{-1}	3.2×10^{-1}		1.27	2.54	3.87	5.08	6.35	7.62	8.89
	1	1.0	6.25×10^{-1}	4.3×10^{-1}		1.71	3.42	5.13	6.84	8.55	10.26	11.97
	2	.5	3.13×10^{-1}	8.7×10^{-1}		3.46	6.92	10.38	13.84	17.30	20.76	24.22
	4	.25	1.56×10^{-1}	1.43		5.69	11.4	17.1	22.8	28.5	34.1	39.8
	6	.166	1.04×10^{-1}	1.79		7.12	14.2	21.4	28.5	35.6	42.7	49.8
	8	.125	7.81×10^{-2}	2.05		8.16	16.3	24.5	32.6	40.8	49.0	57.1
	10	10^{-1}	6.25×10^{-2}	2.26		9.00	18.0	27.0	36.0	45.0	54.0	63.0
	10^2	10^{-2}	6.25×10^{-3}	4.50		17.91	35.8	53.7	71.6	89.6	108	125
	10^3	10^{-3}	6.25×10^{-4}	6.80		27.06	54.1	81.2	108	135	162	189
	10^4	10^{-4}	6.25×10^{-5}	9.10		36.22	72.4	109	145	181	217	254

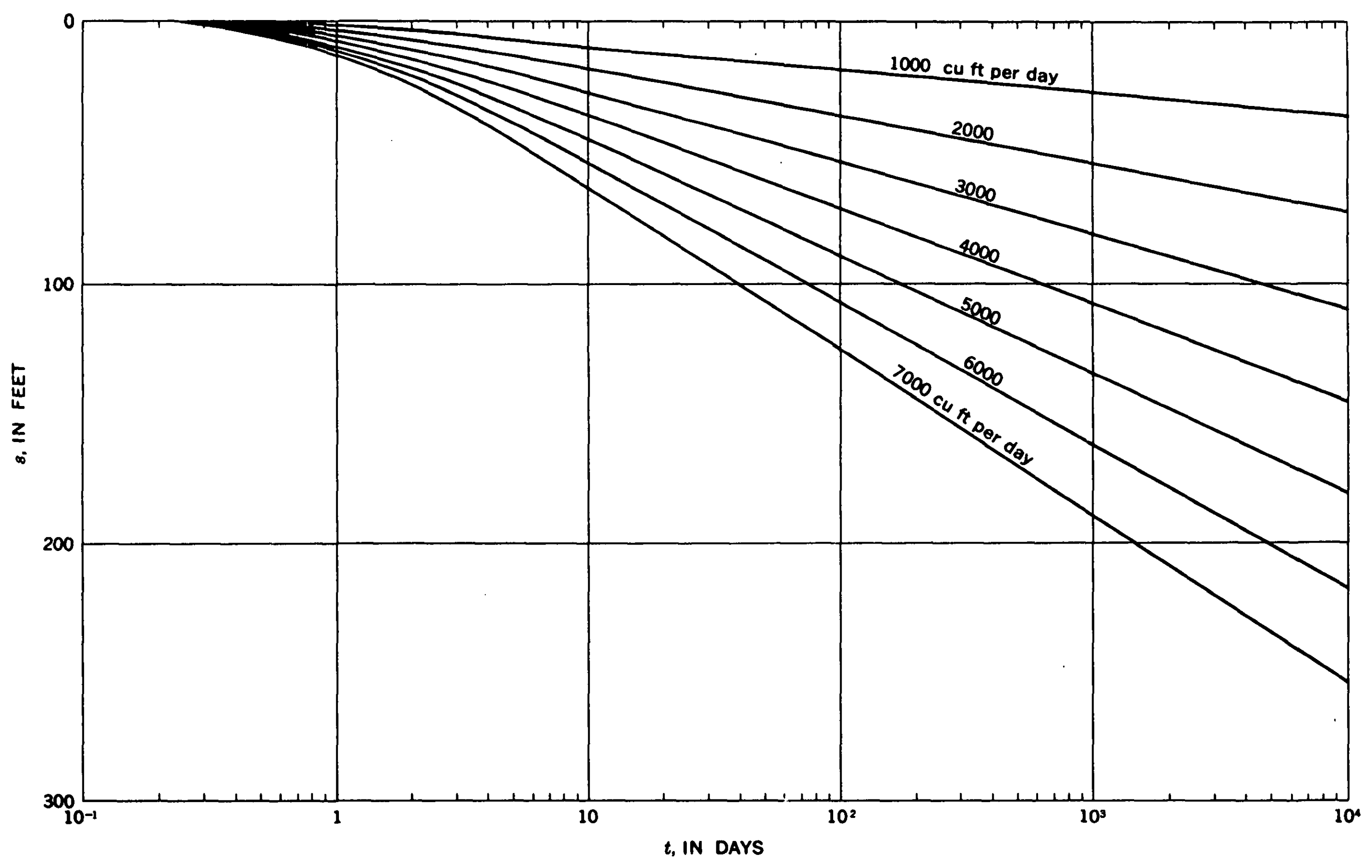

FIGURE 36.—Family of semilogarithmic curves showing the drawdown produced after various times at a distance of 1,000 ft from a well discharging at stated rates from a confined aquifer for which $T = 20$ ft² day⁻¹ and $S = 5\times10^{-5}$.

Thus, single values of $S/4Tt$ or $r^2S/4T$ multiplied by various values of r^2 or $1/t$, respectively, give values of u from which values of $W(u)$ are obtained from table 5. Then, a single value of $1/4\pi T$ multiplied by various values of $W(u)$ and by $Q=10^3$ ft^3 day^{-1} gives the drawdowns for this discharge rate. Then the drawdowns for other values of Q are obtained by simple multiplication by 2 through 7. Note that only a few values establish the straight-line parts of the curves but that more are required for the curvilinear parts.

The examples just given are for constant discharge and variable drawdown, but other equations for different boundary conditions could be similarly used to determine drawdown interference. Thus, for example, equations 66 and 67 and table 7 could be used for discharging wells of constant drawdown and variable discharge. In these examples, discharging wells constitute hydraulic boundaries of the point-sink type; recharging wells constitute boundaries of the point-source type. Other types of boundaries are discussed in later sections.

Figures 35 and 36 show only the drawdown caused by one discharging well at a nondischarging well or point within the cone of depression. Where many discharging wells are mutually interfering with each other and with other points, such as nondischarging wells, the problem becomes much more complex and is best handled by an electric analog-model computer or by a digital computer. The drawdown interference from certain groupings of discharging wells is treated by Hantush (1964, p. 374–382).

RELATION OF STORAGE COEFFICIENT TO SPREAD OF CONE OF DEPRESSION

It has been shown (Lohman, 1965, p. 109, 110) that if equation 51 is solved for r^2, there results

$$r^2=\frac{2.25Tt}{S\log_{10}^{-1}[4\pi Ts/2.30Q]} \quad [L^2]. \qquad (143)$$

For a given set of conditions, all terms except r^2 and S may be considered constant; then, using C as a constant of proportionality,

$$r^2=\frac{C}{S} \quad [L^2]. \qquad (144)$$

For convenience, multiply both sides of equation 144 by π, then

$$\pi r^2=A=\frac{C'}{S} \quad [L^2], \qquad (145)$$

where A = area of influence.

As an example, equation 145 may be used to compare the area of influence, A_1, in a confined aquifer having a storage coefficient of, say 5×10^{-5}, with the area of influence, A_2, in an unconfined aquifer having a specific yield of, say 0.20. Assuming that T, Q, and s are the same for both aquifers, and that t also is the same and is long enough so that $u\leq0.01$ and that the material in the cone of depression in the unconfined aquifer has had time to be drained, then

$$\frac{A_1}{A_2}=\frac{C'/5\times10^{-5}}{C'/0.20}=4\times10^3.$$

Thus, under the assumed conditions, the area of influence in the confined aquifer is 4,000 times larger than that in the unconfined aquifer, or the ratio of the radius extending to the circumference of negligible drawdown in the confined aquifer to that in the unconfined aquifer, r_1/r_2, is $\sqrt{4\times10^3}=63.2$. Thus, changes in water level in a confined aquifer spread outward very rapidly from a discharging well, whereas in an unconfined aquifer changes in water level spread very slowly as gravity drainage takes place.

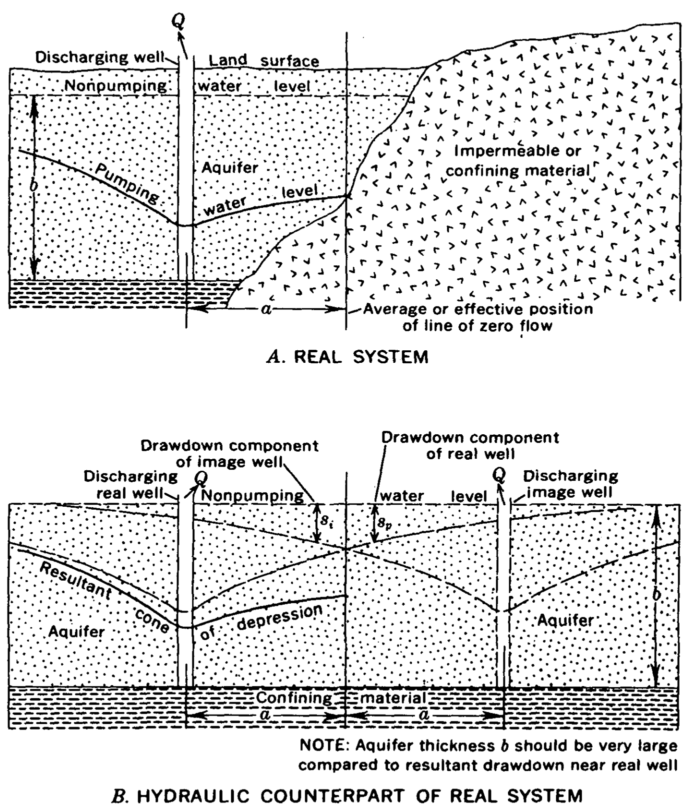

FIGURE 37.—Idealized section views of a discharging well in an aquifer bounded by an "impermeable" barrier and of the equivalent hydraulic system in an infinite aquifer. From Ferris, Knowles, Brown, and Stallman (1962, fig. 37).

AQUIFER BOUNDARIES AND THEORY OF IMAGES

Thus far, the flow equations have all been assumed to be applicable only in aquifers of infinite or semi-infinite areal extent. Many wells are far enough from aquifer boundaries so that this assumption is satisfied reasonably well, but some wells are near boundaries, such as the relatively impermeable bedrock wall of an alluvium-filled valley, a dike, a fault, or a nearby stream or lake. Such boundaries, if close enough to a discharging well, may invalidate the results obtained by use of the flow equations unless suitable adjustments are made.

The method of images used for the solution of boundary problems in the theory of heat conduction in solids has been adapted to the solution of boundary problems in ground-water flow. In this method imagery wells or streams, referred to as images, are placed at proper locations so as to mathematically duplicate the hydraulic effect on ground-water flow caused by the real geologic or hydrologic boundary. Following heat-flow terminology, a discharging image well is regarded as a point sink, a recharging image well as a point source. A discharging image stream or drain is regarded as a line sink, a recharging image stream as a line source. By use of various combinations of such sinks and sources, corrections for almost every conceivable type and shape of linear boundary have been made so as to permit solution of the appropriate ground-water flow equation.

We will take up a few single boundary problems involving single images. Problems involving two or more boundaries at least two of which are parallel have images extending to plus and minus infinity, somewhat like reflections from two facing parallel mirrors. For examples of single and multiple boundary problems, see Ferris, Knowles, Brown, and Stallman (1962, p. 144–166), and Brown (1953).

"IMPERMEABLE" BARRIER

Figure 37*A* (Ferris and others, 1962, fig. 37) shows a discharging well in an aquifer bounded on the right by a barrier of relatively impermeable material. Here it is assumed that no ground water can flow across the barrier.

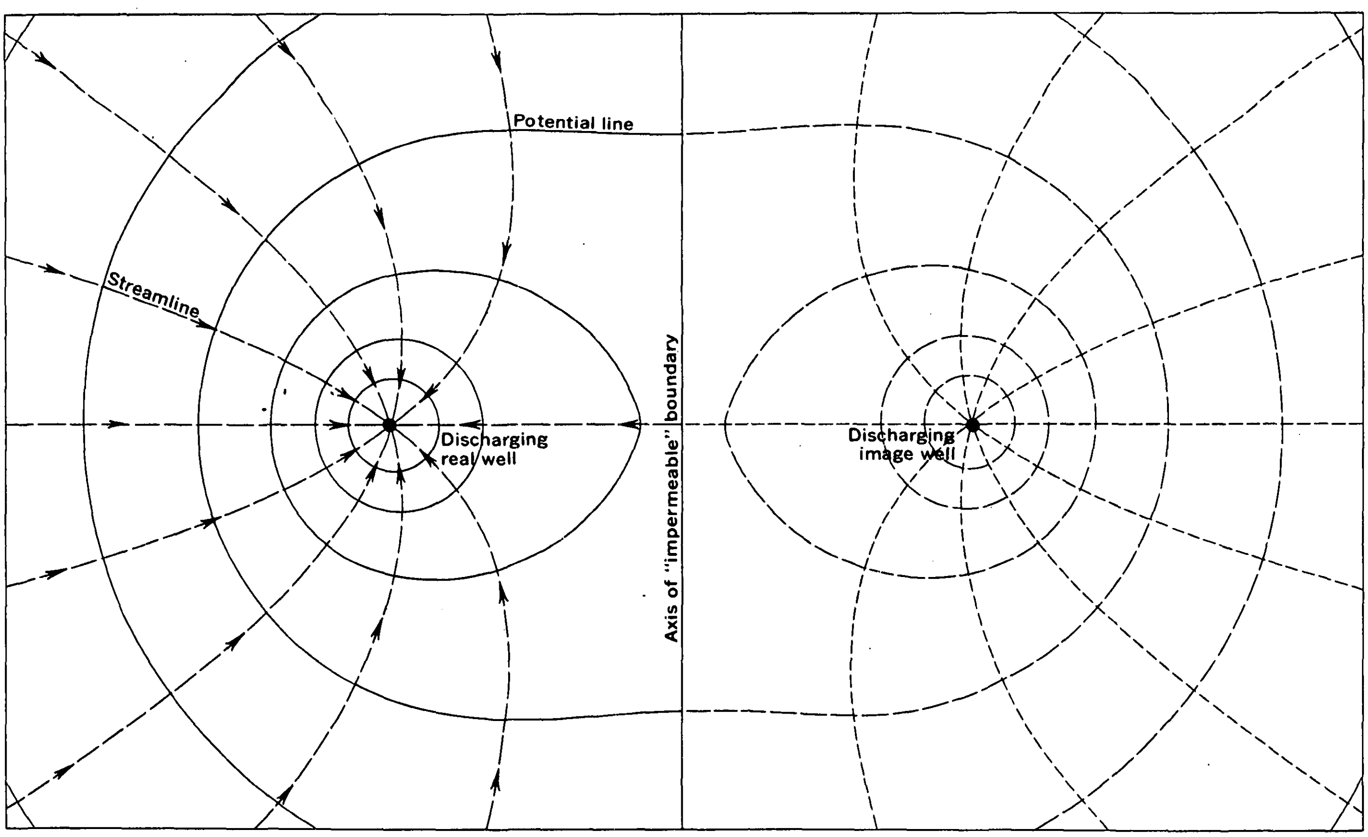

FIGURE 38.—Generalized flow net in the vicinity of discharging real and image wells near an "impermeable" boundary. From Ferris, Knowles, Brown, and Stallman (1962, fig. 38).

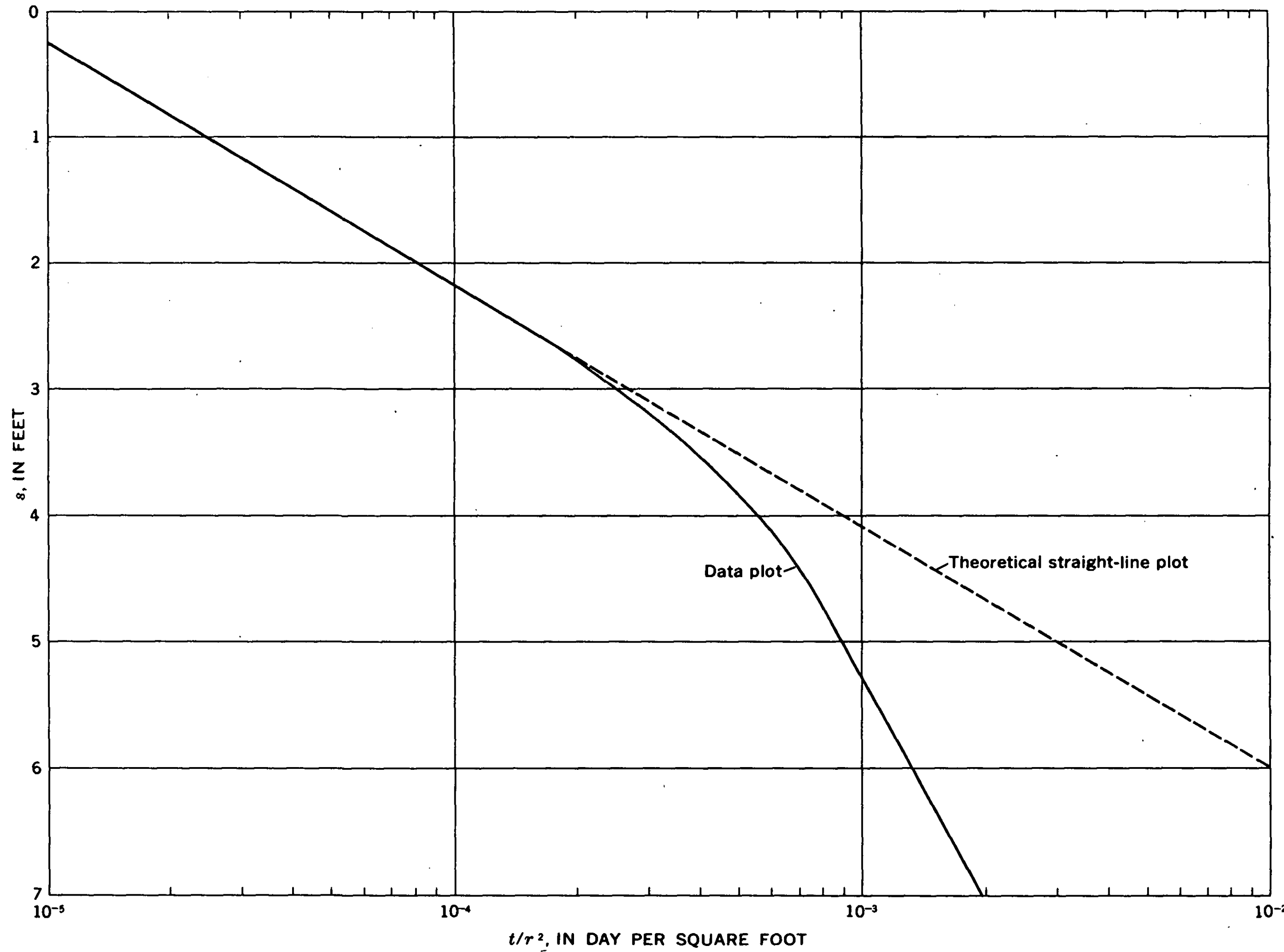

FIGURE 39.—Effect of "impermeable" barrier on semilogarithmic plot of s versus t/r^2.

The image system in the hydraulic counterpart of figure 37*A*, which permits a solution of the real problem by use of the flow equations, is shown in figure 37*B*. Here in an assumed infinite aquifer an image well having a discharge equal to that of the real well is placed the same distance (a) from the now imaginary barrier. The dashed theoretical cones of depression of the real and image wells intersect to form a ground-water divide at the hydraulic barrier, across which no flow can take place, thus satisfying the hydraulic conditions along the barrier. The resultant real cone of depression (heavy line) is the algebraic sum of the theoretical cones of depression (dashed lines) of the real and image wells. Figure 38 depicts the flow net of the two-well system. If the image well and image flow net are removed, the flow net on the left represents the effect of the "impermeable" boundary upon the discharging well.

If the drawdown (s) in an observation well near the real well shown in figures 37 and 38 were plotted against t or t/r^2 on semilogarithmic paper, the curve would deviate downward from the theoretical straight-line plot as shown in figure 39. This shows that the effect of the barrier began to be felt at t/r^2 = about 1.8×10^{-4} day ft^{-2} and that the full effect was apparent at t/r^2 = about 7.4×10^{-4} day ft^{-2}. After the full effect was apparent, the slope of the lower straight line was two times the slope of the theoretical straight line, indicating an apparent transmissivity that was half that of the true value.

LINE SOURCE AT CONSTANT HEAD—PERENNIAL STREAM

If a well in an unconfined aquifer near a large perennial stream hydraulically connected to the aquifer is pumped, obviously the cone of depression cannot extend beyond

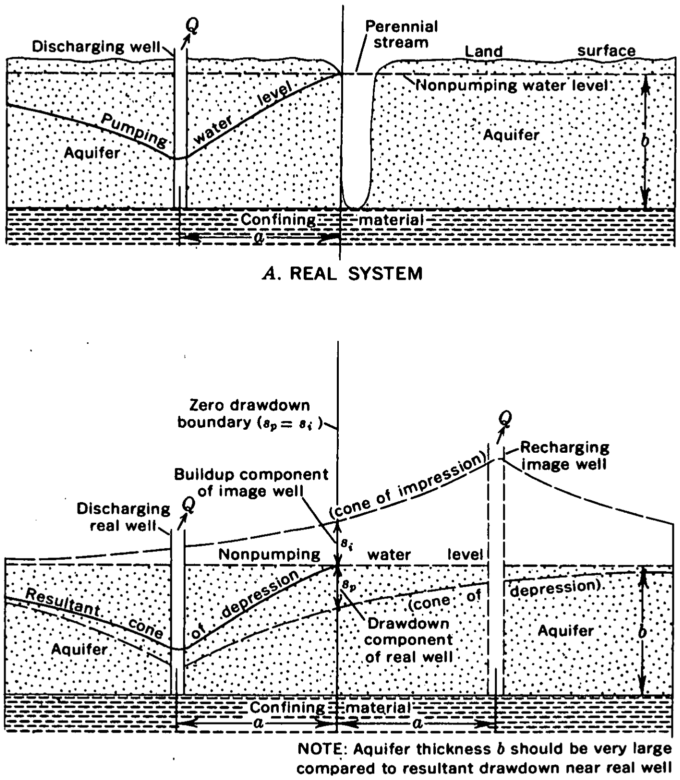

FIGURE 40.—Idealized section views of a discharging well in an aquifer bounded by a perennial stream and of the equivalent hydraulic system in an infinite aquifer. From Ferris, Knowles, Brown, and Stallman (1962, fig. 35).

the stream as the water level in such a stream remains relatively constant, assuming that the well discharge is small relative to the stream discharge. Such a situation is shown in figure 40*A*, in which the nearly straight, partly penetrating perennial stream is shown to be straight and fully penetrating and, hence, is equivalent to a line source at constant head.

The hydraulic counterpart in an assumed infinite aquifer is shown in figure 40*B*, where a recharging image well, or point source, has been placed on a line connecting the real and image wells at right angles to the stream, at the same distance, *a*, from the stream. The recharge and discharge rates, *Q*, are assumed equal. The resulting cone of depression (heavy solid line), which is the algebraic sum of the dashed cones of depression of the real and image wells, intersects the level of the stream, as it should. The flow net for these conditions is shown in figure 41. Note that if the image well and its flow net are removed, the flow net on the left is that of the real well obtaining water by induced infiltration from the stream.

If the drawdown (s) in an observation well near the real well shown in figures 40 and 41 were plotted against t or t/r^2 on semilogarithmic paper, the curve would deviate upward from the theoretical straight-line plot as shown in figure 42. This shows that the effect of the stream began to be felt at t/r^2 = about 2.8×10^{-4} day ft^{-2} and that the full effect was apparent at t/r^2 = about 3.5×10^{-3} day ft^{-2}. After the full effect was apparent, the slope of the upper straight line became horizontal, indicating an apparent infinite transmissivity.

One of the first applications of the image-well theory to ground-water flow was made by Theis (1941), who developed an equation and presented a graph for computing the percentage of the water pumped from a well near a stream that is diverted from the stream at a known distance from the well. (See also Glover and Balmer, 1954; Jenkins, 1968a, 1968b; Theis, 1963b; and Theis and Conover, 1963.)

APPLICATION OF IMAGE THEORY

It was evident in figure 39 that, for a discharging well in an aquifer bounded by a relatively impermeable barrier, a drawdown-time plot for an observation well near the pumped well was steepened (greater drawdown) at the value of t after the cone of depression reached the barrier, and that in figure 42, the converse was true. However, figures 39 and 42 are hypothetical straight-line curves; hence, they would hold true only when $u \leq 0.01$. For larger values of u, as for small values of t, large values of r, or for most unconfined aquifers, it is necessary to plot drawdown-time curves on logarithmic paper for comparison with the Theis curve (fig. 14), for which occur similar but curvilinear deviations from the theoretical curves. Computation of T and S by the usual curve-matching procedure would then be valid only for the early data before the boundary effects changed the slope of the curve. This would offer no problem for most confined aquifers, but in an unconfined aquifer sufficient time might not have elapsed to allow for reasonably complete drainage from storage.

Ferris, Knowles, Brown, and Stallman (1962, p. 161–164) described a method of plotting s versus t or r^2/t for matching with the Theis curve (fig. 14) that permits solving for T and S and also for the distance from the discharging well to the image well, which of course is at twice the distance to the actual boundary. If the boundary is concealed, as a hidden fault, three or more observation wells are required to locate the boundary. (See Ferris and others, 1962, p. 164–166; Moulder, 1963.)

A similar but much simpler method for the solution of single-boundary problems involving either a source or a sink was devised by Stallman (1963b). From figures 37 and 40, it is evident that if s_o is the drawdown in an observation well, and s_p and s_i are the components of that drawdown caused, respectively, by the pumped (real) well and by the discharging or recharging image well, then s_o is the alge-

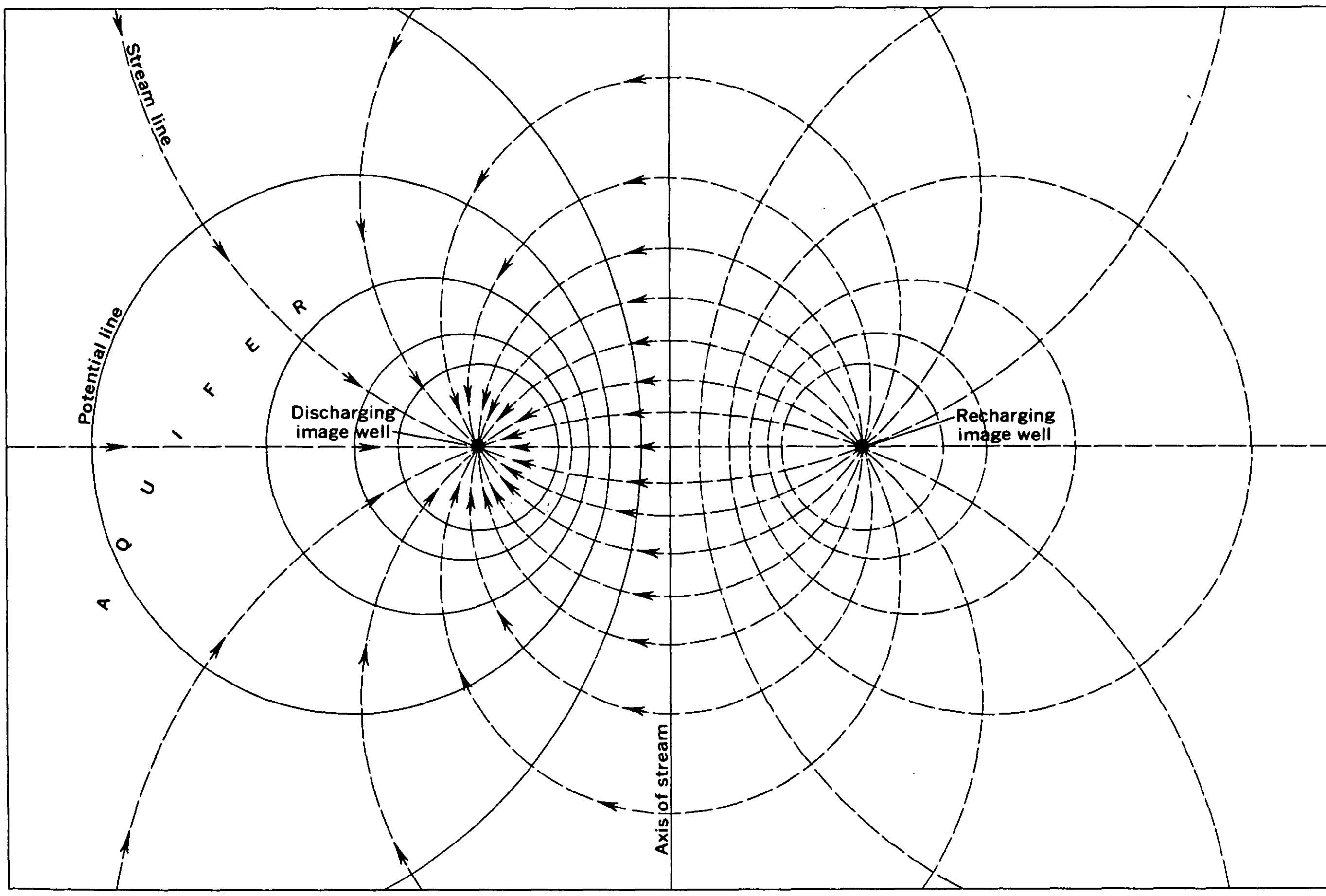

FIGURE 41.—Generalized flow net in the vicinity of a discharging well dependent upon induced infiltration from a nearby stream. From Ferris, Knowles, Brown, and Stallman (1962, fig. 36).

braic sum of s_p and s_i, or

$$s_o = s_p \pm s_i \quad [L]. \tag{146}$$

For this condition, equations 46 and 47 may be rewritten, respectively,

$$s_o = \frac{Q}{4\pi T}[W(u)_p \pm W(u)_i] = \frac{Q}{4\pi T}\sum W(u) \quad [L], \tag{147}$$

and

$$\left.\begin{aligned} u_p &= \frac{r_p^2 S}{4Tt} \\ \text{and} \\ u_i &= \frac{r_i^2 S}{4Tt}, \end{aligned}\right\} \tag{148}$$

in which r_p is the distance from the pumped well to the observation well, and r_i is the distance from the observation well to the image well. In equations 148, u_p and u_i are seen to be related thus:

$$\left.\begin{aligned} u_i &= \left(\frac{r_i}{r_p}\right)^2 u_p \\ \text{or} \\ u_i &= K^2 u_p, \end{aligned}\right\} \tag{149}$$

where

$$K = \frac{r_i}{r_p}. \tag{150}$$

Note: The K in equations 149 and 150 of Stallman is simply a constant of proportionality and is not to be confused with the K used previously to symbolize hydraulic conductivity.

Stallman plotted a family of logarithmic type curves of $\sum W(u)$ versus $1/u_p$ for many values of his $K = r_i/r_p$, as shown on plate 9. For an aquifer in which a single boundary

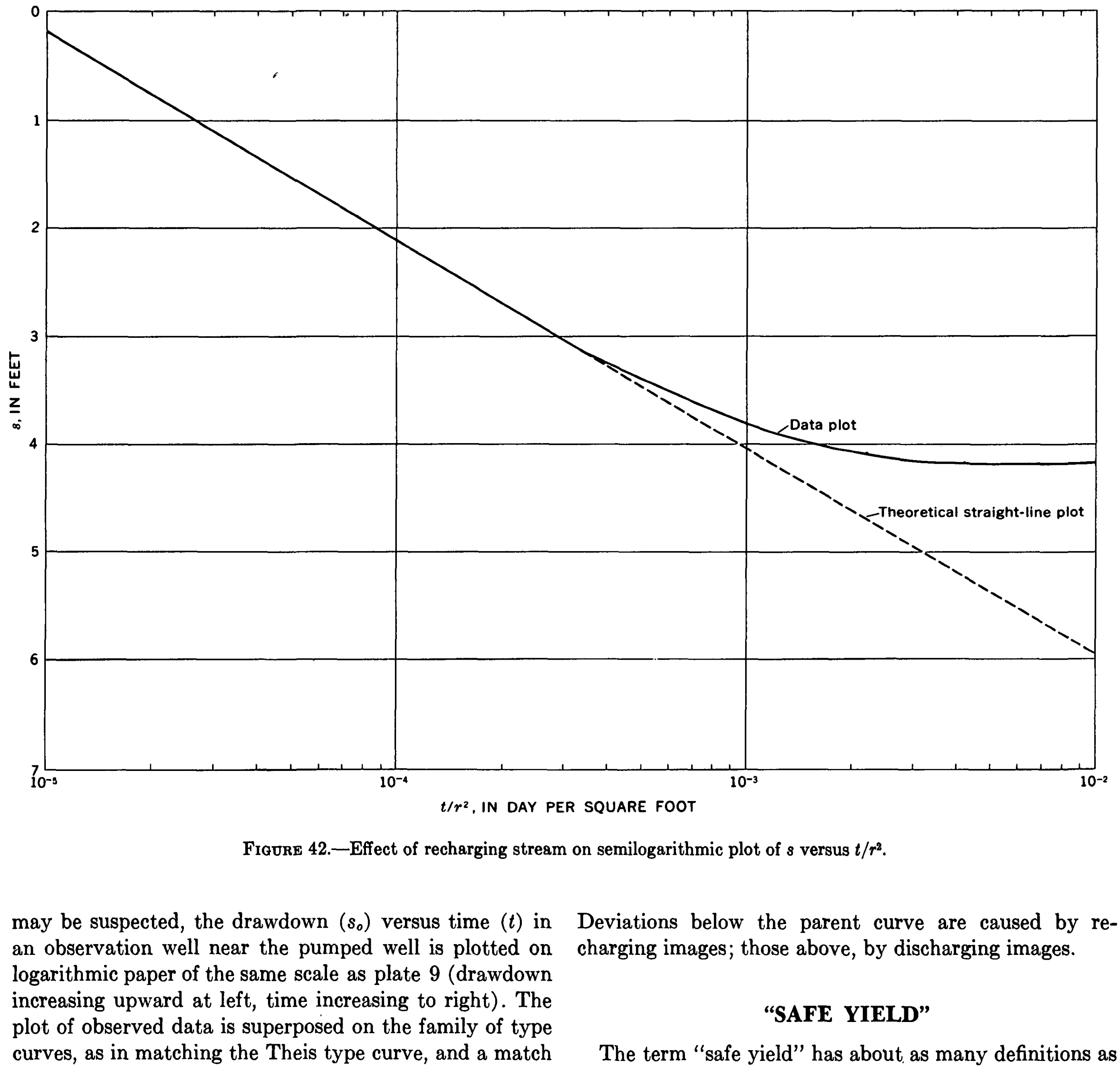

FIGURE 42.—Effect of recharging stream on semilogarithmic plot of s versus t/r^2.

may be suspected, the drawdown (s_o) versus time (t) in an observation well near the pumped well is plotted on logarithmic paper of the same scale as plate 9 (drawdown increasing upward at left, time increasing to right). The plot of observed data is superposed on the family of type curves, as in matching the Theis type curve, and a match point is found for values of $\sum W(u)$ and $1/u_p$ corresponding to values of s_o and t, respectively. Equation 147 can then be solved for T, after which equation 148 can be solved for S by rewriting equations 147 and 148 for solutions of T and S, respectively. From the value of K for the particular modified curve followed by the observed data, the value of r_i can be computed from equation 150.

If a suspected boundary is absent, and, therefore, the aquifer is extensive, the observed data should fit the heavy parent type curve, which is the Theis type curve. If a boundary exists, the observed data will follow the parent curve until the boundary is first "felt," then it will deviate from the parent curve along one of the modified curves. Deviations below the parent curve are caused by recharging images; those above, by discharging images.

"SAFE YIELD"

The term "safe yield" has about as many definitions as the number of people who have defined it. There are questions as to the validity of the term, but if it is valid there remains the question as to who should determine it—ground-water hydrologists or ground-water managers? Let us review briefly the history, meaning, and limitations of the term.

The term "safe yield" seemingly was first defined by Meinzer (1920, p. 330) as "* * * the rate at which the ground water can be withdrawn year after year, for generations to come, without depleting the supply." Later Meinzer (1932, p. 99) modified his definition to "The 'safe yield' of an underground reservoir [is the] practicable rate of withdrawing water from it perennially for human use * * *." Although there was nothing wrong with Meinzer's early definitions, they seemingly did not wholly satisfy all ground-water hydrologists, for beginning about 20 years later many began redefining the concept in more and more precise terms to suit themselves or to suit the particular ground-water conditions with which they were concerned. For a résumé of many of these definitions, see Kazmann (1951, 1956), Hantush (1955, p. 71), and Todd (1959, p. 200–218).

As stated by Kazmann (1956, p. 1103-2), "The thought became current that the 'safe yield' of an aquifer was surely determinable in advance of ground-water development—or even after development had begun." I might add that the thought also became current that one could put a *number* on the safe yield of a ground-water reservoir regardless of its method of development. However, as stated by Thomas (1951, p. 262):

> In a ground-water reservoir where the water is unconfined in certain areas (recharge areas) and under artesian pressure in other places, the safe yield will be a certain quantity if it is specified that all withdrawals must be by flow from artesian wells, a larger quantity if it is permissible to pump water from depths as great as 50 feet, and still more if allowable pumping lifts are as much as 500 feet. But the safe yield will vary also depending upon the locations of wells and the type of wells constructed. Assuming that the well construction and spacing of individual wells are suitable for maximum recovery of water, the safe yield will be a certain quantity if all wells are 40 miles from the recharge area, considerably greater if they are about 10 miles away, and still more if they are adjacent to or within the recharge area.

The multiplicity and looseness of definitions led Kazmann to title his 1956 paper " 'Safe Yield' in Ground-Water Development, Reality or Illusion?," and led Thomas (1951, p. 261) to say: " 'Safe yield.' This term, originated by hydrologists, may well prove awkward for them because of the variety of interpretations possible. 'Safe yield' is an Alice-in-Wonderland term which means whatever its user chooses."

I have a definition which I taught at U.S. Geological Survey Ground Water Short Courses beginning in 1952, namely, "The amount of ground water one can withdraw without getting into trouble." "Withdraw" may mean from flowing or pumped wells, and it may mean continuously, as for many industrial or municipal supplies, or seasonally, as for irrigation. "Trouble" may mean anything under the sun, such as (1) running out of water, (2) drawing in salt water, or other undesirable water, (3) getting shot, or shot at, by an irate nearby wellowner or landowner, (4) getting sued by a less irate neighbor, or (5) getting sued for depleting the flow of a nearby stream for which the water rights have been appropriated. My definition may sound facetious to some, but remembering that I would not attempt to put a number on it before development or in the early stages of development, especially if I did not know where and how the withdrawal would be made, it actually makes more sense than many definitions, does not differ significantly from Meinzer's original definitions, and is very close to the later definition of Todd (1959, p. 200): "The safe yield of a ground water basin is the amount of water which can be withdrawn from it annually without producing an undesired result."

To determine whether or not a desired quantity and quality of water can be withdrawn from a given ground-water reservoir generally requires an adequate knowledge of the geologic framework and its plumbing system plus the application of philosophy, common sense, and knowledge of the proposed type of development that owners or managers have in mind. As problems become more and more complex, however, such as those involving large investments in land and wells, withdrawal of water from both streams and wells that tap a common source, or conflicts in water rights, then the solution may require highly detailed study. The modern approach is for the hydrologist to acquire sufficient detail concerning the combined ground- and surface-water system so aquifer response can be predicted by electric-analog or mathematical models. Then management, such as state or local water-conservation agencies, within the framework of prevailing laws or regulations, may test the response of the system to various assumed stresses and thereby select the most desirable or equitable distribution of available water. Thus the role of the hydrologist is to gather and present the facts; the water manager determines who shall have how much water and from what source. In so doing, the manager generally requires and obtains considerable continuing assistance from the hydrologist. (See Wood and Gabrysch, 1965; Walton and Prickett, 1963; Moore and Wood, 1967.)

In the sections that follow, some additional references will be made to safe yield in discussing examples of different types of ground-water reservoirs.

THE SOURCE OF WATER DERIVED FROM WELLS

Under the above title Theis (1940) stated concisely the hydrologic principles upon which depend much of our present quantitative approach to ground-water problems. The statements that follow are summarized from these principles.

The essential factors that determine the response of aquifers to development by wells are:

1. Distance to, and character of, the recharge.
2. Distance to the locality of natural discharge.
3. Character of the cone of depression in the aquifer, which depends upon the values of T (which contains $\bar{K}$ and b) and S.

Prior to development by wells, aquifers are in a state of dynamic equilibrium, in that over long periods of time recharge and discharge virtually balance. Discharge from wells upsets this balance by producing a loss from storage, and a new state of dynamic equilibrium cannot be reached until there is no further loss from storage. This can only be accomplished by:

1. Increase in recharge (natural or artificial).
2. Decrease in natural discharge.
3. A combination of 1 and 2.

The above statements were put into equation form with terms having the dimensions L^3T^{-1} by Hilton H. Cooper, Jr., U.S. Geological Survey (written commun., April 1967). I have added the $\Delta h/\Delta t$ and have assigned to the terms the dimensions LT^{-1} as follows:

$$R+\Delta R=D+\Delta D+q+S\frac{\Delta h}{\Delta t} \qquad [LT^{-1}], \qquad (151)$$

where

R = recharge rate per unit area,
ΔR = change in recharge rate per unit area,
D = natural discharge rate per unit area,
ΔD = change in discharge rate per unit area,
q = rate of withdrawal from wells per unit area, and

$S\frac{\Delta h}{\Delta t}$ = rate of change in storage per unit area.

Assuming that additions to the aquifer (left-hand terms in eq. 151) are positive, that withdrawals from the aquifer (right-hand terms in eq. 151) are negative, and that over the years $R \approx D$, then solving equation 151 for q, we obtain

$$q \approx \Delta R-(-\Delta D)-\left(-S\frac{\Delta h}{\Delta t}\right) \qquad [LT^{-1}]. \qquad (152)$$

If dynamic equilibrium can be reestablished, there will be no further withdrawals from storage, so that $S\frac{\Delta h}{\Delta t} \approx 0$, and equation 152 becomes

$$q \approx \Delta R-(-\Delta D) \qquad [LT^{-1}]. \qquad (153)$$

We will see in the last section that equation 153 is applicable only to certain types of aquifer systems. For aquifers in which a new dynamic equilibrium is not attainable, continued withdrawal from storage may be greater than changes in recharge or discharge rates, and storage may be the principal or sole source of water. Under such conditions equation 152, or a part of it, is applicable. If we wish to know the total volumes of water involved, each term in equations 152 or 153 must be multiplied by the area (A) over which the changes occur.

Before taking up examples of different types of aquifers, let us review the following summary statements by Theis (1940, p. 280), which are important enough to quote completely:

1. All water discharged by wells is balanced by a loss of water somewhere.
2. This loss is always to some extent and in many cases largely from storage in the aquifer. Some ground water is always mined. The reservoir from which the water is taken is in effect bounded by time and by the structure of the aquifer as well as by material boundaries. The amount of water removed from any area is proportional to the drawdown, which in turn is proportional to the rate of pumping. Therefore, too great concentration of pumping in any area is to be discouraged and a uniform areal distribution of development over the area where the water is shallow should be encouraged, so far as is consistent with soil and marketing or other economic conditions.
3. After sufficient time has elapsed for the cone to reach the area of recharge, further discharge by wells will be made up at least in part by an increase in the recharge if previously there has been rejected recharge. If the recharge was previously rejected through transpiration from nonbeneficial vegetation, no economic loss is suffered. If the recharge was rejected through springs or refusal of the aquifer to absorb surface waters, rights to these surface waters may be injured.
4. Again, after sufficient time has elapsed for the cone to reach the areas of natural discharge, further discharge by wells will be made up in part by a diminution in the natural discharge. If this natural discharge fed surface streams, prior rights to the surface water may be injured.
5. In most artesian aquifers—excluding very extensive ones, such as the Dakota sandstone—little of the water is taken from storage. In these aquifers, because the cones of depression spread with great rapidity, each well in a short time has its maximum effect on the whole aquifer and obtains most of its water by increase of recharge or decrease of natural discharge. Such an artesian basin can be treated as a unit, as is done in the New Mexico ground-water law, and the laws of some other Western States that follow this law. In large nonartesian aquifers, where pumping is done at great distances from the localities of intake or outlet, however, the effects of each well are for a considerable time confined to a rather small radius and the water is taken from storage in the vicinity of the well. Hence these large ground-water bodies cannot be considered a unit in utilizing the ground water. Proper conservation measures will consider such large aquifers to be made up of smaller units, and will attempt to limit the development in each unit. Such procedure would also be advisable, although not as necessary, in an artesian aquifer.
6. The ideal development of any aquifer from the standpoint of the maximum utilization of the supply would follow these points:
   (a) The pumps should be placed as close as economically possible to areas of rejected recharge or natural discharge where ground water is being lost by evaporation or transpiration by nonproductive vegetation, or where the surface water fed by, or rejected by, the ground water cannot be used. By so doing this lost water would be utilized by the pumps with a minimum lowering of the water level in the aquifer.
   (b) In areas remote from zones of natural discharge or rejected recharge, the pumps should be spaced as uniformly as possible throughout the available area. By so

doing the lowering of the water level in any one place would be held to a minimum and hence the life of the development would be extended.

(c) The amount of pumping in any one locality would be limited. For nonartesian aquifers with a comparatively small areal extent and for most artesian aquifers, there is a perennial safe yield equivalent to the amount of rejected recharge and natural discharge it is feasible to utilize. If this amount is not exceeded, the water levels will finally reach an equilibrium stage. If it is exceeded, water levels will continue to decline.

In localities developing water from nonartesian aquifers and remote from areas of rejected recharge or natural discharge, the condition of equilibrium connoted by the concept of perennial safe yield may never be reached in the predictable future and the water used may all be taken from storage. If pumping in such a locality is at a rate that will result in the course of 10 years in a lowering of water level to a depth from which it is not feasible to pump, pumping at half this rate would not cause the same lowering in 100 years. Provided there is no interference by pumping from other wells, in the long run much more water could be taken from the aquifer at less expense.

Thorough knowledge of these hydrologic principles plus the gathering and proper interpretation of the pertinent field data should permit the solution of virtually any quantitative ground-water problem, although in some areas the solution may be very difficult.

EXAMPLES OF AQUIFERS AND THEIR DEVELOPMENT

Let us see how the above principles and equations 152 and 153 apply to several types of aquifers. In the real situations the water would be withdrawn from many wells, but for simplicity only one or two wells are shown in the examples.

VALLEY OF LARGE PERENNIAL STREAM IN HUMID REGION

Setting.—Thick, permeable alluvium filling old valley cut into shale; permeable channel beneath large perennial stream; shallow water table; many phreatophytes; moderately heavy precipitation (fig. 43).

Sources of water.—(1) Withdrawal from storage which creates cone of depression. (2) Salvaged rejected recharge: Lowering of water table provides more room for recharge from precipitation and, hence, reduces or prevents surface runoff to stream. (3) Salvaged natural discharge by (a) lowering water table beneath phreatophytes and (b) decreasing gradient toward stream thus decreasing discharge of ground water into stream. Sources (2) and (3) may suffice for small to moderately large ground-water developments. (4) Recharge directly from stream: Large withdrawal will cause cone of depression to spread until it reaches the stream, then gradient will be reversed and stream water will move toward wells.

Operation of system.—In applying equation 152 to such a development, we assume that $R \approx D$ and that equilibrium has been reestablished so that $S\frac{\Delta h}{\Delta t} \approx 0$; then

$$q \approx \Delta R - (-\Delta D) \qquad [LT^{-1}],$$

equation 153, in which ΔR may suffice for moderately large developments, but ΔD, capture of stream water, becomes the principal source for large developments.

Limitations.—The amount of possible withdrawal is virtually limited to the streamflow. Lowering of the water table may impair or destroy useful phreatophytes or interfere with other ground-water developments, and the reduction in streamflow may interfere with the established rights of others.

VALLEY OF EPHEMERAL STREAM IN SEMIARID REGION

Setting.—Moderately thick, permeable alluvium filling old valley cut into shale; permeable channel beneath ephemeral stream; water table beneath stream channel and below reach of all vegetation except a few cottonwood trees along banks of stream; precipitation, about 15 in. a year; stream dry most of year but floods after heavy rains or cloudbursts in high headwater region (fig. 44).

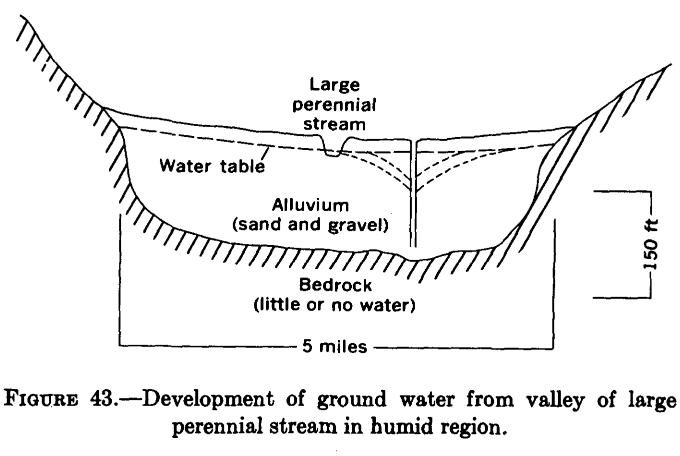

FIGURE 43.—Development of ground water from valley of large perennial stream in humid region.

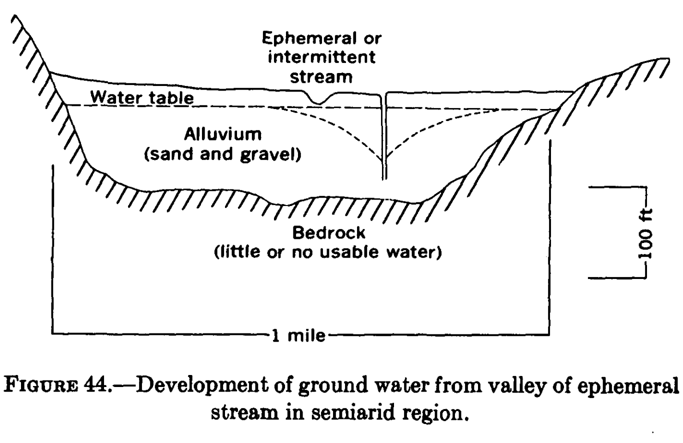

FIGURE 44.—Development of ground water from valley of ephemeral stream in semiarid region.

Sources of water.—(1) Withdrawal from storage which creates cone of depression. (2) Salvaged rejected recharge from precipitation—none; there is more than enough room for the meager recharge from low precipitation. (3) Salvaged natural discharge—very small; the only phreatophytes are a few cottonwoods along banks of stream. (4) Recharge directly from stream: May be very large from floods, provided water table is kept sufficiently lowered by pumping to provide room for all or most of the floodwater, which percolates rapidly downward through the permeable channel. Thus, the system functions effectively as an evaporation-free flood-control reservoir.

Operation of system.—In applying equation 152 to such a development, we assume that $\Delta D \approx 0$ and that equilibrium cannot be reestablished; then

$$q \approx \Delta R - \left(\pm S \frac{\Delta h}{\Delta t}\right) \qquad [LT^{-1}],$$

in which ΔR, capture of floodwater with attendant increase in ground-water storage $\left(+S \frac{\Delta h}{\Delta t}\right)$, becomes the principal source of water. Bewteen floods, when $\Delta R = 0$, loss from storage $\left(-S \frac{\Delta h}{\Delta t}\right)$ is the sole source. This procedure has been successful in some ephemeral stream valleys of eastern Colorado, where water is pumped seasonally for irrigation.

Limitations.—(1) The relation of flood frequency to water needs. (2) The water table must be kept low enough (by pumping) to provide adequate storage space for floodwaters, yet it must be high enough for successful or economical well operation.

CLOSED DESERT BASIN

Setting.—Large desert basin receiving 3–5 in. of precipitation annually, surrounded by high mountains that receive 20–30 in. of precipitation (fig. 45). Aquifer comprises thick bolson deposits built up by coalescing alluvial fans, coarse and permeable near mountains, fine grained and much less permeable at and near playa. Water table at or near surface at playa, deep near mountains. Streams, all ephemeral. Phreatophytes, only near playa in middle of basin.

Sources of water.—(1) Withdrawal from storage creates cone of depression. (2) Salvaged rejected recharge from precipitation—little or none; virtually all the precipitation that falls in the valley evaporates or is transpired in and near playa. Recharge comes mainly at irregular intervals from small ephemeral streams that head in surrounding mountains; incompletely saturated fan areas absorb much of streamflow, but some flood flows reach playa, where

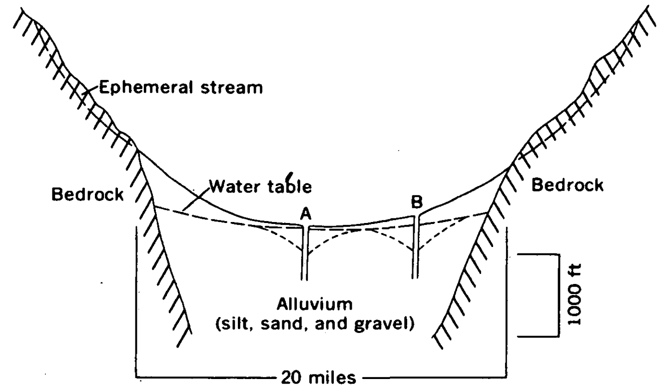

FIGURE 45.—Development of ground water from bolson deposits in closed desert basin.

water evaporates rapidly. (3) Salvaged natural discharge by (a) development near playa (well A), may be large, by lowering water table below reach of evaporation and transpiration by phreatophytes and (b) development along border of basin (well B), where some discharge toward playa may be salvaged by reducing the gradient.

Water near playa (well A) generally is too highly mineralized for most uses, and materials have low permeability. Development of water of better quality in bordering areas (well B) may be limited in quantity and by pumping lift; however, withdrawals from bordering areas can be greatly increased by the construction of retention dams in canyons of bordering mountains so that floodwaters that normally reach and temporarily flood playa may be stored and released slowly for recharge into the heads of alluvial fans, as has been done in California and perhaps elsewhere.

Operation of system.—Assuming that ΔD is small but finite and that ΔR by release from retention dams is fairly constant, then equilibrium can be reestablished $\left(S \frac{\Delta h}{\Delta t} \approx 0\right)$, and equation 152 becomes equation 153,

$$q \approx \Delta R - (-\Delta D) \qquad [LT^{-1}],$$

in which ΔR is the principal source.

Without the retention dams, ΔR will not be constant and, for large withdrawals, equilibrium cannot be reestablished, so equation 152, for well B, becomes

$$q \approx \Delta R - (-\Delta D) - \left(\pm S \frac{\Delta h}{\Delta t}\right) \qquad [LT^{-1}].$$

SOUTHERN HIGH PLAINS OF TEXAS AND NEW MEXICO

Setting.—Remnant of High Plains sloping gently from west to east, cut off from external sources of water by

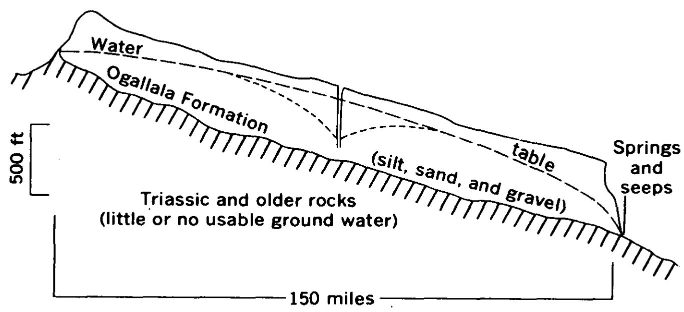

FIGURE 46.—Development of ground water from southern High Plains of Texas and New Mexico.

escarpments upstream and downstream (fig. 46). Water is in Tertiary deposits (Ogallala Formation), which have a maximum thickness of about 600 ft and an average thickness of about 300 ft. The material is moderately permeable and rests on relatively impermeable rocks. The recharge, which is derived solely from scanty precipitation, is estimated to range from 1/20 to 1/2 in. per year, or of the order of 3×10^{9} ft^3 year^{-1}. The natural discharge, of the same estimated order, is from seeps and springs along the eastern escarpment. The storage of ground water prior to development was very large, of the order of 2×10^{13} ft^3. The withdrawal by pumping has increased from about 4×10^{9} ft^3 year^{-1} in 1934 to more than 2×10^{11} ft^3 year^{-1} and is used mainly for irrigation.

Sources of water.—(1) Withdrawal from storage creates cone of depression. (2) Salvaged rejected recharge—virtually none; water table lies 50 ft or more beneath surface in most of the area, so that there is more than ample space for all possible natural recharge. (3) Salvaged natural discharge—virtually none; gradient toward eastern escarpment has been virtually unchanged, but even if all discharge could be salvaged, it would only amount to 1 or 2 percent of the withdrawal rate.

Operation of system.—Assuming that $\Delta R\approx0$ and $\Delta D\approx0$, equation 152 becomes

$$q\approx-\left(-S\frac{\Delta h}{\Delta t}\right)\quad[LT^{-1}],$$

which means that virtually all water is being mined from storage and that equilibrium is not being reestablished. Because ground water is a mineral that is being mined without hope of natural replacement, the Federal courts have affirmed the right of eligible ground-water users (those who have, in effect, paid for the water in the form of land prices higher than that of land lacking a good supply) to claim a depletion allowance for Federal income-tax purposes.

Possible remedial measures.—(1) In the Texas section of the region, a water conservation district, to which most affected counties belong, has sought to retard depletion by encouraging water-saving practices and by requiring proper spacing of wells. In the New Mexico section, the State law based on prior appropriation is applied by allowing, in a particular area, appropriations until the remaining supply is judged sufficient for an additional period (such as 30 or 40 years) to enable recovery of investments in land and wells and the creation of wealth through extraction of this "minable" resource; the area is then declared fully appropriated and additional appropriations are not permitted. (2) Artificial recharge from ephemeral ponds through recharge wells has been tried but so far has not been successful over long periods. Had it been successful, it would have been sufficient only to retard depletion. (3) The importation of water by lifting it several hundred feet from the Canadian River was considered, but was judged too costly. The feasibility of importing water from the distant Mississippi or Missouri River is now (1971) being considered to alleviate water shortages in this and other parts of the High Plains.

GRAND JUNCTION ARTESIAN BASIN, COLORADO

There are three artesian aquifers in the Grand Junction artesian basin (Lohman, 1965), but for simplicity only one, the Jurassic Entrada Sandstone, is taken up here (fig. 47). Conditions here are typical of artesian aquifers of low permeability.

Setting.—Aquifer is fine-grained sandstone partly cemented with calcium carbonate, about 150 ft thick. $T=20$ ft^2 day^{-1}, $S=5\times10^{-5}$. Precipitation is about 7½ inches per year. Recharge occurs only where outcrops are in contact with the alluvium of small ephemeral streams. The alluvium remains partly saturated for short periods after streamflow. Natural discharge is very small and is limited to upward leakage through 500 to 1,000 ft of

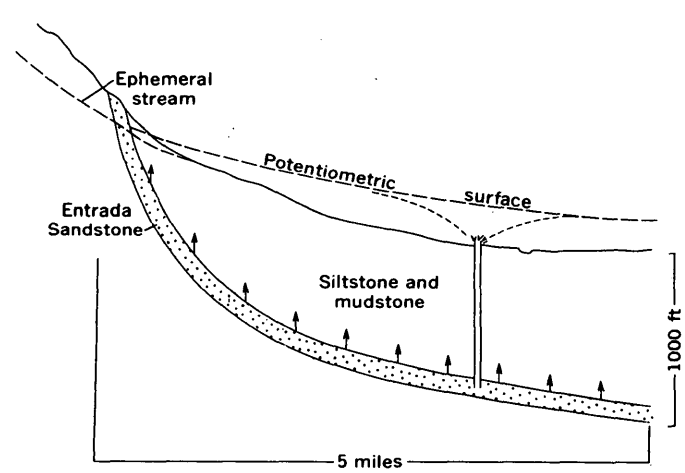

FIGURE 47.—Development of ground water from the Grand Junction artesian basin, Colorado.

relatively impermeable siltstone and mudstone. The artesian head before development was dependent in part upon the local topography but was as much as 160 ft above land surface.

Sources of water.—(1) Withdrawal from artesian storage (no unwatering) creates large, overlapping cones of depression. (2) Salvaged rejected recharge—virtually none; movement of water from already saturated recharge areas is greatly restricted by very low transmitting capacity of aquifer. (3) Salvaged natural discharge—very small; limited to upward leakage through confining beds of very low permeability.

Operation of system.—Assuming that $\Delta R \approx 0$ and $\Delta D \approx 0$, as in the High Plains, equation 152 becomes

$$q \approx -\left(-S\frac{\Delta h}{\Delta t}\right) \qquad [LT^{-1}],$$

which means that virtually all water is being "mined" from artesian storage and that equilibrium cannot be reestablished.

However, there is a significant difference between the mining of water in the High Plains and in the Grand Junction artesian basin. The Ogallala Formation has a specific yield of perhaps 0.15, so for each foot of decline in water level, each cubic foot of drained water-bearing material yields about 0.15 ft^3 of water. For each foot of decline in head, 1 square foot of the Entrada Sandstone yields only about 5×10^{-5} ft^3 of water. Thus it is perhaps fair to say that, in the High Plains, it is mainly the water that has been mined; in the Grand Junction artesian basin, it is mainly the artesian head that has been mined, for the aquifer is still saturated.

Possible remedial measures.—Most of the water withdrawn from the Entrada Sandstone to date has been used for domestic purposes, either by piping to nearby homes or by hauling to the cisterns of other rural residents. Now most rural residents have been supplied by water piped from distant surface sources. If the draft on the wells is thereby reduced sufficiently, the decline in artesian head should be arrested somewhat and eventually the head may slowly recover (Lohman, 1965, p. 122).

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