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Groundedness property and accessibility of ordinal diagrams

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§ 0. Introduction.

The theory of ordinal diagrams as well as its applications and generalizations has been worked out over the years by G. Takeuti and the present author. (See [1]~[6]; all the earlier results in this line are included in [2].) The system of ordinal diagrams was invented by Takeuti as a means for the consistency proofs; that is, the consistency of a subsystem of analysis is reduced to the accessibility of ordinal diagrams. It is therefore of primary importance that we establish an accessibility proof of ordinal diagrams in its strict sense. Such attempts have been made in [2], [7] and [8], but none of them is entirely satisfactory from our standpoint. (We refer the reader to Sections 11 and 26 of [2] for a detailed discussion on the constructive standpoint.)

Let $\mathcal{J}=(J, <)$ be a concretely given linearly ordered structure. An accessibility proof of $\mathcal{J}$ consists in presenting a "concrete" method to establish that there be no infinite $<$-decreasing sequence from J, and J is said to be $<$-accessible if there is such an accessibility proof.

Here in this article the author presents a more constructive accessibility proof of ordinal diagrams.

Let I and A be two accessible sets and let $O(I, A)$ be the system of ordinal diagrams based on I and A. Takeuti originally proved the accessibility of $O(I, A)$ by making use of a subset of it, which he named F_i, for each i an element of $I\cup\{\infty\}$ (Section 26 of [2]). He later became declined to accept F_i as a concrete object, and he and the present author proposed another version in [7], in which the theory of fundamental sequences in $O(I, A, S)$ (an extended system of ordinal diagrams) for some S and the notion of strong accessibility stand essential. Since the construction of fundamental sequences is finitary (see [3]), the problem of this approach can be pinned down to the notion of strong accessibility, which is defined in terms of arbitrary well-ordered sets.

Takeuti has recently revisited the theme of accessibility in the appendix of [8], resorting to the sets F_i's once again. In [8] he defines F_i for each i by the condition that, for an α to be in F_i, "there be a method" to show that every

k-section of α is $<_k$-accessible in F_k for every $k<i$.

The author believes that the problem is not whether or not it is acceptable to consider F_i; we can first introduce it as an abstract concept, work out an accessibility proof by using it as an auxiliary means, and give substance to it later. What are not concrete in the proof of [8] are the existence statements (of decreasing sequences) in the proofs of Propositions 2 and 3, and it is here that the theory of fundamental sequences can be applied. These propositions are listed as Propositions 3.3 and 4.4 respectively in this paper, and they follow immediately from Propositions 3.2 and 4.3 respectively, which are our major results.

We first define the following for each j an element of $I\cup\{\infty\}$.

$$I\langle j\rangle=\{l\in I\,;\ l\geqq j\},$$

$$J_j=\{\gamma[j]\,;\ \gamma=(k,b,\beta),\ k<j,\ \beta \text{ is } k\text{-accessible in } F_k \text{ and } \gamma[j] \text{ is a new symbol corresponding to } \gamma\},$$

$$\prec_j:\ \text{the order of } J_j \text{ induced by } <_j,$$

$$\boldsymbol{O}(j)=\boldsymbol{O}(I\langle j\rangle, A, J_j).$$

An ordinal diagram of $\boldsymbol{O}(I, A)$ will be said to be i-grounded if it is $<_i$-accessible in F_i. The i-groundedness property is reduced to the $<_i$-accessibility in $\boldsymbol{O}(i)$. Through certain constructions induced from the fundamental sequences in $\boldsymbol{O}(j)$ for some $j<i$, we demonstrate that every ordinal diagram is i-grounded, and is hence $<_i$-accessible, for every i. Notice that j is the least indicator in $\boldsymbol{O}(j)$ even if it is not in the original system $\boldsymbol{O}(I, A)$.

We do not explicitly state that "there be concrete methods" in defining various accessible sets; such methods are implied in the notion of accessibility, and the entire proof is so designed that when completed there be concrete methods as desired.

Our proof is an improvement over [7] also in the sense that in a way the "arbitrary well-ordered sets S" in the definition of strong accessibility have been replaced by a concrete accessible set J_j for each j.

Let us emphasize that we are not giving an alternative proof to the preceding ones, but our approach serves as a conclusive version of the accessibility proof of ordinal diagrams.

The full knowledge of [2], [3] and [7] are assumed. In particular, Section 26 of [2] will be assumed without quotations.

Let us take this opportunity to correct an error in [3]: (*) of page 8 should be read as follows.

$$\{\tilde{\alpha}_m\}_m \text{ is } <_{j_0}\text{-increasing and } \forall m(\tilde{\alpha}_m<_{j_0}\tilde{\alpha}).$$

(The corresponding propositions and theorems in the main context are stated properly.)

The author is grateful to Gaisi Takeuti, who suggested her to take up the accessibility problem.

§1. Ordinal diagrams, their projections and elevations.

DEFINITION 1.1. 1) Let I and A be two accessible sets, both of whose orders will be denoted by $<$. The system of ordinal diagrams based on I and A, $\boldsymbol{O}(I, A)$, is defined as in Definition 26.1 of [**2**]. For each i, which is either an indicator (an element of I) or ∞, the order of the ordinal diagrams with respect to i is defined as in Definition 26.7 of [**2**] and is denoted by $<_i$. See Section 26 of [**2**] for all the definitions, the technical terms and the properties concerning the ordinal diagrams.

2) Let S be a well-ordered set. The system of ordinal diagrams based on I, A and S, $\boldsymbol{O}(I, A, S)$, is defined as in Definitions 1.1~1.3 of [**7**]. $\boldsymbol{O}(I, A, S)$ is the system of ordinal diagrams where S forms an initial segment for any order $<_i$. When I and A are supposed to be fixed, we may write $\boldsymbol{O}(S)$ for $\boldsymbol{O}(I, A, S)$. See Sections 1 and 2 of [**7**] for details.

REMARK. 1) We shall call the elements of $\boldsymbol{O}(I, A)$ or of $\boldsymbol{O}(S)$ simply the diagrams.

2) In the course of our accessibility proof, it will become clear that the set S functions as a parameter for accessible sets. That is, if an accessible set J is defined, we can substitute it for S to obtain a concrete system $\boldsymbol{O}(J)$. It is thus appropriate to say that $\boldsymbol{O}(S)$ serves as an auxiliary system. In this section we develop the general theory of projections and elevations of the diagrams in $\boldsymbol{O}(S)$ with S a parameter.

DEFINITION 1.2. 1) The letters $i, j, k, l, \cdots$ will stand for the elements of $I\cup\{\infty\}$, $a, b, c, \cdots$ for the elements of A and $\alpha, \beta, \gamma, \cdots$ for the diagrams. $i+1$ will stand for the successor of i. In case i is the maximal indicator, $i+1$ is ∞.

2) When a diagram α is accessible in a subsystem F of $\boldsymbol{O}(I, A)$ or $\boldsymbol{O}(I, A, S)$ with respect to $<_i$, we say that α is i-accessible in F. F is said to be i-accessible if every element of F is i-accessible in F. (See Introduction for the notion of accessibility.)

3) For every i, we define two subsets of $\boldsymbol{O}(I, A, S)$, F_i and G_i, as follows.

$$F_i(S)=F_i=\{\alpha\in\boldsymbol{O}(S)\,;\ \forall j<i \text{ (every } j\text{-section of } \alpha \text{ is } j\text{-accessible in } F_j)\},$$

$$G_i(S)=G_i=\{\alpha\in F_i\,;\ \alpha \text{ is } i\text{-accessible in } F_i\}.$$

4) If α is in F_i, we say that α is i-fit. If α is in G_i, we say that α is

i-grounded.

5) A sequence of diagrams, say $\{\alpha_n\}_n$, is said to be i-decreasing if $\alpha_{n+1} <_i \alpha_n$ for every n. It is said to be led by α if $\alpha_0 \leqq_i \alpha$.

NOTE. 1) From the definition it follows that $F_o = \boldsymbol{O}(S)$ and $G_o = \{\alpha;\ \alpha$ is o-accessible$\}$, where o designates the least element of I. We shall use the letter o for the least element of A also.

2) See [8] for the original definition of F_i. Here we do not explicitly demand the concreteness of the accessibility, since the entire accessibility proof is designed so that there be a concrete method for the accessibility whenever it is asserted.

PROPOSITION 1.1. 1) *$\{F_i\}_i$ is decreasing (non-increasing) as a (transfinite) sequence of sets.*

2) *α is $(i+1)$-fit if and only if α is i-fit and every i-section of α is i-grounded.*

3) *If α is i-fit and σ is i-active in α, then σ is i-fit.*

4) *If α is i-grounded and σ is i-active in α, then σ is i-grounded.*

5) *α is i-fit if and only if each component of α is i-fit.*

6) *G_i is i-accessible.*

The facts claimed above will be used in the subsequent material without quotations.

DEFINITION 1.3. 1) For each $i > o$, we define H_i a subset of $\boldsymbol{O}(S)$ as follows.

$$H_i(S) = H_i = \{(k, b, \beta);\ k < i \text{ and } \beta \in G_k\}.$$

2) For each $i > o$, we assume new, distinct symbols for the elements of H_i. Let $\gamma[i]$ represent the symbol for γ. We define J_i and $\prec_i$ as follows.

$$J_i(S) = J_i = \{\gamma[i];\ \gamma \in H_i\}.$$

$\prec_i$ is defined to be the order of J_i so that $(H_i, <_i)$ and $(J_i, \prec_i)$ are isomorphic.

PROPOSITION 1.2. 1) *H_i is a subset of F_i, and H_i is i-accessible, and hence J_i is accessible with respect to $\prec_i$.* (See (1) in Proposition 8.1 of [7].)

2) *If $j < i$, then H_j is a subset of H_i, and H_j forms an initial segment of H_i with respect to $<_i$. Furthermore $<_i$ is an extension of $<_j$ on H_i. These facts imply that $(J_j, \prec_j)$ can be regarded as an initial segment of $(J_i, \prec_i)$ if $j < i$.*

DEFINITION 1.4. 1) $\boldsymbol{O}^*(i;\ S) = \{\gamma \in \boldsymbol{O}(S);\ \gamma$ does not contain any indicator below $i\}$.

2) $I\langle i \rangle = \{j \in I;\ j \geqq i\}$.

3) $\boldsymbol{O}(i;\ I, A, S) = \boldsymbol{O}(i;\ S) = \boldsymbol{O}(I\langle i \rangle, A, S)$.

4) For any i-fit α, $\alpha[i]$ is defined to be the figure obtained from α by replacing in it each i-active element of H_i by its corresponding symbol in J_i.

($\alpha[i]$ can be defined inductively; see Proposition 1.3 below.) The operation to obtain $\alpha[i]$ from α will be called the i-projection of α.

5) $E_i=\{\alpha[i]\,;\ \alpha\in F_i\}$.

6) $S*J_i$ is defined as in Definition 6.1 of [7]; it is the set $S\cup J_i$ with the order which puts the elements of S in precedence to those of J_i.

7) $D_i=\{\kappa\in \boldsymbol{O}(i\,;\ I,\ A,\ S*J_i)\,;\ \kappa$ is i-accessible in $\boldsymbol{O}(i\,;\ I,\ A,\ S*J_i)\}$.

NOTE. For $i=o$, we can define $\alpha[o]=\alpha$, $\boldsymbol{O}^*(o\,;\ S)=\boldsymbol{O}(S)=\boldsymbol{O}(o\,;\ S)$, $E_0=\boldsymbol{O}(S)=F_o$ and $D_o=G_o$.

PROPOSITION 1.3. 1) *$\boldsymbol{O}^*(i\,;\ S)$ is closed with respect to the formation rules of diagrams as well as the subdiagram property.*

2) *For every $j\leqq i$, the orders $<_j$ and $<_i$ are isomorphic on $\boldsymbol{O}^*(i\,;\ S)$.*

3) *The system $\boldsymbol{O}^*(i\,;\ S)$ can be identified with $\boldsymbol{O}(i\,;\ I,\ A,\ S)$. We can therefore develop the theory of ordinal diagrams for $\boldsymbol{O}^*(i\,;\ S)$. In particular, i is the least indicator in this theory.*

4) *For every i-fit α, $\alpha[i]$ is uniquely determined, and it can be defined inductively as follows. Let σ be an i-active subdiagram of α. If σ is $(k,\ b,\ \beta)$ where $k<i$, then it is an element of H_i, and hence $\sigma[i]$ is its corresponding symbol in J_i. If $k\geqq i$, then $\sigma[i]=(k,\ b,\ \beta[i])$. If σ is $\beta_1\#\cdots\#\beta_m$ where $\beta_1,\cdots,\beta_m$ are connected, then $\sigma[i]=\beta_1[i]\#\cdots\#\beta_m[i]$.*

5) *For every κ in $\boldsymbol{O}^*(i\,;\ S*J_i)$, one can find a unique i-fit α such that $\alpha[i]=\kappa$.*

6) *$E_i=\boldsymbol{O}^*(i\,;\ S*J_i)=\boldsymbol{O}(i\,;\ I,\ A,\ S*J_i)$. The orders $<_j$ for $j\geqq i$ can therefore be introduced to the elements of E_i.*

7) *$D_i=\{\alpha[i]\,;\ \alpha\in F_i$ and $\alpha[i]$ is i-accessible in $E_i\}$.*

8) *For every j, $(\boldsymbol{O}(S),\ <_j)$ can be embedded in $(\boldsymbol{O}(S*J_i),\ <_j)$.*

9) *For every $j\geqq i$, $(E_i,\ <_j)$ can be embedded in $(\boldsymbol{O}(S*J_i),\ <_j)$.*

10) *If κ is in D_i and δ is a subdiagram of κ, then δ is also in D_i.*

DEFINITION 1.5. Let κ be a diagram of $\boldsymbol{O}^*(i\,;\ S*J_i)$. Then the map from κ to the unique α which has been claimed to exist in 5) of Proposition 1.3 will be called the i-elevation of κ. We shall write $\kappa\{i\}$ to denote this α.

PROPOSITION 1.4. *Assume that α and β are i-fit (diagrams of $\boldsymbol{O}(S)$).*

1) *For any $j\geqq i$ and δ a j-section of $\alpha[i]$, there is a j-section of α, say σ, such that $\sigma[i]=\delta$. (σ is also i-fit.) Conversely, if σ is a j-section of α, then $\sigma[i]$ is a j-section of $\alpha[i]$.*

2) *For every j, $\alpha[i]\leqq_j\alpha$ in $\boldsymbol{O}(S*J_i)$. For every $j\geqq i$ and every δ a j-section of $\alpha[i]$, there is a σ a j-section of α satisfying $\delta\leqq_j\sigma$ in $\boldsymbol{O}(S*J_i)$.* (See 9) of Proposition 1.3.)

3) *For every $j\geqq i$, $\alpha<_j\beta$ in $\boldsymbol{O}(S)$ implies $\alpha[i]<_j\beta[i]$ in $\boldsymbol{O}(i\,;\ S*J_i)$, and $\alpha<_i\beta$ implies $\alpha[i]<_j\beta[i]$ for every $j<i$.*

4) *Suppose $l\leqq i$. Then, for every $j\geqq l$, $\alpha<_j\beta$ (in $\boldsymbol{O}(S)$) if and only if*

$\alpha[l]<_j\beta[l]$ *in* E_l. *(Since* α *is i-fit and* $l\leqq i$, α *is l-fit, and hence* $\alpha[l]$ *is well-defined.)*

5) *The i-projection and the i-elevation are mutually inverse operations between* $(F_i, <_j)$ *and* $(E_i, <_j)$ *for every* $j\geqq i$.

PROOF. 2) The proof of this proposition goes exactly parallel to that of (2) in Proposition 8.1 of [7].

3) The proof of (3) in Proposition 8.1 of [7] goes through. Consult Proposition 1.3, and 1) and 2) above.

4) The "only if" part follows from 3), and the "if" part follows from the "only if" part.

5) This follows from 4), 5) and 6) of Proposition 1.3, and 4) above.

PROPOSITION 1.5. *For any i-fit* α, α *is i-grounded if and only if* $\alpha[i]$ *is in* D_i. *In view of* 5) *and* 6) *of Proposition* 1.3, *we can thus deduce that* $D_i=\{\alpha[i]\,;\,\alpha\in G_i\}$.

By virtue of this proposition, we can say that "α is i-grounded" when $\alpha[i]\in D_i$ is established.

PROOF. This follows from 5) of Proposition 1.4.

PROPOSITION 1.6. 1) *Let* α *and* β *be* $(i+1)$*-fit. If* $\beta<_i\alpha$ *and* β *is not i-grounded, then* $\beta<_{i+1}\alpha$. *This can also be stated that, if* $\beta[i]<_i\alpha[i]$ *and* $\beta[i]$ *does not belong to* D_i, *then* $\beta<_{i+1}\alpha$.

2) α *is* $(i+1)$*-fit if and only if* α *is i-fit and every i-section of* $\alpha[i]$ *is an element of* D_i.

PROOF. 1) Suppose $\alpha<_{i+1}\beta$. Then there is an i-section of α, say σ, satisfying $\beta\leqq_i\sigma$. But σ is i-grounded, and hence so must be β.

2) By 2) of Proposition 1.1, 1) of Proposition 1.4 and Proposition 1.5.

PROPOSITION 1.7. *Suppose* $j\geqq i+1$. *Let* κ *be in* $\boldsymbol{O}(i\,;\,S*J_i)$. *Then* $\kappa\{i\}[j]$ *is well-defined if and only if* $\kappa\{i\}$ *is j-fit. In particular,* $\kappa\{i\}[i+1]$ *is well-defined if and only if every i-section of* κ *is i-accessible in* $\boldsymbol{O}(i\,;\,S*J_i)$.

DEFINITION 1.6. We shall henceforth write $\boldsymbol{O}(i)$ for $\boldsymbol{O}(i\,;\,S*J_i)=\boldsymbol{O}(i\,;\,I,A,S*J_i)$.

§ 2. Fundamental sequences.

DEFINITION 2.1. Consider $\boldsymbol{O}(i)=\boldsymbol{O}(i\,;\,S*J_i)$. For every κ in $\boldsymbol{O}(i)$, the fundamental sequence for (i, κ) is defined as in Definition 3.2 of [7]. Recall that here i is the least indicator.

Various notions and properties concerning the fundamental sequences were studied in detail in [3]. Let us quote one proposition from there, which will play an important role in our accessibility proof. Although it is stated and proved for the system $\boldsymbol{O}(I, A)$, it is valid also for the extended system $\boldsymbol{O}(I, A, S)$.

PROPOSITION 2.1 OF [3]. *Let j_0 be an indicator, α be a diagram of $\boldsymbol{O}(I, A)$, (j, γ) be a scanned pair* (sp) *of (j_0, α) and $\{\gamma_m\}_m$ be the reduction sequence for γ. Suppose $h=h(\gamma)\leqq l\leqq j$. Then $1\sim5$ below hold.*

1. $\gamma_m<_l\gamma$ *for every m.*
2. *The maximum, h-active value of $\gamma_m\leqq$the corresponding value of γ.*
3. *If σ is an l-section of γ_m, then $\sigma<_l\gamma$.*
4. $\gamma_m<_l\gamma_{m+1}$.
5. *If σ is an l-section of γ_m, then $\sigma<_l\gamma_{m+1}$.*

Our present version of the loan proposition above can be stated as follows.

PROPOSITION 2.1. *We consider the fundamental sequence of (i, κ) in $\boldsymbol{O}(i)$. Let (j, γ) be a scanned pair of (i, κ), and let $\{\gamma_m\}_m$ be the reduction sequence for γ. Suppose $i\leqq l\leqq j$. Then $1\sim5$ above hold.*

§ 3. Descent of groundedness: the successor case.

The groundedness property is descending; if α is i-grounded, then α is j-grounded for every $j<i$. This fact is established in the next section. Here in this section we shall show the descent for one step. First, we need a pre liminary proposition.

PROPOSITION 3.1. *Let $(J, <)$ be a linearly ordered structure, and let $(J^\#, <)$ be its $\#$-extension.* (See Definition 5.1 of [7] for the $\#$-extension.) *For every element of $J^\#$, say α, α is $<$-accessible if and only if each component of α is $<$-accessible (in $J^\#$).*

PROOF. The "only if" part is obvious. Suppose α is not accessible. There is an infinite decreasing sequence led by α. Let $\beta_1, \cdots, \beta_m$ be all the components of α arranged in the non-increasing order, and suppose each β_p is accessible, $1\leqq p\leqq m$. Each component of every entry of the supposed sequence is $\leqq\beta_1$. Since β_1 is accessible, every entry of the sequence must have β_1 as its maximal component. The sequence obtained from the given one by knocking off one occurrence of β_1 from each entry is decreasing and is led by $\beta_2\#\cdots\#\beta_m$. Applying the same reasoning repeatedly, we eventually obtain a decreasing sequence led by β_m. But this is impossible since β_m is assumed to be accessible. Thus follows the "if" part.

DEFINITION 3.1. Let κ be any element of $\boldsymbol{O}(i)=\boldsymbol{O}(i; S*J_i)$, and let ζ be any $(i+1)$-active subdiagram of κ. We define $s(\kappa; \zeta)$ and then put $s(\kappa)=s(\kappa; \kappa)$.

$s(\kappa; \zeta)=0$ if ζ is an element of $S*J_i$ or is of the form (i, a, η).

$s(\kappa; \zeta_1\#\cdots\#\zeta_m)=s(\kappa; \zeta_1)+\cdots+s(\kappa; \zeta_m)$ if $\zeta_1, \cdots, \zeta_m$ are connected.

$s(\kappa; (k, a, \eta))=s(\kappa; \eta)+1$ if $k>i$.

We now come to one of our main propositions.

PROPOSITION 3.2. *Let* κ *be a connected element of* $\boldsymbol{O}(i)$. *If* $\kappa\{i\}$ *is* $(i+1)$*-fit (and hence* $\kappa\{i\}[i+1]$ *is well-defined by virtue of Proposition* 1.7) *and* $\kappa\{i\}[i+1]$ *is in* D_{i+1}, *or is* $(i+1)$*-accessible in* $\boldsymbol{O}(i+1)$, *then* κ *is in* D_i.

As a corollary of this, we obtain Proposition 2 in the appendix of [8].

PROPOSITION 3.3. *If* α *a diagram of* $\boldsymbol{O}(S)$ *is* $(i+1)$*-grounded, then* α *is* i*-grounded.*

PROOF. By virtue of Proposition 3.1, it suffices to consider a connected α. Notice that, under the assumption, α is $(i+1)$-fit, and hence is i-fit. So $\kappa=\alpha[i]$ is a connected element of $\boldsymbol{O}(i)$. Since α is $(i+1)$-grounded, $\alpha[i+1]=\kappa\{i\}[i+1]$ is in D_{i+1} by Proposition 1.5. Proposition 3.2 then implies that $\kappa=\alpha[i]$ is in D_i, or α is i-grounded by Proposition 1.5.

PROOF OF PROPOSITION 3.2. First notice that any element of $S*J_i$ is in D_i. Let (k, a) be the least value such that $(k, a)\geqq(i, o)$ and, for some η in D_i, $\kappa=(k, a, \eta)$ (which is also a diagram of $\boldsymbol{O}(i)$) satisfies that $\kappa\{i\}[i+1]$ is in D_{i+1}, but κ is not in D_i (supposing that such a κ exists). Notice that the i-accessibility of η and 10) of Proposition 1.3 imply that

$C(i, \eta)$: every subdiagram of η (inclusive) is in D_i.

REMARK. Let δ be an i-section of η. $C(i, \eta)$ assures us that δ is in D_i. Thus, by virtue of 2) of Proposition 1.6 and Definition 1.5, $\kappa\{i\}$ is $(i+1)$-fit, and so by Proposition 1.7 $\kappa\{i\}[i+1]$ is well-defined as an element of $\boldsymbol{O}(i+1)$. The assumption that $\kappa\{i\}[i+1]$ be in D_{i+1} is therefore meaningful.

We shall show that actually every such κ must be in D_i. (When $k=i$, $\kappa\{i\}[i+1]\in D_{i+1}$ directly implies $\kappa\in D_i$.) Then by induction on $s(\kappa)$, we can conclude the proposition.

Suppose $\kappa=(k, a, \eta)$ as above is not in D_i. We shall define another diagram λ of $\boldsymbol{O}(i)$ which is

1) of the form (k, a, ξ),

and which satisfies

2) $\lambda<_i\kappa$, λ is not in D_i, ξ is in D_i, and hence $C(i, \xi)$ holds, and $\lambda\{i\}$ is $(i+1)$-fit.

(We need not claim that $\lambda\{i\}$ is $(i+1)$-grounded.) Thus we obtain $\lambda\{i\}$ so that

3) $\lambda\{i\}$ and $\kappa\{i\}$ are $(i+1)$-fit, $\lambda<_i\kappa$ and λ is not in D_i.

3) above and 1) of Proposition 1.6 imply that $\lambda\{i\}<_{i+1}\kappa\{i\}$. This, 3) of Proposition 1.4 and 9) of Proposition 1.3 imply that $\lambda\{i\}[i+1]<_{i+1}\kappa\{i\}[i+1]$ in $\boldsymbol{O}(i+1)$.

It will become clear that the construction of λ does not depend on the assumption that $\kappa\{i\}[i+1]$ belong to D_i+1. Since λ satisfies the same condition as κ save for that $\kappa\{i\}[i+1]$ be in D_{i+1}, we can repeat the same argument to obtain a sequence $\{\lambda_n\}_n$ from $O(i)$ such that $\lambda_0=\kappa$, $\lambda_1=\lambda$, $(\lambda_n, \lambda_{n+1})$ satisfies the same conditions as (κ, λ) (see 1)$\sim$3) above), and hence

$$\lambda_{n+1}\{i\}[i+1]<_{i+1}\lambda_n\{i\}[i+1]$$

in $O(i+1)$. But then $\kappa\{i\}[i+1]$ cannot be in D_{i+1}, contradicting the assumption. κ must therefore be in D_i.

NOTE. When $k=i$, the situation is slightly different from the general cases; the details will be seen below.

The construction of λ is based on the theory of fundamental sequences in the system $O(i)$, for which we refer the reader to Definitions 1.1, 1.3, 1.4, 4.2 and 4.3 of [3] and IX in Section 9 of [7].

I. $\kappa=(i, a, \eta)$. Recall that η is in D_i. Let $\{\kappa_m\}_m$ be the fundamental sequence for (i, κ) in $O(i)$. Since κ is assumed not to be i-accessible, there is an m such that κ_m is not i-accessible. This is so, because the fundamental sequence of (i, κ) converges from below to κ with respect to $<_i$. (See Theorem 2 in Section 3 of [3].) We can find the desired λ, and hence ξ, as a subdiagram of κ_m in the same manner as in [7]. Notice that i is the least indicator in the theory $O(i)$. We shall go over all the relevant cases, consulting Section 9 of [7].

$a=o$. It is an easy matter to see that the following are the only cases that apply. Notice that $h=i$ (which is the least indicator).

(2) $\kappa=(i, o, \zeta+1)$.

c.2. 3) $\kappa_m=\mu\#\cdots\#\mu$, where $\mu=(i, o, \zeta)$. In order that κ_m be not i-accessible, μ should not be either. (See Proposition 3.1 above.) Put $\lambda=\mu$.

[2°] of (3.3) $\kappa=\text{apr}(0, i, \kappa)$ and not all the components of η are marked. Suppose $\eta=\gamma\#\delta$, where (i, δ) is the next scanned pair. $\kappa_m=(i, o, \gamma\#\delta_m)$. $\delta_m<_i\delta$ by Proposition 2.1, and hence $\gamma\#\delta_m<_i\eta$. Thus, $\xi=\gamma\#\delta_m$ is in D_i. Put $\lambda=\kappa_m$.

$a>o$. We shall work a few cases as examples.

(1) $\kappa=(i, a, s_0)$, where s_0 denotes the least element of S.

a. 2) $a=b+1$ and $\kappa_m=(i, b, \cdots, (i, b, s_0)\cdots)$. By induction on m, we can show that κ_m is i-accessible due to the choice of a. So this case is non-applicable.

c.2. 1) of (2) $\kappa=(i, a, \zeta+1)$, $\{a_m\}_m$ tends to a from below and $\kappa_m=(i, a_m, \mu)$, where $\mu=(i, a, \zeta)$. If μ is in D_i, then so is κ_m due to the choice of (i, a). So μ is not in D_i. Take μ as λ.

[1°] of (3.3) $\kappa=(i, a, \gamma\#\delta)$ and $\kappa_m=\nu_m\#\mu_m$, where $\mu_m=(i, a, \gamma\#\delta_m)$ and all the values in ν_m which are connected to $\gamma\#\delta$ are below (i, a). So μ_m cannot be in D_i. Take μ_m as λ. $\delta_m<_i\delta$ by Proposition 2.1, and hence δ_m is in D_i.

[2°] of (3.3) Take κ_m as λ.

Other cases are either non-applicable or dealt with similarly to earlier cases.

Now, $\lambda=(i, a, \xi)<_i\kappa=(i, a, \eta)$ implies $\xi<_i\eta$, where ξ and η are in D_i. Repeating the same argument, we obtain a sequence $\{\xi_n\}_n$ from D_i, where $\xi_0=\eta$,

$\xi_1=\xi$ and $\xi_{n+1}<_i\xi_n$ for every n. But this is impossible, since D_i is i-accessible. Thus follows that κ is in fact i-accessible.

II. $\kappa=(k, a, \eta)$ where $k>i$. Let $\{\kappa_m\}_m$ be the fundamental sequence for (i, κ) in $\boldsymbol{O}(i)$. We shall define $\lambda=(k, a, \xi)$ as an $(i+1)$-active part of κ_m, where κ_m is assumed not to be in D_i. For the condition 2) required of λ, it suffices to show that λ is not in D_i, while ξ is in D_i, since $\lambda<_i\kappa$ automatically holds, and every i-section of λ, which is a subdiagram of ξ, belongs to D_i.

a) of (1) Consider, as an example, a. 2). $\kappa_m=(k, b, \cdots, (k, b, s_0)\cdots)$, where $a=b+1$. Due to the choice of (k, a), we can show by induction on m that κ_m is i-accessible for every m, and so this case is non-applicable.

a. 3) is non-applicable, since $t>k$ does not apply in our case.

(2) $\kappa=(k, a, \zeta+1)$. Let μ be (k, a, ζ).

(2.1; 1) Case 1: μ is in D_i. Take $\kappa_m=(k, o, \zeta_m\#(l, o, \mu))$ as λ, where $k=l+1$. $\zeta_m<_i\zeta$ by Proposition 2.1, and hence ζ_m is in D_i. (l, o, μ) is in D_i, since μ is in D_i and $(l, o)<(k, a)$. Thus follows that $\zeta_m\#(l, o, \mu)$ is in D_i.

Case 2: μ is not in D_i. Take μ as λ.

(2.1; 2) $\kappa_m=\mu\#\cdots\#\mu\#(k, o, \zeta_m\#(l, o, \mu))$, where $k=l+1$.

Case 1: μ is in D_i. Take $(k, o, \zeta_m\#(l, o, \mu))$ as λ.

Case 2: μ is not in D_i. Take μ as λ.

(2.2) Due to the choice of (k, a), in order that κ_m be not in D_i, μ cannot be either. Take μ as λ.

[1°] of (3.3) See the same case for $k=i$.

[2°] of (3.3) Take μ_m as λ.

(4.1) Take $(k, o, \eta_m\#(l, o, \eta))$ as λ, where $k=l+1$.

(4.2) is non-applicable, since all the values in κ_m which are connected to η are below (k, a).

The remaining cases can be dealt with in a manner similar to the earlier ones.

§ 4. Descent of groundedness: the general cases.

We now conclude the descending property of groundedness as was announced in the preceding section.

Let i denote an element of $I\cup\{\infty\}$ throughout.

DEFINITION 4.1. Suppose $p<i$ and κ is an element of $\boldsymbol{O}(p+1)$. κ is said to be (p, i)-free if, for every j such that $p<j<i$, j does not occur in κ. (For technical reasons, we include the case where κ is an element of $\boldsymbol{O}(S)$, in which case we assume $p=-1$ and $p+1=o$, the least indicator in I.)

PROPOSITION 4.1. *Suppose* $p<i$.

1) *If* κ *is an element of* $\boldsymbol{O}(p+1)$ *and is* (p, i)*-free, then* κ *is an element of* $\boldsymbol{O}(j)$ *and the order* $<_i$ *restricted to such elements is identical in* $\boldsymbol{O}(j)$ *and in*

$O(p+1)$ *for every* j *such that* $p<j\leqq i$.

2) *If* κ *is an element of* $O(p+1)$ *and is* (p, i)*-free, then* $\kappa\{j\}=\kappa\{l\}$ *and* $\kappa=\kappa\{j\}[l]$ *for every pair of* j *and* l *such that* $p<j,\ l\leqq i$.

DEFINITION 4.2. 1) Suppose $p<i$ and κ is an element of $O(p+1)$. When κ is (p, i)-free (and hence, by Proposition 4.1 above, κ is an element of $O(j)$ if $p<j\leqq i$), κ is said to be (p, i)-secure if κ is j-accessible in $O(j)$ (that is, κ is in D_j) for every j such that $p<j<i$.

2) Let α be an element of $O(S)$.

$$\iota(i, \alpha)=\max\{j<i\,;\ j \text{ is an index of } \alpha\}.$$

(If there are no indices of α below i, then let $l=\iota(i, \alpha)$ be -1, and hence let $l+1$ be o.)

PROPOSITION 4.2. *Let* α *be i-fit and let* p *be* $\iota(i, \alpha)$. *Then* $\alpha[i]=\alpha[j]$ *for every* j *such that* $p<j<i$, *and* $\alpha[i]$ *is in* $O(p+1)$, *is* (p, i)*-free, and hence is an element of* $O(j)$ *for every* j *such that* $p<j\leqq i$ (Proposition 4.1). $\alpha[i]\{j\}=\alpha$ *for every* j *satisfying* $p<j\leqq i$.

The proposition below is our second main result.

PROPOSITION 4.3. *Assume the five conditions below for* i, p *and* κ.

(i) i *is limitary.*

(ii) *For every* $q<i$ *and for every* ρ *in* D_q, $\rho\{q\}[j]$ *belongs to* D_j *for every* $j<q$.

(iii) $p<i$.

(iv) κ *is in* $O(p+1)$ *and is* (p, i)*-free (and hence, by virtue of Proposition* 4.1, *is in* $O(j)$ *if* $p<j\leqq i$).

(v) κ *is in* D_i.

Then we are led to the conclusion:

$$\kappa \text{ is } (p, i)\text{-secure.}$$

As a corollary of Propositions 3.3 and 4.3, we obtain Proposition 3 in the appendix of [8].

PROPOSITION 4.4. *For every* i *and every* α *a diagram of* $O(S)$, *if* α *is i-grounded, then* α *is j-grounded for every* $j<i$.

PROOF. Suppose the contrary and let i be the least element of $I\cup\{\infty\}$ such that there is an α which is i-grounded but not j-grounded for some $j<i$. Take this i and such an α. We shall deduce a contradiction via several steps below.

1°. i is limitary. For, otherwise, the i-groundedness of α implies the k-groundedness of α, where $i=k+1$ (Proposition 3.3). But then, by the choice of i, α is j-grounded for every $j<k$. In other words, α is j-grounded for every $j<i$, contradicting the hypothesis on i and α.

2°. Let p be $\iota(i, \alpha)$. Suppose we have shown that α is l-grounded for every l such that $p<l\leqq i$. Then, since $p+1<i$ (by 1° above), α is j-grounded for every $j\leqq p$ due to the choice of i. So in fact α is j-grounded for every $j\leqq i$. That is, there cannot be any i and α as was supposed to exist at the start.

3°. What must be proved is, therefore, the following.

(*) For every limitary i and every α, if α is i-grounded, then, for every l such that $p=\iota(i, \alpha)<l<i$, α is l-grounded.

If we let κ be $\alpha[i]$ and p be $\iota(i, \alpha)$, then obviously (i) and (iii)~(v) in Proposition 4.3 are satisfied. (See 1° above, and Propositions 4.2 and 1.5.) For (ii), suppose $q<i$ and ρ is in D_q. Then $\rho\{q\}$ is q-grounded, and hence, by the choice of i, $\rho\{q\}$ is j-grounded for every $j<q$, or $\rho\{q\}[j]$ is in D_j. So, Proposition 4.3 applies to $\alpha[i]$, and $\alpha[i]$ is (p, i)-secure, or $\alpha[i]$ is in D_l if $p<l<i$, or $\alpha[i]\{l\}$ is l-grounded. Since α is i-grounded, α is i-fit, and hence by Proposition 4.2 $\alpha[i]\{l\}=\alpha$. That is, α is l-grounded for every l such that $p<l<i$. This proves (*).

PROOF OF PROPOSITION 4.3. Suppose there are i, p and κ which satisfy (i)~(v) but not the conclusion; that is, κ is not (p, i)-secure. In view of Proposition 3.1, we may assume that κ is connected. It is obvious that κ cannot be an element of $S*J_{p+1}$. Let (k, a) be the least value such that $(k, a)\geqq(i, o)$ and $\kappa=(k, a, \eta)$ is such a diagram for some (p, i)-secure η. (Notice that if κ satisfies (i)~(v), then so does every subdiagram of η.) We shall find an indicator r and a diagram $\lambda=(k, a, \xi)$ of $\boldsymbol{O}(r+1)$ satisfying

1) $p\leqq r<i$

and

2) λ is (r, i)-free, ξ is (r, i)-secure, λ is not (r, i)-secure and $\lambda<_{r+1}\kappa$.

The last condition implies that $\lambda<_i\kappa$, and hence λ is in D_i. (λ and κ can be compared with respect to i in $\boldsymbol{O}(j)$ for every j such that $r<j\leqq i$; see Proposition 4.1.) So (r, λ) satisfies the same conditions as (p, κ).

Repeating the same argument, we obtain an i-decreasing sequence of diagrams $\{\lambda_n\}_n$ from $\boldsymbol{O}(i)$, where $\lambda_0=\kappa$ and $\lambda_1=\lambda$. But this contradicts (v). So such a κ must in fact be (p, i)-secure. From this fact follows the proposition for every κ; use induction on the construction of κ.

The construction of λ is based on the theory of fundamental sequences, and is quite similar to the construction in the proof of Proposition 3.2. Suppose $\kappa=(k, a, \eta)$ as above is not j-accessible in $\boldsymbol{O}(j)$ for a j satisfying $p<j<i$. We consider the fundamental sequence for (j, κ) in $\boldsymbol{O}(j)$, say $\{\kappa_m\}_m$. There is an m such that κ_m is not in D_j. We shall find the desired r and λ from κ_m by going over all the cases as before. As for the non-applicable cases, we shall take up a few of them as examples.

(1) $\kappa=(k, a, s_0)$.

a. 1) $\kappa_m=(k, a_m, s_0)$ where $a_m<a$. κ_m satisfies (i)$\sim$(v) with the same p. Due to the choice of (k, a), κ_m must be (p, i)-secure. So this case is non-applicable.

a. 2) $\kappa_m=(k, b, \cdots, (k, b, s_0)\cdots)$ where $a=b+1$. We can show, by induction on m, that κ_m is (p, i)-secure. So this case is non-applicable.

b. 1) $\kappa_m=(k_m, o, s_0)$ where k_m tends to k from below. If $k>i$, then we may assume $k_m>i$, and hence κ_m satisfies the same condition as κ. But $(k_m, o)<(k, a)$, and so this case is non-applicable. If $k=i$, then $j\leqq k_m<i$, κ_m is in $\boldsymbol{O}(k_m)$ and in fact κ_m is k_m-accessible there. (Notice that k_m is the least indicator in this system.) So, by (ii) applied to $q=k_m$ and $\rho=\kappa_m$, $\kappa_m\{k_m\}[l]$ is in D_l for every $l\leqq k_m$; in particular $\kappa_m=\kappa_m\{k_m\}[j]$ is in D_j for every m. So this case too is non-applicable.

(2) $\kappa=(k, a, \zeta+1)$. Let μ denote (k, a, ζ).

(2.1) $a=o$ and $k=l+1$. Notice that $l\geqq i$.

(2.1; 1) $\kappa_m=(k, o, \zeta_m\#(l, o, \mu))$. Case 1: μ is (p, i)-secure. Then (l, o, μ) is also (p, i)-secure by the choice of (k, a). Let r be

$$\max\{j\leqq t<i;\ t \text{ is an index of } \zeta_m\}.$$

Then $p\leqq j\leqq r<i$ and κ_m is (r, i)-free. (If there is no such t, let r be j.)

We here insert a lemma, which is valid also for a few other cases.

LEMMA. *Any indicator t which occurs in a proper subdiagram of κ_m and which is below i arises from one of the following cases.*

b. 1) *$t=i_m$ and $\nu_m=(i_m, o, o)$, where (i, ν) is the last reduction pair and $\nu=(i, o, o)$.*

d. 4. 1) *$t=i_m$ and $\nu_m=\alpha\#\cdots\#\alpha\#(i_m, o, \alpha)$, where (i, ν) is the last reduction pair and $\nu=(i, o, \alpha)$.*

e. 4. 1) *Similarly to* d. 4. 1).

In any case, a t-section of κ_m is a subdiagram of η, and hence is (p, i)-secure.

The lemma can be easily checked by going over the definitions of the reduction sequences in [**3**], if we note that the indicators occurring in κ are $\geqq i$, i is limitary and that $\{\kappa_m\}_m$ is considered in $\boldsymbol{O}(j)$.

We can claim that $\zeta_m\{j\}$ is $(r+1)$-fit by virtue of the lemma above, Proposition 4.1 applied to the t-sections of ζ_m, $j\leqq t\leqq r$, and Proposition 1.5. So

$$\lambda=\kappa_m\{j\}[r+1]=(k, o, \zeta_m\{j\}[r+1]\#(l, o, \mu))$$

is well-defined and is (r, i)-free. (l, o, μ) is (r, i)-secure as was claimed previously. If we can show that

$$\zeta_m\{j\}[r+1]<_q\zeta \quad \text{in } \boldsymbol{O}(r+1) \text{ for every } q \text{ such that } r<q<i,$$

then, since ζ is (r, i)-secure, so is $\zeta_m\{j\}[r+1]$. (See Proposition 4.1.)

$$\zeta_m <_t \zeta \quad \text{in } \boldsymbol{O}(j) \text{ for every } t \text{ satisfying } j \leqq t < i < k$$

(Proposition 2.1). So, by Proposition 1.4, $\zeta_m\{j\} <_t \zeta\{j\}$ for every such t. This, together with Propositions 1.4 and 4.1, in turn implies $\zeta_m\{j\}[r+1] <_{r+1} \zeta\{j\}[r+1] = \zeta$. Since ζ has no indicators below i, this implies that $\zeta_m\{j\}[r+1] <_q \zeta$ for every q as above.

It thus follows that

$$\xi = \zeta_m\{j\}[r+1] \# (l, o, \mu)$$

is (r, i)-secure.

Next we shall show that $\lambda <_{r+1} \kappa$. From $\kappa_m <_j \kappa$ follows $\kappa_m\{j\} <_j \kappa\{j\}$ (Proposition 1.4), and hence $\kappa_m\{j\} <_{r+1} \kappa\{j\}$ due to the condition (iv). So, by Propositions 1.4 and 4.1, we obtain $\lambda = \kappa_m\{j\}[r+1] <_{r+1} \kappa\{j\}[r+1] = \kappa$. Finally, let us show that λ is not $(r+1)$-accessible in $\boldsymbol{O}(r+1)$; that is, λ is not (r, i)-secure. Suppose λ is in D_{r+1}. Then by (ii) applied to $q = r+1$ and $\rho = \lambda$, $\lambda\{r+1\}[j] = \kappa_m$ must be in D_j, contradicting the choice of κ_m. So λ cannot be $(r+1)$-accessible in $\boldsymbol{O}(r+1)$. We have thus completed the proof of 1) and 2) for r and λ.

(2.1; 1) Case 2: μ is not (p, i)-secure. Take μ as λ and p as r.

(2.1; 2) $\kappa_m = \mu \# \cdots \# \mu \# \omega_m$, where $\omega_m = (k, o, \zeta_m \# (l, o, \mu))$.

Case 1: μ is (p, i)-secure. Put $\lambda = \omega_m\{j\}[r+1]$. (See Proposition 3.1 and Case 1 of (2.1; 1) for the details.)

Case 2: μ is not (p, i)-secure. Take μ as λ.

(2.2) c. 2. 1) $\kappa_m = (k, a_m, \mu)$. If μ is (p, i)-secure, then so is κ_m since κ_m satisfies (i)$\sim$(v) and $(k, a_m) < (k, a)$. So μ cannot be (p, i)-secure. Take μ as λ.

Similarly for c. 2. 2).

c. 2. 3) $a = o$, $t = k$ and $\kappa_m = \mu \# \cdots \# \mu \# \rho_m$. If μ is (p, i)-secure, then so is ρ_m. So μ cannot be (p, i)-secure. Take μ as λ.

Similarly for c. 2. 4) and c. 2. 5).

(3.1) and (3.2) are non-applicable.

[1°] of (3.3) $\kappa = (k, a, \zeta \# \nu)$ and $\kappa_m = \omega_m \# \mu_m$, where $\mu_m = (k, a, \zeta \# \nu_m)$ and the values in ω_m which are connected to $\zeta \# \nu$ are below (k, a). So ω_m is (p, i)-secure, and hence μ_m cannot be (p, i)-secure. Let r be

$$\max\{j \leqq t < i;\ t \text{ is an index of } \nu_m\}.$$

In a manner similar to Case 1 of (2.1; 1), we can claim that this r and

$$\lambda = \mu_m\{j\}[r+1] = (k, a, \zeta \# \nu_m\{j\}[r+1])$$

satisfy 1) and 2).

[2°] of (3.3) $\kappa = (k, a, \eta)$ and $\kappa_m = (k, a, \eta_m)$. Let r be

$$\max\{j \leqq t < i;\ t \text{ is an index of } \eta_m\}.$$

This r and $\lambda=\kappa_m\{j\}[r+1]$ will do.

(4.1) $k=l+1$ and $\kappa=(k, o, \eta)$. Put $\omega_m=(k, o, \eta_m \# (l, o, \eta))$.

(4.1; 1) $\kappa_m=\omega_m$.

(4.1; 2) $\kappa_m=\eta \# \cdots \# \eta \# \omega_m$.

In both cases, let r be

$$\max\{j \leqq t < i\,;\ t \text{ is an index of } \eta_m\}$$

and put $\lambda=\omega_m\{j\}[r+1]$.

(4.2) is non-applicable.

The remaining cases can be dealt with similarly to the earlier cases.

This completes the proof of Proposition 4.3.

§ 5. Accessibility.

PROPOSITION 5.1. *Let* $(J, <)$ *and* $(J^{\#}, <)$ *be as in Proposition* 3.1. *If* $(J, <)$ *is accessible, then so is* $(J^{\#}, <)$.

PROOF. Suppose there is an infinite, $<$-decreasing sequence from $J^{\#}$, say $\{\alpha_n\}_n$, where $\beta(n, 1) \# \cdots \# \beta(n, m_n)$ is the canonical representation of α_n with the components $\beta(n, 1), \cdots, \beta(n, m_n)$. We can then induce an infinite, $<$-decreasing sequence from $\mathcal{J}$, say $\{\gamma_l\}_l$, as follows.

Put $\gamma_0=\beta(0, 1)$, and assume that $\gamma_0, \cdots, \gamma_l$ have been defined so that $\gamma_l=\beta(p, q)$ for some p and q, $1 \leqq q \leqq m_p$. Let n be the least number such that $p<n \leqq p+n+1$ and, for some k,

$$(1) \qquad 1 \leqq k \leqq m_n \quad \text{and} \quad \beta(n, k)<\gamma_l\,.$$

(We can easily show the existence of such n and k.) Let k be the least number satisfying (1), and put $\gamma_{l+1}=\beta(n, k)$.

PROPOSITION 5.2 (Proposition 1 in the appendix of [8]). *F_∞ is ∞-accessible.*

This is an immediate consequence of Proposition 5.1 and the definition of F_∞.

PROPOSITION 5.3 (Proposition 4 in the appendix of [8]). *For every i, $F_i=G_i$.*

PROOF. F_∞ is cofinal with F_i with respect to i. (See Lemma 26.37 of [2] for the proof.) Let α be i-fit. Then there is an ∞-fit β such that $\alpha<_i\beta$. By Proposition 5.2 above, β is ∞-grounded, and hence is i-grounded by Proposition 4.4. So, α is also.

We now conclude our accessibility proof.

THEOREM. *Every ordinal diagram (of $\boldsymbol{O}(I, A)$) is i-accessible for every i an indicator or ∞.*

PROOF. We shall show that $\boldsymbol{O}(I, A)=F_i$ ($=G_i$ by Proposition 5.3). Suppose otherwise, and let i be the least element of $I \cup \{\infty\}$ such that there is an α which is not i-fit. For every $j<i$, every j-section of α is j-fit due to the choice of i,

and so it is j-grounded. This, however, forces that α be i-fit. We are therefore led to the conclusion that $\boldsymbol{O}(I, A)=F_i=G_i$ for every i.

References

[1] G. Takeuti and M. Yasugi, The ordinals of the systems of second order arithmetic with the provably Δ_2^1-comprehension axiom and with the Δ_2^1-comprehension axiom respectively, Japan. J. Math., **41** (1973), 1-67.

[2] G. Takeuti, Proof theory, North-Holland, Amsterdam, 1975.

[3] G. Takeuti and M. Yasugi, Fundamental sequences of ordinal diagrams, Comment. Math. Univ. St. Pauli, **25** (1976), 1-80.

[4] M. Yasugi, Some properties of ordinal diagrams, Nagoya Math. J., **70** (1978), 143-155.

[5] M. Yasugi, Self-iterating schemes of ordinal diagrams, Comment. Math. Univ. St. Pauli, **27** (1978), 11-24.

[6] M. Yasugi, A progression of consistency proofs, Comment. Math. Univ. St. Pauli, **28** (1980), 31-49.

[7] G. Takeuti and M. Yasugi, An accessibility proof of ordinal diagrams, J. Math. Soc. Japan, **33** (1981), 1-21.

[8] G. Takeuti, Proof theory and set theory, to appear.

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