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GROUP ALGEBRAS WITH UNITS SATISFYING A GROUP IDENTITY II

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Abstract. We classify group algebras of torsion groups over a field of characteristic $p > 0$ with units satisfying a group identity.

1. Introduction

A group U is said to satisfy a group identity if there exists a nontrivial word $w = w(x_1, \dots, x_n)$ in the free group generated by $x_1, \dots, x_n$ such that $w(u_1, \dots, u_n) = 1$ for all $u_i \in U$. In early 1980s, Brian Hartley made the conjecture that if the units of the group algebra of a torsion group G over a field K satisfy a group identity, then the group algebra $K[G]$ satisfies a polynomial identity. This was settled recently for group algebras over infinite fields in [GSV97], and completely solved in [Liu]. Some natural questions we can ask are: "If the group algebra satisfies a polynomial identity, does the unit group satisfy a group identity? If not, what additional conditions are required to make it true?" After [GSV97] appeared, these questions were answered in [Pas97] for group algebras over infinite fields. Indeed, the paper showed that, for the group algebra $K[G]$ of a torsion group G over an infinite field K of characteristic $p > 0$, the unit group satisfies a group identity if and only if $K[G]$ satisfies a polynomial identity and G' is a p-group of bounded period. The proof given in [Pas97] uses two facts: [GJV94, Proposition 1] and [GSV97, Lemma 2.3]. [GJV94, Proposition 1] basically says that if units of an algebra over an infinite field satisfy a group identity, then the product of any two square zero elements is nilpotent of bounded degree. This proposition was modified and extended to algebras over an arbitrary field in [Liu, Lemmas 3.1, 3.2], and thus it is natural to expect that the results in [Pas97] can be extended to group algebras over finite fields. On the other hand, [GSV97, Lemma 2.3] asserts that for any nonabelian finite group G and any infinite field K of characteristic $p > 0$, if the units of the group algebra $K[G]$ satisfy a group identity, then G' must be a finite p-group. This is no longer true when K is finite. Actually, if G' is a p-group, then we do obtain the same result as in [Pas97].

Theorem 1.1. *Let $K[G]$ be the group algebra of a torsion group G over a field K of characteristic $p > 0$ and let $U(K[G])$ be the group of units of $K[G]$. If G' is a p-group, then the following are equivalent.*

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1. *$U(K[G])$ satisfies a group identity.*
2. *G has a normal p-abelian subgroup of finite index, and G' has bounded period.*
3. *$U(K[G])$ satisfies $(x,y)^{p^k} = 1$ for some integer $k \geq 0$.*

Surprisingly, if G' is not a p-group, then not only can the period of G' be bounded, but also the period of the whole group G can be bounded.

Theorem 1.2. *Let $K[G]$ be the group algebra of a torsion group G over a field K of characteristic $p > 0$ and let $U(K[G])$ be the group of units of $K[G]$. If G' is not a p-group, then the following are equivalent.*

1. *$U(K[G])$ satisfies a group identity.*
2. *G has a normal p-abelian subgroup of finite index, G has bounded period and K is finite.*
3. *$U(K[G])$ satisfies $x^n = 1$ for some integer n.*

2. Proofs of the theorems

The implications $3 \Rightarrow 1$ are trivial. The implication $2 \Rightarrow 3$ in Theorem 1.1 has been proved by [Pas97, Section 3] whether the field K is infinite or finite. The implication $2 \Rightarrow 3$ in Theorem 1.2 can be obtained from the proof of [Coe82, Theorem A]. So we need to prove $1 \Rightarrow 2$ in both theorems.

We assume that G is a torsion group and that K is a field of characteristic $p > 0$. Also, we assume that the group of units $U(K[G])$ of the group algebra $K[G]$ satisfies the group identity $w = 1$. In view of [Liu, Theorem 1.1] and [Pas85, Corollary 5.3.10], G has a normal p-abelian subgroup A of finite index. In particular, G is locally finite.

Let us record some lemmas we need. The following is from [Liu, Lemma 2.3].

Lemma 2.1. *Let $R = K[H]$ be the group algebra of a locally finite group H and assume that the group of units $U(R)$ satisfies $w = 1$. If S is any subalgebra of R or $\bar{R}$ is any homomorphic image of R, then $U(S)$ and $U(\bar{R})$ also satisfy $w = 1$.*

The following lemma is from [Liu, Lemma 3.2]. Note that this result is an analogue of [GJV94, Proposition 1] for algebras over arbitrary fields and plays a crucial role in our proofs.

Lemma 2.2. *Let R be an algebra over a field K and suppose $U(R)$ satisfies $w = 1$. Let $a, b \in R$ such that $a^2 = b^2 = 0$. If ab is nilpotent, then $(ab)^d = 0$ for some integer d determined by w.*

For the rest of the paper, we fix notation so that d will be as in the above lemma. If $M_n(F)$ is the n by n matrix algebra over a field F and $U(M_n(F))$ satisfies $w = 1$, then we have the following bounds on the size of the field and the degree n as shown in [Liu, Lemma 3.3].

Lemma 2.3. *Let F be any field. If $U(M_n(F))$ satisfies $w = 1$ and $n \geq 2$, then*

1. *$|F| \leq d$ and hence F is a finite field.*
2. *$n < 2\log_{|F|} d + 2 \leq 2\log_2 d + 2$.*

Let m be the smallest integer not less than $2\log_2 d + 2$ and define

$$N = \prod_{|F| \leq d} |U(M_m(F))|.$$

Certainly, N is finite and determined by d.

Lemma 2.4. *Let x be a nonidentity p'-element in G', and let y be a nonidentity p'-element in a normal p'-subgroup of G. If $U(K[G])$ satisfies $w = 1$, then $y^N = 1$.*

Proof. Suppose by way of contradiction that $y^N \neq 1$. Since $x \in G'$, we can write

$$x = (x_1, y_1)(x_2, y_2) \cdots (x_n, y_n) \neq 1.$$

Note that $x^{-1}y^N$ is a p'-element since y is in a normal p'-subgroup of G. If $x \neq y^N$, let $\alpha = (1 - x^{-1})(1 - y^N)$, then α is not a nilpotent by [Pas85, Lemma 2.3.3]. If $x = y^N$, let $\alpha = 1 - x$, so that α is also not nilpotent in this case. Observe that $H = \langle x_1, y_1, \ldots, x_n, y_n, y\rangle$ is a finite subgroup of G since G is locally finite. If K is infinite, then G' is a p-group by [Pas97, Theorem 1.1]. Thus K is finite here. Let $J = J(K[H])$, the Jacobson radical of $K[H]$, and now write $K[H]/J = \bigoplus \sum_i M_{n_i}(F_i)$ where the F_i are fields since K is finite. Now α is not nilpotent, so $\alpha + J$ is not zero in $K[H]/J$. Hence there exists a natural map

$$\theta : K[H]/J \to M_{n_j}(F_j)$$

for some j with $\theta(\alpha + J) \neq 0$. In particular, $\theta(1 - x^{-1} + J) \neq 0$ and $\theta(1 - x + J) \neq 0$. If $n_j = 1$, then

$$\theta(x + J) = \prod_{i=1}^{n} \theta((x_i, y_i) + J) = \prod_{i=1}^{n} (\theta(x_i + J), \theta(y_i + J)) = 1$$

since F_j is commutative. But $\theta(1 - x^{-1} + J) \neq 0$, and hence $n_j \geq 2$. Also $U(M_{n_j}(F_j))$ satisfies $w = 1$ by Lemma 2.1. Hence $n_j \leq m$ and $|F_j| \leq d$ by Lemma 2.3. So we get $\theta(y^N + J) = \theta(y + J)^N = 1$ since $\theta(y + J) \in U(M_{n_j}(F_j)) \hookrightarrow U(M_m(F_j))$. This implies that $\theta(\alpha + J) = 0$, a contradiction. Therefore, $y^N = 1$. □

The following is an analogue of [Pas97, Lemma 2.3].

Lemma 2.5. *Suppose that $G = \langle A, t\rangle$ where A is a normal abelian p-subgroup and t has finite order q. If $U(K[G])$ satisfies $w = 1$, then G' has finite period.*

Proof. The proof given in [Pas97, Lemma 2.3] basically works here. First, [Pas97, Lemma 2.1] holds for group algebras over arbitrary fields by Lemma 2.1. The argument given in the proof of [Pas97, Lemma 2.3] shows that we can assume G is the semidirect product of A by $\langle t\rangle$ and that t has prime order q. So the only concern now is how we use Lemma 2.2, an analogue of [GJV94, Proposition 1].

If $q \neq p$, we take two square zero elements $\alpha = \tau a^{-1}(1 - t^{-1})$ and $\beta = (qa - tr(a))\tau$ as in the proof of [Pas97, Lemma 2.3]. Notice that $qa - tr(a)$ has augmentation 0 hence is in the augmentation ideal $\omega(K[A])$. But now A is a locally finite normal p-subgroup of G of finite index, so we have $\omega(K[A]) = J(K[A])$ and $J(K[A])K[G] \subseteq J(K[G])$ by [Pas85, Lemma 8.1.17] and [Pas85, Theorem 7.2.7]. This implies that β and hence $\alpha\beta$ are in $J(K[G])$. Also, $J(K[G])$ is nil since G is locally finite and we see that $\alpha\beta$ is nilpotent. Therefore, we can apply Lemma 2.2 to conclude that $(\alpha\beta)^d = 0$ for some integer d depending on the group identity.

If $q = p$, both τ and $a^{-1}\tau a$ have square 0 and augmentation 0, so the product $\tau a^{-1}\tau a$ is in $\omega(K[G])$. But now G is a locally finite p-group, so $\omega(K[G])$ is nil and Lemma 2.2 implies that $(\tau a^{-1}\tau a)^d = 0$.

Therefore, the proof of [Pas97, Lemma 2.3] applies here and we deduce that G' has finite period. □

Lemma 2.6. *Suppose that A is a normal abelian p-subgroup of G of finite index. If $U(K[G])$ satisfies $w = 1$, then G' has finite period.*

Proof. Use Lemma 2.5 and the proof of [Pas97, Lemma 2.4]. □

Lemma 2.7. *If $U(K[G])$ satisfies $w = 1$ and G' is not a p-group, then the p'-elements of G have finite period.*

Proof. Since $U(K[G])$ satisfies $w = 1$, [Liu, Theorem 1.1] and [Pas85, Corollary 5.3.10] imply that G has a normal p-abelian subgroup A of finite index. Note that A' is a finite normal p-subgroup of G, $(G/A')'$ is not a p-group, and $U(K[G/A'])$ satisfies $w = 1$ by Lemma 2.1. Thus it suffices to consider G/A', or equivalently, we may assume that A is abelian. Write $A = P \times Q$ where P is the set of p-elements of A and Q is the set of p'-elements of A. Since A is a normal abelian subgroup of G, P and Q are normal subgroups of G. Also, A is a subgroup of G of finite index, so it suffices to bound the period of Q. Now since G' is not a p-group, there exists a nonidentity p'-element x in G'. For any nonidentity y in Q, we have $y^N = 1$ by Lemma 2.4. This shows that Q has finite period and hence the p'-elements of G have finite period. □

Lemma 2.8. *If $U(K[G])$ satisfies $w = 1$, then G' has finite period.*

Proof. As in the proof of Lemma 2.7, we can assume that A is abelian and write $A = P \times Q$. If G' is a p-group, it suffices to consider G/Q since Q is a p'-group. If G' is not a p-group, Lemma 2.7 implies that Q has finite period, hence it still suffices to consider G/Q in this case. We can now assume that A is a p-group. Therefore G' has finite period by Lemma 2.6. □

Lemma 2.9. *If $U(K[G])$ satisfies $w = 1$ and G' is not a p-group, then the p-elements of G have bounded period.*

Proof. As usual, we can assume that A is abelian and write $A = P \times Q$. If $B = (P, G)$, then B is a normal subgroup of G contained in $P \cap G'$. Thus B is a p-group of finite period by Lemma 2.8. Therefore, it suffices to consider G/B, or equivalently we can assume that P is central in G. Now, notice that A has finite index in G, hence it suffices to bound the period of P.

Since G' is not a p-group, we can find a p'-element in G' with

$$x = (x_1, y_1)(x_2, y_2) \cdots (x_n, y_n) \neq 1.$$

Let $H = \langle x_1, y_1, \ldots, x_n, y_n \rangle$; then $x \in H'$ and H is finite since G is locally finite. If $C = H \cap P$, then C is a finite normal p-subgroup of G since P is central. It suffices to consider G/C, or equivalently we can assume $H \cap P = 1$. G' is not a p-group, so K is finite by [Pas97, Theorem 1.1]. Let $J = J(K[H])$ and write $K[H]/J = \bigoplus \sum_i M_{n_i}(F_i)$ where F_i are fields since K is finite. If all $n_i = 1$, then $K[H]/J$ is commutative and $x + J = 1 + J$. Since J is nil, we get that x is a p-element, a contradiction. Therefore, there exists some $n_j \geq 2$. Since finite fields are perfect, by Wedderburn's Principal Theorem [Row91, Theorem 2.5.37], $K[H]$ contains a copy of $K[H]/J$ and hence it contains a copy of $M_2(K)$. Note that $P \times H \cong PH$ since P is central and $H \cap P = 1$. Thus we have

$$\begin{aligned} M_2(K[P]) \cong K[P] \otimes_K M_2(K) \hookrightarrow K[P] \otimes_K K[H] \\ \cong K[P \times H] \cong K[PH] \subseteq K[G]. \end{aligned}$$

Since $U(K[G])$ satisfies $w = 1$, $U(M_2(K[P]))$ also satisfies $w = 1$. If y is any element in P, then $1 - y$ is nilpotent since P is a p-group. Let $a = \begin{pmatrix} 0 & 1-y \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $a, b \in M_2(K[P])$ and $ab = \begin{pmatrix} 1-y & 0 \\ 0 & 0 \end{pmatrix}$ is nilpotent since $1 - y$ is. Lemma 2.2 now implies that $(ab)^d = 0$. Fix an integer k so that $p^k \geq d$. Then $(ab)^{p^k} = 0$, so we get $(1 - y)^{p^k} = 0$ and $y^{p^k} = 1$. Hence P has finite period dividing p^k. This completes the proof. □

Lemma 2.10. $1 \Rightarrow 2$.

Proof. [Pas85, Corollary 5.3.10] and [Liu, Theorem 1.1] imply that G has a normal p-abelian subgroup of finite index.

If G' is a p-group, then Lemma 2.8 implies that G' has finite period.

If G' is not a p-group, [Pas97, Theorem 1.1] implies that K must be finite. Since G is a torsion group, Lemma 2.7 and 2.9 imply that the whole group G has finite period. □

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