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Group cocycles and the ring of affiliated operators

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Abstract In this article we study cocycles of discrete countable groups with values in $\ell^2 G$ and the ring of affiliated operators $\mathcal{U}G$. We clarify properties of the first cohomology of a group G with coefficients in $\ell^2 G$ and answer several questions from De Cornulier et al. (Transform. Groups 13(1):125–147, 2008). Moreover, we obtain strong results about the existence of free subgroups and the subgroup structure, provided the group has a positive first ℓ^2-Betti number. We give numerous applications and examples of groups which satisfy our assumptions.

1 Introduction

Let G be a discrete countable group and let M be a G-module. A *cocycle* $c\colon G \to M$ is a map which satisfies

$$c(gh) = g \cdot c(h) + c(g).$$

It is called *inner*, if there exists $\xi \in M$, such that $c(g) = (g-1)\xi$. We denote by $Z^1(G;M)$ the space of cocycles, by $B^1(G;M)$ the subspace of inner cocycles and by $H^1(G;M)$ the first cohomology of the group G with coefficients in M, i.e. the quotient of $Z^1(G;M)$ by $B^1(G;M)$. Many properties of G can

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be phrased in terms of cocycles and cohomology; and with this article we support the view that certain G-modules of functional analytic nature turn out to be particularly useful in the study of infinite groups. The closer study of the special case $M = \ell^2 G$ is usually called the theory of ℓ^2-invariants of groups.

Here, we denote by $\ell^2 G$ the Hilbert space with basis G, and by $B(\ell^2 G)$ the Banach space of bounded linear endomorphisms of $\ell^2 G$. The left and right translation of G on itself extend to two commuting unitary representations:

$$\lambda, \rho \colon G \to U(\ell^2 G) = \{u \in B(\ell^2 G) \mid uu^* = u^* u = 1\},$$

and endow $\ell^2 G$ with a left and a right G-module structure. It is well-known that the generated von Neumann algebras $LG = \lambda(G)''$ and $RG = \rho(G)''$ are commutants of each other. Here $S' = \{t \in B(\ell^2 G) \mid st = ts,\ \forall s \in S\}$ denotes the commutant of the set S. LG is called the group von Neumann algebra of G. We frequently identify RG with LG^{op} and consider $\ell^2 G$ as a LG-bimodule.

The theory of ℓ^2-invariants was started in the seminal work of M. Atiyah in [2] and developed further by J. Dodziuk, see [16], J. Cheeger and M. Gromov in [13]. Among many others, major contributions were obtained by W. Lück, see [31], and D. Gaboriau, see [19].

In our study, the ring $\mathcal{U}G$ of densely defined, closed operators on $\ell^2 G$, which are affiliated with LG, is of major importance. For details about its definition consult [45, Chap. IX]. We naturally have the following chain of inclusions of G-modules: $LG \subset \ell^2 G \subset \mathcal{U}G$, which induce maps on cocycles and cohomology.

The first ℓ^2-Betti number $\beta_1^{(2)}(G)$ is defined to be a certain dimension of either $H_1(G, \ell^2 G)$ or $H^1(G, \ell^2 G)$, see Sect. 2. It turns out to be useful to study $Z^1(G; \ell^2 G)$ through its map to $Z^1(G; \mathcal{U}G)$. In the case where the group G in non-amenable we show that the first ℓ^2-Betti number vanishes if and only if $H^1(G, \ell^2 G) = 0$ which was previously shown for finitely generated groups in [5]. In case the group G is amenable, $Z^1(G; \mathcal{U}G) = B^1(G; \mathcal{U}G)$, and we can show that each element in $c \in Z^1(G; \ell^2 G)$ is either bounded or proper on G, depending on whether the vector $\xi \in \mathcal{U}G$, for which $c(g) = (g-1)\xi$, is in $\ell^2 G$ or not.

Theorem (See Theorem 2.5) *Let G be an countable and discrete group which is amenable. Every 1-cocycle with values in $\ell^2 G$ is either bounded or proper.*

Moreover, the existence of co-cycles which are neither proper nor bounded is proved for non-amenable G, with the necessary condition of non-vanishing first ℓ^2-Betti number, and provided there exists an infinite amenable subgroup of G.

Providing a large of examples of groups with a positive first ℓ^2-Betti number, we prove:

Theorem (See Theorem 3.2) *Let G be an infinite countable discrete group. Assume that*

$$G = \langle g_1, \dots, g_n \mid r_1^{w_1}, \dots, r_k^{w_k} \rangle,$$

for elements $r_1, \dots, r_k \in \mathbb{F}_n = \langle g_1, \dots, g_n \rangle$ and positive integers $w_1, \dots, w_k$. We assume that the presentation is irredundant in the sense that $r_i^l \neq e \in G$, for $1 < l < w_i$ and $1 \leq i \leq k$. Then, the following inequality holds:

$$\beta_1^{(2)}(G) \geq n - 1 - \sum_{j=1}^{k} \frac{1}{w_j}.$$

Denis Osin has used this theorem in [34] to construct n-generated torsion groups with first ℓ^2-Betti number greater than $n - 1 - \varepsilon$.

We study the existence of free subgroups in torsionfree discrete groups. It is well-known that non-amenability is not sufficient to ensure the existence of a free subgroup. We show that a non-vanishing first ℓ^2-Betti number is sufficient, provided G is torsionfree and satisfies a weak form of Atiyah's Conjecture, see Sect. 4. More precisely,

Theorem (See Theorem 4.1) *Let G be a torsionfree discrete countable group. There exists a family of subgroups $\{G_i \mid i \in I\}$, such that*

(i) *We can write G as the disjoint union:*

$$G = \{e\} \cup \dot{\bigcup_{i \in I}} G_i.$$

(ii) *The groups G_i are mal-normal in G, for $i \in I$.*
(iii) *If G satisfies a weak form of the Atiyah Conjecture, then G_i is free from G_j, for $i \neq j$.*
(iv) $\beta_1^{(2)}(G_i) = 0$, *for all $i \in I$.*

Moreover, we obtain a strong structure theorem for such groups. The techniques allow to generalize a recent result of J. Wilson, see [48]. This also gives a new estimate on the exponential growth rate in terms of the first ℓ^2-Betti number; proving a generalized form of Conjecture 5.14 of Gromov from [23].

D. Gaboriau proved in [19] that an infinite, normal, infinite index subgroup H of a group G with positive first ℓ^2-Betti number cannot have a finite first ℓ^2-Betti number, and in particular cannot be finitely generated. Assuming infinite index, we extend this result to subgroups H, for which $H \cap H^g$ is infinite, for all $g \in G$. (See Sect. 5.1 for the definition of wq-normality and ws-normality.) In particular, it applies to all subgroups which contain an infinite normal subgroup. This covers classical results by Karrass-Solitar [28], Griffiths [22] and Baumslag [3], as well as more recent results by Bridson-Howie [8] and Kapovich [26]. More precisely, we prove:

Theorem (See Theorems 5.6 and 5.12) *Let G be a countable discrete group and suppose H is an infinite subgroup.*

(1) *If H is a wq-normal subgroup, then $\beta_1^{(2)}(H) \geq \beta_1^{(2)}(G)$.*
(2) *If is ws-normal, has infinite index, and $\beta_1^{(2)}(H) < \infty$, then $\beta_1^{(2)}(G) = 0$.*

Among the corollaries, we prove that if H, K are infinite, finitely generated subgroups of G, so that $H \cap K$ is of finite index in H and K, then: the index of $H \cap K$ in $\langle H, K \rangle$ is finite, if the first ℓ^2-Betti number of $\langle H, K \rangle$ is non-zero, see Theorem 7.3. Many of these results are well-known for free groups and were proved by several authors in various other cases, which are mostly covered by our result. The particular case of limit groups was studied in [26]. Our proof is using concrete computations with cocycles with values in $\mathcal{U}G$ and results from ergodic theory. It was D. Gaboriau in his groundbreaking work [19], who was the first to use ergodic theory to obtain striking results in the theory of ℓ^2-invariants with applications to infinite groups.

The article is organized as follows:

Section 1 is the Introduction. In Sect. 2 we recall the program of W. Lück and introduce the algebra $\mathcal{U}G$ of densely defined closed operators, which are affiliated with the group von Neumann algebra LG. Several algebraic properties of $\mathcal{U}G$ are recalled and their implications are clarified. In Theorem 2.2, we show that Lück's generalized dimension of the first cohomology with coefficients in either LG, $\ell^2 G$ or $\mathcal{U}G$ coincides with the first ℓ^2-Betti number. Moreover, for $H \subset G$, we show that $H^1(H; \mathcal{U}G) = 0$, whenever the first ℓ^2-Betti number of H vanishes. This will turn out to be very useful in algebraic computations.

Using these results we prove that a ℓ^2-cocycle on an amenable group is either bounded or proper. This is Conjecture 2 from [14]. Theorem 2.6 clarifies the existence of co-cycles which are neither bounded nor proper for general groups (admitting an infinite amenable subgroup).

In Sect. 3 we give examples of groups with non-vanishing first ℓ^2-Betti number. We give lower bounds on the first ℓ^2-Betti numbers for amalgamated free products, HNN-extensions and various other more elaborate constructions. We hope that this section provides some useful tools, for example Theorem 3.2, to estimate the first ℓ^2-Betti in some interesting cases.

In Sect. 4, we examine torsionfree groups which satisfy a weak form of the Atiyah Conjecture. It turns out that a positive first ℓ^2-Betti number has strong implications on the structure of such groups. In Theorem 4.1, we show that any such group decomposes *as a pointed set* into malnormal, mutually free subgroups with vanishing first ℓ^2-Betti number. This result is used to prove that in this case the reduced group C^*-algebra is simple with a unique trace state. Moreover, the techniques imply a Freiheitssatz, see Corollary 4.7, and give estimates about the exponential growth rate. In particular, assuming a weak form of Atiyah's Conjecture, we obtain a new proof of Conjecture 5.14 of M. Gromov in [23] about the exponential growth rate of a finitely presented group with fewer relations than generators.

Section 5 contains the main results of this article. We introduce various notions of normality, see the definition in Sect. 5.1, and study the existence of infinite subgroups of a group G with infinite index, sharing one of the normality properties, see Theorems 5.6 and 5.12. In particular, if $\beta_1^{(2)}(G) \neq 0$, we can exclude the existence of a finitely generated subgroup of infinite index, which contains an infinite normal subgroup. Several other corollaries can be found in this section, whereas various other applications of the main theorems are contained in Sect. 7. The proof of Theorem 5.12 relies on discrete measured groupoids and ergodic theory.

The necessary results from ergodic theory and the theory of discrete measured groupoids are collected in Sect. 6, where we extend some of our results from Sect. 5 to discrete measured groupoids.

In Sect. 7 we collect results about various classes of groups. In particular, we study boundedly generated groups, certain groups which are generated by a family of subgroups, limit groups, groups which are measure equivalent to free groups and so-called powerabsorbing subgroups. We are able to reprove and generalize several results from the literature. Most notably, we prove Proposition 7.3, which is a generalization of Theorem C of I. Kapovich in [26].

2 ℓ^2-cohomology, cocycles and Betti numbers

The computations of ℓ^2-homology have been algebraized through the seminal work of W. Lück, which is summarized and explained in detail in his nice compendium [31]. The basic observation is that through a dimension function, which is defined for all modules over the group von Neumann algebra, entirely algebraic objects give rise to numerical invariants. One of the main results is the following equality:

$$\beta_n^{(2)}(G) = \dim_{LG} H_n(G; LG), \qquad (1)$$

where G is a countable discrete group and $\beta_n^{(2)}(G)$ denotes the n-th ℓ^2-Betti number in the sense of M. Atiyah and as generalized by J. Cheeger and

M. Gromov, see [2, 13]. We are freely using $\dim_{LG}$, Lück's dimension function, which is defined for all LG-modules. We are frequently using that for an extension

$$0 \to M' \to M \to M'' \to 0$$

of LG-modules, the following formula holds:

$$\dim_{LG} M = \dim_{LG} M' + \dim_{LG} M''.$$

For more details about the definition of the dimension function and its properties consult [31].

In our study of ℓ^2-cohomology, the ring of closed and densely defined operators on $\ell^2 G$, which are affiliated with LG, plays a prominent role. This ring is denoted by $\mathcal{U}G$. Those rings were the motivating examples for J. von Neumann to study rings with a remarkably strong algebraic property, which was later named *von Neumann regularity*. A ring R is said to be von Neumann regular, if for each $a \in R$, there exists $b \in R$, such that $aba = a$. Alternatively, von Neumann regular rings are precisely those, for which all modules are flat. Recall, a module M over a ring R is called *flat*, if the functor $? \otimes_R M$ is exact.

In the process of algebraization of ℓ^2-homology, it was P. Linnell in [30], who reintroduced the ring of affiliated operators and studied its nice algebraic properties. Some ring theoretic properties were studied towards applications to ℓ^2-invariants and K-theory in [42, 47].

To our knowledge, this following lemma was first observed by S.K. Berberian (see [6]) in the context of finite von Neumann algebras and shortly afterwards by K.R. Goodearl in [21] in the more general context of metrically complete von Neumann regular rings.

Lemma 2.1 *Let (M, τ) be a finite tracial von Neumann algebra. The ring $\mathcal{U}G$ of operators affiliated with M is self-injective.*

Recall, a ring R is called *self-injective*, if the functor $\hom_R(?, R)$ is exact. (Note that $R \cong R^{\mathrm{op}}$ for all our rings, so that we do not need to talk about *left* self-injectivity etc.) Although, Lemma 2.1 has been around for more than 20 years, its consequences for the computation of ℓ^2-invariants have not been fully exploited. There are indications, that the context of metrically complete modules over metrically complete rings is indeed a useful context to study ℓ^2-invariants. Indeed, in [46], the second author gave a conceptual and short proof of D. Gaboriau's result about invariance of ℓ^2-Betti numbers under orbit equivalence. Moreover in [43], using essential properties of the category of metrically complete modules, R. Sauer and the second author constructed a Hochschild-Serre spectral sequence for extensions of discrete measurable groupoids.

In this note, we are basically interested in the first ℓ^2-Betti number of a countable discrete group. Right from the beginning of the study of ℓ^2-homology and ℓ^2-cohomology of discrete groups, it was observed that those are only dual to each other under some finiteness assumptions on the group, i.e. the group was assumed to have a finite classifying space. However, using the self-injectivity of $\mathcal{U}G$, it is obvious that always:

$$\hom_{\mathcal{U}G}(H_n(G;\mathcal{U}G),\mathcal{U}G) \cong H^n(G;\mathcal{U}G). \tag{2}$$

Moreover, $LG \subset \mathcal{U}G$ is a flat ring extension and $? \otimes_{LG} \mathcal{U}G$ preserves the dimension, see [42, 46]. Hence

$$\dim_{LG} H_n(G, LG) = \dim_{LG} H_n(G, LG) \otimes_{LG} \mathcal{U}G = \dim_{LG} H_n(G, \mathcal{U}G).$$

Note that, by Corollary 3.4 in [47], also dualizing a $\mathcal{U}G$-module preserves its dimension. We conclude that

$$\beta_n^{(2)}(G) = \dim_{LG} H^n(G;\mathcal{U}G). \tag{3}$$

The computations with this cohomology group simplify the picture drastically since they have the nice property that they vanish unless their dimension is non-zero. This follows from Corollary 3.3 in [47]. We will use this fact frequently.

2.1 A cocycle description

It is well-known that the first group cohomology with coefficients in a module M can be computed as the vector space of M-valued 1-cocycles on G modulo inner cocycles. A 1-cocycle with values in the G-module M is a map

$$c\colon G \to M, \quad \text{with } c(gh) = gc(h) + c(g),\ \forall g, h \in G.$$

It is called inner, if there exists $\xi \in M$, such that $c(g) = (g-1)\xi$, for all $g \in G$. We denote the space of M-valued 1-cocycles by $Z^1(G; M)$ and the space of inner cocycles by $B^1(G; M)$. There is an exact sequence

$$0 \to B^1(G; M) \to Z^1(G; M) \to H^1(G; M) \to 0.$$

Our first theorem gives an identification of dimensions of cohomology groups, where the coefficients vary among the canonical choices LG, $\ell^2 G$ and $\mathcal{U}G$.

Theorem 2.2 *Let G be a countable discrete group.*

$$\beta_k^{(2)}(G) = \dim_{LG} H^k(G,\mathcal{U}G) = \dim_{LG} H^k(G,\ell^2 G) = \dim_{LG} H^k(G, LG).$$

Moreover, if $\beta_k^{(2)}(G) = 0$ for some k, then $H^k(G,\mathcal{U}G) = 0$.

Proof We give the proof only for $k = 1$, since we are mainly concerned with the first ℓ^2-cohomology. A similar argument can be found for $k \neq 1$. There exists a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^1(G; LG) & \longrightarrow & Z^1(G; LG) & \longrightarrow & H^1(G; LG) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1(G; \ell^2 G) & \longrightarrow & Z^1(G; \ell^2 G) & \longrightarrow & H^1(G; \ell^2 G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1(G; \mathcal{U}G) & \longrightarrow & Z^1(G; \mathcal{U}G) & \longrightarrow & H^1(G; \mathcal{U}G) & \longrightarrow & 0.
\end{array}$$

Recall, if $M_1 \subset M_2$ is an inclusion of LG module, then it is called *rank dense*, if for every $\xi \in M_2$, there exists an increasing sequence of projections $p_n \uparrow 1$, such that $\xi p_n \in M_1$, for all $n \in \mathbb{N}$. It was shown in [46], that a rank dense inclusion is a dimension isomorphism.

If G is infinite, the left column identifies with the inclusions $LG \subset \ell^2 G \subset \mathcal{U}G$, which are well-known to be dimension isomorphisms since LG is rank dense in $\mathcal{U}G$.

The column in the middle also consists of inclusions and we claim that the images are rank dense as well. Indeed, every 1-cocycle with values in $\mathcal{U}G$ can be cut by a projection of trace bigger than $1 - \varepsilon$ to take values in LG. For each $g \in G$, $c(g) \in \mathcal{U}G$ and we find a projection p_g of trace bigger $1 - \varepsilon_g$, so that $c(g)p_g \in LG$. Taking the infimum over all p_g, we obtain a projection p of trace bigger than $1 - \sum_{g \in G} \varepsilon_g$. Hence, choosing a suitable sequence ε_g proves the claim.

The vanishing of $H^1(G; \mathcal{U}G)$ in case of vanishing first ℓ^2-Betti number follows since it is the dual of the $\mathcal{U}G$-module $H_1(G; \mathcal{U}G)$. It was shown in [47], that the dual is zero if and only if the dimension is zero. This finishes the proof. $\square$

Remark 2.3 Let $H \subset G$ be a subgroup. It follows from standard computations that

$$\beta_1^{(2)}(H) = \dim_{LG} H^1(H, \mathcal{U}G)$$

and $H^1(H, \mathcal{U}G) = 0$, if and only if the first ℓ^2-Betti number of H vanishes.

In [5] it is shown that for a finitely generated non-amenable discrete group, the first ℓ^2-Betti number vanishes if and only if the first cohomology group with values in the left regular representation vanishes (see also Corollary 3.2 in [32]). We will now show that we may drop the assumption that the group is finitely generated.

Corollary 2.4 *Let G be a non-amenable countable discrete group, then $\beta_1^{(2)}(G) = 0$ if and only if $H^1(G, \ell^2 G) = 0$.*

Proof First let us suppose that $H^1(G, \ell^2 G) = 0$. Let $c : G \to \mathcal{U}G$ be a 1-cocycle, we must show that there is an affiliated operator $\xi \in \mathcal{U}G$ such that $c(g) = (g-1)\xi$, for each $g \in G$. Given $\varepsilon > 0$, an affiliated operator η and a projection $p \in RG$ we may find a projection $q \in RG$, $q \leq p$ such that $q\eta \in \ell^2 G$ and $\tau(p-q) < \varepsilon$. From this fact we may construct a partition of unity $\{p_n\}_{n\in\mathbb{N}}$ in RG such that

$$p_n c(g) \in \ell^2 G, \quad \forall g \in G, n \in \mathbb{N}.$$

Since $H^1(G, \ell^2 G) = 0$ we conclude that there exist $\xi_n \in \ell^2 G$ such that

$$p_n c(g) = (g-1)\xi_n, \quad \forall g \in G, n \in \mathbb{N}.$$

Moreover we may assume $p_n \xi_n = \xi_n$, $\forall n \in \mathbb{N}$ so that $\xi = \sum_{n\in\mathbb{N}} \xi_n \in \mathcal{U}G$ is well defined and has the desired properties.

Next let us suppose that $\beta_1^{(2)}(G) = 0$. From Theorem 2.2, we conclude that if $c : G \to \ell^2 G$ is a 1-cocycle then there exists and affiliated operator $\xi \in \mathcal{U}G$ such that $c(g) = (g-1)\xi$, $\forall g \in G$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of projections in RG which increase to 1 such that $p_n \xi \in \ell^2 G$, for each $n \in \mathbb{N}$. Then since RG acts normally on $\ell^2 G$ we conclude that

$$\lim_{n\to\infty} \|(1-p_n)c(g)\| = 0, \quad \forall g \in G.$$

Hence c is approximately inner which shows that $\overline{H^1}(G, \ell^2 G) = 0$. As $\ell^2 G$ does not weakly contain the trivial representation it is a well known result that we must also have that $H^1(G, \ell^2 G) = 0$. □

2.2 Dichotomy of ℓ^2-cocycles on amenable groups

The following theorem is an affirmative answer to Conjecture 2 in [14].

Theorem 2.5 *Let G be an countable and discrete group which is amenable. Every 1-cocycle with values in $\ell^2 G$ is either bounded or proper.*

Proof Let $c : G \to \ell^2 G$ be a 1-cocycle. We need to show that either $\sup_{g\in G} \|c(g)\|_2$ is finite or $\{\|c(g_n)\|_2, n \in \mathbb{N}\}$ is unbounded for every sequence $\{g_n\}_{n\in\mathbb{N}}$ in G that goes to infinity.

It is well-known, that $\beta_1^{(2)}(G) = 0$ if G is amenable. We conclude from Theorem 2.2, that there exists an affiliated operator $\xi \in \mathcal{U}G$, such that

$$c(g) = (g-1)\xi,$$

for all $g \in G$. Let $\{p_m\}_{m\in\mathbb{N}}$ be a partition of unity of RG, which has the property that $p_m\xi \in \ell^2 G$, for all $m \in \mathbb{N}$.

Since the left-regular representation is mixing, for every sequence $\{g_n\}_{n\in\mathbb{N}}$ that goes to infinity

$$\lim_{n\to\infty} \|(g_n - 1)p_m\xi\|_2^2 = 2\|p_m\xi\|_2^2.$$

We consider two cases depending whether $\sum_{m\in\mathbb{N}} \|p_m\xi\|_2^2$ is finite or not. (a) If it is finite, then $\xi \in \ell^2 G$ and the cocycle will be bounded. (b) If it is infinite we aim to show that the cocycle is proper. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in G that tends to infinity. Given $C > 0$, there exists $k \in \mathbb{N}$, such that

$$\sum_{m=1}^{k} 2\|p_m\xi\|_2^2 \geq C + 1,$$

and there exists some $l \in \mathbb{N}$, such that

$$\|(g_j - 1)p_m\xi\|_2^2 \geq 2\|p_m\xi\|_2^2 - k^{-1}$$

for all $1 \leq m \leq k$ and all $j \geq l$. This implies that for every $j \geq l$ we have

$$\|(g_j - 1)\xi\|_2^2 \geq \sum_{m=1}^{k} \|(g_j - 1)p_m\xi\|_2^2 \geq \sum_{m=1}^{k} 2\|p_m\xi\|_2^2 - k^{-1} \geq C.$$

This finishes the proof. □

We remark that it was previously shown in [32] that if G is a countable discrete group and $c : G \to \ell^2 G$ is an unbounded 1-cocycle then c is also unbounded on any infinite subgroup of G. The above theorem states that this is true for infinite subsets as well. There is a partial converse to the preceding result.

Theorem 2.6 *Let G be a group with $\beta_1^{(2)}(G) \neq 0$ and assume that there exists an infinite amenable sub-group. There exists a 1-cocycle with values in $\ell^2 G$ on G which is neither bounded nor proper.*

Proof Let $H \subset G$ be an infinite amenable subgroup. Since $\beta_1^{(2)}(H) = 0 < \beta_1^{(2)}(G)$ the restriction map $H^1(G, \ell^2 G) \to H^1(H, \ell^2 G)$ cannot be injective. Hence there is an unbounded ℓ^2-cocycle on G which is bounded on H. □

Remark 2.7 Note that for a non-amenable group with vanishing first ℓ^2-Betti number, all ℓ^2-cocycles are automatically bounded.

The following corollary answers Question 1 in [14] negatively.

Corollary 2.8 *If $G = H_1 * H_2$ with H_j non-trivial, $j = 1, 2$ and $H_1 \neq \mathbb{Z}/2\mathbb{Z}$, then there exists an ℓ^2-cocycle, which is neither proper nor bounded. In particular, this is the case for $PSL_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.*

3 Examples of groups with positive first ℓ^2-Betti number

This class of groups (or rather its complement) was also studied by B. Bekka and A. Valette in [5]. Among the classical examples, there are free groups, surface groups and groups containing such groups with finite index. W. Dicks and P. Linnell (see [15]) computed that the first ℓ^2-Betti number of a n-generated one-relator group is $n - 2$. It was shown in [37], that the class of finitely generated groups with first ℓ^2-Betti number greater or equal than ε is closed in Grigorchuk's space of marked groups. This implies in particular, that limit groups have positive first ℓ^2-Betti number, see Sect. 7.3.

D. Gaboriau showed in [19] that the non-vanishing of the n-th ℓ^2-Betti number does only depend on the group up to measure equivalence. This provides a class of examples which we study more closely in Sect. 7.5.

Throughout this section, we are using the convention that $|G|^{-1} = 0$, if G is infinite.

3.1 Amalgamated free products

In the case of amalgamated free products, we state the following well-known result:

Proposition 3.1 *Let G be a discrete countable group. If G is an amalgamated free product $G = A *_C B$, then*

$$\beta_1^{(2)}(G) \geq \left(\beta_1^{(2)}(A) - \frac{1}{|A|}\right) + \left(\beta_1^{(2)}(B) - \frac{1}{|B|}\right) - \left(\beta_1^{(2)}(C) - \frac{1}{|C|}\right). \quad (4)$$

Proof This follows from the long exact sequence of homology with coefficients in $\mathcal{U}G$, which is associated to an amalgamated free product (see [10, Chap. VII.9]) and an easy dimension count. □

Note, if G acts on a tree, similar estimates can be found in terms of the order and the first ℓ^2-Betti numbers of the stabilizer groups. In particular, if $G = A*_B$ is an HNN-extension, then:

$$\beta_1^{(2)}(G) \geq \left(\beta_1^{(2)}(A) - \frac{1}{|A|}\right) - \left(\beta_1^{(2)}(B) - \frac{1}{|B|}\right).$$

Of course, in special cases, much more can be said. For example, from Appendix A of [11] we see that the inequality in (4) is actually an equality in the case where $\beta_1^{(2)}(C) = 0$.

3.2 Triangle groups and related constructions

We now want to provide more non-trivial examples of groups which have a positive first ℓ^2-Betti number. Our main result in this direction is the following theorem:

Theorem 3.2 *Let G be an infinite countable discrete group. Assume that there exist subgroups $G_1, \ldots, G_n$, such that*

$$G = \langle G_1, \ldots, G_n \mid r_1^{w_1}, \ldots, r_k^{w_k} \rangle,$$

for elements $r_1, \ldots, r_k \in G_1 * \cdots * G_n$ and positive integers $w_1, \ldots, w_k$. We assume that the presentation is irredundant in the sense that $r_i^l \neq e \in G$, for $1 < l < w_i$ and $1 \leq i \leq k$. Then, the following inequality holds:

$$\beta_1^{(2)}(G) \geq n - 1 + \sum_{i=1}^{n} \left(\beta_1^{(2)}(G_i) - \frac{1}{|G_i|} \right) - \sum_{j=1}^{k} \frac{1}{w_j}.$$

Proof There is an exact sequence

$$0 \to Z^1(G; \mathcal{U}G) \xrightarrow{p} Z^1(G_1 * \cdots * G_n; \mathcal{U}G) \xrightarrow{q} \bigoplus_{j=1}^{k} \mathcal{U}G,$$

where p is given by the composition

$$Z^1(G; \mathcal{U}G) \to \bigoplus_{i=1}^{n} Z^1(G_i; \mathcal{U}G) \cong Z^1(G_1 * \cdots * G_n; \mathcal{U}G),$$

and q is given by the sum of the evaluation maps at $r_j^{w_j}$. Indeed, exactness at $Z^1(G; \mathcal{U}G)$ is clear and it remains to prove exactness in the middle. Again, it is obvious that the composition is zero so that we only have to show that any element in the kernel of the evaluation maps defines a cocycle on G. If a cocycle on $G_1 * \cdots * G_n$ vanishes on a relator $r_i^{w_i}$, then it also vanishes on any of its conjugates, since:

$$c(g r_i^{w_i} g^{-1}) = (1 - g r_i^{w_i} g^{-1}) c(g) + g c(r_i^{w_i}) = (1 - g r_i^{w_i} g^{-1}) c(g) = 0.$$

Here, we are using that $g r_i^{w_i} g^{-1}$ acts trivially on $\mathcal{U}G$.

Clearly,

$$c(r_j^{w_j}) = \sum_{l=1}^{w_j} r_j^l \cdot c(r_j).$$

Thus the dimension of the image of the evaluation map at $r_j^{w_j}$ is less than $1/w_j$, since $1/w_j \cdot \sum_{l=1}^{w_j} r_j^l$ is a projection of trace $1/w_j$. Here, we are using that the presentation is irredundant. Hence, the dimension of the image of q is less than $\sum_{j=1}^{k} 1/w_j$. The claim follows by noting that

$$\dim_{LG} Z^1(G_i; \mathcal{U}G) = \beta_1^{(2)}(G_i) - \frac{1}{|G_i|} + 1. \qquad \square$$

The theorem covers generalized triangle groups and so-called generalized tetrahedron groups. Let us spell out what the theorem says in the case of generalized triangle groups. Let us first recall the definition.

Definition 3.3 A group G is called a *generalized triangle group*, if it admits a representation

$$G = \langle a, b \mid a^p = b^q = w(a, b)^r \rangle,$$

where $w(a, b)$ is a cyclically reduced word of length at least 2 in $C_p * C_q$. We call

$$\kappa(G) = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1$$

the curvature of G.

The following result is an immediate consequence of Theorem 3.2.

Corollary 3.4 *Let G be a triangle group. The following inequality holds*:

$$\beta_1^{(2)}(G) \geq -\kappa(G).$$

In particular, if G is negatively curved, i.e. $\kappa(G) < 0$, then $\beta_1^{(2)}(G) \neq 0$.

3.3 The relation module

Another result which also estimates the first ℓ^2-Betti number in terms of more algebraic data is given by the following theorem:

Proposition 3.5 *Let G be a finitely generated group with a presentation $0 \to R \to F \to G$ with F free of rank n. Then the following inequality holds*:

$$\beta_1^{(2)}(G) \geq n - 1 - \dim_{LG}(R_{ab} \otimes_{\mathbb{Z}G} \mathcal{U}G),$$

where $R_{ab} = R/[R, R]$ is the relation G-module (*induced by the conjugation action of G on R*).

Proof Underlying the computation in the proof of Theorem 3.2, there is a Lyndon-Serre spectral sequence with a low degree exact sequence. Let $0 \to R \to F \to G \to 0$ be a presentation of the group G with F free of rank n. Then,

$$0 \to H^1(G; \mathcal{U}G) \to H^1(F; \mathcal{U}G) \to H^1(R; \mathcal{U}G)^G \to H^2(G; \mathcal{U}G) \to 0$$

is an exact sequence. Note that $\mathcal{U}G$ is a trivial R-module, so that

$$H^1(R; \mathcal{U}G) = \hom_{\mathbb{Z}}(R_{ab} \otimes_{\mathbb{Z}} \mathbb{C}, \mathcal{U}G),$$

where G acts diagonally with the conjugation action on R and on the left on $\mathcal{U}G$. Hence,

$$H^1(R; \mathcal{U}G)^G = \hom_{\mathbb{Z}G}(R_{ab}, \mathcal{U}G).$$

Writing everything out, we get:

$$0 \to H^1(G; \mathcal{U}G) \to \mathcal{U}G^{\oplus n-1} \to \hom_{\mathbb{Z}G}(R_{ab}, \mathcal{U}G) \to H^2(G; \mathcal{U}G) \to 0.$$

Taking dimensions, this implies the claim. □

4 Free subgroups

4.1 Restriction maps and free subgroups

Throughout this section, we are assuming that G is a torsionfree discrete countable group and most of the time also that it satisfies the following condition:

($\star$) Every non-trivial element of $\mathbb{Z}G$ acts without kernel on $\ell^2 G$.

This condition is satisfied if G satisfies the Atiyah conjecture but is *a priori* weaker. Recall, the Atiyah Conjecture for torsionfree groups predicts the existence of a skew-field $\mathbb{Z}G \subset K \subset \mathcal{U}G$. Note that the Atiyah Conjecture was established for a large class of torsionfree groups, see the results and references in [17]. In particular, condition ($\star$) is known to hold for all right orderable groups and all residually torsionfree elementary amenable groups. The subset of Grigorchuk's space of marked groups for which the conjecture holds is closed and hence contains for example all limit groups. No counterexample is known.

Let G be a discrete group, we use the notation $\dot{G}$ to denote the set $G \setminus \{e\}$. In our computations we are exploiting the basic fact that $1-g$ is invertible as an affiliated operator if g is not torsion. Of course, this observation does not rely on condition ($\star$). The main result here is the following theorem.

Theorem 4.1 *Let G be a torsionfree discrete countable group. There exists a family of subgroups $\{G_i \mid i \in I\}$, such that*

(i) *We can write G as the disjoint union*:

$$G = \{e\} \cup \bigcup_{i \in I} \dot{G}_i.$$

(ii) *The groups G_i are mal-normal in G, for $i \in I$.*
(iii) *If G satisfies condition* ($\star$), *then G_i is free from G_j, for $i \neq j$.*
(iv) $\beta_1^{(2)}(G_i) = 0$, *for all $i \in I$.*

Proof We partition $\dot{G}$ according to the following equivalence relation

$$g \sim h \quad \Longleftrightarrow \quad c(g) = 0 \quad \text{if and only if} \quad c(h) = 0, \quad \forall c \in Z^1(G; \mathcal{U}G).$$

First of all, one direction is sufficient to imply the *if and only if* in the definition of the equivalence relation. Indeed, assume $c(g) = 0 \Rightarrow c(h) = 0$, but there exists some cocycle, such that $c(h) = 0$ but $c(g) \neq 0$. If $c(g) \neq 0$, then $c(g) = (g-1)\xi$ for some $0 \neq \xi \in \mathcal{U}G$ and the cocycle $k \mapsto (k-1)\xi - c(k)$ vanishes on g. This implies $c(h) = (h-1)\xi \neq 0$, which is a contradiction.

If $c(g) = c(h) = 0$, then also $c(gh) = 0$ and $c(g^{-1}) = 0$, so that the equivalence classes together with the unit form subgroups. Denote by $[g]_1 = \{h \in \dot{G} \mid h \sim g\} \cup \{e\}$. If $\beta_1^{(2)}([g]_1) \neq 0$, we continue with the partitioning into subsets and proceed by transfinite induction. This implies claims (i) and (iv).

Claim (ii) is proved by the following argument. If $hgh^{-1} \in [g]_1$ and $c(g) = 0$, then $0 = c(hgh^{-1}) = (1 - hgh^{-1})c(h)$, and hence $c(h) = 0$. We conclude that $h \in [g]_1$.

We now prove (iii) under the assumption of condition ($\star$). Let $h = w_1 v_1 w_2 v_2 \cdots v_n$ be a shortest (with respect to block length) trivial word consisting of non-trivial words $w_k \in G_i$ and $v_k \in G_j$ with $i \neq j$. Let c be a co-cycle which vanishes on G_i, but not on G_j. Then:

$$0 = c(h) = \{w_1(v_1 - 1) + \cdots + w_1 v_1 \cdots w_n (v_n - 1)\}\xi,$$

for some $\xi \in \mathcal{U}G$. We conclude by $(\star)$ that

$$w_1(v_1 - 1) + \cdots + w_1 v_1 \cdots w_n(v_n - 1) = 0 \in \mathbb{C}G$$

and hence there has to be a shorter trivial word. This contradicts the assumption and hence G_i is free from G_j. This finishes the proof. □

Remark 4.2 It follows from Theorem 7.1, that the set I is infinite if the first ℓ^2-Betti number of G does not vanish. Indeed, if I were finite, then G would be boundedly generated by subgroups G_i with vanishing first ℓ^2-Betti number. This contradicts Theorem 7.1.

The following lemma is useful to exploit the technique further.

Lemma 4.3 *Let G be a torsionfree discrete countable group satisfying condition $(\star)$. Let $H \subset G$ be a subgroup and assume that the restriction map*

$$\operatorname{res}_H^G \colon H^1(G, \mathcal{U}G) \to H^1(H, \mathcal{U}G)$$

is not injective. Then, there exists $h \in G$, such that the natural map

$$\pi : \mathbb{Z} * H \to \langle h, H \rangle \subset G$$

is an isomorphism. Moreover, if $G = \langle h, H \rangle$, then h is free from H.

Proof Since res_H^G is not injective, there exists a non-trivial co-cycle $c \colon G \to \mathcal{U}G$ which is inner on H. Subtracting this inner co-cycle, we can assume that the restriction vanishes. Since c was non-trivial, there exists $h \in G$, such that $c(h) = (h-1)\xi \neq 0$. The proof proceeds as before. □

In the next two subsections we collect some corollaries of the results of the preceding section.

4.2 Simplicity of the reduced group C^*-algebra

The following corollary shows that a non-trivial first Betti number of a torsionfree group implies the existence of free subgroups. The proof uses the validity of condition $(\star)$ for the group. It would be desirable to remove this assumption.

Corollary 4.4 *Let G be a discrete countable group satisfying condition $(\star)$. Assume that the first ℓ^2-Betti number does not vanish. Let F be a finite subset of G. There exists $g \in G$, such that g is free from each element in F. In particular, G contains a copy of F_2.*

Proof In view of Theorem 4.1, this can fail only if the index set I in the proof of Theorem 4.1 is finite. Hence, the result follows from Remark 4.2. □

Remark 4.5 Corollary 4.4 confirms the feeling that a sufficiently non-amenable group contains a free subgroup. Note, that various weaker conditions like *non-amenability* itself or *uniform non-amenability* have been proved to be insufficient to ensure the existence of free subgroups, at least in the presence of torsion.

Using results from [4] we obtain the following result.

Corollary 4.6 *Let G be a torsionfree discrete countable group satisfying condition $(\star)$. If the first ℓ^2-Betti number does not vanish, then the reduced group C^*-algebra $C^*_{red}(G)$ is simple and carries a unique trace.*

Proof This follows from Lemmas 2.2 and 2.1 in [4]. Indeed, assuming condition $(\star)$ and non-vanishing first ℓ^2-Betti number, Corollary 4.4 verifies Condition P_{nai} from Definition 4 of [4]. □

4.3 Freiheitssatz and uniform exponential growth

The following result is a generalization of the main result of J. Wilson in [48] for torsionfree groups which satisfy $(\star)$. For this, note that a group G with n generators and m relations satisfies $\beta_1^{(2)}(G) \geq n - m - 1$.

Corollary 4.7 (Freiheitssatz) *Let G be a torsionfree discrete countable group which satisfies $(\star)$. Assume that $a_1, \dots, a_n \in G$ generate G and $\lceil \beta_1^{(2)}(G) \rceil \geq k$. There exist $k+1$ elements $a_{i_0}, \dots, a_{i_k}$ among the generators such that the natural map*

$$\pi : F_{k+1} \to \langle a_{i_0}, \dots, a_{i_k} \rangle \subset G$$

is an isomorphism.

Proof We proof this result by induction over n. The case $n = 1$ is obvious, since $n \geq k+1$ and there is nothing to prove. For the induction step, consider the restriction map

$$\mathrm{res}_1 : H^1(G, \mathcal{U}G) \to H^1(\langle a_2, \dots, a_n \rangle, \mathcal{U}G).$$

If res_1 is injective, then we can pass to the subgroup $G' = \langle a_2, \dots, a_n \rangle$. Note that $\lceil \beta_1^{(2)}(G') \rceil \geq k$. In this case the proof is finished by induction since we decreased the number of generators by 1.

Hence, we can assume that the map res_1 is not injective and there exists a cocycle on G which is inner on $G' = \langle a_2, \ldots, a_n \rangle$. Lemma 4.3 implies that $G = \langle a_1 \rangle * G'$. Now, the number of generators of G' is $n-1$ and $\lceil \beta_1^{(2)}(G') \rceil \geq k-1$. Again, the proof is finished by induction. □

Following the work of J. Wilson in [48], this gives also an easy proof of Conjecture 5.14 of M. Gromov in [23], saying that the exponential growth rate of a group with n generators and m relations is bigger than $2(n-m)-1$. Recall, the exponential growth rate is defined as

$$e_S(G) = \lim_{n\to\infty} \sqrt[n]{\# B_S(e,n)},$$

where $B_S(e,n)$ denotes the ball of radius n with respect to the word length metric coming from a generating set S. In general, we obtain the following result about the exponential growth rate:

Corollary 4.8 *Let G be a finitely generated torsionfree discrete countable group which satisfies* $(\star)$. *Then*

$$e_S(G) \geq 2\lceil \beta_1^{(2)}(G) \rceil + 1,$$

for any generating set S.

Proof This is obvious, since Corollary 4.7 says that a generating set S contains the base of a free group of rank $\lceil \beta_1^{(2)}(G) \rceil + 1$. □

In particular, a torsionfree group satisfying condition $(\star)$ has uniform exponential growth if its first ℓ^2-Betti number is positive.

5 Results about the subgroup structure

5.1 Various notions of normality

We first want to review some notions of normality of subgroups and introduce some notation. A subgroup $H \subset G$ is called:

(i) normal iff $gHg^{-1} = H$, for all $g \in G$,
(ii) s-normal iff $gHg^{-1} \cap H$ is infinite for all $g \in G$, and
(iii) q-normal iff $gHg^{-1} \cap H$ is infinite for elements $g \in G$, which generate G.

We say that a subgroup inclusion $H \subset G$ satisfies one of the normality properties from above *weakly*, iff there exists an ordinal number α, and an ascending α-chain of subgroups, such that $H_0 = H$, $H_\alpha = G$, and $\bigcup_{\beta<\gamma} H_\beta \subset H_\gamma$ has the required normality property.

Clearly, normal implies s-normal implies q-normal; and similarly for the weak notions. The notion of (*weakly*) q-normal subgroups were introduced by Popa (Definition 2.3 in [38]) in order to "untwist" cocycles from the subgroup to the whole group under certain weak mixingness conditions. This method was also used quite successfully by Popa in subsequent works [39–41]. Theorem 5.6 below gives another instance where this notion is useful in untwisting cocycles. See also Definition 1.2 in [24] for a von Neumann analogue of this notion.

A weakly q-normal subgroup is called wq-normal in [40] and we follow this convention. In analogy, we call weakly s-normal subgroups ws-normal. For obvious reasons s- and q-normality are considered only for infinite subgroups.

Weakly normal subgroups are usually called *descendent*. Every subgroup is an descendent subgroup of a self-normalizing subgroup. Remark 5.3 will clarify the corresponding observation in case of wq-normality. There are various other notions of normality. For example, P. Kropholler studies the notion of *near normality* in [29]. A subgroup $H \subset G$ is said to be near normal, if $H^g \cap H$ has finite index in H, for all $g \in G$. Clearly, near normality implies s-normality.

Example 5.1 The inclusions

$$GL_n(\mathbb{Z}) \subset GL_n(\mathbb{Q}), \quad \text{and} \quad \mathbb{Z} = \langle x\rangle \subset \langle x, y \mid yx^p y^{-1} = x^q\rangle = BS_{p,q}$$

are inclusions of s-normal subgroups. The inclusion

$$F_2 = \langle a, b^2\rangle \subset \langle a, b\rangle = F_2$$

is q-normal but not s-normal.

Lemma 5.2 [38] *Let G be a discrete countable group and H be an infinite subgroup. The subgroup H is wq-normal in G if and only if given any intermediate subgroup $H \subset K \subsetneq G$ there exists $g \in G \setminus K$ with $gKg^{-1} \cap K$ infinite.*

Proof One direction is obvious, since one can perform a transfinite induction to produce the chain of subgroups with the desired properties.

We prove the converse: Consider the least β, such that H_β is not contained in K. Then $\bigcup_{\gamma<\beta} H_\gamma \subset K$ and there exists $g \in H_\beta \setminus K \subset G \setminus K$, such that $g(\bigcup_{\gamma<\beta} H_\gamma)g^{-1} \cap (\bigcup_{\gamma<\beta} H_\gamma)$ is infinite. Hence, $gKg^{-1} \cap K$ is infinite as well. □

Remark 5.3 The notion of wq-normal subgroup is rather general. The following fact is easily deduced from the previous lemma. If G is torsionfree and $H \subset G$, then H is wq-normal in a malnormal subgroup of G. For general G, almost malnormal subgroups have to be considered.

Corollary 5.4 *If $H \subset G$ is wq-normal, and $H \subset K \subset G$, then $K \subset G$ is wq-normal. In particular if H contains an infinite group which is normal in G then H is wq-normal.*

Proof This is an immediate consequence of Lemma 5.2. □

We see from the next lemma, that s-normality shares slightly better inheritance properties than q-normality. However, the following lemma does not seem to extend to the notion of ws-normality.

Lemma 5.5 *If $H \subset G$ is s-normal, and $H \subset K \subset G$, then $H \subset K$ and $K \subset G$ are inclusions of s-normal subgroups.*

Proof This is obvious. □

5.2 ℓ^2-invariants and normal subgroups

The two main results in this subsection are Theorems 5.6 and 5.12. We derive several corollaries about the structure of groups G with $\beta_1^{(2)}(G) \neq 0$.

Theorem 5.6 *Let G be a countable discrete group and suppose H is an infinite wq-normal subgroup. We have $\beta_1^{(2)}(H) \geq \beta_1^{(2)}(G)$.*

Proof According to Theorem 2.2, the ℓ^2-Betti-numbers are the $\mathcal{U}G$-dimension of the spaces $H^1(H, \mathcal{U}G)$ and $H^1(G, \mathcal{U}G)$. In order to show the inequality, we show that the restriction map $H^1(G, \mathcal{U}G) \to H^1(H, \mathcal{U}G)$ is injective. Let $c\colon G \to \mathcal{U}G$ be a 1-cocycle which is inner on H. We may subtract the inner cocycle and assume that $c(h) = 0$, $\forall h \in H$. Let $K = \{g \in G \mid c(g) = 0\}$, then $H \subset K \subset G$ and so if $K \neq G$ then there exists $g \in G \setminus K$ with $gKg^{-1} \cap K$ is infinite. However, for each $k \in gKg^{-1} \cap K$ we have $c(g) - kc(g) = c(k) - gc(g^{-1}kg) = 0$. Hence, if $gKg^{-1} \cap K$ is infinite we conclude that $c(g) = 0$. Indeed, this follows for $c(g) \in \ell^2 G$ from strong mixing of the regular representation. The result extends to the general case by approximation in rank metric. Thus $g \in K$ which gives a contradiction. Thus we conclude that $K = G$ which finishes the proof. □

Remark 5.7 The inequality in Theorem 5.6 is sharp. Indeed $\langle a, b^2 \rangle \subset \langle a, b \rangle = F_2$ is wq-normal and the restriction map in ℓ^2-cohomology is an isomorphism.

Corollary 5.8 *Let $H \subset K \subset G$ be a chain of subgroups and assume that $H \subset G$ is wq-normal and $[K : H] < \infty$. Then*

$$[K : H] \cdot \beta_1^{(2)}(G) \leq \beta_1^{(2)}(H).$$

Proof This follows immediately from the proof of Theorem 5.6, since the restriction map factorizes through $H^1(K, \mathcal{U}G)$. This $\mathcal{U}G$-module has dimension $[K : H]^{-1}\beta_1^{(2)}(H)$, if the index $[K : H]$ is finite. Alternatively, one can use Corollary 5.4. □

Corollary 5.9 *Let G be a torsionfree discrete countable group and let $H \subset G$ be an infinite subgroup. If $\beta_1^{(2)}(H) < \beta_1^{(2)}(G)$, then there exists a proper malnormal subgroup $K \subset G$, such that $H \subset K$.*

Proof This follows from Theorem 5.6 and Lemma 5.2. □

Remark 5.10 Assume that G is finitely presented of deficiency d and that H is finitely generated with n generators. Note that the hypothesis of Corollary 5.9 is satisfied whenever $n < d$ or H amenable and $0 < d$. Moreover, the example in Remark 5.7 shows that the assumption of a strict inequality cannot be improved.

Corollary 5.11 *Let G be a countable discrete group and let $H \subset G$ be an infinite wq-normal subgroup. Let $K \subset G$ be a subgroup with $H \subset K$ and assume that $\beta_1^{(2)}(G) > n$. Then, K is not generated by n or less elements.*

The second main result in this section is the following.

Theorem 5.12 *Let G be a countable discrete group and suppose H is an infinite index, infinite ws-normal subgroup. If $\beta_1^{(2)}(H) < \infty$, then $\beta_1^{(2)}(G) = 0$.*

Obviously, for the proof we can restrict to the case of a s-normal subgroup. The proof of this theorem requires the introduction of some tools from ergodic theory and dynamical systems. It will be carried out in the next section. Note that the result follows from Theorem 5.6, in case there are finite index subgroups G' of G, which have arbitrary high index and contain H. Indeed, in this case

$$\beta_1^{(2)}(H) \geq \beta_1^{(2)}(G') = [G, G'] \cdot \beta_1^{(2)}(G),$$

by Theorem 5.6, since Lemma 5.5 implies that H is also s-normal and hence q-normal in G'. This implies $\beta_1^{(2)}(G) = 0$, under the assumption $\beta_1^{(2)}(H) < \infty$.

Although the existence of such families of finite index subgroups seems to be rare, it can always be achieved in the setting of discrete measured groupoids, see Lemma 6.2. After having established the analogue of Theorem 5.6 for discrete measured groupoids, the proof of Theorem 5.12 follows as before.

Corollary 5.13 *Let G be a countable discrete group with $\beta_1^{(2)}(G) > 0$. Suppose that $H \subset G$ is an infinite, finitely generated ws-normal subgroup. Then H has to be of finite index.*

Note that the result applies in case H contains an infinite normal subgroup. Hence, this result is a generalization of the classical results by A. Karrass and D. Solitar in [28], H. Griffiths in [22], and B. Baumslag in [3]. A weaker statement with additional hypothesis was proved as Theorem 1(2) in [7].

Corollary 5.14 (Gaboriau) *Let G be a group with an infinite index, infinite, normal subgroup H with $\beta_1^{(2)}(H) < \infty$, then $\beta_1^{(2)}(G) = 0$.*

Remark 5.15 A generalization of Gaboriau's result to higher ℓ^2-Betti numbers was obtained by R. Sauer and the second author in [43]. There it was shown that for a normal subgroup $N \subset G$ with all $\beta_p^{(2)}(N) = 0$, for $p < q$, and $\beta_q^{(2)}(N)$ finite, it follows that $\beta_p^{(2)}(G) = 0$, for $p \leq q$. The proof uses a Hochschild-Serre spectral sequence for discrete measured groupoids. For more results in this direction, see [43].

6 Discrete measured groupoids

6.1 Infinite index subgroups

After the statement of Theorem 5.12, we outlined a proof in the presence of a descending chain of finite index subgroups. In this subsection, we prove that such a chain exists as soon as we pass to a suitable setting of discrete measured groupoids.

Lemma 6.1 *Let G be a countable discrete group and let $H \subset G$ be a subgroup of infinite index. There exists a standard probability space (X, μ) and an ergodic m.p. action of G on X, such that the restriction of the action to H has a continuum of ergodic components.*

Proof We set $X = \prod_{gH \in G/H} [0, 1]$ and let μ be the product of the Lebesgue measure. Then $G \curvearrowright X$ is ergodic, since $G \curvearrowright G/H$ is transitive with one infinite orbit. Moreover, the restriction of the action to H leaves the first factor invariant, and hence the space of ergodic components with respect to the H-action is continuous. □

The following lemma is another instance, where the flexibility of measure spaces and discrete measured groupoids allows for constructions which are not possible in the realm of groups.

Lemma 6.2 *Let G be a countable discrete group and let $H \subset G$ be a subgroup of infinite index. There exists a standard probability space (X, μ), on which G acts by m.p. Borel isomorphisms, such that the translation groupoid $X \rtimes G$ has finite index subgroupoids of arbitrary index which contain $X \rtimes H$.*

Proof Consider the space (X, μ) obtained from Lemma 6.1. Consider a partition $Y = \bigcup_{i=1}^{n} Y_i$ of the space of ergodic components with respect to the action H. Assume that $\mu(Y_i) = n^{-1}$, for all $1 \le i \le n$. Consider the subgroupoid $\mathcal{K} \subset \mathcal{G}$, which consists of those elements in $\mathcal{G}$, which preserve the partition of Y. Lemma 3.7 of [43] implies that the index of $\mathcal{K}$ in $\mathcal{G}$ is n. □

6.2 Notions of normality for groupoids

In [43], following the work of [18], the notion of strong normality of subgroupoids has been identified to be the right notion if one wants to construct quotient groupoids. For our purposes, a weaker notion of normality is of importance.

Definition 6.3 Let $(\mathcal{G}, \mu)$ be a discrete measured groupoid and let $\mathcal{R} \subset \mathcal{G}$ be a subgroupoid. The subgroupoid $\mathcal{R}$ is said to be s-normal, if for every local section ϕ of $\mathcal{G}$, the set $\phi \mathcal{R} \phi^{-1} \cap \mathcal{R}$ has infinite measure. The notion of ws-normality is defined similarly.

The following lemma is the analogue of Lemma 5.5 for discrete measured groupoids. The proof is straightforward and we leave it as an exercise.

Lemma 6.4 *Let $(\mathcal{G}, \mu)$ be a discrete measure groupoid, let $A \subset \mathcal{G}^0$ have positive measure and let $\mathcal{H} \subset \mathcal{K} \subset \mathcal{G}$ be subgroupoids. If $\mathcal{H} \subset \mathcal{G}$ is a s-normal inclusion, then $\mathcal{H}_A \subset \mathcal{G}_A$, $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{K} \subset \mathcal{G}$ are s-normal inclusions as well.*

In order to relate s-normality for groups to s-normality for groupoids, we need the following technical lemma.

Lemma 6.5 *Let G be an infinite countable discrete group which acts by m.p. Borel automorphisms on a probability space (X, μ). Let $A \subset X$ be a Borel subset such that $\mu(A) \neq 0$ then* $\limsup_{g \in G} \mu(A \cap gA) > 0$.

Proof Take $0 < \varepsilon < \mu(A)$ and suppose that $F = \{g \in G \mid \mu(A \cap gA) \geq \varepsilon\}$ is finite. Then for all $g \in G \setminus F$ we have $\mu(A^c \cap gA) \geq \mu(A) - \varepsilon$. Let $n \in \mathbb{N}$ be such that

$$n(\mu(A) - \varepsilon) > \mu(A^c)$$

and let $d > 0$ be such that

$$\binom{n}{2} d < n(\mu(A) - \varepsilon) - \mu(A^c).$$

Let $F' = \{g \in G \mid \mu(A \cap gA) \geq d\}$. If F' is finite, then we may take $g_1, \dots, g_n \in G \setminus F$ such that

$$g_j \notin \bigcup_{i<j} g_i F', \quad \text{for all } 1 \leq j \leq n.$$

Then

$$\begin{aligned} \sum_{1 \leq i < j \leq n} \mu(g_i A \cap g_j A) &\geq \sum_{1 \leq i \leq n} \mu(g_i A \cap A^c) - \mu\left(\bigcup_i g_i A \cap A^c\right) \\ &\geq n(\mu(A) - \varepsilon) - \mu(A^c) \\ &> \binom{n}{2} d. \end{aligned}$$

Hence there exists $i < j$ such that $\mu(A \cap g_i^{-1} g_j A) \geq d$. This contradicts the fact that $g_j \notin g_i F'$ and hence we must have that F' is infinite, i.e. $\limsup_{g \in G} \mu(A \cap gA) \geq d > 0$. □

The next theorem shows that an s-normal inclusion of groups leads to an s-normal inclusion of translation groupoids.

Theorem 6.6 *Let G be a countable discrete group and let $H \subset G$ be a ws-normal subgroup. Moreover, let (X, μ) be a standard probability space on which G acts by m.p. Borel automorphisms. Then, $X \rtimes H \subset X \rtimes G$ is an inclusion of a ws-normal subgroupoid.*

Proof It is enough to treat the case of an s-normal subgroup. Each local section of $X \rtimes G$ is a countable sum of sections of the form $\chi_A g$, where $A \subset X$ is Borel of positive measure and $g \in G$. It is enough to prove the assertion for local sections of the form $\chi_A g \colon g^{-1}A \to A$. In which case $(\chi_A g)(X \rtimes H)(\chi_A g)^{-1} \cap (X \rtimes H)$ contains

$$\left(g^{-1}A \times \{g\}\right) \cdot (X \times \{h\}) \cdot (A \times \{g^{-1}\}) = \left((A \cap (gh^{-1}g^{-1})A) \times \{ghg^{-1}\}\right),$$

for each $h \in H \cap g^{-1}Hg$. Hence its measure is infinite by applying Lemma 6.5 to the infinite group $H \cap g^{-1}Hg$. □

6.3 ℓ^2-invariants of discrete measured groupoids

We define the first complete ℓ^2-cohomology of $\mathcal{G}$ to be

$$H^1(\mathcal{G}, \mathcal{U}(\mathcal{G}, \mu)) = \underline{\mathrm{Ext}}^1_{\mathcal{R}(\mathcal{G},\mu)}(L^\infty(\mathcal{G}^0), \mathcal{U}(\mathcal{G})),$$

where $\underline{\mathrm{Ext}}$ denotes the derived functor from the abelian category of $L^\infty(X)$-complete $\mathcal{R}(\mathcal{G})$-modules. All these notions were explained in great detail in [43]. What is important for our purposes is the following concrete and familiar description of $H^1(\mathcal{G}, \mathcal{U}(\mathcal{G}, \mu))$ as a space cocycles modulo inner co-cycles.

A $\mathcal{G}$-cocycle with values in $\mathcal{U}(\mathcal{G}, \mu)$ is an assignment c of an element in $\mathcal{U}(\mathcal{G}, \mu)$ to every local section, such that

(1) $c(\phi) \in \mathrm{ran}(\phi)\mathcal{U}(\mathcal{G}, \mu)$,
(2) c is compatible with countable decompositions, and
(3) $c(\phi \circ \psi) = \phi \cdot c(\psi) + \mathrm{ran}(\phi \circ \psi) \cdot c(\phi)$.

A $\mathcal{G}$-cocycle with values in $\mathcal{U}(\mathcal{G}, \mu)$ is said to be *inner*, if there exists $\xi \in \mathcal{U}(\mathcal{G}, \mu)$, such that $c(\phi) = (\phi - \mathrm{ran}(\phi)) \cdot \xi$, for all local sections ϕ.

Clearly, the vector space of $\mathcal{G}$-cocycles forms a right module over $\mathcal{U}(\mathcal{G}, \mu)$. We denote this module by $Z^1(\mathcal{G}, \mathcal{U}(\mathcal{G}, \mu))$.

Proposition 6.7 *Let $(\mathcal{G}, \mu)$ be an infinite discrete measured groupoid. The following sequence of $\mathcal{U}(\mathcal{G}, \mu)$-modules is exact*

$$0 \to \mathcal{U}(\mathcal{G}, \mu) \to Z^1(\mathcal{G}, \mathcal{U}(\mathcal{G}, \mu)) \to H^1(\mathcal{G}, \mathcal{U}(\mathcal{G}, \mu)) \to 0.$$

Proof The proof follows the standard arguments in group cohomology, which are used to identify the first cohomology with the space of co-cycles modulo inner co-cycles. □

Lemma 6.8 *Let G be a countable discrete group and let (X, μ) be a probability space, on which G acts by m.p. Borel automorphisms. Then,*

$$\beta_1^{(2)}(X \rtimes G, \mu) = \beta_1^{(2)}(G).$$

Proof D. Gaboriau found this result for free actions in his groundbreaking work [19]. Later, again through a process of algebraization, it turned out that freeness is not needed. A proof can be found in [46] or [43]. □

6.4 The analogue of Theorem 5.6 for groupoids

We are now proving the analogue of Theorem 5.6 in the setup of discrete measured groupoids. In view of the remarks after the statement of Theorem 5.12, this finishes the proof of Theorem 5.12.

Theorem 6.9 *Let $(\mathcal{G}, \mu)$ be a discrete measured groupoid and let $\mathcal{H}$ be a ws-normal subgroupoid. Then the restriction map*

$$H^1(\mathcal{G}, \mathcal{U}(\mathcal{G}, \mu)) \to H^1(\mathcal{H}, \mathcal{U}(\mathcal{G}, \mu))$$

is injective.

Proof For simplicity we will restrict to the case when $\mathcal{H}$ is a s-normal subgroupoid. Let c be a $\mathcal{G}$-cocycle with values in $\mathcal{U}(\mathcal{G}, \mu)$ and suppose that c is inner when restricted to $\mathcal{H}$. By subtracting an inner cocycle we may assume that $c(\phi) = 0$ for all local sections of $\mathcal{H}$.

Let ψ be a local section for $\mathcal{G}$, and let p be the maximal projection in $L^\infty(\mathcal{G}^0)$ such that $pc(\psi) = 0$, note that $p \geq 1 - \text{ran}(\psi)$. Set $\chi_A = 1 - p$. If $\chi_A \neq 0$, then by Lemma 6.4 the inclusion $\mathcal{H}_A \subset \mathcal{G}_A$ is also s-normal and thus the set $(\chi_A\psi)^{-1}\mathcal{H}_A(\chi_A\psi) \cap \mathcal{H}_A$ has infinite measure. Thus there exist local sections ϕ_n for $\mathcal{H}_A$ which have large support (i.e. $\liminf_{n\to\infty} \mu(\text{ran}(\phi_n)) \neq 0$), converge weakly to 0 as partial isometries acting on $L^2(L(\mathcal{G}, \mu))$ and such that $(\chi_A\psi)^{-1}\phi_n\chi_A\psi$ is again a local section for $\mathcal{H}_A$. By the Banach-Alaoglu theorem we may take a subsequence and suppose that $\text{ran}(\phi_n)$ converges weakly to a non-zero positive element $x \leq \chi_A$.

Since $L^2(L(\mathcal{G}, \mu))$ is rank dense in $\mathcal{U}(\mathcal{G}, \mu)$ we may use the normality of the action and the cocycle relation to show that for a $\|\cdot\|_2$-dense collection of vectors $\xi \in L^2(L(\mathcal{G}, \mu))$ we have

$$\begin{aligned}
\langle xc(\psi), \xi\rangle &= \lim_{n\to\infty} \langle r(\phi_n)c(\psi), \xi\rangle \\
&= \lim_{n\to\infty} \langle c(\psi(\chi_A\psi)^{-1}\phi_n\chi_A\psi), \xi\rangle \\
&= \lim_{n\to\infty} \langle \phi_n c(\psi), \xi\rangle = 0,
\end{aligned}$$

hence $xc(\psi) = 0$ contradicting the maximality of p because $0 \lneq x \leq \chi A$. Thus we must have that $c = 0$ which completes the proof. □

7 Applications

In this section we collect applications of Theorems 5.6 and 5.12. In particular, we find upper bounds on the first ℓ^2-Betti number of a group in terms of the first ℓ^2-Betti numbers of the constituents of the group. We reprove and generalize several results about non-existence of certain infinite index subgroups in certain situations.

7.1 Boundedly generated groups

Let G be a discrete group and $G_1, \dots, G_n$ be sub-groups. The group G is said to be *boundedly generated* by the subgroups $G_1, \dots, G_n$, if there exists an integer $k \in \mathbb{N}$, such that every element in G is a product of less than k elements from $G_1, \dots, G_n$. The following theorem is based on an idea of A. Ioana.

Proposition 7.1 *Let G be a countable discrete group. If G is boundedly generated by subgroups $G_1, \dots, G_n$, then the following relation holds*:

$$\beta_1^{(2)}(G) \leq \sum_{i=1}^{n} \beta_1^{(2)}(G_i).$$

Proof It suffices to show that the restriction map

$$H^1(G, \mathcal{U}G) \to \bigoplus_{i=1}^{n} H^1(G_i, \mathcal{U}G)$$

is injective. Indeed, $\dim_{LG} H^1(G_i, \mathcal{U}G) = \beta_1^{(2)}(G_i)$, and therefore:

$$\dim_{LG} \bigoplus_{i=1}^{n} H^1(G_i, \mathcal{U}G) = \sum_{i=1}^{n} \beta_1^{(2)}(G_i).$$

Let $c \colon G \to \mathcal{U}G$ be a cocycle which is in the kernel. The cocycle c is inner on G_i. Being inner, we find a projection $p_i \in RG$ with $\tau(p_i^{\perp}) \leq \epsilon/n$, such that $cp_i \colon G_i \to \mathcal{U}G$ will be uniformly bounded in the 2-norm. We consider $p = \inf p_i$. Since G is boundedly generated by the $G_1, \dots, G_n$, we conclude, using the cocycle relation, that cp is uniformly bounded in the 2-norm on G. Hence it is inner and there exists $\xi_\epsilon \in \ell^2 G$ such that $c(g)p = (g-1)\xi_\epsilon p$, for all $g \in G$. It follows that ξ_ϵ converges in rank metric to some vector $\xi \in \mathcal{U}G$ and $c(g) = (g-1)\xi$, for all $g \in G$. This finishes the proof. □

This generalizes Proposition 5 in [1]. A particular case of the preceding theorem is $G = SL_n(\mathbb{Z})$, for $n \geq 3$, which is boundedly generated by copies of $\mathbb{Z}$.

7.2 Certain groups generated by a family of subgroups

Let G be a countable discrete group and let $\{G_\alpha \mid \alpha \in V\}$ be a family of subgroups. We define a graph $V_G = (V, E)$ with vertices V and an edge between α and β, if and only if the intersection $G_\alpha \cap G_\beta$ is infinite.

Proposition 7.2 *Let G be a group and let $\{G_\alpha \mid \alpha \in V\}$ be a family of subgroups. Assume that*

(i) *$\bigcup_{\alpha\in V} G_\alpha$ generates G as a group, and*
(ii) *the graph V_G is connected.*

Then,

$$\beta_1^{(2)}(G) \le \sum_{\alpha\in V} \beta_1^{(2)}(G_i).$$

Proof Following the ideas in the proof of Theorem 7.1, one can show that the restriction map

$$H^1(G, \mathcal{U}G) \to \bigoplus_{\alpha\in V} H^1(G_\alpha, \mathcal{U}G)$$

is injective. □

7.3 Limit groups

The notion of *limit groups* or *fully residually free groups* was introduced by Z. Sela in [44]. A countable discrete group Γ is said to be a limit group, if it is finitely generated and for every finite set $T \subset \Gamma$, there exists a group homomorphisms $\phi\colon \Gamma \to F_2$, which is injective on T. It was shown by C. Champetier and V. Guirardel in [12], that limit groups are precisely the limits of free groups in R. Grigorchuk's space of marked groups. Moreover, M. Pichot showed in [37], that a semi-continuity property holds for ℓ^2-Betti numbers in the space of marked groups. In particular, as noted in [37]:

$$\beta_1^{(2)}(\Gamma) \ge 1, \quad \text{for all non-abelian limits groups } \Gamma.$$

Hence, our results apply and in particular a generalization of Theorem 3.1 and Corollary 3.4 in [8] follows from Theorems 5.6 and 5.12.

Another implication of our results are Theorems *B* and *C* of I. Kapovich in [26], which were partially withdrawn in [27]. Indeed Theorem *B* follows from Corollary 5.13. For convenience, we restate Theorem *C* of [26] in the generality to which we can extend this result:

Proposition 7.3 *Let G be a countable discrete group and let $H, K \subset G$ be finitely generated infinite subgroups. Assume that $[H : H \cap K]$ and $[K : H \cap K]$ are finite. If $\beta_1^{(2)}(\langle H, K\rangle) \neq 0$, then the inclusion $H \cap K \subset \langle H, K\rangle$ has finite index.*

Proof We show that $H \subset \langle H, K\rangle$ is a s-normal inclusion. Note that $(H \cap K) \cap (H \cap K)^g$ is of finite index in $H \cap K$ for all elements $g \in \langle H, K\rangle$. Hence $H \cap H^g$ is infinite, for all $g \in \langle H, K\rangle$. Since H is s-normal and finitely generated, $[\langle H : K\rangle, H]$ has to be finite, by Corollary 5.13. This finishes the proof. □

7.4 Power-absorbing subgroups

The following definition has been studied in [22, 28, 33]. We reprove most of the results and generalize to groups with non-vanishing first ℓ^2-Betti number.

Definition 7.4 Let G be a torsionfree discrete countable group. A subgroup $H \subset G$ is called *power-absorbing*, if for every $g \in G$, there exists $n \in \mathbb{N}$, such that $g^n \in H$.

It has been studied under which conditions a finitely generated normal power-absorbing subgroup has to be of finite index.

Proposition 7.5 *Let G be a torsionfree discrete countable group and $H \subset G$ be a power-absorbing finitely generated subgroup. If $\beta_1^{(2)}(G) \neq 0$, then the subgroup H has to be of finite index.*

Proof Clearly, H is s-normal in G. Hence, the claim follows from Corollary 5.13. □

7.5 Groups measure equivalent to free groups

In [20], D. Gaboriau investigates groups which are measure equivalent to free groups. Examples of such groups include amenable groups, lattices in $SL(2, \mathbb{R})$, elementarily free groups [9], and is stable under taking free products. Gaboriau in particular shows that this class is stable under taking subgroups and that a group in this class with vanishing first ℓ^2-Betti number is amenable.

Recall that if H is a subgroup of G then the *commensurator subgroup* $\mathrm{comm}_G(H)$ of H in G is the group of all $g \in G$ such that $H \cap H^g$ has finite index in H and H^g. From the above proof we can show the following:

Proposition 7.6 *Let G be measure equivalent to a free group and H a finitely generated subgroup of G then either* $\mathrm{comm}_G(H)$ *is amenable or else H has finite index in* $\mathrm{comm}_G(H)$.

We can also show that groups in this class have unique maximal amenable extensions:

Proposition 7.7 *Let G be measure equivalent to a free group and H an infinite amenable subgroup, then H has a unique maximal amenable extension.*

Proof Suppose that H_1, and H_2 are amenable subgroups which contain H, we must show that $\langle H_1, H_2\rangle$ is also amenable. As H is infinite it follows from Theorem 7.2 that

$$\beta_1^{(2)}(\langle H_1, H_2\rangle) = 0$$

and hence $\langle H_1, H_2\rangle$ is indeed amenable. □

Note that the above theorems will also hold for the class of groups which admit a proper ℓ^2-co-cycle. Also note that the above theorem should have a von Neumann algebra analog. Specifically, it should be the case that if $Q \subset L\mathbb{F}_2$ is a diffuse amenable von Neumann subalgebra of the free group factor, then Q has a unique maximal amenable extension in $L\mathbb{F}_2$. Some evidence for this appears in [25, 36], and in [35].

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