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GROUP-INVARIANT SOLUTIONS OF DIFFERENTIAL EQUATIONS*

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Abstract. We introduce the concept of a *weak symmetry group* of a system of partial differential equations, that generalizes the "nonclassical" method introduced by Bluman and Cole for finding group-invariant solutions to partial differential equations. Given any system of partial differential equations, it is shown how, in principle, to construct group-invariant solutions for *any* group of transformations by reducing the number of variables in the system. Conversely, *every* solution of the system can be found using this reduction method with some weak symmetry group.

Key words. symmetry group, group-invariant solution, partial differential equation, weak symmetry group

AMS(MOS) subject classifications. 58G35, 35C05, 22E70

1. Background. By a classical or *strong symmetry group* of a system of partial differential equations we mean a continuous group of transformations acting on the space of independent and dependent variables which transforms solutions of the system to other solutions. As is well known ([4], [11], [13]) solutions of a system of partial differential equations which are invariant under a continuous symmetry group are all found by solving a reduced system of differential equations involving fewer independent variables. Included among these solutions are the important classes of traveling wave solutions and similarity solutions, as well as many other explicit solutions of direct physical significance. What is not well known is that the basic reduction method originated with Sophus Lie himself. Lie was interested in solutions to systems of partial differential equations invariant under groups of contact transformations, but his results include the local versions of the present-day reduction theorem. In § 65 of [8], Lie proves that the solutions to a partial differential equation in two independent variables, which are invariant under a one-parameter symmetry group, can all be found by solving a "reduced" ordinary differential equation. The generalization to systems of partial differential equations, invariant under multi-parameter groups, is stated and proved in § 76 of the same paper, but, as far as we are aware, has never before been referred to in any of the literature on this subject!

In [3], Bluman and Cole proposed a generalization of Lie's method for finding group-invariant solutions, which they named the "nonclassical" method. The method also appears in Ames [2, § 2.10]. In this approach one replaces the conditions for the invariance of the given system of differential equations by the weaker conditions for the invariance of the combined system consisting of the original differential equations along with the equations requiring the group-invariance of the solutions. By this device, a much wider class of groups is potentially available, and hence there is the possibility of further kinds of explicit solutions being found by the same reduction techniques. In practice, however, the determining equations for a nonclassical symmetry group of Bluman and Cole type may be too difficult to explictly solve; nevertheless, as is shown here, even finding particular nonclassical groups can lead to new explicit solutions of the system. (Admittedly, at first sight, the fact that one can *expand* the range of possible symmetry groups by adding in more equations seems contradictory. The explanation from a geometrical point of view is that the additional equations restrict us to a smaller

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subclass of solutions, and, while the entire set of solutions may not be invariant under some transformation group, an appropriately smaller subclass may be invariant. From a more technical standpoint, the additional equations can be seen to reduce the total number of constraints expressed by the defining equations for the group, and hence allow a much larger class of groups which satisfy them.)

The purpose of this paper is to show that the generalization of Bluman and Cole is in fact far broader than was previously thought; in principle, not only can equations for invariant solutions corresponding to arbitrary transformation groups be found by the reduction method, but every possible solution of the system can be found by using some such group! In other words, there are *no* conditions that need to be placed on the transformation group in order to apply the basic reduction procedure. In light of this result, we can define two types of "symmetry groups" of a system of partial differential equations.

DEFINITION. Let Δ be a system of partial differential equations. A *strong symmetry group* of Δ is a group of transformations G on the space of independent and dependent variables which has the following two properties:

(a) The elements of G transform solutions of the system to other solutions of the system.

(b) The G-invariant solutions of the system are found from a reduced system of differential equations involving a fewer number of independent variables than the original system Δ. (The degree of reduction is determined by the dimension of the orbits of G; see § 3.)

A *weak symmetry group* of the system Δ is a group of transformations which satisfies the reduction property (b), but no longer transforms solutions to solutions.

In addition, one can extend these concepts to include groups of generalized symmetries (also known as Lie-Bäcklund transformations), [11], leading to both weak and strong generalized symmetry groups of the system Δ.

Thus, for the problem of constructing explicit solutions of partial differential equations, a strong symmetry group can be employed in two distinct ways—either by transforming known solutions by group elements, or, by reduction, constructing invariant solutions—whereas for a weak symmetry group only the latter option is available. (However, a weak symmetry group *can* map subclasses of solutions to solutions, an aspect of the subject we hope to fully investigate in a future paper.)

In this paper, we are only concerned with groups of point transformations, leaving aside problems involving generalized symmetry groups. Strong generalized symmetry groups are those used in the general version of Noether's theorem and in the study of soliton equations [11, Chap. 5]. The theory of weak generalized symmetry groups is equivalent to the recently introduced concept of a differential equation with *side conditions*, which is discussed in detail in [12]. As is shown there, besides the group-invariant solutions of the type discussed here, the solutions obtained through weak generalized symmetry groups include those arising from separation of variables, partially-invariant solutions [13], and many others. Fokas [5], has used special types of weak generalized symmetries, under the name "conditionally admissible operators," for constructing Bäcklund transformations of nonlinear partial differential equations.

Returning to point transformational groups, our basic result is that every group of transformations is a weak symmetry group, and, conversely, every solution can be obtained from some weak symmetry group. There is, however, one important caveat. Although one can apply the general reduction procedure for any transformation group whatsoever, the resulting system of differential equations may turn out to be incompatible, and so there will not be any invariant solutions for the given group. (This can happen even in the case of strong symmetry groups; see [11, Chap. 3].) Therefore, one should distinguish those symmetry groups which have some invariant solutions from the others. The procedure for determining whether or not a given group is of this class is straightforward and described here; however, for a given system of differential equations, the determination of the most general weak symmetry group which possesses invariant solutions is a very difficult, if not impossible, problem.

The proposed reduction method is illustrated by a number of examples, including the heat equation, a nonlinear wave equation and a version of the Boussinesq equation. The paper is divided into two parts: §§ 2 and 3 present the method and illustrative examples in a form that can be appreciated by the reader whose primary interests are in applications, while §§ 4 and 5 recapitulate the theory of symmetry groups and group-invariant solutions, and prove the basic theorems that rigorously justify the method. The essential computational techniques which one needs in order to apply our method to specific partial differential equations *all* appear in the first half of the paper, with the second half being devoted to the more rigorous, mathematical aspects of the analysis. Finally, § 6 draws some general conclusions and outlines some further directions for research that are suggested by our approach.

2. Illustrative examples. One of the annoying features of the nonclassical method as presented in the above-mentioned references has been that all the solutions that have been found could, in fact, already have been found by the classical group-invariant reduction method, leading one to question whether this generalization of the classical method is, in fact, of any real use. Therefore, to illustrate the method, we begin with an example where this is not the case. Consider the nonlinear wave equation

$$u_{tt} = u \cdot u_{xx}, \tag{1}$$

whose classical symmetry group consists solely of translations in x and t, and the two-parameter scaling group $(x, t, u) \mapsto (\lambda\mu x, \lambda t, \mu^2 u)$, λ, $\mu > 0$, (cf. [4, p. 301]). The one-parameter group G of Galilean boosts

$$(x, t, u) \mapsto (x + 2\varepsilon t + \varepsilon^2, t + \varepsilon, u + 8\varepsilon t + 4\varepsilon^2),$$

where $\varepsilon \in \mathbb{R}$ is the group parameter, does not appear among the classical symmetries, and so is not a candidate for the usual method of finding group-invariant solutions. Nevertheless, we can find G-invariant solutions as follows. The infinitesimal generator of G is the vector field

$$\mathbf{v} = 2t\partial_x + \partial_t + 8t\partial_u \qquad (\partial_x \equiv \partial/\partial x, \text{etc.}),$$

so a function $u = f(x, t)$ is invariant under the group G if and only if it satisfies the first order partial differential equation

$$8t = 2tu_x + u_t. \tag{2}$$

Using the basic infinitesimal method of Lie and Ovsiannikov (see below), one easily checks that even though the wave equation (1) is not invariant under G, the combined pair of differential equations (1)–(2) *is* invariant. This is precisely what is needed to apply the nonclassical method, and hence we can find G-invariant solutions to (1) by solving an ordinary differential equation. According to the basic method, [3], we first find the independent invariants of the group, which are

$$y = x - t^2 \quad \text{and} \quad w = u - 4t^2.$$

Treating w as a function of y, so

$$u = 4t^2 + w(x - t^2),$$

we use the chain rule to compute formulae for the relevant derivatives of u in terms of derivatives of w with respect to y:

$$u_{tt}=8+4t^2w_{yy}-2w_y, \qquad u_{xx}=w_{yy}.$$

Substituting these into (1),we see that w must satisfy the reduced ordinary differential equation

$$ww_{yy}+2w_y=8$$

if u is to be a solution to (1). This ordinary differential equation can be integrated by Lie's method for ordinary differential equations (cf. [11, § 2.5]), using the obvious translational symmetry group: we let $z=w_y$, and treat z as a function of w. The resulting equation

$$wzz_w+2z=8$$

readily separates, leading to the implicit equation

$$(w_y-4)^{-2}\,e^{-w_y/2}=cw,$$

where c is an arbitrary constant of integration. For each solution $w=h(y)$ of this latter first order equation, we obtain an explicit G-invariant solution $u=4t^2+h(x-t^2)$ of the original equation (1): most of these do not appear among the group-invariant solutions computed using the ordinary symmetry groups of (1), and are thus genuinely new invariant solutions not obtainable by the classical method.

A similar construction is valid in the case of the Boussinesq equation

$$u_{tt}=u_{xx}+\beta(u^2)_{xx}+\gamma u_{xxxx}, \tag{3}$$

β, γ constant, which is a soliton equation arising in water waves and plasma physics. For $\gamma\neq 0$, its classical symmetry group consists of just translations in x and t and the group generated by $x\partial_x+2t\partial_t-(2u+\beta^{-1})\partial_u$. Let a be a constant and consider the one-parameter Galilean group

$$(x,t,u)\mapsto(x-2a\beta\varepsilon t-a\beta\varepsilon^2,\ t+\varepsilon,\ u+4a^2\beta\varepsilon t+2a^2\beta\varepsilon^2),$$

which is generated by

$$\mathbf{v}=-2a\beta t\partial_x+\partial_t+4a^2\beta t\partial_u.$$

For $a\neq 0$ this is not a symmetry group; nevertheless there do exist group-invariant solutions to the Boussinesq equation. The independent invariants of this group are $y=x+a\beta t^2$ and $w=u-2a^2\beta t^2$, so any invariant solution has the form $w=h(y)$, or, equivalently,

$$u=2a^2\beta t^2+h(x+a\beta t^2).$$

We are thus led to the ansatz originally proposed by Tomotika and Tamada [15] for a nonlinear wave equation arising in transonic gas flow, which corresponds to the special case $\gamma=0$, and extended to the Boussinesq equation in [10], [14]. To find the ordinary differential equation satisfied by w, we compute the derivatives of u in terms of those of w:

$$u_{tt}=4a^2\beta+4a^2\beta^2t^2w_{yy}+2a\beta w_y, \qquad u_{xx}=w_{yy},$$
$$(u^2)_{xx}=4a^2\beta t^2w_{yy}+(w^2)_{yy}, \qquad u_{xxxx}=w_{yyyy}.$$

Therefore, w satisfies the fourth order ordinary differential equation

$$\gamma w_{yyyy}+2\beta ww_{yy}+2\beta w_y^2-2a\beta w_y=4a^2\beta,$$

and every solution of this will provide an invariant solution of the Boussinesq equation. Integrating once, and using the substitution

$$\tilde{w} = w + ay,$$

we are left with the interesting third order equation

$$\gamma \tilde{w}_{\tilde{y}\tilde{y}\tilde{y}} + 2\beta \tilde{w}\tilde{w}_{\tilde{y}} - 2a\beta \tilde{y}\tilde{w}_{\tilde{y}} - 4a\beta \tilde{w} = 0,$$

where $\tilde{y}$ is a translate of y depending on the constant of integration. This last equation arises in the study of the scale-invariant solutions to the Korteweg–de Vries equation, and is closely related to the second Painlevé transcendent [1], [9]. Hence there are solutions of the Boussinesq equation which can be written in terms of the solutions to this Painlevé equation. (The appearance of Painlevé transcendents, cf. [6], for the reduced equations for invariant solutions for the Boussinesq equation should come as no surprise to readers familiar with the Painlevé conjecture (cf. [1], [9], [17]) for soliton equations.)

3. The reduction method. Let us now review the original Lie method for finding group-invariant solutions of partial differential equations in order to see how to generalize it to other types of transformation groups. (For ease of exposition, we will gloss over some of the more technical points in this construction; see [11], [13] for a more rigorous discussion.) Consider an nth order system of differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l \tag{4}$$

in p independent variables $x = (x^1, \cdots, x^p)$ and q dependent variables $u = (u^1, \cdots, u^q)$. Here $u^{(n)}$ stands for all the derivatives of the dependent variables u with respect to the independent variables x up to order n, and the functions Δ_ν are, for simplicity, assumed to depend smoothly on their arguments for x, u in some open set M of the total space $X \times U = \mathbb{R}^p \times \mathbb{R}^q$ of independent and dependent variables.

Let G be a local group of transformations acting on $M \subset X \times U$. The group elements act on functions $u = f(x)$ by pointwise transformation of their graphs. Specifically, if g is a group transformation, and $u = f(x)$ is any function whose graph $\Gamma_f \equiv \{(x, f(x))\}$ lies in the domain of g, then the transformed function $\tilde{f} \equiv g \cdot f$ has graph $\Gamma_{\tilde{f}} = g \cdot \Gamma_f = \{g \cdot (x, f(x))\}$. (It may be necessary to restrict the domain of definition of f in order that $g \cdot f$ be well defined.) The transformation group G is said to be a *symmetry group* of the system of differential equations Δ if each group element $g \in G$ transforms solutions of Δ to other solutions to Δ. (The basic Lie–Ovsiannikov infinitesimal method discussed below allows one to explicitly compute the most general (connected) symmetry group of practically any given system of differential equations.)

If G is a symmetry group of the system of differential equations Δ, then the solutions which are actually invariant under G are of especial interest. By definition, a function $u = f(x)$ is called *G-invariant* if all the transformations in G leave it unchanged, so whenever $g \in G$, $g \cdot f = f$ on their common domains of definition; equivalently, the graph of f is a (locally) G-invariant subset of M. Lie's reduction method for finding G-invariant solutions to the system Δ relies on the introduction of invariants of the group action. Here a real-valued function $\eta(x, u)$ is called an *invariant* of G if it is unchanged by the group action: $\eta(g \cdot (x, u)) = \eta(x, u)$ for all $(x, u) \in M$ and all $g \in G$ such that $g \cdot (x, u)$ is defined. Assume that G has a complete set of globally defined, functionally independent invariants

$$y^1 = \eta^1(x, u), \cdots, y^{p-r} = \eta^{p-r}(x, u), \quad w^1 = \zeta^1(x, u), \cdots, \quad w^q = \zeta^q(x, u),$$

an assumption that can always be realized by suitably shrinking the domain of definition M of the group action. (In particular, this requires all the orbits of G to have dimension r.) The invariants are parceled into two sets, with $y=(y^1,\cdots,y^{p-r})$ representing new independent variables and $w=(w^1,\cdots,w^q)$ representing new dependent variables. Each G-invariant function $u=f(x)$ on M can be re-expressed in terms of the invariants of G:

$$\zeta(x,u)=h[\eta(x,u)], \tag{5}$$

and hence is uniquely determined by some function $w=h(y)$ involving the new variables.

The reduced system of differential equations Δ/G for the G-invariant solutions to Δ will involve just the new variables y, w formed from the invariants of G. To find Δ/G we need to express the derivatives of u with respect to x in terms of derivatives of w with respect to y. We split the variables $x=(x^1,\cdots,x^p)$ into *parametric* $\tilde{x}=(x^{i_1},\cdots,x^{i_r})$ and *principal* $\hat{x}=(x^{j_1},\cdots,x^{j_{p-r}})$ subsets chosen so that the system of $p+q-r$ equations

$$y=\eta(x,u),\qquad w=\zeta(x,u)$$

can be solved, via the implicit function theorem, for the variables $\hat{x}$ and u in terms of the new variables y and w and the remaining parametric variables $\tilde{x}$:

$$\hat{x}=\gamma(\tilde{x},y,w),\qquad u=\delta(\tilde{x},y,w). \tag{6}$$

As in the above examples, we can differentiate these expressions with respect to x to find corresponding formulae for the nth order derivatives of u,

$$u^{(n)}=\delta^{(n)}(\tilde{x},y,w^{(n)}) \tag{7}$$

in terms of y, w, the derivatives of w with respect to y up to order n, plus the ubiquitous parametric variables $\tilde{x}$. The formulae (6)–(7) are then substituted into the original system (4), leading to a system of equations

$$\tilde{\Delta}_\nu(\tilde{x},y,w^{(n)})=0,\qquad \nu=1,\cdots,l \tag{8}$$

still involving y, w, derivatives of w and the parametric variables $\tilde{x}$. Provided G is a symmetry group of the system, it can be proved that this latter system is, in fact, algebraically equivalent to a system of differential equations

$$(\Delta/G)_\nu(y,w^{(n)})=0,\qquad \nu=1,\cdots,l \tag{9}$$

in y and w that no longer involves the parametric variables. (For example, Δ_ν might be the product of a nonvanishing function of $\tilde{x}$ with a function of y, $w^{(n)}$, in which case $(\Delta/G)_\nu$ would be the latter function.) The system of differential equations (9), which has r fewer independent variables, constitutes the reduced system Δ/G. Every solution $w=h(y)$ to (9) gives rise to a G-invariant solution $u=f(x)$ to Δ, determined implicitly from (5), and, moreover, every G-invariant solution to Δ can be thus found.

If G is not a symmetry group to the original system (4), we can still ask whether Δ has any G-invariant solutions. The same reduction procedure, using the independent invariants y, w of G, can still be applied, resulting in a system (8) involving the chosen parametric variables $\tilde{x}$. At this point, there are two possibilities. In the first, which is the nonclassical method as envisaged by Bluman and Cole, it happens that even though G is not a symmetry group to Δ, nevertheless the system (8) is still algebraically equivalent to a system of differential equations (9) involving only the new variables y, w and their derivatives. (This is precisely what happens in the examples of § 2.) As in the case when G is a symmetry group, every solution of the reduced system gives rise to a G-invariant solution to Δ. The second possibility is that the system (8) depends in some essential way on the parametric variables $\tilde{x}$. In this case we can still reduce (8) to a system of differential equations for $w(y)$ by requiring y, $w^{(n)}$ to satisfy all possible conditions so that (8) becomes an identity in $\tilde{x}$. (These can be found, for instance, by expanding (8) in powers of $\tilde{x}$.) We are then left with an overdetermined system of reduced differential equations for w as a function of y; any solution will, just as before, lead to a G-invariant solution to the original system (4). Of course, the last reduced system might be incompatible, implying that there are *no* G-invariant solutions to Δ, so an important question is which transformation groups lead to compatible reduced equations. (It is possible, though unlikely, that even a classical symmetry group can lead to incompatible reduced equations: consider the translation symmetry group $(x, y, u) \mapsto (x+\varepsilon, y+\varepsilon, u)$ for the equation $u_x + u_y = 1$.)

Example. Consider the heat equation

$$u_t = u_{xx}. \tag{10}$$

The one-parameter group

$$G: (x, t, u) \mapsto (\lambda x, \lambda^{-1} t, u + x^3(\lambda^3 - 1)), \qquad \lambda > 0$$

is not a classical symmetry group of the heat equation. Nor is it of the form amenable to the nonclassical method given by Bluman and Cole [3]. Indeed, its infinitesimal generator is the vector field

$$\mathbf{v} = x\partial_x - t\partial_t + 3x^3\partial_u,$$

so, using their notation on p.1041, we would have $X = x/t$, $U = 3x^3/t$ (cf. their equation (90)), but these two functions do not satisfy their defining equations (94)-(96). Nevertheless, there do exist G-invariant solutions of the heat equation, and we can construct them as follows.

The basic invariants of G are the functions

$$y = xt, \qquad w = u - x^3,$$

so the most general G-invariant function is of the form $w = h(y)$, or

$$u = x^3 + h(xt).$$

Let us treat x as the parametric variable, and find expressions for t, u and derivatives of u in terms of y, w and derivatives of w with respect to y. We have

$$t = \frac{y}{x}, \qquad u = x^3 + w,$$

and, using the chain rule,

$$u_t = xw_y, \qquad u_{xx} = 6x + x^{-2}y^2 w_{yy}.$$

Substituting these latter expressions into the heat equation (10), we are left with the equation

$$xw_y = 6x + x^{-2}y^2 w_{yy}, \tag{11}$$

which is equation (8) in this particular example. If G were a classical symmetry group, or a nonclassical group of the type considered by Bluman and Cole, then (11) would be equivalent to an ordinary differential equation just involving w and y. In the present case, though, we treat (11) as an identity in the parametric variable x; separating the coefficients of x and x^{-2}, we are left with *two* ordinary differential equations

$$w_y = 6, \qquad w_{yy} = 0, \tag{12}$$

which must be simultaneously satisfied for a G-invariant solution to the heat equation to exist. In this case, the equations are compatible, with general solution $w = 6y + c$, where c is an arbitrary constant. Thus we obtain a one-parameter family of G-invariant solutions to the heat equation

$$u = x^3 + 6xt + c, \tag{13}$$

which do not appear among the classical group-invariant solutions (although they are, of course, linear combinations of two such solutions).

Example. The heat equation example, while reasonably easy to understand, leads to a fairly trivial family of solutions; even though they do not explicitly appear among the group-invariant solutions, they can be easily derived from them by superposition. We therefore give an illustration of the application of the method to a nonlinear partial differential equation and construct some new solutions to the Boussinesq equation (3). First consider the scaling group

$$(x, t, u) \mapsto (\lambda x, \lambda t, u), \qquad \lambda > 0,$$

which is *not* a symmetry group of the equation unless $\gamma = 0$. The similarity variables (invariants) are $y = x/t$ and $w = u$ (at least when $t > 0$). To apply the above reduction method, we view w as a function of y only, and substitute into the equation (3). We find the equation

$$t^{-2}(y^2 w_{yy} + y w_y) = t^{-2}(w_{yy} + \beta(w^2)_{yy}) + \gamma t^{-4} w_{yyyy}, \tag{14}$$

in which we have chosen t to be the parametric variable. Again, when $\gamma \neq 0$, (14) (which corresponds to (8) in this example) is not equivalent to an ordinary differential equation for $w(y)$ which does not involve the parametric variable t, so this scaling group is not even a symmetry group of the type discussed by Bluman and Cole. Nevertheless, there still exist similarity solutions of the Boussinesq equation invariant under this group. We note that (14) requires that the pair of ordinary differential equations

$$y^2 w_{yy} + y w_y = w_{yy} + \beta(w^2)_{yy}, \qquad w_{yyyy} = 0$$

be satisfied. From the latter equation, we see that

$$w = ay^3 + by^2 + cy + d$$

must be a cubic function of y; substituting into the former, we find that $a = c = 0$, and either $b = (3\beta)^{-1}$, $d = -(2\beta)^{-1}$, or else $b = 0$, and d is arbitrary. The similarity solutions are thus the constants and the special solution $u = x^2/(3\beta t^2) - 1/(2\beta)$. The lesson here is that even though a partial differential equation may not admit any scaling groups of symmetries, nevertheless there still may exist similarity solutions to it, and these can be found by the present reduction method.

In § 2 we obtained solutions of the Boussinesq equation which are invariant under a certain group of Galilean boosts by using Bluman and Cole's nonclassical method. We find further Galilean-invariant solutions for a different one-parameter group in which the dependent variable u is unaffected by the boost. Consider the group with infinitesimal generator $\mathbf{v} = -2at\partial_x + \partial_t$. Invariants are provided by $y = x + at^2$ and u itself. Treating u as a function of y, we are led to the equation

$$4a^2t^2 u_{yy} + 2a u_y = u_{yy} + \beta(u^2)_{yy} + \gamma u_{yyyy}, \tag{15}$$

with t again the parametric variable. As this is not equivalent to a single equation not involving t, *this* Galilean group is not of Bluman and Cole type. However, as above, there still exist invariant solutions. We need only solve the pair of equations

$$u_{yy}=0, \qquad 2au_y = u_{yy}+\beta(u^2)_{yy}+\gamma u_{yyyy},$$

stemming from (15). The general solution is easily seen to be $u=a\beta^{-1}y+c$, where c is an arbitrary constant, leading to a second family of Galilean-invariant solutions $u=a\beta^{-1}(x+at^2)+c$ of the Boussinesq equation.

The difference between the three methods for finding invariant solutions to differential equations is made crystal clear if we look at just one-parameter groups. Suppose for simplicity that we have a single partial differential equation in two independent variables and one dependent variable. As long as the infinitesimal generator $\mathbf{v}$ of G does not vanish, we can always locally choose new coordinates (s, y, w) such that $\mathbf{v}=\partial_s$ (and hence y, w are the independent invariants of G), so that G becomes a translation group $(s, y, w)\mapsto(s+\varepsilon, y, w)$ in the new coordinates. In the new variables, the differential equation has the form

$$\Delta(s, y, w, w_s, w_y, w_{ss}, w_{sy}, w_{yy}, \cdots)=0.$$

There are now three distinct cases pertaining to the construction of G-invariant solutions to Δ.

Case 1. If G is a strong symmetry group of Δ, then Δ is equivalent to an equation which does not depend explicitly on s:

$$\Delta(y, w, w_s, w_y, w_{ss}, w_{sy}, w_{yy}, \cdots)=0.$$

G-invariant solutions $w=w(y)$ are determined by the simple translational invariance condition $w_s=0$. Substituting, we immediately obtain the reduced ordinary differential equation for the G-invariant solutions

$$\Delta(y, w, 0, w_y, 0, 0, w_{yy}, \cdots)=0.$$

Case 2. In the nonclassical method of Bluman and Cole, Δ need no longer be independent of s, but when we substitute the invariance condition $w_s=0$ into Δ, we obtain an equation that is equivalent to an s-independent ordinary differential equation for $w(y)$:

$$\Delta(s, y, w, 0, w_y, 0, 0, w_{yy}, \cdots)=F(s)\cdot\tilde{\Delta}(y, w, w_y, w_{yy}, \cdots)=0.$$

In this case $\tilde{\Delta}=0$ constitutes the reduced ordinary differential equation for the G-invariant solutions to Δ.

Case 3. In the most general case, proceeding as in Case 2, we obtain an equation of the form

$$\Delta(s, y, w, 0, w_y, 0, 0, w_{yy}, \cdots)=0,$$

which must hold identically in s. Expanding Δ in powers of s or in a Fourier series in s, say, we will obtain a collection of ordinary differential equations for $w(y)$. If these are compatible, each solution will determine a G-invariant solution to the original system Δ; otherwise, Δ has no G-invariant solutions.

In both Case 2 and Case 3 the group is only a weak symmetry group; the main difference is the increase in the number of reduced equations needed for the last case.

As an illustration of this approach, in the heat equation example we would straighten the vector field $\mathbf{v}$ by introducing the new variables

$$s=-\log t, \qquad y=xt, \quad w=u-x^3,$$

so that $\mathbf{v}=\partial_s$ in these coordinates. In the s, y, w-coordinates, the heat equation takes the form

$$e^{-s}w_s+ye^{-s}w_y=e^{2s}w_{yy}+6ye^{-s}.$$

Any G-invariant solution has $w_s=0$, and hence must satisfy

$$ye^{-s}w_y=e^{2s}w_{yy}+6ye^{-s}$$

for all s. This is equivalent to the reduced system

$$yw_y=6y, \qquad w_{yy}=0,$$

from which we once again obtain the general G-invariant solution (13) to the heat equation.

4. Symmetry groups of differential equations. We now turn to a proof of the key result that shows that the above reduction procedure will work for any group of transformations whatsoever. We will follow the development of the general theory of symmetry groups of differential equations presented in [11], which the reader should consult for more details; see also [13].

Consider a system of differential equations of the form (4). We will assume (without essential loss of generality) that the system (4) is of *maximal rank*, meaning that the Jacobian matrix of the Δ_ν with respect to all the variables $(x, u^{(n)})$ has rank l at every solution to the system. Let G be a (connected) local group of transformations acting on an open subset $M\subset X\times U=\mathbb{R}^p\times\mathbb{R}^q$ of the space of independent and dependent variables (x, u). Rather than trying to treat the group transformations directly, we look at the infinitesimal generators of G, which are vector fields

$$\mathbf{v}=\sum_{i=1}^{p}\xi^i(x,u)\frac{\partial}{\partial x^i}+\sum_{\alpha=1}^{q}\varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{16}$$

on M. Since the equations (4) involve not only x and u but also derivatives of u, we need to know how the group transforms these derivatives; this is covered by the theory of *prolongation* [11], [13]. The infinitesimal generators of G will have corresponding prolongations, telling how G acts "infinitesimally" on the derivatives of u. The general prolongation of the vector field (16) is given by the formula

$$\mathrm{pr}\,\mathbf{v}=\mathrm{pr}\,\mathbf{v}_Q+\sum_{i=1}^{p}\xi^i\cdot D_i, \tag{17}$$

where D_i is the total derivative with respect to x^i, $Q=(Q_1,\cdots,Q_q)$ is the *characteristic* of $\mathbf{v}$, with entries

$$Q_\alpha=\varphi^\alpha(x,u)-\sum_{i=1}^{p}\xi^i(x,u)\frac{\partial u^\alpha}{\partial x^i} \tag{18}$$

and $\mathbf{v}_Q\equiv\sum Q_\alpha\partial/\partial u^\alpha$ is the corresponding *evolutionary vector field*, with prolongation

$$\mathrm{pr}\,\mathbf{v}_Q=\sum_{J,\alpha}(D_JQ_\alpha)\frac{\partial}{\partial u_J^\alpha}, \qquad u_J^\alpha\equiv D_Ju^\alpha. \tag{19}$$

(In (19) each multi-index J refers to a specific partial derivative of u^α, with D_J denoting the corresponding higher order total derivative.) The fundamental observation of Lie, allowing one to explicitly compute the general symmetry groups of differential equations, was that the complicated nonlinear conditions for G to be a symmetry group to the system (4) could be replaced by equivalent, *linear* conditions using the infinitesimal generators of G.

THEOREM 1. *Let Δ be a system of differential equations of maximal rank. A connected group of transformations G is a strong symmetry group of Δ if and only if for every infinitesimal generator $\mathbf{v}$ of G*

$$\text{pr}\,\mathbf{v}(\Delta_\nu) = 0, \qquad \nu = 1, \cdots, l \tag{20}$$

for all solutions $u = f(x)$ to Δ.

In practice, the *determining equations* (20) of the symmetry group of Δ constitute a large number of elementary *linear* differential equations for the coefficients ξ^i, φ^α of $\mathbf{v}$, which can be explicitly solved, giving the most general (continuous) symmetry group. (See [4], [11], [13] for the above theory and illustrative examples of this procedure.)

Often one says, somewhat loosely, that the infinitesimal invariance conditions (20) should hold "whenever $(x, u^{(n)})$ satisfy (4)." There is, in fact, a subtle distinction between the way the theorem is stated and the latter statement. It is *not* true in general that if x_0 is a point in $X = \mathbb{R}^p$ and $u_0^{(n)}$ a collection of prescribed values of the derivatives of u at x_0 that satisfy the algebraic conditions imposed by the system (4), then there exists a smooth solution $u = f(x)$ to the system whose derivatives at the point x_0 agree with the values $u_0^{(n)}$. A point $(x_0, u_0^{(n)})$, which *does* satisfy this condition, and so pertains to an actual solution $u = f(x)$ of the system, is said to be a point of *local solvability* of the system [11]. Theorem 1 says that G is a strong symmetry group of the system (4) provided the infinitesimal invariance condition (20) holds only at the points of local solvability of the system, and not necessarily *all* $(x, u^{(n)})$ satisfying (4).

There are two principal causes of nonsolvability of systems of partial differential equations. The first, which will not concern us so much here, are those smooth but nonanalytic systems which, like the example due to Lewy [7], have no solutions. More interesting for our purposes is the nonsolvability due to integrability conditions coming from cross-differentiating the equations in the system. For example, the system

$$u_x = yu, \qquad u_y = 0$$

is not locally solvable at any point

$$(x^0, y^0, u^0, u_x^0, u_y^0) = (x^0, y^0, u^0, y^0 \cdot u^0, 0)$$

where $u^0 \neq 0$, e.g. $(0, 0, 1, 0, 0)$. Indeed, cross-differentiation shows that

$$0 = u_{xy} = yu_y + u = u,$$

and so the only solution is the trivial one $u \equiv 0$. For analytic systems of differential equations, a theorem of Tresse [16] says that all the integrability conditions, and hence the equations for the points of local solvability, can be found by a finite number of such cross-differentiations. (See [11] for a fuller discussion of these issues.)

5. Group-invariant solutions. Given a system of partial differential equations (4), if we are looking at the solutions which are invariant under some transformation group G, then there will be further restrictions on the possible values of the variables $(x, u^{(n)})$ which can be assumed by such solutions. In our first example, these were encapsulated in the relation (2) between the first order derivatives. The general result is similar:

LEMMA 2. *Let G be a connected group of transformations on M, with infinitesimal generators $\mathbf{v}_1, \cdots, \mathbf{v}_s$. Let $Q^1, \cdots, Q^s$ be the corresponding characteristics. Then any G-invariant function $u = f(x)$ must satisfy the first order system of $s \cdot q$ equations*

$$Q_\alpha^\mu(x, u^{(1)}) = 0, \qquad \mu = 1, \cdots, s, \quad \alpha = 1, \cdots, q, \tag{21}$$

determining by the vanishing of the characteristics for G.

(In fact, if G acts regularly [11] or, more restrictively, has a complete set of functionally independent invariants defined over all of M, then (21) is both necessary and sufficient for u to be a G-invariant function.)

Any G-invariant solution $u=f(x)$ to the system (4) will also be a solution to (21), and hence a solution to the combined system

$$\begin{aligned} &\Delta_\nu(x, u^{(n)})=0, \qquad \nu=1,\cdots,l, \\ &Q^\mu_\alpha(x, u^{(1)})=0, \qquad \mu=1,\cdots,s, \quad \alpha=1,\cdots,q, \end{aligned} \tag{22}$$

as in (1) and (2) in the illustrative example. (In the context of [12], the characteristics (21) provide the side conditions that are to be appended to the system in order to determine the G-invariant solutions.)

As detailed in [11], the key to the reduction method lies, not in the invariance of the system (4) under the group G, but rather in the invariance of the combined system (22).

THEOREM 3. *Let Δ be a system of differential equations on $M\subset X\times U$. Let G be a transformation group acting on M which has a complete set of globally defined, functionally independent invariants $y^1=\eta^1(x,u),\cdots,y^{p-r}=\eta^{p-r}(x,u)$, $w^1=\zeta^1(x,u),\cdots,w^q=\zeta^q(x,u)$, which provide local coordinates on an open subset $M/G\subset Y\times W=\mathbb{R}^{p-r}\times\mathbb{R}^q$ (the quotient manifold of M by G). Let $Q^1,\cdots,Q^r$ be the characteristics for a basis of the space of infinitesimal generators of G. If the combined system* (22) *consisting of the equations in Δ plus the vanishing of the characteristics is invariant under the prolonged action of G, then G is a weak symmetry group of Δ, and the basic reduction procedure of* § 3 *will lead to a well-defined system of differential equations Δ/G in the new variables y, w. Each solution $w=h(y)$ of this reduced system Δ/G leads, via* (5), *to a G-invariant solution to Δ and, conversely, each G-invariant solution to Δ arises from a solution to Δ/G.*

The key to the proof of this theorem [11], [13] is that the combined system (22) be invariant under G, so that the infinitesimal conditions of Theorem 1 hold for all $(x, u^{(n)})$ satisfying (22). In particular, if G is a strong symmetry group to the original system (4), then it is automatically a symmetry group of the combined system (22), since it is trivially a symmetry group of the characteristic system (21). For more general transformation groups, we might not expect the entire system (22) to be invariant under G; indeed, Bluman and Cole's nonlinear conditions for G to be a nonclassical symmetry group, (cf. [3]) are the same as the infinitesimal invariance conditions that the *entire* subvariety

$$\{(x, u^{(n)}):\Delta_\nu(x,u^{(n)})=0,\ \nu=1,\cdots,l,\quad Q^\mu_\alpha(x,u^{(1)})=0,\ \mu=1,\cdots,s,\ \alpha=1,\cdots,q\}$$

be invariant under the prolonged group action (cf. (20)). However, as we saw in our discussion following Theorem 1, as far as the solutions of (22) (i.e. the G-invariant solutions to (4)) are concerned, we really need only look at the points $(x, u^{(n)})$ of local solvability of (22), and the imposition of the infinitesimal invariance conditions just at these points will impose less stringent requirements on the group G.

We can now state the basic result of this paper, which is, perhaps surprisingly, that the subset consisting of the points of local solvability of (22) is *always* invariant under G. In other words, no matter what the group G is, the combined system (22) *always* admits G as a symmetry group in the sense that G transforms solutions $u=f(x)$ to solutions, and hence G is *always* a weak symmetry group of the original system. Therefore, provided we are in the domain of applicability of Tresse's theorem, once we append to (22) all the integrability conditions coming from cross-differentiations, we are left with a system of the same form which *is* invariant under G, and hence gives rise to a reduced system in the new variables whose solutions corresponding to all the G-invariant solutions to (4). Put another way, the reason why Bluman and Cole find nontrivial conditions on their groups in order to apply their nonclassical method is that they fail to take into account the additional restrictions on the derivatives of u coming from these integrability conditions.

THEOREM 4. *Let G be any group of transformations acting on $M \subset X \times U$. Let Δ be any system of differential equations defined over M. Then G is always a symmetry group of the combined system* (22) *consisting of the equations in Δ and the vanishing of the characteristics of the infinitesimal generators of G, and hence is always a weak symmetry group to Δ.*

Proof. Let $\mathbf{v} = \mathbf{v}_k$ be an infinitesimal generator of G with characteristic $Q = Q^k$. Writing out the infinitesimal conditions of invariance (20) for the system (22), we find

$$\begin{aligned} \operatorname{pr} \mathbf{v}(\Delta_\nu) &= \operatorname{pr} \mathbf{v}_Q(\Delta_\nu) + \sum \xi^i D_i \Delta_\nu = 0, \\ \operatorname{pr} \mathbf{v}(Q^\mu_\alpha) &= \operatorname{pr} \mathbf{v}_Q(Q^\mu_\alpha) + \sum \xi^i D_i Q^\mu_\alpha = 0, \end{aligned} \tag{23}$$

which must hold for all solutions to (22). Note first that the coefficients of the prolonged evolutionary form $\operatorname{pr} \mathbf{v}_Q$ of any infinitesimal generator $\mathbf{v}$ of G are just total derivatives of the entries Q_α of the characteristic Q of $\mathbf{v}$ (cf. (19)) and hence vanish on solutions to (22). Second, the remaining terms in (23) just involve the first order derivatives $D_i \Delta_\nu$, $D_i Q^\mu_\alpha$ of the equations in (22), and hence also vanish on solutions. Thus the infinitesimal criterion of invariance (20) for the combined system (22) is verified, proving the theorem.

The discussion preceding the theorem indicates the possibility of a second approach to determining G-invariant solutions to a system of partial differential equations. Namely, one writes down the combined system (22), and then differentiates to find the relevant integrability conditions. According to Tresse's theorem, this process will, in a finite number of steps, lead to a G-invariant system of differential equations, from which the reduced equations Δ/G for the G-invariant solutions to the original system can be determined. For example, consider the case of the heat equation (10) discussed above. In terms of the invariants y, w of the one-parameter group G, we have

$$\begin{aligned} &u_t = xw_y, \qquad u_x = tw_y + 3x^2, \qquad u_{tt} = x^2 w_{yy}, \\ &u_{xt} = xtw_{yy} + w_y, \qquad u_{xx} = t^2 w_{yy} + 6x, \qquad u_{ttt} = x^3 w_{yyy}, \\ &u_{xtt} = x^2 t w_{yyy} + 2xw_{yy}, \quad u_{xxt} = xt^2 w_{yyy} + 2tw_{yy}, \quad u_{xxx} = t^3 w_{yyy} + 6, \end{aligned} \tag{24}$$

and so on. If we substitute these expressions into just the heat equation $u_t = u_{xx}$ itself, we obtain the equation

$$xw_y = t^2 w_{yy} + 6x, \tag{25}$$

which, as we noted above, is *not* an ordinary differential equation for w as a function of y. However, if we append the equations for the first prolongation of the heat equation, namely

$$u_{tt} = u_{xxt}, \qquad u_{xt} = u_{xxx}, \tag{26}$$

and substitute according to (24), we find

$$x^2 w_{yy} = xt^2 w_{yyy} + 2tw_{yy}, \qquad xtw_{yy} + w_y = t^3 w_{yyy} + 6.$$

Eliminating w_{yyy} from this latter pair of equations, we have

$$xw_y = 6x - 2t^2 w_{yy}. \tag{27}$$

Comparing with (25), we see that w has to satisfy the same pair of ordinary differential equations we deduced earlier in (12). We again have found the one-parameter family of G-invariant solutions (13).

Note that, in contrast to Bluman and Cole's procedure, here we were forced to use the first derivatives of the differential equation in order to uncover the reduced ordinary differential equations for the invariant solutions. Another way of seeing this is to look at the combined system

$$u_t = u_{xx}, \qquad xu_x - tu_t - 3x^3 = 0,$$

which is (22) in this particular case. This system is not G-invariant, but a suitable prolongation of it is. In fact, differentiating the second equation once with respect to t and twice with respect to x, we have

$$xu_{xt} - tu_{tt} - u_t = 0, \qquad xu_{xxx} + 2u_{xx} - tu_{xxt} - 18x = 0.$$

Substituting into the latter equation according to (26), we find

$$xu_{xt} + 2u_{xx} - tu_{tt} - 18x = 0;$$

comparison with the former equation yields the G-invariant system

$$u_t = 6x = u_{xx},$$

from which we can also deduce the general G-invariant solution (13).

Now that we know how to implement the reduction method for an arbitrary group of transformations on the space of independent and dependent variables, it is easy to prove that *any* solution $u = f(x)$ to a system of differential equations can be found by these group-theoretic methods. All we need to do is to find some local group of transformations G that leaves the graph of f invariant. In fact, there are many such groups; a specific example would be the one-parameter group with infinitesimal generator $\mathbf{v}_i = \partial/\partial x^i + \sum_\alpha (\partial f^\alpha/\partial x^i)\partial/\partial u^\alpha$ for any $1 \leqq i \leqq p$, or, more generally, any variable-coefficient linear combination of these generators.

THEOREM 5. *If Δ is any system of partial differential equations and $u = f(x)$ is any solution, then there exists a weak symmetry group G of Δ such that f is invariant under G and hence f can be obtained by the reduction method of the preceding theorem.*

This substantiates our claim that any given solution could be found by the general reduction method for weak symmetry groups. However, if one has already obtained the solution by some other method, the reasoning in Theorem 5 is, perhaps, of an a posteriori nature. In other words, Theorem 5, while certainly of interest, is not meant to supplant other valid and useful methods for finding explicit solutions to partial differential equations. Moreover, as shown in [12], while one can always derive individual solutions from the group reduction method, the same cannot be said for parametrized families of solutions such as those arising from separation of variables; they may not all come from one and the same symmetry group.

6. Conclusions. We have shown how the basic group reduction method for finding group-invariant solutions to systems of partial differential equations can be applied to any group of transformations whatsoever, without regard for any underlying symmetry conditions imposed by the system itself. On the one hand, this observation is liberating, in that one is no longer shackled by possibly artificial symmetry constraints in the search for explicit group-invariant solutions. On the other hand, this appears to open up a whole Pandora's box: how is one to determine which groups will actually be useful, a) in the sense that the resulting reduced system is compatible and hence invariant solutions do exist, or b) more restrictively, in the sense that the reduced system can be explicitly solved to determine the solutions in closed form? It would be quite enlightening to determine the answer to these questions, even in just one specific example, such as the heat equation, but this we leave to future research. The chances are that the conditions, like those of Bluman and Cole, are extremely complicated, so one can never know in full detail the entire range of possible reductions which are available. Indeed, since in principle one can determine *any* solution by a suitably clever choice of weak symmetry group, one would scarcely be able to determine all possible weak symmetry groups having invariant solutions unless one explicitly knew all possible solutions.

An alternative tactic, which seems more practical, is to specify the group by external symmetry considerations; for example, one might try symmetries relevant to the physical problem that the system is modeling (whether or not these are symmetries of the system itself), or symmetries which preserve any boundary conditions that are present in the problem. Once the group has been prescribed, one can algorithmically implement the reduction procedure presented here, and thereby determine all solutions which are invariant under the given group. If the combined system (22) is compatible, invariant solutions to the system will exist, despite the fact that the given group is not a symmetry group of the system. Alternatively, one may find (22) to be an incompatible overdetermined system of differential equations, and hence there are *no* solutions to the system that are invariant under the given group. (As remarked above, this latter possibility exists even for strong symmetry groups; see [11, Chap. 3] for physical examples.) Even this information, we believe, could be important for the analysis or physical applications of the problem at hand. At the moment, the principal direction of research should be on applying the method to specific, physically interesting examples, thereby gaining an appreciation of its usefulness and range of applicability.

Finally, it is worth mentioning that these results are subsumed under the more general concept of a differential equation with side conditions proposed in [12]. This idea not only includes group-invariant solutions of all the above types, but also separable solutions and more general types of special solutions to partial differential equations. As discussed in detail in [12], side conditions, and not group theory, appear to provide the real unifying framework for all the methods for finding special solutions to differential equations. Nevertheless, simple group invariance can, as we have demonstrated, still lead to many new, explicit solutions of physical importance, and retains its validity as a practical method for the study of partial differential equations.

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