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Group schemes and local densities of ramified hermitian lattices in residue characteristic 2 Part I

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The obstruction to the local-global principle for a hermitian lattice (L, H) can be quantified by computing the mass of (L, H). The mass formula expresses the mass of (L, H) as a product of local factors, called the local densities of (L, H). The local density formula is known except in the case of a ramified hermitian lattice of residue characteristic 2.

Let F be a finite unramified field extension of $\mathbb{Q}_2$. Ramified quadratic extensions E/F fall into two cases that we call *Case 1* and *Case 2*. In this paper, we obtain the local density formula for a ramified hermitian lattice in *Case 1*, by constructing a smooth integral group scheme model for an appropriate unitary group. Consequently, this paper, combined with the paper of W. T. Gan and J.-K. Yu (*Duke Math. J.* **105** (2000), 497–524), allows the computation of the mass formula for a hermitian lattice (L, H) in *Case 1*.

1. Introduction

1A. ***Introduction.*** The subject of this paper is old and has intrigued many mathematicians. If (V, H) and (V', H') are two hermitian k'-spaces (or quadratic k-spaces), where k is a number field and k' is a quadratic field extension of k, then it

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is well known that they are isometric if and only if for all places v, the localizations (V_v, H_v) and (V'_v, H'_v) are isometric. That is, the local-global principle holds for hermitian spaces and quadratic spaces. It is natural to ask whether the local-global principle holds for a hermitian R'-lattice or quadratic R-lattice (L, H), where R' and R are the rings of integers of k' and k, respectively. In general, the answer to this question is no. However, there is a way, namely, the mass of (L, H), to quantify the obstruction to the local-global principle. An essential tool for computing the mass of a quadratic or hermitian lattice is the mass formula. The mass formula expresses the mass of (L, H) as a product of local factors, called the local densities of (L, H).

Therefore, it suffices to find the explicit local density formula in order to obtain the mass formula and thus quantify the obstruction to the local-global principle.

For a quadratic lattice, the local density formula was first computed by G. Pall [1965] (for $p \neq 2$) and G. L. Watson [1976] (for $p = 2$). For an expository sketch of their approach, see [Kitaoka 1993]. There is another proof of Y. Hironaka and F. Sato [2000] computing the local density when $p \neq 2$. They treat an arbitrary pair of lattices, not just a single lattice, over $\mathbb{Z}_p$ (for $p \neq 2$). J. H. Conway and J. A. Sloane [1988] further developed the formula for any p and gave a heuristic explanation for it. Later, W. T. Gan and J.-K. Yu [2000] (for $p \neq 2$) and S. Cho [2015a] (for $p = 2$) provided a simple and conceptual proof of Conway and Sloane's formula by explicitly constructing a smooth affine group scheme $\underline{G}$ over $\mathbb{Z}_2$ with generic fiber $\mathrm{Aut}_{\mathbb{Q}_2}(L, H)$, which satisfies $\underline{G}(\mathbb{Z}_2) = \mathrm{Aut}_{\mathbb{Z}_2}(L, H)$.

There has not been as much work done in computing local density formulas for hermitian lattices as in the case of quadratic lattices. Although the local density formula for a quadratic lattice with $p = 2$ was first proved in the author's paper [2015a], the formula was proposed in Conway and Sloane's paper [1988]. However, the local density formula for a ramified hermitian lattice with $p = 2$ has not been proposed yet and therefore, the mass formula, when the ideal (2) is ramified in k'/k, is not known.

Hironaka [1998; 1999] obtained the local density formula for an unramified hermitian lattice. In addition, M. Mischler [2000] computed the formula for a ramified hermitian lattice ($p \neq 2$) under restricted conditions. Later, Gan and Yu [2000] found a conceptual and elegant proof of the local density formula for an unramified hermitian lattice without any restriction on p, and for a ramified hermitian lattice with the restriction $p \neq 2$, by explicitly constructing certain smooth affine group schemes (called smooth integral models) of a unitary group.

As discussed further on p. 456, we distinguish two cases for a ramified quadratic extension E/F, where F is an unramified finite extension of $\mathbb{Q}_2$, depending on the lower ramification groups G_i of the Galois group $\mathrm{Gal}(E/F)$. The division is as follows:

$$\begin{cases} \textit{Case 1}: & G_{-1} = G_0 = G_1,\ G_2 = 0; \\ \textit{Case 2}: & G_{-1} = G_0 = G_1 = G_2,\ G_3 = 0. \end{cases}$$

These two cases should be handled independently because of technical difficulty and complexity. The methodologies of the two cases are basically the same, but *Case 2* is much more difficult than *Case 1*.

The main contribution of this paper is to get an explicit formula for the local density of a hermitian B-lattice (L, h) in *Case 1*, by explicitly constructing a certain smooth group scheme associated to it that serves as an integral model for the unitary group associated to $(L \otimes_A F, h \otimes_A F)$ and by investigating its special fiber, where B is a ramified quadratic extension of A and A is an unramified finite extension of $\mathbb{Z}_2$ with F as the quotient field of A. The local density formula in *Case 2* is handled in [Cho 2015b].

In conclusion, this paper, combined with [Gan and Yu 2000] and [Cho 2015a], allows the computation of the mass formula for a hermitian R'-lattice (L, H) when $k_v/\mathbb{Q}_2$ is unramified, and $k'_{v'}/k_v$ satisfies *Case 1* or is unramified. Here, $k'_{v'}$ (resp. k_v) is the completion of k' (resp. k) at the place v' (resp. v), where v' lies over v and v lies over the ideal (2). As the simplest case, we can compute the mass formula for an arbitrary hermitian lattice explicitly when k is $\mathbb{Q}$ and k' is any quadratic field extension of $\mathbb{Q}$ such that the completion of k' at any place lying over the ideal (2) satisfies *Case 1* or is unramified over $\mathbb{Q}_2$.

Let us briefly comment on the proofs. A key input into the local density formula is

$$\lim_{N\to\infty} f^{-N \dim G} \# \underline{G}'(A/\pi^N A), \tag{1-1}$$

where f is the cardinality of the residue field of A, π is a uniformizer in A, and $\underline{G}'$ is the naive integral model for the unitary group G associated to $(L \otimes_A F, h \otimes_A F)$, which represents the functor $R \mapsto \mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h \otimes_A R)$.

Now if we are lucky enough that $\underline{G}'$ is smooth, then the limit in (1-1) would stabilize at $N = 1$, which would reduce us to simply finding $\underline{G}'(\kappa)$, where κ denotes the residue field of A. A key observation of Gan and Yu is that, even when $\underline{G}'$ is not smooth, one can employ a certain smooth group scheme $\underline{G}$ lurking in the background, which is a smooth integral model of G that satisfies $\underline{G}(R) = \underline{G}'(R)$ for every étale A-algebra R. The existence and uniqueness of such a $\underline{G}$ is guaranteed by the general theory of group smoothening. Then the problem essentially reduces to constructing $\underline{G}$ explicitly, so that one can compute the cardinality of the group $\underline{G}(\kappa)$ of κ-points of its special fiber. This tells us what the analog of (1-1) for $\underline{G}$ is, and further, it so turns out that one can deduce the expression (1-1) from its analog for $\underline{G}$. For a detailed explanation about this, see Section 3 of [Gan and Yu 2000].

Let us now describe, therefore, how we construct $\underline{G}$ and study its special fiber. As $\underline{G}'$ fails to be smooth, one must impose more equations than merely the ones related to the preservation of $(L \otimes_A R, h \otimes_A R)$. Towards this, note that there exist several sublattices L' of L such that any element of $\mathrm{Aut}_B(L, h)$ automatically also preserves L' (and such that, for any étale A-algebra R, any element of

$\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ automatically also preserves $L'\otimes_A R$). For instance, the sublattice L' of elements $x\in L$ such that $h(x, L)$ belongs to a given ideal of B necessarily satisfies this property. This gives us additional equations to impose — these equations leave the group of R-points for any étale A-algebra R untouched, while taking us closer to smoothness. It so happens that taking sufficiently many sublattices L' into consideration, and imposing further restrictions arising from the behavior of an element of $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ on some of their quotients, do leave us with enough equations to ensure that the group scheme $\underline{G}$ defined by them is smooth. This step already turns out to be much harder for $p=2$ than for odd p, since in this case there are many more isomorphism classes of hermitian lattices. Another source of complications is the fact that the equations involve quadratic forms over the residue field κ of A that arise as quotients of some of the lattices L' mentioned above (the theory of quadratic forms over finite fields is more complicated in characteristic 2 than in other characteristics).

Now let us describe some of the ideas involved in the computation of the special fiber $\tilde{G}$ of $\underline{G}$. Since the quotients of some pairs of lattices of the form L' alluded to in the previous paragraph naturally support symplectic or quadratic forms, it is not hard to construct a map φ from $\tilde{G}$ to a suitable product of symplectic and orthogonal groups. This step occurs in [Gan and Yu 2000], too. However, p being even for us poses at least two new difficulties. Firstly, although this product of symplectic and orthogonal groups contains the identity component of the maximal reductive quotient of $\tilde{G}$, this fact seems to be difficult to prove directly. Rather, we prove this fact indirectly, by explicitly computing the dimension of the kernel of φ. Secondly, φ does not quite define the maximal reductive quotient of $\tilde{G}$: this maximal reductive quotient is built up from φ together with a few additional homomorphisms $\tilde{G}\to\mathbb{Z}/2\mathbb{Z}$.

Our construction of these homomorphisms $\tilde{G}\to\mathbb{Z}/2\mathbb{Z}$ is quite indirect. A typical homomorphism is constructed in the following manner. We define a certain new hermitian lattice, say (L'', h''), starting from (L, h). This lattice naturally gives us a homomorphism $\tilde{G}\to\tilde{G}''$, where $\tilde{G}''$ is the special fiber of the smooth integral model obtained by applying our construction to (L'', h'') in place of (L, h). The analog φ'' of φ defines a map from $\tilde{G}''$ to (a product of symplectic and orthogonal groups, and in particular) an orthogonal group, and, by composing with the Dickson invariant, one gets a homomorphism $\tilde{G}''\to\mathbb{Z}/2\mathbb{Z}$. Precomposing this with the homomorphism $\tilde{G}\to\tilde{G}''$ yields a homomorphism $\tilde{G}\to\mathbb{Z}/2\mathbb{Z}$. All our homomorphisms $\tilde{G}\to\mathbb{Z}/2\mathbb{Z}$ are constructed in this way.

To show that the candidate for the maximal reductive quotient of $\tilde{G}$ obtained from φ and the morphisms $\tilde{G}\to\mathbb{Z}/2\mathbb{Z}$ is indeed the maximal reductive quotient, one shows that its kernel is isomorphic, as an affine variety, to an affine space over κ. This implies by a theorem of Lazard that the kernel of our candidate for maximal reductive quotient is indeed a connected unipotent group scheme, as desired.

Our main results are Theorem 3.8, Theorem 4.12 and Theorem 5.2. Theorem 3.8 shows that the group scheme $\underline{G}$ we construct is indeed the sought after smooth group scheme over A, Theorem 4.12 gives the maximal reductive quotient of $\widetilde{G}$, and Theorem 5.2 (supplemented by Remark 5.3) gives us the final local density formulas as follows. The local density of (L, h) is

$$\beta_L = f^N \cdot f^{-\dim G} \# \widetilde{G}(\kappa).$$

Here, N is a certain integer which can be found in Theorem 5.2 and $\#\widetilde{G}(\kappa)$ can be computed explicitly based on Remark 5.3(1) and Theorem 4.12.

Appendix B is devoted to illustrating our method with a simple example: the case where $L = B \cdot e$ is of rank one and h is defined by $h(le, l'e) = \sigma(l)l'$, σ being the unique nontrivial element of $\mathrm{Gal}(E/F)$. Section B.1 describes how the usual approach that works when $p \neq 2$ (and yields the obvious integral model for the "norm one" torus associated to B/A) fails when $p = 2$, and how one may fix this from "first principles", without using any of our techniques. We hope this helps clarify some of the issues involved. Section B.2 illustrates how our construction specializes to this case; we hope that the simplicity of this case may better motivate our general construction. Some readers may therefore prefer to look at Appendix B before perusing the general constructions of Sections 3 and 4 and Appendix A.

This paper is organized as follows. We first state a structure theorem for integral hermitian forms in Section 2. We then give an explicit construction of $\underline{G}$ (in Section 3) and study its special fiber (in Section 4) in *Case 1*. Finally, we obtain an explicit formula for the local density in Section 5 in *Case 1*. In Appendix B, we provide an example to describe the smooth integral model and its special fiber and to compute the local density for a unimodular lattice of rank 1.

The reader might want to skip to Appendix B and at least go to Section B.1 to get a first glimpse into why the case of $p = 2$ is really different. Some of the ideas behind our construction can be seen in the simple example illustrated in Section B.2.

The construction of smooth integral models and the investigation of their special fibers in this paper basically follow the arguments in [Gan and Yu 2000] and [Cho 2015a]. As in [Gan and Yu 2000], the smooth group schemes constructed in this paper should be of independent interest.

2. Structure theorem for hermitian lattices and notations

2A. ***Notation.*** Notation and definitions in this section are taken from [Cho 2015a; Gan and Yu 2000; Jacobowitz 1962].

- Let F be an unramified finite extension of $\mathbb{Q}_2$ with A its ring of integers and κ its residue field.

- Let E be a ramified quadratic field extension of F with B its ring of integers.

- Let σ be the nontrivial element of the Galois group $\mathrm{Gal}(E/F)$.

- The lower ramification groups G_i of the Galois group $\mathrm{Gal}(E/F)$ satisfy one of the following:

$$\begin{cases} \textit{Case 1}: & G_{-1}=G_0=G_1,\ G_2=0; \\ \textit{Case 2}: & G_{-1}=G_0=G_1=G_2,\ G_3=0. \end{cases}$$

We explain the above briefly. Based on Section 6 and Section 9 of [Jacobowitz 1962], we can select a suitable choice of a uniformizer π of B in the following way. In *Case 1*, $E=F(\sqrt{1+2u})$ for some unit u of A and $\pi=1+\sqrt{1+2u}$. Then $\sigma(\pi)=\epsilon\pi$, where $\epsilon\equiv 1 \bmod \pi$ and $\frac{\epsilon-1}{\pi}$ is a unit in B. So we have that $\sigma(\pi)+\pi, \sigma(\pi)\cdot\pi \in (2)\backslash(4)$. In *Case 2*, $E=F(\pi)$. Here, $\pi=\sqrt{2\delta}$, where $\delta\in A$ and $\delta\equiv 1 \bmod 2$. Then $\sigma(\pi)=-\pi$.

From now on, a uniformizing element π of B, u, and δ are fixed as explained above throughout this paper. The constructions of smooth integral models associated to these two cases are different and we will treat them independently.

- We consider a B-lattice L with a hermitian form

$$h: L\times L\to B,$$

where $h(a\cdot v, b\cdot w)=\sigma(a)b\cdot h(v,w)$ and $h(w,v)=\sigma(h(v,w))$. Here, $a,b\in B$ and $v,w\in L$. We denote by a pair (L,h) a hermitian lattice. We assume that $V=L\otimes_A F$ is nondegenerate with respect to h.

- We denote by (ϵ) the B-lattice of rank 1 equipped with the hermitian form having Gram matrix (ϵ). We use the symbol $A(a,b,c)$ to denote the B-lattice $B\cdot e_1+B\cdot e_2$ with the hermitian form having Gram matrix $\begin{pmatrix} a & c \\ \sigma(c) & b\end{pmatrix}$. For each integer i, the lattice of rank 2 having Gram matrix $\begin{pmatrix} 0 & \pi^i \\ \sigma(\pi^i) & 0\end{pmatrix}$ is called the π^i-modular hyperbolic plane and denoted by $H(i)$.

- A hermitian lattice L is the orthogonal sum of sublattices L_1 and L_2, written $L=L_1\oplus L_2$, if $L_1\cap L_2=0$, L_1 is orthogonal to L_2 with respect to the hermitian form h, and L_1 and L_2 together span L.

- The ideal in B generated by $h(x,x)$ as x runs through L will be called the norm of L and written $n(L)$.

- By the scale $s(L)$ of L, we mean the ideal generated by the subset $h(L,L)$ of B.

- We define the dual lattice of L, denoted by $L^{\perp}$, as

$$L^{\perp}=\{x\in L\otimes_A F : h(x,L)\subset B\}.$$

Definition 2.1. Let L be a hermitian lattice. Then:

(a) For any nonzero scalar a, define $aL=\{ax \mid x\in L\}$. It is also a lattice in the space $L\otimes_A F$. Call a vector x of L maximal in L if x does not lie in πL.

(b) The lattice L will be called π^i-modular if the ideal generated by the subset $h(x, L)$ of E is $\pi^i B$ for every maximal vector x in L. Note that L is π^i-modular if and only if $L^\perp = \pi^{-i}L$. We can also see that $H(i)$ is π^i-modular.

(c) Assume that i is even. A π^i-modular lattice L is of *parity type I* if $n(L) = s(L)$, and of *parity type II* otherwise. The zero lattice is considered to be *of parity type II*. We caution that we do not assign a *parity type* to a π^i-modular lattice L with i odd.

2B. ***A structure theorem for integral hermitian forms.*** We state a structure theorem for π^i-modular lattices in this subsection. Note that if L is π^{2i}-modular (resp. π^{2i+1}-modular), then $\pi^{-i}L \subset L \otimes_A F$ is π^0-modular (resp. π^1-modular). We will emphasize this in Remark 2.3(a) again. Thus it is enough to provide a structure theorem for π^0-modular or π^1-modular lattices.

Theorem 2.2. *Let $i = 0$ or 1.*

(a) *Let L be a π^i-modular lattice of rank at least 3. Then $L = \bigoplus_\lambda H_\lambda \oplus K$, where K is π^i-modular of rank 1 or 2, and each $H_\lambda = H(i)$.*

(b) *We denote by (1) or (2) the ideal of B generated by the element 1 or 2, respectively. Assume that K is π^i-modular of rank 1 or 2. Then, depending on i, the rank of K, the case that E/F falls into, the parity type of L (when applicable), and $n(L)$ which is the norm of L, we may take K to be of the following form*:

Rank of K	i	E/F	Parity type of L	$n(L)$	Form for K
1	0	Case 1	I*	(1)*	(a), $a \in A$, $a \equiv 1 \bmod 2$
1	0	Case 2	I*	(1)*	(a), $a \in A$, $a \equiv 1 \bmod 2$
2	0	Case 1	I	(1)*	$A(1, 2b, 1)$, $b \in A$
2	0	Case 2	I	(1)*	$A(1, 2b, 1)$, $b \in A$
2	0	Case 1	II	(2)*	$H(0)$
2	0	Case 2	II	(2)*	$A(2\delta, 2b', 1)$, $b' \in A$
2	1	Case 1		(2)*	$A(2, 2a, \pi)$, $a \in A$
2	1	Case 2		(2)	$A(4a, 2\delta, \pi)$, $a \in A$
2	1	Case 2		(4)	$H(1)$

Here, the superscript $*$ indicates the value in the table necessarily holds.

Proof. Part (a) is proved in Proposition 10.3 of [Jacobowitz 1962].

For part (b), when the rank of K is 1, it is clear that $K \cong (a')$ for a certain unit $a' \in A$ with a basis e. Since the residue field κ is perfect, there is a unit element a'' in A such that $a' \equiv a''^2 \bmod 2$. The reader can check that replacing e by $(1/a'')e$ realizes K in the manner dictated by the theorem.

From now on, we assume that the rank of K is 2. Suppose that $i = 0$. Then $n(K) = (1)$ or $n(K) = (2)$ since $n(K) \supseteq n(H(0)) = (2)$ (Proposition 9.1(a) and

Equation 9.1 with $k = 0$ in [Jacobowitz 1962]). If $n(K) = (1)$, then we can use Proposition 10.2 of [Jacobowitz 1962] to get $K \cong A(1, a, 1)$ with respect to a basis (e_1, e_2). Furthermore, the determinant $a - 1$ is a unit in A. To show this, we observe that K has an orthogonal basis, since $n(K) = s(K) = (1)$ (Proposition 4.4 in [Jacobowitz 1962]), and so the determinant should be a unit in order for K to be π^0-modular. Since the residue field κ is perfect, there is a unit element β in A such that $a - 1 \equiv \frac{1}{\beta^2} \bmod 2$. We now choose another basis $(e_1, (1-\beta)e_1 + \beta e_2)$. With this basis, it is easy to see that $K \cong A(1, 2b, 1)$ for a certain $b \in A$.

Now assume that $n(K) = (2)$ so that we cannot use Proposition 10.2 of [Jacobowitz 1962]. We choose a basis of K so that $K \cong A(x, y, 1)$ for some $x, y \in A$. Since $n(K) = (2)$, both x and y should be contained in the ideal (2). Thus $K \cong A(2a, 2b, 1)$ for some $a, b \in A$. Furthermore, in *Case 1*, if $n(K) = (2)$ then $K \cong H(0)$ by parts (a) and (b) of Proposition 9.2 of [Jacobowitz 1962].

The remaining case we need to prove when $i = 0$ is then that

$$K = A(2\delta, 2b', 1)$$

for certain $b' \in A$, in *Case 2* if $n(K) = (2)$. By Proposition 9.2(a) of [Jacobowitz 1962], if K is isotropic then $K \cong H(0)$ so that we can choose $b' = 0$. Furthermore, the lattice K with $n(K) = (2)$ is determined by its determinant up to isomorphism (Proposition 10.4 in [Jacobowitz 1962]). Since the determinant $d(K)$ of K is a unit and is well-defined modulo $N_{B/A}B^\times$, there are at most two cases of $d(K)$ because $|A^\times/N_{B/A}B^\times| = 2$. Here, $B^\times$ and $A^\times$ are the unit groups of B and A, respectively, and $N_{B/A}B^\times$ is the norm of $B^\times$. We observe that $d(A(2\delta, 0, 1))$ and $d(A(2\delta, 2d/\delta, 1))$, which are clearly π^0-modular, give different classes in $A^\times/N_{B/A}B^\times$, where d is as defined in Lemma 2.4. Thus, a lattice K with $n(K) = (2)$ in *Case 2* should be isomorphic to one of these two. In other words, such K is isomorphic to $K = A(2\delta, 2b', 1)$ with $b' = 0$ or $b' = d/\delta$.

We next suppose that $i = 1$. In *Case 1*, $n(H(1)) = (2)$ and so $n(K) = (2)$ since $s(K) \supseteq n(K) \supseteq n(H(1))$. Thus K is also determined by its determinant up to isomorphism (Proposition 10.4 in [Jacobowitz 1962]). This fact implies that there are at most two cases for K since the determinant of K divided by 2 is a unit in A and the cardinality of $A^\times/N_{B/A}B^\times$ is 2. By Lemma 2.5, $d(A(2, 0, \pi))$ and $d(A(2, 2ud, \pi))$, which are clearly π^1-modular, give different classes in $A^\times/N_{B/A}B^\times$, where u and d are as defined in Lemma 2.5. Thus, a lattice K with $n(K) = (2)$ in *Case 1* should be isomorphic to one of these two. In other words, such K is isomorphic to $d(A(2, 0, \pi))$ or $d(A(2, 2ud, \pi))$.

In *Case 2*, $n(H(1)) = (4)$ and so $n(K) = (2)$ or $n(K) = (4)$. If $n(K) = (2)$, then we can use Proposition 10.2(b) of [Jacobowitz 1962] (take $m = 1$) to get $K \cong A(2\delta, 4a, \pi)$ with basis (e_1, e_2). If we use a basis $(e_2, -e_1)$, then $K \cong$

$A(4a, 2\delta, \pi)$. If $n(K) = (4)$, then by Proposition 9.2(a–b) of [Jacobowitz 1962], $K \cong H(1) \cong A(0, 4\delta, \pi)$.

These complete the proof. □

Remark 2.3. (a) If L is π^i-modular, then $\pi^j L$ is π^{i+2j}-modular for any integer j. Thus, the above theorem implies its obvious generalization to the case where i is allowed to be any element of $\mathbb{Z}$.

(b) [Jacobowitz 1962, Section 4] For a general lattice L, we have a Jordan splitting, namely $L = \bigoplus_i L_i$ such that L_i is $\pi^{n(i)}$-modular and such that the sequence $\{n(i)\}_i$ increases. Two Jordan splittings $L = \bigoplus_{1 \leqq i \leqq t} L_i$ and $K = \bigoplus_{1 \leqq i \leqq T} K_i$ will be said to be of the same type if $t = T$ and, for $1 \leqq i \leqq T$, the following conditions are satisfied: $s(L_i) = s(K_i)$, rank L_i = rank K_i, and $n(L_i) = s(L_i)$ if and only if $n(K_i) = s(K_i)$. Jordan splitting is not unique but partially canonical in the sense that two Jordan splittings of isometric lattices are always of the same type.

(c) If we allow some of the L_i's to be zero, then we may assume that $n(i) = i$ for all i. In other words, for all $i \in \mathbb{N} \cup \{0\}$ we have $s(L_i) = (\pi^i)$, and, more precisely, L_i is π^i-modular. Then we can rephrase part (b) above as follows. Let $L = \bigoplus_i L_i$ be a Jordan splitting with $s(L_i) = (\pi^i)$ for all $i \geq 0$. Then the scale, rank and parity type of L_i depend only on L. We will deal exclusively with a Jordan splitting satisfying $s(L_i) = (\pi^i)$ from now on.

Lemma 2.4. *Assume that B/A satisfies Case 2. Then there is an element $d \in A^\times$ such that $1 - 4d$ and 1 give different classes in $A^\times / N_{B/A} B^\times$.*

Proof. Using our knowledge of the lower ramification groups G_i for $\mathrm{Gal}(E/F)$, we can compute the higher ramification groups G^i for the same extension:

$$G^{-1} = G^0 = G^1 = G^2 \quad \text{and} \quad G^3 = 0.$$

Let $U^i = 1 + (2)^i$ be the i-th higher unit group in F with $i \geq 1$. Then by *local class field theory*, the image of G^i under the isomorphism $\mathrm{Gal}(E/F) \cong F^*/N_{E/F}E^*$ is $U^i/(U^i \cap N_{E/F}E^*)$. We apply this when i is 2. Then we can easily verify the existence of a d as stated in the lemma. □

Lemma 2.5. *Assume that B/A satisfies Case 1. Then there is an element $d \in A^\times$ such that $1 + 2d$ and 1 give different classes in $A^\times / N_{B/A} B^\times$.*

Proof. The proof of this lemma is similar to that of the above lemma. In this case the higher ramification groups are as follows:

$$G^{-1} = G^0 = G^1 \quad \text{and} \quad G^2 = 0.$$

Again we use *local class field theory* as explained in the proof of the above lemma but with $i = 1$. Then we can easily verify the existence of a d as stated in the lemma. □

2C. ***Lattices.*** In this subsection, we will define several lattices and associated notation. Fix a hermitian lattice (L, h). We denote by (π^l) the scale $s(L)$ of L.

(1) Define $A_i = \{x \in L \mid h(x, L) \in \pi^i B\}$.

(2) Define $X(L)$ to be the sublattice of L such that $X(L)/\pi L$ is the radical of the symmetric bilinear form $\frac{1}{\pi^l}h \bmod \pi$ on $L/\pi L$.

Let $l = 2m$ or $l = 2m - 1$. We consider the function defined over L by

$$\frac{1}{2^m}q : L \to A, \quad x \mapsto \frac{1}{2^m}h(x, x).$$

Then $\frac{1}{2^m}q \bmod 2$ defines a quadratic form $L/\pi L \to \kappa$. It can be easily checked that $\frac{1}{2^m}q \bmod 2$ on $L/\pi L$ is an additive polynomial if $l = 2m$, or if $l = 2m - 1$ and E/F satisfies *Case 2*. Otherwise, that is, if $l = 2m - 1$ and E/F satisfies *Case 1*, it is not additive. We define a lattice $B(L)$ as follows.

(3) If $\frac{1}{2^m}q \bmod 2$ on $L/\pi L$ is an additive polynomial, then $B(L)$ is defined to be the sublattice of L such that $B(L)/\pi L$ is the kernel of the additive polynomial $\frac{1}{2^m}q \bmod 2$ on $L/\pi L$. If $\frac{1}{2^m}q \bmod 2$ on $L/\pi L$ is not an additive polynomial, then $B(L) = L$.

To define a few more lattices, we need some preparation as follows. For the remainder of the paper, set

$$\xi := \pi \cdot \sigma(\pi).$$

Assume $B(L) \subsetneq L$ and l is even. Then the bilinear form $\xi^{-l/2}h \bmod \pi$ on the κ-vector space $L/X(L)$ is nonsingular symmetric and nonalternating. It is well known that there is a unique vector $e \in L/X(L)$ such that

$$\left(\xi^{-l/2}h(v, e)\right)^2 = \xi^{-l/2}h(v, v) \bmod \pi$$

for every vector $v \in L/X(L)$. Let $\langle e \rangle$ denote the 1-dimensional vector space spanned by the vector e and denote by $e^{\perp}$ the 1-codimensional subspace of $L/X(L)$ which is orthogonal to the vector e with respect to $\xi^{-l/2}h \bmod \pi$. Then

$$B(L)/X(L) = e^{\perp}.$$

If $B(L) = L$, the bilinear form $\xi^{-l/2}h \bmod \pi$ on the κ-vector space $L/X(L)$ is nonsingular symmetric and alternating. In this case, we put $e = 0 \in L/X(L)$ and note that it is characterized by the same identity.

The remaining lattices we need for our definition are:

(4) Define $W(L)$ to be the sublattice of L such that

$$\begin{cases} W(L)/X(L) = \langle e \rangle & \text{if } l \text{ is even;} \\ W(L) = X(L) & \text{if } l \text{ is odd.} \end{cases}$$

(5) Define $Y(L)$ to be the sublattice of L such that $Y(L)/\pi L$ is the radical of

$$\begin{cases} \text{the form } \frac{1}{2^m}h \bmod \pi \text{ on } B(L)/\pi L & \text{if } l = 2m; \\ \text{the form } \frac{1}{\pi}\cdot\frac{1}{2^{m-1}}h \bmod \pi \text{ on } B(L)/\pi L & \text{if } l = 2m-1 \text{ in } \textit{Case 2}. \end{cases}$$

Both forms are alternating and bilinear.

(6) Define $Z(L)$ to be the sublattice of L such that $Z(L)/\pi L$ in *Case 1* or $Z(L)/\pi B(L)$ in *Case 2* is the radical of

$$\begin{cases} \text{the form } \frac{1}{2^m}q \bmod 2 \text{ on } L/\pi L & \text{if } l = 2m-1 \text{ in } \textit{Case 1}; \\ \text{the form } \frac{1}{2^{m+1}}q \bmod 2 \text{ on } B(L)/\pi B(L) & \text{if } l = 2m \text{ in } \textit{Case 2}. \end{cases}$$

Both forms are quadratic.

See, e.g., page 813 of [Sah 1960] for the notion of the radical of a quadratic form on a vector space over a field of characteristic 2.

Remark 2.6. (a) We can associate the 5 lattices $(B(L), W(L), X(L), Y(L), Z(L))$ above with (A_i, h) in place of L. Let B_i, W_i, X_i, Y_i, Z_i denote the resulting lattices.

(b) As κ-vector spaces, the dimensions of A_i/B_i and W_i/X_i are at most 1.

Let $L = \bigoplus_i L_i$ be a Jordan splitting. We assign a type to each L_i as follows:

parity of i	type of L_i	condition
even	I	L_i is of parity type I
even	I^o	L_i is of parity type I and the rank of L_i is odd
even	I^e	L_i is of parity type I and the rank of L_i is even
even	II	L_i is of parity type II
odd	II	E/F satisfies *Case 1* or E/F satisfies *Case 2* with $A_i = B_i$
odd	I	E/F satisfies *Case 2* and $A_i \supsetneq B_i$

In addition, we assign a subtype to L_i in the following manner:

parity of i	subtype of L_i	condition
even	bound of type I	L_i is of type I and either L_{i-2} or L_{i+2} is of type I
even	bound of type II	L_i is of type II and either L_{i-1} or L_{i+1} is of type I
odd	bound	either L_{i-1} or L_{i+1} is of type I

In all other cases, L_i is called *free*.

Notice that the type of each L_i is determined canonically regardless of the choice of a Jordan splitting.

2D. ***Sharpened structure theorem for integral hermitian forms.*** While Theorem 2.2 lets us work with a restricted set of candidates for each L_i, further pruning is facilitated by the type of each L_i. For this, we need a series of lemmas.

Lemma 2.7 [Jacobowitz 1962, Proposition 9.2]. *Let L be a π^i-modular lattice of rank* 2 *with $n(L) = n(H(i))$. Then $L \cong H(i)$ in Case* 2 *with i odd and in Case* 1 *with i even.*

Lemma 2.8 [Jacobowitz 1962, Proposition 4.4]. *A π^i-modular lattice L has an orthogonal basis if $n(L) = s(L)$.*

Lemma 2.9. *Assume that E/F satisfies Case* 2.

(1) *Let $L = A(4a, 2\delta, \pi) \oplus (2c)$ with respect to a basis (e_1, e_2, e_3), where $c \equiv 1 \bmod 2$. Then $L \cong H(1) \oplus (2c')$ where $c' \equiv 1 \bmod 2$.*

(2) *Let $L = A(4a, 2\delta, \pi) \oplus (c)$ with respect to a basis (e_1, e_2, e_3), where $c \equiv 1 \bmod 2$. Then $L \cong H(1) \oplus (c')$ where $c' \equiv 1 \bmod 2$.*

Proof. For (1), we work with the basis $(e_1 - (2a\pi/\delta)e_2,\ e_2 + e_3,\ (c\pi/\delta)e_1 + e_3)$ of L. With respect to this basis, $L \cong A(-4a - 16a^2, 2(\delta + c), \pi(1+4a)) \oplus (2c(1 - 4ac/\delta))$. Moreover, $n(A(-4a - 16a^2, 2(\delta + c), \pi(1+4a))) = n(H(1)) = (4)$. Combined with the lemma above, this completes the proof.

For (2), we note that the sublattice of L spanned by $(e_1, e_2, \pi e_3)$ is isomorphic to $A(4a, 2\delta, \pi) \oplus (2c')$ where $c' \equiv 1 \bmod 2$. If we apply (1) to this sublattice by choosing a basis $(e_1 - (2a\pi/\delta)e_2,\ e_2 + \pi e_3,\ (c'\pi/\delta)e_1 + \pi e_3)$, then $A(4a, 2\delta, \pi) \oplus (2c')$ is isomorphic to $H(1) \oplus (2c'')$ where $c'' \equiv 1 \bmod 2$. Now the sublattice of L spanned by $(e_1 - (2a\pi/\delta)e_2,\ e_2 + \pi e_3,\ \frac{1}{\pi}((c'\pi/\delta)e_1 + \pi e_3))$, which is the same as L, is isomorphic to $H(1) \oplus (-c''/\delta)$. □

The above lemmas will contribute to the proof of Theorem 2.10 below in the following manner. For a given Jordan splitting $L = \bigoplus_i L_i$ in *Case 2*, assume that L_1 is bound of type I. Theorem 2.2 tells us that there are two different possibilities for L_1 as a hermitian lattice and if $L_1 = \bigoplus H(1)$ then the conclusion of the as yet unstated Theorem 2.10, for $i = 1$, will follow. If $L_1 = \bigoplus H(1) \oplus A(4a, 2\delta, \pi)$ and either L_0 or L_2 is of type I^o, then by Lemma 2.9 and the above paragraph, $L_0 \oplus L_1 \oplus L_2 = L_0' \oplus L_1' \oplus L_2'$ such that $L_1' = \bigoplus H(1)$ and the types of L_0 and L_2 are the same as those of L_0' and L_2', respectively. In case either L_0 or L_2 is of type I^e, say L_2 is of type I^e, $L_2 = (\bigoplus H(2)) \oplus (2a) \oplus (2b)$ where $a, b \equiv 1 \bmod 2$ by Lemma 2.8. Then we use Lemma 2.9 on $L_1 \oplus (2b)$ to get $L_1 \oplus (2b) = (\bigoplus H(1)) \oplus (2b')$ with $b' \equiv 1 \bmod 2$. Thus $L_1 \oplus L_2 = L_1' \oplus L_2'$ where $L_1' = \bigoplus H(1)$ and the type of $L_2' = (\bigoplus H(2)) \oplus (2a) \oplus (2b')$ is the same as that of L_2. We conclude that $L = L_0' \oplus L_1' \oplus L_2' \oplus (\bigoplus_i L_i)$ is another Jordan splitting of L and in this case, $L_1' = \bigoplus H(1)$. Therefore, if L_1 is bound of type I in *Case 2*, then L_1 can always be replaced by $\bigoplus H(1)$. Combined with Theorem 2.2, this yields the following structure theorem:

Theorem 2.10. *There exists a suitable choice of a Jordan splitting of the given lattice $L = \bigoplus_i L_i$ such that $L_i = \bigoplus_\lambda H_\lambda \oplus K$, where each $H_\lambda = H(i)$ and K is π^i-modular of rank* 1 *or* 2, *with the following descriptions. Let $i = 0$ or $i = 1$. Then*

(a) *In Case* 1,

$$K = \begin{cases} (a) \text{ where } a \equiv 1 \bmod 2 & \text{if } i = 0 \text{ and } L_0 \text{ is of type } I^o; \\ A(1, 2b, 1) & \text{if } i = 0 \text{ and } L_0 \text{ is of type } I^e; \\ H(0) & \text{if } i = 0 \text{ and } L_0 \text{ is of type } II; \\ A(2, 2b, \pi) & \text{if } i = 1. \end{cases}$$

(b) *In Case* 2,

$$K=\begin{cases}(a)\ \textit{where}\ a\equiv 1 \bmod 2 & \textit{if } i=0 \textit{ and } L_0 \textit{ is of type } I^o;\\ A(1,2b,1) & \textit{if } i=0 \textit{ and } L_0 \textit{ is of type } I^e;\\ A(2\delta,2b,1) & \textit{if } i=0 \textit{ and } L_0 \textit{ is of type } II;\\ A(4a,2\delta,\pi) & \textit{if } i=1 \textit{ and } L_1 \textit{ is free of type } I;\\ H(1) & \textit{if } i=1, \textit{ and } L_1 \textit{ is bound of type } I \textit{ or of type } II.\end{cases}$$

Here, $a, b \in A$ *and* δ, π *are explained in Section 2A.*

From now on, the pair (L, h) is fixed throughout this paper.

Remark 2.11. Working with a basis furnished by Theorem 2.10, we can describe our lattices A_i through Z_i more explicitly. We use the following conventions. Let $\mathcal{L}_i$ denote $\bigoplus_{j\neq i}\pi^{\max\{0,i-j\}}L_j$. Further, the $\bigoplus_\lambda H_\lambda$ will be denoted by $\mathcal{H}_i$. Theorem 2.10 involves a basis for a lattice K, which we will write as $\{e_1^{(i)}, e_2^{(i)}\}$ according to the ordering contained therein. For all cases, we have $A_i=\mathcal{L}_i\oplus L_i$ and $X_i=\mathcal{L}_i\oplus\pi L_i$.

In order to write W_i, we should first find the vector $e\in A_i/X_i$ explained in the paragraph right after the definition of $B(L)$ in Section 2C. In order to simplify notations, let us work with one example. Assume that (e_1, e_2, e_3, e_4) is a B-basis of L with respect to which h is represented by the matrix

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&1\\0&0&1&2\end{pmatrix}.$$

So $L\ (=L_0=A_0)$ is of type I^e and our basis is as explained in Theorem 2.10. Now, in order to find W_0, we should find the vector $e\in L/\pi L$ explained in Section 2C (after the definition of $B(L)$). If $v=(x,y,z,w)$ is a vector in $L/\pi L$, then $h(v,v) \bmod \pi = z^2$. On the other hand, if $e=(0,0,0,1)\in L/\pi L$, then $(h(v,e))^2 \bmod \pi = z^2$. Therefore, by uniqueness of the vector e, $(0,0,0,1)\in L/\pi L$ is the vector e we are looking for.

Since W_0 is the sublattice of L such that $W_0/X_0=W_0/\pi L$ is the subspace of $L/\pi L$ spanned by the vector e, W_0 is spanned by $(\pi e_1, \pi e_2, \pi e_3, e_4)$, and it is easy to see that B_0 is spanned by $(e_1, e_2, \pi e_3, e_4)$.

To describe all lattices, it is good to start with the matrix of our fixed hermitian form h with respect to a basis furnished by Theorem 2.10.

Case 1, i even: For type I, $e=(0,\cdots,0,1)\in A_i/X_i$. The following table describes the lattices:

Type	B_i	W_i	Y_i
I^o	$\mathcal{L}_i\oplus\mathcal{H}_i\oplus(\pi)e_1^{(i)}$	$\mathcal{L}_i\oplus\pi\cdot\mathcal{H}_i\oplus Be_1^{(i)}$	X_i
I^e	$\mathcal{L}_i\oplus\mathcal{H}_i\oplus(\pi)e_1^{(i)}\oplus Be_2^{(i)}$	$\mathcal{L}_i\oplus\pi\cdot\mathcal{H}_i\oplus(\pi)e_1^{(i)}\oplus Be_2^{(i)}$	W_i
II	A_i	X_i	X_i

Case 1, i odd. We have $B_i = A_i$, $W_i = X_i$, and Y_i is not defined. Also, Z_i is a sublattice of A_i and so we should have congruence conditions for L_j. Namely,

$$Z_i = \bigoplus_{j\notin\{i\}\cup\mathcal{E}} \pi^{\max\{0,i-j\}} L_j \oplus \pi L_i \oplus \bigoplus_{j\in\mathcal{E}} \pi^{\max\{0,i-j\}}\big(\mathcal{H}_j \oplus Be_2^{(j)}\big)$$
$$\oplus \left\{ \sum_{j\in\mathcal{E}} \pi^{\max\{0,i-j\}} \cdot a_j e_1^{(j)} \;\middle|\; \text{for each } a_j \in B, \sum_{j\in\mathcal{E}} a_j \in (\pi) \right\}.$$

Here, $\mathcal{E} = \{j \in \{i-1, i+1\} \mid L_j \text{ is of type } I\}$ and the $e_2^{(j)}$ factor should be ignored for those $j \in \mathcal{E}$ such that L_j is of type I^o.

The following example would be helpful to have a better understanding of the notions of "bound" and "free" and of the notion of type when i is odd. Let $L = L_1 \oplus L_2 = A(0, 0, \pi) \oplus (2)$, so that L_1 is bound of type I (since $A_1 \neq B_1$) and L_2 is free of type I.

Case 2, i even. The B_i, W_i, and Y_i are exactly as in the table given for *Case 1*. The lattice Z_i is a little complicated. Note that when L_i is of type I or bound of type II, the dimension of Y_i/Z_i as a κ-vector space is 1. We describe it case by case below.

- Let $\mathcal{E}' = \{j \in \{i-2, i+2\} \mid L_j \text{ is of type } I\}$. If L_i is of type I so that L_{i-1} and L_{i+1} are bound,

$$Z_i = \bigoplus_{j\notin\{i,i\pm2\}} \pi^{\max\{0,i-j\}} L_j \oplus \bigoplus_{j\in\{i\pm2\}} \pi^{\max\{0,i-j\}}\big(\mathcal{H}_j \oplus Be_2^{(j)}\big) \oplus \pi\mathcal{H}_i$$
$$\oplus \left\{ \left(\sum_{j\in\mathcal{E}'} \pi^{\max\{0,i-j\}} \cdot a_j e_1^{(j)}\right) + \big(\pi \cdot a_i e_1^{(i)} + b \cdot b_i e_2^{(i)}\big) \;\middle|\; \text{for each } a_j \in B, \left(\sum_{j\in\mathcal{E}'} a_j\right) + a_i + b \cdot b_i \in (\pi) \right\},$$

  where the $e_2^{(j)}$ (resp. $e_2^{(i)}$) factor should be ignored for those $j \in \{i \pm 2\}$ (resp. i) such that L_j (resp. L_i) is not of type I^e, and $b \in B$ is such that $L_i = \pi^{i/2}(\mathcal{H}_i \oplus A(1, 2b, 1))$ when L_i is of type I^e.

- If L_i is free of type II (so that all of $L_{i\pm2}$ and $L_{i\pm1}$ are of type II), then $Z_i = X_i$.
- If L_i is bound of type II, then with $\mathcal{E}_1 = \{j \in \{i-1, i+1\} \mid L_j \text{ is free of type } I\}$ and $\mathcal{E}_2 = \{j \in \{i-2, i+2\} \mid L_j \text{ is of type } I\}$, we have

$$Z_i = \bigoplus_{j\notin\{i,i\pm1,i\pm2\}} \pi^{\max\{0,i-j\}} L_j \oplus \pi L_i$$
$$\oplus \bigoplus_{j\in\{i\pm1\}} \pi^{\max\{0,i-j\}}\big(\mathcal{H}_j \oplus Be_1^{(j)}\big) \oplus \bigoplus_{j\in\{i\pm2\}} \pi^{\max\{0,i-j\}}\big(\mathcal{H}_j \oplus Be_2^{(j)}\big)$$
$$\oplus \left\{ \left(\sum_{j\in\mathcal{E}_1} \pi^{\max\{0,i-j\}} \cdot a_j e_2^{(j)}\right) + \left(\sum_{j\in\mathcal{E}_2} \pi^{\max\{0,i-j\}} \cdot a_j e_1^{(j)}\right) \;\middle|\; \text{for each } a_j \in B, \left(\sum_{j\in\mathcal{E}_1\cup\mathcal{E}_2} a_j\right) \in (\pi) \right\}.$$

  For example, if $i+1 \in \mathcal{E}_1$, then $i+2 \notin \mathcal{E}_2$. And if $i+2 \in \mathcal{E}_2$, then $i+1 \notin \mathcal{E}_1$.

Case 2, i odd. In this case, $W_i = X_i$ and Z_i is not defined.

Type	B_i	Y_i
free of type I	$\mathcal{L}_i \oplus \mathcal{H}_i \oplus Be_1^{(i)} \oplus (\pi)e_2^{(i)}$	$\mathcal{L}_i \oplus \pi\mathcal{H}_i \oplus Be_1^{(i)} \oplus (\pi)e_2^{(i)}$
bound of type I	see below	see below
type II	A_i	X_i

When L_i is bound of type I, the dimension of A_i/B_i as κ-spaces is 1.

$$B_i = \bigoplus_{j\notin\{i\}\cup\mathcal{E}} \pi^{\max\{0,i-j\}} L_j \oplus L_i \oplus \bigoplus_{j\in\mathcal{E}} \pi^{\max\{0,i-j\}}\big(\mathcal{H}_j \oplus Be_2^{(j)}\big) \oplus \Big\{\sum_{j\in\mathcal{E}} \pi^{\max\{0,i-j\}}\cdot a_j e_1^{(j)} \;\Big|\; \text{for each } a_j \in B, \sum_{j\in\mathcal{E}} a_j \in (\pi)\Big\},$$

$$Y_i = \bigoplus_{j\notin\{i\}\cup\mathcal{E}} \pi^{\max\{0,i-j\}} L_j \oplus \pi L_i \oplus \bigoplus_{j\in\mathcal{E}} \pi^{\max\{0,i-j\}}\big(\mathcal{H}_j \oplus Be_2^{(j)}\big) \oplus \Big\{\sum_{j\in\mathcal{E}} \pi^{\max\{0,i-j\}}\cdot a_j e_1^{(j)} \;\Big|\; \text{for each } a_j \in B, \sum_{j\in\mathcal{E}} a_j \in (\pi)\Big\}.$$

Here, $\mathcal{E} = \{j \in \{i-1, i+1\} \mid L_j \text{ is of type } I\}$.

3. The construction of the smooth model

Let $\underline{G}'$ be the naive integral model of the unitary group $U(V,h)$, where $V = L\otimes_A F$, such that for any commutative A-algebra R,

$$\underline{G}'(R) = \mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R).$$

The scheme $\underline{G}'$ is then an (possibly nonsmooth) affine group scheme over A with smooth generic fiber $U(V,h)$. Then by Proposition 3.7 in [Gan and Yu 2000], there exists a unique smooth integral model, denoted by $\underline{G}$, with generic fiber $U(V,h)$, characterized by

$$\underline{G}(R) = \underline{G}'(R)$$

for any étale A-algebra R. Note that every étale A-algebra is a finite product of finite unramified extensions of A. This section, Section 4 and Appendix A are devoted to gaining an explicit knowledge of the smooth integral model $\underline{G}$ in *Case 1*, which will be used in Section 5 to compute the local density of (L,h) (again, in *Case 1*). For a detailed exposition of the relation between the local density of (L,h) and $\underline{G}$, see [Gan and Yu 2000, Section 3].

In this section, we give an explicit construction of the smooth integral model $\underline{G}$ when E/F satisfies *Case 1*. The construction of $\underline{G}$ is based on that of Section 5 in [Gan and Yu 2000] and Section 3 in [Cho 2015a]. Since the functor $R\mapsto \underline{G}(R)$ restricted to étale A-algebras R determines $\underline{G}$, we first list out some properties that are satisfied by each element of $\underline{G}(R) = \underline{G}'(R)$.

We choose an element $g\in\underline{G}(R)$ for an étale A-algebra R. Then g is an element of $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$. Here we consider $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ as a subgroup of $\mathrm{Res}_{E/F}\,\mathrm{GL}_E(V)(F\otimes_A R)$. To ease the notation, we say $g\in\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ stabilizes a lattice $M\subseteq V$ if $g(M\otimes_A R) = M\otimes_A R$.

3A. ***Main construction.*** Let R be an étale A-algebra. In this subsection, as mentioned above, we observe properties of elements of $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ and their matrix interpretations. We choose a Jordan splitting $L=\bigoplus_i L_i$ and a basis of L as explained in Theorem 2.10 and Remark 2.3(a). Let $n_i=\mathrm{rank}_B L_i$, and $n=\mathrm{rank}_B L=\sum n_i$. Assume that $n_i=0$ unless $0\leq i<N$. Let g be an element of $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$. We always divide a matrix g of size $n\times n$ into N^2 blocks such that the block in position (i,j) is of size $n_i\times n_j$. For simplicity, the row and column numbering starts at 0 rather than 1.

(1) First of all, g stabilizes A_i for every integer i. In terms of matrices, this fact means that the (i,j)-block has entries in $\pi^{\max\{0,j-i\}}B\otimes_A R$. From now on, we write
$$g=\left(\pi^{\max\{0,j-i\}}g_{i,j}\right).$$

(2) The element g stabilizes A_i, B_i, W_i, X_i and induces the identity on A_i/B_i and W_i/X_i. We also interpret these facts in terms of matrices as described below:

(a) If i is odd or L_i is of type *II*, then $A_i=B_i$ and $W_i=X_i$ and so there is no contribution.

(b) If L_i is of type I^o, the diagonal (i,i)-block $g_{i,i}$ is of the form
$$\begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1+\pi z_i \end{pmatrix}\in \mathrm{GL}_{n_i}(B\otimes_A R),$$
where s_i is an $(n_i-1)\times(n_i-1)$-matrix, etc.

(c) If L_i is of type I^e, the diagonal (i,i)-block $g_{i,i}$ is of the form
$$\begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1+\pi x_i & \pi z_i \\ v_i & u_i & 1+\pi w_i \end{pmatrix}\in \mathrm{GL}_{n_i}(B\otimes_A R),$$
where s_i is an $(n_i-2)\times(n_i-2)$-matrix, etc.

3B. ***Construction of $\underline{M}$.*** We define a functor from the category of commutative flat A-algebras to the category of monoids as follows. For any commutative flat A-algebra R, let
$$\underline{M}(R)\subset\{m\in \mathrm{End}_{B\otimes_A R}(L\otimes_A R)\}$$
to be the set of $m\in \mathrm{End}_{B\otimes_A R}(L\otimes_A R)$ satisfying the following conditions:

(1) m stabilizes $A_i\otimes_A R$, $B_i\otimes_A R$, $W_i\otimes_A R$, $X_i\otimes_A R$ for all i.

(2) m induces the identity on $A_i\otimes_A R/B_i\otimes_A R$, $W_i\otimes_A R/X_i\otimes_A R$ for all i.

Remark 3.1. We give another description for the functor $\underline{M}$ and using this, we show that it is represented by a polynomial ring. Let us define a functor from the category of commutative flat A-algebras to the category of rings as follows:

For any commutative flat A-algebra R, define
$$\underline{M}'(R)\subset\{m\in \mathrm{End}_{B\otimes_A R}(L\otimes_A R)\}$$
to be the set of $m\in \mathrm{End}_{B\otimes_A R}(L\otimes_A R)$ satisfying the following conditions:

(1) m stabilizes $A_i\otimes_A R$, $B_i\otimes_A R$, $W_i\otimes_A R$, $X_i\otimes_A R$ for all i.

(2) m maps $A_i\otimes_A R$, $W_i\otimes_A R$ into $B_i\otimes_A R$, $X_i\otimes_A R$, respectively.

Then, by Lemma 3.1 of [Cho 2015a], $\underline{M}'$ is represented by a unique flat A-algebra $A(\underline{M}')$ which is a polynomial ring over A of $2n^2$ variables. Moreover, it is easy to see that $\underline{M}'$ has the structure of a scheme of rings since $\underline{M}'(R)$ is closed under addition and multiplication.

We consider a scheme $\mathrm{Res}_{B/A}\,\mathrm{End}_B(L)$ such that the associated set to a commutative flat A-algebra R is $\mathrm{End}_{B\otimes_A R}(L\otimes_A R)$. Indeed, $\mathrm{Res}_{B/A}\,\mathrm{End}_B(L)$ is a group scheme under addition. But at this moment, we consider it as a scheme of sets so as to embed $\underline{M}$ into this. Let us consider both $\underline{M}$ and $\underline{M}'$ as functors from the category of commutative flat A-algebras to the category of sets. Then they are subfunctors of $\mathrm{Res}_{B/A}\,\mathrm{End}_B(L)$. Furthermore, the functor $\underline{M}$ (viewed as valued in sets) is the same as the functor $1+\underline{M}'$, where $(1+\underline{M}')(R)=\{1+m : m\in \underline{M}'(R)\}$. Here, the set $\mathrm{End}_{B\otimes_A R}(L\otimes_A R)$ has an obvious additive structure and the addition in the description of $(1+\underline{M}')(R)$ comes from this.

Therefore, $\underline{M}$ and $\underline{M}'$ are equivalent, as subfunctors of $\mathrm{Res}_{B/A}\,\mathrm{End}_B(L)$. This fact induces that the functor $\underline{M}$ is also represented by a unique flat A-algebra $A[\underline{M}]$ which is a polynomial ring over A of $2n^2$ variables. Moreover, it is easy to see that $\underline{M}$ has the structure of a scheme of monoids since $\underline{M}(R)$ is closed under multiplication.

We can therefore now talk of $\underline{M}(R)$ for any (not necessarily flat) A-algebra R. However, for a general R, the above description for $\underline{M}(R)$ will no longer be true. For such R, we use our chosen basis of L to write each element of $\underline{M}(R)$ formally. We describe each element of $\underline{M}(R)$ as a formal matrix $\left(\pi^{\max\{0,j-i\}}m_{i,j}\right)$. Here, $m_{i,j}$, when $i\neq j$, is an $(n_i\times n_j)$-matrix with entries in $B\otimes_A R$ and

$$m_{i,i}=\begin{cases}\begin{pmatrix} s_i & \pi y_i\\ \pi v_i & 1+\pi z_i\end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^o;\\ \begin{pmatrix} s_i & r_i & \pi t_i\\ \pi y_i & 1+\pi x_i & \pi z_i\\ v_i & u_i & 1+\pi w_i\end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^e;\\ m_{i,i} & \text{otherwise, i.e., if } L_i \text{ is of type } \mathit{II}.\end{cases}$$

Here, s_i is an $(n_i-1\times n_i-1)$-matrix (resp. $(n_i-2\times n_i-2)$-matrix) with entries in $B\otimes_A R$ if L_i of type I^o (resp. of type I^e) and $y_i, v_i, z_i, r_i, t_i, y_i, x_i, u_i, w_i$ are matrices of suitable sizes with entries in $B\otimes_A R$. Similarly, if L_i is of type II, then $m_{i,i}$ is an $(n_i\times n_j)$-matrix with entries in $B\otimes_A R$. To simplify notation, each element

$$((m_{i,j})_{i\neq j},(m_{i,i})_{L_i \text{ of type II}},(s_i, y_i, v_i, z_i)_{L_i \text{ of type } I^o},(s_i, v_i, z_i, r_i, t_i, y_i, x_i, u_i, w_i)_{L_i \text{ of type } I^e}).$$

of $\underline{M}(R)$ is denoted by $(m_{i,j}, s_i\cdots w_i)$.

In the next section, we need a description of an element of $\underline{M}(R)$ and its multiplication for a κ-algebra R. In order to prepare for this, we describe the multiplication explicitly only for a κ-algebra R. To multiply $(m_{i,j}, s_i\cdots w_i)$ and $(m'_{i,j}, s'_i\cdots w'_i)$, we form the matrices $m=\left(\pi^{\max\{0,j-i\}}m_{i,j}\right)$ and $m'=\left(\pi^{\max\{0,j-i\}}m'_{i,j}\right)$ with $s_i\cdots w_i$ and $s'_i\cdots w'_i$ and write the formal matrix product $\left(\pi^{\max\{0,j-i\}}m_{i,j}\right)\cdot\left(\pi^{\max\{0,j-i\}}m'_{i,j}\right)=\left(\pi^{\max\{0,j-i\}}\tilde{m}''_{i,j}\right)$ with

$$\tilde{m}''_{i,i}=\begin{cases}\begin{pmatrix} \tilde{s}''_i & \pi \tilde{y}''_i\\ \pi \tilde{v}''_i & 1+\pi \tilde{z}''_i\end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^o;\\ \begin{pmatrix} \tilde{s}''_i & \tilde{r}''_i & \pi \tilde{t}''_i\\ \pi \tilde{y}''_i & 1+\pi \tilde{x}''_i & \pi \tilde{z}''_i\\ \tilde{v}''_i & \tilde{u}''_i & 1+\pi \tilde{w}''_i\end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^e.\end{cases}$$

Let $(m''_{i,j}, s''_i \cdots w''_i)$ be formed by letting π^2 be zero in each entry of $(\tilde{m}''_{i,j}, \tilde{s}''_i \cdots \tilde{w}''_i)$. Then each matrix of $(m''_{i,j}, s''_i \cdots w''_i)$ has entries in $B \otimes_A R$ and so $(m''_{i,j}, s''_i \cdots w''_i)$ is an element of $\underline{M}(R)$ and is the product of $(m_{i,j}, s_i \cdots w_i)$ and $(m'_{i,j}, s'_i \cdots w'_i)$. More precisely,

(1) If $i \neq j$ or if $i = j$ and L_i is of type II,

$$m''_{i,j} = \sum_{k=1}^{N} \pi^{(\max\{0,k-i\}+\max\{0,j-k\}-\max\{0,j-i\})} m_{i,k} m'_{k,j};$$

(2) For L_i of type I^o, we write $m_{i,i-1}m'_{i-1,i} + m_{i,i+1}m'_{i+1,i} = \begin{pmatrix} a''_i & b''_i \\ c''_i & d''_i \end{pmatrix}$ and $m_{i,i-2}m'_{i-2,i} + m_{i,i+2}m'_{i+2,i} = \begin{pmatrix} \tilde{a}''_i & \tilde{b}''_i \\ \tilde{c}''_i & \tilde{d}''_i \end{pmatrix}$ where a''_i and $\tilde{a}''_i$ are $(n_i - 1) \times (n_i - 1)$-matrices, etc. Then

$$\begin{cases} s''_i = s_i s'_i + \pi a''_i; \\ y''_i = s_i y'_i + y_i + b''_i + \pi(y_i z'_i + \tilde{b}''_i); \\ v''_i = v_i s'_i + v'_i + c''_i + \pi(z_i v'_i + \tilde{c}''_i); \\ z''_i = z_i + z'_i + d''_i + \pi(z_i z'_i + v_i y'_i + \tilde{d}''_i). \end{cases}$$

(3) When L_i is of type I^e, we write

$$m_{i,i-1}m'_{i-1,i} + m_{i,i+1}m'_{i+1,i} = \begin{pmatrix} a''_i & b''_i & c''_i \\ d''_i & e''_i & f''_i \\ g''_i & h''_i & k''_i \end{pmatrix}$$

and

$$m_{i,i-2}m'_{i-2,i} + m_{i,i+2}m'_{i+2,i} = \begin{pmatrix} \tilde{a}''_i & \tilde{b}''_i & \tilde{c}''_i \\ \tilde{d}''_i & \tilde{e}''_i & \tilde{f}''_i \\ \tilde{g}''_i & \tilde{h}''_i & \tilde{k}''_i \end{pmatrix}$$

where a''_i and $\tilde{a}''_i$ are $(n_i - 2) \times (n_i - 2)$-matrices, etc. Then

$$\begin{cases} s''_i = s_i s'_i + \pi(r_i y'_i + t_i v'_i + a''_i); \\ r''_i = s_i r'_i + r_i + \pi(r_i x'_i + t_i u'_i + b''_i); \\ t''_i = s_i t'_i + r_i z'_i + t_i + c''_i + \pi(t_i w'_i + \tilde{c}''_i); \\ y''_i = y_i s'_i + y'_i + z_i v'_i + d''_i + \pi(x_i y'_i + \tilde{d}''_i); \\ x''_i = x_i + x'_i + z_i u'_i + y_i r'_i + e''_i + \pi(x_i x'_i + \tilde{e}''_i); \\ z''_i = z_i + z'_i + f''_i + \pi(y_i t'_i + x_i z'_i + z_i w'_i + \tilde{f}''_i); \\ v''_i = v_i s'_i + v'_i + \pi(u_i y'_i + w_i v'_i + g''_i); \\ u''_i = u_i + u'_i + v_i r'_i + \pi(u_i x'_i + w_i u'_i + h''_i); \\ w''_i = w_i + w'_i + v_i t'_i + u_i z'_i + k''_i + \pi(w_i w'_i + \tilde{k}''_i). \end{cases}$$

Remark 3.2. We let d be the determinant homomorphism on the algebraic monoid $\mathrm{Res}_{B/A}\,\mathrm{End}_B(L)$. We consider the inclusion

$$\iota : \underline{M} \longrightarrow \mathrm{Res}_{B/A}\,\mathrm{End}_B(L)$$

between functors of sets on the category of commutative flat A-algebras. Note that this inclusion is a morphism of schemes by Yoneda's lemma since $\underline{M}$ is flat over A. It is not an immersion as schemes since the special fiber of $\underline{M}$ is no longer embedded into that of $\operatorname{Res}_{B/A}\operatorname{End}_B(L)$. For a commutative flat A-algebra R, the multiplication on $\underline{M}(R)$ is induced from that on $\operatorname{Res}_{B/A}\operatorname{End}_B(L)(R)$ under ι. Thus the morphism ι is a morphism of monoid schemes.

We consider d as the restriction of the determinant homomorphism under ι. Then $\operatorname{Spec}(A[\underline{M}]_d)$ is an open subscheme of $\underline{M}$, where $A[\underline{M}]_d$ is the localization of the ring $A[\underline{M}]$ at d. Note that $\operatorname{Spec}(A[\underline{M}]_d)(R)$, the set of R-points of $\operatorname{Spec}(A[\underline{M}]_d)$ for a commutative A-algebra R, is characterized by

$$\{m \in \underline{M}(R) : \text{there exists } \tilde{m}' \in \operatorname{End}_{B\otimes_A R}(L \otimes_A R) \text{ such that } \iota_R(m)\cdot\tilde{m}' = \tilde{m}'\cdot\iota_R(m) = 1\}.$$

Here, $\iota_R : \underline{M}(R) \to \operatorname{Res}_{B/A}\operatorname{End}_B(L)(R)$ is a morphism of monoids induced by ι. It is easy to see that the above set $\operatorname{Spec}(A[\underline{M}]_d)(R)$ is a monoid, and hence $\operatorname{Spec}(A[\underline{M}]_d)$ is a scheme of monoids.

We define a functor $\underline{M}^*$ from the category of commutative A-algebras to the category of groups as follows. For a commutative A-algebra R, set

$$\underline{M}^*(R) = \{m \in \underline{M}(R) : \text{there exists } m' \in \underline{M}(R) \text{ such that } m\cdot m' = m'\cdot m = 1\}.$$

We claim that $\underline{M}^*$ is representable by $\operatorname{Spec}(A[\underline{M}]_d)$. For any commutative A-algebra R, the inclusion $\underline{M}^*(R) \subseteq \operatorname{Spec}(A[\underline{M}]_d)(R)$ is obvious.

In order to show $\operatorname{Spec}(A[\underline{M}]_d)(R) \subseteq \underline{M}^*(R)$, we first prove that $\tilde{m}'$ ($\in \operatorname{End}_{B\otimes_A R}(L\otimes_A R)$) associated to $m \in \underline{M}(R)$ is an element of $\underline{M}(R)$ for every flat A-algebra R. To verify this statement, it suffices to show that $\tilde{m}'$ satisfies conditions (1) and (2) defining $\underline{M}$. This follows from the following fact: if L' is a sublattice of L and m is an element of $\operatorname{Spec}(A[\underline{M}]_d)(R)$ for a flat A-algebra R which stabilizes $L' \otimes_A R$, then $L' \otimes_A R$ is stabilized by $\tilde{m}'$ as well. This can be easily proved as in Lemma 3.2 of [Cho 2015a] and so we skip the proof. Thus $\underline{M}^*(R)$ is the same as $\operatorname{Spec}(A[\underline{M}]_d)(R)$ for a flat A-algebra R. In order to show $\underline{M}^*(R) = \operatorname{Spec}(A[\underline{M}]_d)(R)$ for any commutative A-algebra R, we consider the following well-defined map, for any flat A-algebra R:

$$\begin{aligned}\operatorname{Spec}(A[\underline{M}]_d)(R) &\to \operatorname{Spec}(A[\underline{M}]_d)(R) \times \operatorname{Spec}(A[\underline{M}]_d)(R)\\ m &\mapsto (m, \tilde{m}').\end{aligned}$$

Since $\operatorname{Spec}(A[\underline{M}]_d)$ is flat, this map is represented by a morphism of schemes by Yoneda's lemma. On the other hand, since $\operatorname{Spec}(A[\underline{M}]_d)$ is a scheme of monoids, the map

$$\begin{aligned}\operatorname{Spec}(A[\underline{M}]_d)(R) \times \operatorname{Spec}(A[\underline{M}]_d)(R) &\to \operatorname{Spec}(A[\underline{M}]_d)(R)\\ (m, m') &\mapsto mm'\end{aligned}$$

is represented by a morphism of schemes. We consider the composite of these two morphisms. It is the constant map (at the identity) at least at the level of R-points, for a flat A-algebra R. To show that the composite is the constant morphism of schemes (at the identity), it suffices to show that it is uniquely determined at the level of R-points, for a flat A-algebra R. Note that $\operatorname{Spec}(A[\underline{M}]_d)$ is an irreducible smooth affine scheme. We consider the open subscheme of $\operatorname{Spec}(A[\underline{M}]_d)$ which is the complement of the closed subscheme of $\operatorname{Spec}(A[\underline{M}]_d)$ determined by the prime ideal (2). This open subscheme of $\operatorname{Spec}(A[\underline{M}]_d)$

is then nonempty and dense since $\mathrm{Spec}(A[\underline{M}]_d)$ is reduced and irreducible. Furthermore, all R-points of $\mathrm{Spec}(A[\underline{M}]_d)$, for a flat A-algebra R, factor through this open subscheme. Since a morphism of schemes is continuous, the above composite is uniquely determined at the level of R-points, for a flat A-algebra R.

Thus, the inverse of $m \in \mathrm{Spec}(A[\underline{M}]_d)(R)$, for any commutative A-algebra R, is also contained in $\mathrm{Spec}(A[\underline{M}]_d)(R) \subseteq \underline{M}(R)$. This fact implies $\underline{M}^*(R) \supseteq \mathrm{Spec}(A[\underline{M}]_d)(R)$. Consequently, for any commutative A-algebra R, we have

$$\underline{M}^*(R) = \mathrm{Spec}(A[\underline{M}]_d)(R).$$

Therefore, we conclude that $\underline{M}^*$ is an open subscheme of $\underline{M}$ (since $\underline{M}^* = \mathrm{Spec}(A[\underline{M}]_d)$, which is an open subscheme of $\underline{M}$), with generic fiber $M^* = \mathrm{Res}_{E/F}\,\mathrm{GL}_E(V)$, and that $\underline{M}^*$ is smooth over A. Moreover, $\underline{M}^*$ is a group scheme since $\underline{M}$ is a scheme in monoids.

3C. *Construction of $\underline{H}$.* Recall that the pair (L, h) is fixed throughout this paper and the lattices A_i, B_i, W_i, X_i only depend on the hermitian pair (L, h). For any flat A-algebra R, let $\underline{H}(R)$ be the set of hermitian forms f on $L \otimes_A R$ (with values in $B \otimes_A R$) such that f satisfies the following conditions:

(a) $f(L \otimes_A R, A_i \otimes_A R) \subset \pi^i B \otimes_A R$ for all i.

(b) $\xi^{-m} f(a_i, a_i) \bmod 2 = \xi^{-m} h(a_i, a_i) \bmod 2$, where $a_i \in A_i \otimes_A R$, and $i = 2m$.

(c) $\frac{1}{\pi^i} f(a_i, w_i) = \frac{1}{\pi^i} h(a_i, w_i) \bmod \pi$, where $a_i \in A_i \otimes_A R$ and $w_i \in W_i \otimes_A R$, and $i = 2m$.

We interpret the above conditions in terms of matrices. The matrix forms are taken with respect to the basis of L fixed in Theorem 2.10 and Remark 2.3(a). A matrix form of the given hermitian form h is described in Remark 3.3(1) below. We use σ to mean the automorphism of $B \otimes_A R$ given by $b \otimes r \mapsto \sigma(b) \otimes r$. For a flat A-algebra R, $\underline{H}(R)$ is the set of hermitian matrices

$$\left(\pi^{\max\{i,j\}} f_{i,j}\right)$$

of size $n \times n$ satisfying the following:

(1) $f_{i,j}$ is an $(n_i \times n_j)$-matrix with entries in $B \otimes_A R$.

(2) If i is even and L_i is of type I^o, then $\pi^i f_{i,i}$ is of the form

$$\xi^{i/2} \begin{pmatrix} a_i & \pi b_i \\ \sigma(\pi \cdot {}^t b_i) & 1 + 2c_i \end{pmatrix}.$$

Here, the diagonal entries of a_i are divisible by 2, where a_i is an $(n_i - 1) \times (n_i - 1)$-matrix with entries in $B \otimes_A R$, etc.

(3) If i is even and L_i is of type I^e, then $\pi^i f_{i,i}$ is of the form

$$\xi^{i/2} \begin{pmatrix} a_i & b_i & \pi e_i \\ \sigma({}^t b_i) & 1 + 2f_i & 1 + \pi d_i \\ \sigma(\pi \cdot {}^t e_i) & \sigma(1 + \pi d_i) & 2c_i \end{pmatrix}.$$

Here, the diagonal entries of a_i are divisible by 2, where a_i is an $(n_i - 2) \times (n_i - 2)$-matrix with entries in $B \otimes_A R$, etc.

(4) Assume that L_i is of type II. The diagonal entries of $f_{i,i}$ (resp. $\pi f_{i,i}$) are divisible by 2 if i is even (resp. odd).

(5) Since $\left(\pi^{\max\{i,j\}} f_{i,j}\right)$ is a hermitian matrix, its diagonal entries are fixed by the nontrivial Galois action over E/F and hence belong to R.

Let us consider the *hermitian functor* from the category of commutative flat A-algebras to the category of sets such that the associated set to R is the set of hermitian forms f on $L \otimes_A R$ (with values in $B \otimes_A R$). Indeed, this functor is represented by a commutative group scheme since it is closed under addition. Then $\underline{H}$ is a subfunctor of the hermitian functor. We consider another functor $\underline{H}'$ such that $\underline{H}'(R) = \{f - h : f \in \underline{H}(R)\}$. Note that h is the fixed hermitian form and the notion of $f - h$ follows from the additive structure of the hermitian functor. For a matrix interpretation of h, we refer to Remark 3.3(1) below.

Then by Lemma 3.1 of [Cho 2015a], $\underline{H}'$ is represented by a flat A-scheme which is isomorphic to an affine space. Since $\underline{H}$ and $\underline{H}'$ are equivalent as subfunctors of the hermitian functor, the functor $\underline{H}$ is also represented by a flat A-scheme which is isomorphic to an affine space.

To compute the dimension of $\underline{H}$, we see that each entry of the upper triangular matrix of an element of $\underline{H}(R)$, for a flat A-algebra R, gives two variables and each diagonal entry gives one variable. Furthermore, each lower triangular entry of the matrix representing an element of $\underline{H}(R)$ is completely determined by the corresponding upper triangular entry. Thus the dimension of $\underline{H}$ is $2 \cdot n(n-1)/2 + n = n^2$. This is also the same as $2n^2 - \dim U(V, h) = n^2$.

Now suppose that R is any (not necessarily flat) A-algebra. Recall that ϵ is a unit in B such that $\sigma(\pi) = \epsilon\pi$ and $(\epsilon - 1)/\pi$ is a unit in B. We also use ϵ to mean $\epsilon \otimes 1$ in $B \otimes_A R$. We again use σ to mean the automorphism of $B \otimes_A R$ given by $b \otimes r \mapsto \sigma(b) \otimes r$. By choosing a B-basis of L as explained in Theorem 2.10 and Remark 2.3(a), we describe each element of $\underline{H}(R)$ formally as a matrix $\left(\pi^{\max\{i,j\}} f_{i,j}\right)$ with the following:

(1) When $i \neq j$, $f_{i,j}$ is an $(n_i \times n_j)$-matrix with entries in $B \otimes_A R$ and $\epsilon^{\max\{i,j\}} \sigma({}^t f_{i,j}) = f_{j,i}$.

(2) Assume that $i = j$ is even. Then

$$\pi^i f_{i,i} = \begin{cases} \xi^{i/2} \begin{pmatrix} a_i & \pi b_i \\ \sigma(\pi \cdot {}^t b_i) & 1 + 2c_i \end{pmatrix} & \text{if } L_i \text{ is of type } I^o; \\ \xi^{i/2} \begin{pmatrix} a_i & b_i & \pi e_i \\ \sigma({}^t b_i) & 1 + 2f_i & 1 + \pi d_i \\ \sigma(\pi \cdot {}^t e_i) & \sigma(1 + \pi d_i) & 2c_i \end{pmatrix} & \text{if } L_i \text{ is of type } I^e; \\ \xi^{i/2} a_i & \text{if } L_i \text{ is of type } \mathit{II}. \end{cases}$$

Here, a_i is a formal $(n_i - 1 \times n_i - 1)$-matrix (resp. $(n_i - 2 \times n_i - 2)$-matrix or $(n_i \times n_i)$-matrix) when L_i is of type I^o (resp. of type I^e or of type II). Nondiagonal entries of a_i are in $B \otimes_A R$ and the j-th diagonal entry of a_i is of the form $2x_i^j$ with $x_i^j \in R$. In addition, for nondiagonal entries of a_i, we have the relation $\sigma({}^t a_i) = a_i$. And b_i, d_i, e_i are matrices of suitable sizes with entries in $B \otimes_A R$ and c_i, f_i are elements in R.

(3) Assume that $i = j$ is odd. Then

$$\pi^i f_{i,i} = \xi^{(i-1)/2} \pi a_i,$$

where a_i is a formal $(n_i \times n_i)$-matrix. Here, nondiagonal entries of a_i are in $B \otimes_A R$ and the j-th diagonal entry of a_i is of the form $\epsilon\pi x_i^j$ with $x_i^j \in R$. In addition, for nondiagonal entries of a_i, we have the relation $\epsilon \cdot \sigma({}^t a_i) = a_i$.

To simplify notation, each element

$$((f_{i,j})_{i<j}, (a_i, x_i^j)_{L_i \text{ of type } II}, (a_i, x_i^j, b_i, c_i)_{L_i \text{ of type } I^o}, (a_i, x_i^j, b_i, c_i, d_i, e_i, f_i)_{L_i \text{ of type } I^e})$$

of $\underline{H}(R)$ is denoted by $(f_{i,j}, a_i \cdots f_i)$.

Remark 3.3. (1) Note that the given hermitian form h is an element of $\underline{H}(A)$. We represent the given hermitian form h by a hermitian matrix $(\pi^i \cdot h_i)$ whose (i,i)-block is $\pi^i \cdot h_i$ for all i, and all of whose remaining blocks are 0. Then:

(a) If i is even and L_i is of type I^o, then $\pi^i \cdot h_i$ has the following form (with $\gamma_i \in A$):

$$\xi^{i/2}\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & & 1+2\gamma_i \end{pmatrix}.$$

(b) If i is even and L_i is of type I^e, then $\pi^i \cdot h_i$ has the following form (with $\gamma_i \in A$):

$$\xi^{i/2}\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & & \begin{pmatrix} 1 & 1 \\ 1 & 2\gamma_i \end{pmatrix} \end{pmatrix}.$$

(c) If i is even and L_i is of type II, then $\pi^i \cdot h_i$ has the following form:

$$\xi^{i/2}\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}.$$

(d) If i is odd, then $\pi^i \cdot h_i$ has the following form (with $\gamma_i \in A$):

$$\xi^{(i-1)/2}\begin{pmatrix} \begin{pmatrix} 0 & \pi \\ \sigma(\pi) & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & \pi \\ \sigma(\pi) & 0 \end{pmatrix} & \\ & & & \begin{pmatrix} 2 & \pi \\ \sigma(\pi) & 2\gamma_i \end{pmatrix} \end{pmatrix}.$$

(2) Let R be a κ-algebra. We also denote by h the element of $\underline{H}(R)$ which is the image of $h \in \underline{H}(A)$ under the natural map from $\underline{H}(A)$ to $\underline{H}(R)$. Recall that we denote each element of $\underline{H}(R)$ by $(f_{i,j}, a_i \cdots f_i)$. Then the tuple $(f_{i,j}, a_i \cdots f_i)$ denoting $h \in \underline{H}(R)$ is defined by the conditions:

(a) If $i \neq j$, then $f_{i,j} = 0$.

(b) If i is even, then

$$a_i = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}, \quad \text{thus } x_i^j = 0,$$

$$b_i = 0, d_i = 0, e_i = 0, f_i = 0, c_i = \bar{\gamma}_i.$$

Here, $\bar{\gamma}_i \in \kappa$ is the reduction of $\gamma_i \mod 2$.

(c) If i is odd, then

$$a_i = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ \bar{\epsilon} & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ \bar{\epsilon} & 0 \end{pmatrix} & \\ & & & \begin{pmatrix} \pi \cdot \bar{\epsilon}\bar{\zeta} & 1 \\ \bar{\epsilon} & \pi \cdot \bar{\epsilon}\bar{\zeta}\bar{\gamma}_i \end{pmatrix} \end{pmatrix}.$$

Here, $\bar{\epsilon}$ is the reduction of $\epsilon \mod 2$, not mod π, so that $\bar{\epsilon}$ is an element of $B \otimes_A R$. In addition, $\bar{\zeta} \in \kappa$ is the reduction of $\zeta \mod 2$, where $\zeta \in A$ is the unit satisfying $2 = \xi \cdot \zeta$. Thus, $x_i^j = 0$ for all $1 \leq j \leq n_i - 2$ and $x_i^{n_i-1} = \bar{\zeta}$ and $x_i^{n_i} = \bar{\zeta}\bar{\gamma}_i$.

3D. *The smooth affine group scheme $\underline{G}$.*

Theorem 3.4. *For any flat A-algebra R, the group $\underline{M}^*(R)$ acts on $\underline{H}(R)$ on the right by $f \circ m = \sigma({}^t m) \cdot f \cdot m$. This action is represented by an action morphism*

$$\underline{H} \times \underline{M}^* \longrightarrow \underline{H}.$$

Proof. We start with any $m \in \underline{M}^*(R)$ and $f \in \underline{H}(R)$. In order to show that $\underline{M}^*(R)$ acts on the right of $\underline{H}(R)$ by $f \circ m = \sigma({}^t m) \cdot f \cdot m$, it suffices to show that $f \circ m$ satisfies conditions (a) to (c) given in Section 3C. Since elements of $\underline{M}(R)$ preserve $L \otimes_A R$ and $A_i \otimes_A R$, $f \circ m$ satisfies condition (a). That $f \circ m$ satisfies condition (b) follows from the fact that m stabilizes A_i and B_i and induces the identity on A_i/B_i.

For condition (c), it suffices to show that $\frac{1}{\pi^i} f(ma_i, mw_i) \equiv \frac{1}{\pi^i} f(a_i, w_i) \mod \pi$. We denote $ma_i = a_i + b_i$ and $mw_i = w_i + x_i$, where $b_i \in B_i \otimes_A R$, $x_i \in X_i \otimes_A R$. Hence it suffices to show $\frac{1}{\pi^i} f(a_i + b_i, x_i) + \frac{1}{\pi^i} f(b_i, w_i) \mod \pi \equiv 0$. Firstly, $\frac{1}{\pi^i} f(a_i + b_i, x_i) \mod \pi \equiv 0$ due to the definition of the lattice X_i. Secondly, if $B_i \subsetneq A_i$, then $\frac{1}{\pi^i} f(b_i, w_i) \mod \pi \equiv 0$ because $\frac{1}{\pi^i} f(b_i, w_i) = \frac{1}{\pi^i} h(b_i, w_i) \mod \pi$ and $(\xi^{-m} h(b_i, e))^2 \equiv \xi^{-m} h(b_i, b_i) \equiv 0 \mod \pi$, where e is the unique vector chosen earlier. If $B_i = A_i$, then $W_i = X_i$ and thus $\frac{1}{\pi^i} f(b_i, w_i) \bmod \pi \equiv 0$.

We now show that this action of the group $\underline{M}^*(R)$ on the right of $\underline{H}(R)$ is represented by an action morphism of schemes. We observe that the action map $\underline{H}(R) \times \underline{M}^*(R) \longrightarrow \underline{H}(R)$, $(f, m) \mapsto \sigma({}^t m) \cdot f \cdot m$ is given by polynomials over A. Thus it induces a ring homomorphism over A from the coordinate ring of $\underline{H}$ to the coordinate ring of $\underline{H} \times \underline{M}^*$, which accordingly induces a morphism from $\underline{H} \times \underline{M}^*$ to $\underline{H}$ such that the action map induced by this morphism at the level of R-points, for a flat A-algebra R, is the same as the action given in the theorem. □

Remark 3.5. Let R be a κ-algebra. We explain the above action morphism in terms of R-points. Choose an element $(m_{i,j}, s_i \cdots w_i)$ in $\underline{M}^*(R)$ as explained in Section 3B and express this element formally as a matrix $m = \left(\pi^{\max\{0, j-i\}} m_{i,j}\right)$. We also choose an element $(f_{i,j}, a_i \cdots f_i)$ of $\underline{H}(R)$ and express this element formally as a matrix $f = \left(\pi^{\max\{i,j\}} f_{i,j}\right)$ as explained in Section 3C.

We then compute the formal matrix product $\sigma({}^t m) \cdot f \cdot m$ and denote it by the formal matrix $\left(\pi^{\max\{i,j\}} \tilde{f}'_{i,j}\right)$ with $(\tilde{f}'_{i,j}, \tilde{a}'_i \cdots \tilde{f}'_i)$. Here, the description of the formal matrix $\left(\pi^{\max\{i,j\}} \tilde{f}'_{i,j}\right)$ with $(\tilde{f}'_{i,j}, \tilde{a}'_i \cdots \tilde{f}'_i)$ is as explained in Section 3C.

We now let π^2 be zero in each entry of the formal matrices $(\tilde{f}'_{i,j})_{i<j}$, $(\tilde{b}'_i)_{L_i \text{ of type } I^o}$, $(\tilde{b}'_i, \tilde{d}'_i, \tilde{e}'_i)_{L_i \text{ of type } I^e}$ and in each nondiagonal entry of the formal matrix $(\tilde{a}'_i)$. Then these entries are elements in $B \otimes_A R$. We also let π^2 be zero in $(\tilde{x}_i^j)'$, $(\tilde{c}'_i)_{L_i \text{ of type } I^o}$, $(\tilde{f}'_i, \tilde{c}'_i)_{L_i \text{ of type } I^e}$. Note that $(\tilde{x}_i^j)'$ is a diagonal entry of a formal matrix $\tilde{a}'_i$. Then these entries are elements in R.

Let $(f'_{i,j}, a'_i \cdots f'_i)$ be the reduction of $(\tilde{f}'_{i,j}, \tilde{a}'_i \cdots \tilde{f}'_i)$ as explained above, i.e., by letting π^2 be zero in the entries of formal matrices as described above. Then $(f'_{i,j}, a'_i \cdots f'_i)$ is an element of $\underline{H}(R)$ and the composition $(f_{i,j}, a_i \cdots f_i) \circ (m_{i,j}, s_i \cdots w_i)$ is $(f'_{i,j}, a'_i \cdots f'_i)$.

We can also write $(f'_{i,j}, a'_i \cdots f'_i)$ explicitly in terms of $(f_{i,j}, a_i \cdots f_i)$ and $(m_{i,j}, s_i \cdots w_i)$ like the product of $(m_{i,j}, s_i \cdots w_i)$ and $(m'_{i,j}, s'_i \cdots w'_i)$ explained in Section 3B. However, this is complicated and we do not use it in this generality. On the other hand, we explicitly calculate $(f_{i,j}, a_i \cdots f_i) \circ (m_{i,j}, s_i \cdots w_i)$ when $(f_{i,j}, a_i \cdots f_i)$ is the given hermitian form h and $(m_{i,j}, s_i \cdots w_i)$ satisfies certain conditions on each block. This explicit calculation will be done in Appendix A.

Theorem 3.6. *Let ρ be the morphism $\underline{M}^* \to \underline{H}$ defined by $\rho(m) = h \circ m$, which is induced by the action morphism of Theorem 3.4. Then ρ is smooth of relative dimension* $\dim U(V, h)$.

Proof. The theorem follows from Lemma 5.5.1 of [Gan and Yu 2000] and the following lemma. □

Lemma 3.7. *The morphism $\rho \otimes \kappa : \underline{M}^* \otimes \kappa \to \underline{H} \otimes \kappa$ is smooth of relative dimension* $\dim U(V, h)$.

Proof. The proof is based on Lemma 5.5.2 in [Gan and Yu 2000]. It is enough to check the statement over the algebraic closure $\bar{\kappa}$ of κ. By [Hartshorne 1977, Proposition III.10.4], it suffices to show that, for any $m \in \underline{M}^*(\bar{\kappa})$, the induced map on the Zariski tangent space $\rho_{*,m} : T_m \to T_{\rho(m)}$ is surjective.

We define the two functors from the category of commutative flat A-algebras to the category of abelian groups as follows:

$$T_1(R) = \{m - 1 : m \in \underline{M}(R)\},$$
$$T_2(R) = \{f - h : f \in \underline{H}(R)\}.$$

The functor T_1 (resp. T_2) is representable by a flat A-algebra which is a polynomial ring over A of $2n^2$ (resp. n^2) variables by Lemma 3.1 of [Cho 2015a]. Moreover, each of them is represented by a commutative group scheme since they are closed under addition. In fact, T_1 is the same as the functor $\underline{M}'$ in Remark 3.1 and T_2 is the same as the functor $\underline{H}'$ in Section 3C.

We still need to introduce another functor on flat A-algebras. Define $T_3(R)$ to be the set of all $(n \times n)$-matrices y over $B \otimes_A R$ satisfying the following conditions:

(a) The (i, j)-block of y has entries in $\pi^{\max\{i,j\}} B \otimes_A R$ so that

$$y = \left(\pi^{\max\{i,j\}} y_{i,j}\right).$$

Here, the size of $y_{i,j}$ is $n_i \times n_j$.

(b) If i is even and L_i is of type I^o, then $y_{i,i}$ is of the form

$$\begin{pmatrix} s_i & \pi y_i \\ \pi v_i & \pi z_i \end{pmatrix} \in M_{n_i}(B \otimes_A R)$$

where s_i is an $(n_i - 1) \times (n_i - 1)$-matrix, etc.

(c) If i is even and L_i is of type I^e, then $y_{i,i}$ is of the form

$$\begin{pmatrix} s_i & r_i & \pi t_i \\ y_i & x_i & \pi z_i \\ \pi v_i & \pi u_i & \pi w_i \end{pmatrix} \in M_{n_i}(B \otimes_A R)$$

where s_i is an $(n_i - 2) \times (n_i - 2)$-matrix, etc.

The functor T_3 is represented by a flat A-scheme which is isomorphic to an affine space by Lemma 3.1 of [Cho 2015a]. Moreover it is represented by a commutative group scheme since it is closed under addition. So far, we have defined three functors T_1, T_2, T_3 and these are represented by schemes. Therefore, we can talk about their $\bar{\kappa}$-points.

We now compute the map $\rho_{*,m}$ explicitly. We first describe an element of the tangent space T_m. Since $\underline{M}^*$ is an open subscheme of $\underline{M}$, the tangent space T_m may and shall be identified with the set of elements of $\underline{M}(\bar{\kappa}[\epsilon]/(\epsilon^2))$ whose reduction to $\underline{M}(\bar{\kappa})$ induced by the obvious map $\bar{\kappa}[\epsilon]/(\epsilon^2) \to \bar{\kappa}$ is m, by considering m as an element of $\underline{M}(\bar{\kappa})$. Recall from Remark 3.1 that we defined the functor $\underline{M}'$ such that $(1 + \underline{M}')(R) = \underline{M}(R)$ inside $\mathrm{End}_{B\otimes_A R}(L \otimes_A R)$ for a flat A-algebra R. Thus there is an isomorphism of schemes (as set valued functors)

$$1+ : \underline{M}' \longrightarrow \underline{M}.$$

Let m' be an element of $\underline{M}'(\bar{\kappa})$ which maps to m under the morphism $1+$ at the level of $\bar{\kappa}$-points. Then each element of the tangent space of $\underline{M}'$ at m' is of the form $m' + \epsilon X \in$

$\underline{M}'(\bar{\kappa}[\epsilon]/(\epsilon^2))$ for $X \in \underline{M}'(\bar{\kappa})$. We denote by $m+\epsilon X$ the image of $m'+\epsilon X$ under the morphism $1+$ at the level of $\bar{\kappa}[\epsilon]/(\epsilon^2)$-points. Thus we can express an element of T_m formally as $m+\epsilon X$ where $X \in \underline{M}'(\bar{\kappa})$. Similarly, an element of $T_{\rho(m)}$ can be expressed formally as $\rho(m)+\epsilon Y$ where $Y \in \underline{H}'(\bar{\kappa})$, by using an isomorphism of schemes (as set valued functors)

$$h+ : \underline{H}' \longrightarrow \underline{H}.$$

Here, $\underline{H}'$ is defined in Section 3C.

Before observing the image of $m+\epsilon X$ under the morphism ρ at the level of $\bar{\kappa}[\epsilon]/(\epsilon^2)$-points, we lift $m+\epsilon X$ to an element of $\underline{M}(R[\epsilon]/(\epsilon^2))$ as follows, where R is a local A-algebra whose residue field is $\bar{\kappa}$. Let $\tilde{m}' \in \underline{M}'(R)$ (resp. $\widetilde{X} \in \underline{M}'(R)$) be a lift of m' (resp. X) so that $\tilde{m}'+\epsilon\widetilde{X} \in \underline{M}'(R[\epsilon]/(\epsilon^2))$ is a lift of $m'+\epsilon X \in \underline{M}'(\bar{\kappa}[\epsilon]/(\epsilon^2))$. Let $\tilde{m} \in \underline{M}(R)$ be the image of $\tilde{m}'$ under the morphism $1+$. Then $\tilde{m}+\epsilon\widetilde{X}$ is an element of $\underline{M}(R[\epsilon]/(\epsilon^2))$ whose reduction to $\underline{M}(\bar{\kappa}[\epsilon]/(\epsilon^2))$ induced by the map $R[\epsilon]/(\epsilon^2) \to \bar{\kappa}[\epsilon]/(\epsilon^2)$ is $m+\epsilon X$. Here, the addition in $\tilde{m}+\epsilon\widetilde{X}$ is the addition inside $\mathrm{End}_{B\otimes_A R[\epsilon]/(\epsilon^2)}(L\otimes_A R[\epsilon]/(\epsilon^2))$ since $R[\epsilon]/(\epsilon^2)$ is flat over A (cf. Remark 3.1). This is illustrated in the following commutative diagrams:

$$\begin{array}{ccc}
\underline{M}'(R[\epsilon]/(\epsilon^2)) & \xrightarrow{1+} & \underline{M}(R[\epsilon]/(\epsilon^2)) \\
\downarrow & & \downarrow \\
\underline{M}'(\bar{\kappa}[\epsilon]/(\epsilon^2)) & \xrightarrow{1+} & \underline{M}(\bar{\kappa}[\epsilon]/(\epsilon^2))
\end{array}$$

$$\begin{array}{ccc}
\tilde{m}'+\epsilon\widetilde{X} & \longmapsto & \tilde{m}+\epsilon\widetilde{X} \\
\downarrow & & \downarrow \\
m'+\epsilon X & \longmapsto & m+\epsilon X
\end{array}$$

Note that the proof of Theorem 3.4 also gives the existence of the morphism $\underline{H}\times\underline{M} \to \underline{H}$, defined by $(f, m) \mapsto f\circ m = \sigma({}^t m)\cdot f\cdot m$, where $f \in \underline{H}(R)$ and $m \in \underline{M}(R)$ for a flat A-algebra R. This morphism induces the morphism $\underline{M} \to \underline{H}$ with $m \mapsto h\circ m$ whose reduction to $\underline{M}^*$ is the same as ρ. Thus the above morphism $\underline{M} \to \underline{H}$ can also be denoted by ρ. We can now talk about the image of $\tilde{m}+\epsilon\widetilde{X}$ under the morphism ρ at the level of $R[\epsilon]/(\epsilon^2)$-points. Since $R[\epsilon]/(\epsilon^2)$ is a flat A-algebra, the image of $\tilde{m}+\epsilon\widetilde{X}$ comes from a usual matrix product

$$\sigma(\tilde{m}+\epsilon\widetilde{X})^t\cdot h\cdot(\tilde{m}+\epsilon\widetilde{X}) = \sigma(\tilde{m})^t\cdot h\cdot\tilde{m}+\epsilon(\sigma(\tilde{m})^t\cdot h\cdot\widetilde{X}+\sigma(\widetilde{X})^t\cdot h\cdot\tilde{m}). \tag{3-1}$$

Thus the image of $m+\epsilon X$ under the morphism ρ at the level of $\bar{\kappa}[\epsilon]/(\epsilon^2)$-points is the reduction of $\sigma(\tilde{m})^t\cdot h\cdot\tilde{m}+\epsilon(\sigma(\tilde{m})^t\cdot h\cdot\widetilde{X}+\sigma(\widetilde{X})^t\cdot h\cdot\tilde{m})$ to $\underline{H}(\bar{\kappa}[\epsilon]/(\epsilon^2))$. It is obvious that $\rho(m)$ $(\in \underline{H}(\bar{\kappa}))$ is the reduction of $\sigma(\tilde{m})^t\cdot h\cdot\tilde{m}$ $(\in \underline{H}(R))$ since $\tilde{m}$ is a lift of m and ρ is a morphism of schemes. To observe the reduction of $\sigma(\tilde{m})^t\cdot h\cdot\widetilde{X}+\sigma(\widetilde{X})^t\cdot h\cdot\tilde{m}$ $(\in \underline{H}'(R))$ to $\underline{H}'(\bar{\kappa})$, we consider a morphism $\underline{M}\times\underline{H}' \to \underline{H}'$ such that $(\tilde{m}, \widetilde{X})$ maps to $\sigma(\tilde{m})^t\cdot h\cdot\widetilde{X}+\sigma(\widetilde{X})^t\cdot h\cdot\tilde{m}$, where $(\tilde{m}, \widetilde{X}) \in \underline{M}(R)\times\underline{H}'(R)$ for a flat A-algebra R. To show that this map is well-defined, we need to show that $\sigma(\tilde{m})^t\cdot h\cdot\widetilde{X}+\sigma(\widetilde{X})^t\cdot h\cdot\tilde{m}$ is an element of $\underline{H}'(R)$. This can be easily shown by considering the morphism of tangent spaces induced from ρ at $\tilde{m} \in \underline{M}(R)$ (cf. Equation (3-1)). Since this morphism is representable, we can denote by $\sigma(m)^t \cdot h \cdot X + \sigma(X)^t \cdot h \cdot m$ ($\in \underline{H}'(\bar{\kappa})$) the reduction of $\sigma(\tilde{m})^t \cdot h \cdot \widetilde{X} + \sigma(\widetilde{X})^t \cdot h \cdot \tilde{m}$ ($\in \underline{H}'(R)$) to $\underline{H}'(\bar{\kappa})$. Then the image of $m + \epsilon X$ is a formal sum $\rho(m) + \epsilon(\sigma(m)^t \cdot h \cdot X + \sigma(X)^t \cdot h \cdot m)$ ($\in \underline{H}(\bar{\kappa}[\epsilon]/(\epsilon^2))$).

Thus if we identify T_m with $T_1(\bar{\kappa})$ and $T_{\rho(m)}$ with $T_2(\bar{\kappa})$, then

$$\rho_{*,m} : T_m \to T_{\rho(m)}$$
$$X \mapsto \sigma(m)^t \cdot h \cdot X + \sigma(X)^t \cdot h \cdot m.$$

We explain how to compute $X \mapsto \sigma(m)^t \cdot h \cdot X + \sigma(X)^t \cdot h \cdot m$ explicitly. Recall that for a κ-algebra R, we denote an element m of $\underline{M}(R)$ by $(m_{i,j}, s_i \cdots w_i)$ with a formal matrix interpretation $m = (\pi^{\max\{0,j-i\}} m_{i,j})$ (cf. Section 3B) and we denote an element f of $\underline{H}(R)$ by $(f_{i,j}, a_i \cdots f_i)$ with a formal matrix interpretation $f = (\pi^{\max\{i,j\}} f_{i,j})$ (cf. Section 3C). Similarly, we can also denote an element X of $T_1(\bar{\kappa})$ by $(m'_{i,j}, s'_i \cdots w'_i)$ with a formal matrix interpretation $X = (\pi^{\max\{0,j-i\}} m'_{i,j})$ and an element Z of $T_2(\bar{\kappa})$ by $(f'_{i,j}, a'_i \cdots f'_i)$ with a formal matrix interpretation $Z = (\pi^{\max\{i,j\}} f'_{i,j})$. Then we formally compute $X \mapsto \sigma(m^t) \cdot h \cdot X + \sigma(X^t) \cdot h \cdot m$ and consider the reduction of the formal matrix $\sigma(m^t) \cdot h \cdot X + \sigma(X^t) \cdot h \cdot m$ in a manner similar to that of the reduction explained in Remark 3.5. We denote this reduction by $(f''_{i,j}, a''_i \cdots f''_i)$ with a formal matrix interpretation $(\pi^{\max\{i,j\}} f''_{i,j})$. This $(f''_{i,j}, a''_i \cdots f''_i)$ may and shall be identified with an element of $T_2(\bar{\kappa})$ in the manner just described. Then $\rho_{*,m}(X)$ is the element $Z = (f''_{i,j}, a''_i \cdots f''_i)$ of $T_2(\bar{\kappa})$.

To prove the surjectivity of $\rho_{*,m} : T_1(\bar{\kappa}) \to T_2(\bar{\kappa})$, it suffices to show the following three statements:

(1) $X \mapsto h \cdot X$ defines a bijection $T_1(\bar{\kappa}) \to T_3(\bar{\kappa})$;

(2) for any $m \in \underline{M}^*(\bar{\kappa})$, $Y \mapsto \sigma({}^t m) \cdot Y$ defines a bijection from $T_3(\bar{\kappa})$ to itself;

(3) $Y \mapsto \sigma({}^t Y) + Y$ defines a surjection $T_3(\bar{\kappa}) \to T_2(\bar{\kappa})$.

Here, all the above maps are interpreted as in Remark 3.5 (if they are well-defined). Then $\rho_{*,m}$ is the composite of these three. Condition (3) is direct from the construction of $T_3(\bar{\kappa})$. Hence we provide the proof of (1) and (2).

For (1), suppose that the two functors $T_1(R) \longrightarrow T_3(R)$, $X \mapsto h \cdot X (\in M_{n\times n}(B \otimes_A R))$ and $T_3(R) \longrightarrow T_1(R)$, $Y \mapsto h^{-1} \cdot Y (\in M_{n\times n}(B \otimes_A R))$ are well-defined for all flat A-algebras R. In other words, suppose that $h \cdot X \in T_3(R)$ and $h^{-1} \cdot Y \in T_1(R)$. These functors are then represented by morphisms of schemes by an argument similar to that used in the proof of Theorem 3.4, so we skip it. Thus they give maps at the level of κ-algebra points. Furthermore, the composition of these two maps at the level of κ-algebra points is the identity. To show this, it suffices to prove that the composition of two morphisms given by the actions of h and h^{-1} is uniquely determined at the level of R-points, for a flat A-algebra R. This is proved in Remark 3.2.

We now show that these two functors are well-defined for a flat A-algebra R. We represent h by a hermitian block matrix $\left(\pi^i \cdot h_i\right)$ with a matrix $\left(\pi^i \cdot h_i\right)$ for the (i,i)-block and 0 for the remaining blocks as in Remark 3.3(1).

For the first functor, it suffices to show that $h \cdot X$ satisfies the three conditions defining the functor T_3. Here, $X \in T_1(R)$ for a flat A-algebra R. We write

$$X = \left(\pi^{\max\{0, j-i\}} x_{i,j}\right).$$

Then

$$h \cdot X = \left(\pi^{\max(i,j)} y_{i,j}\right).$$

Here, $y_{i,i} = h_i \cdot x_{i,i}$. Therefore, it suffices to show that $y_{i,i} = h_i \cdot x_{i,i}$ satisfies conditions (b) and (c) in the description of $T_3(R)$ when L_i is of type I.

If L_i is of type I^o, then we express $x_{i,i}$ as a matrix $\left(\begin{smallmatrix} s_i & \pi y_i \\ \pi v_i & \pi z_i \end{smallmatrix}\right)$. The matrix form of h_i is

$$\epsilon^{i/2} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & & 1+2\gamma_i \end{pmatrix}$$

as in Remark 3.3(1). Here, ϵ is a unit in B such that $\sigma(\pi) = \epsilon\pi$, as explained in Section 2A. To simplify our notation, write $h_i = \epsilon^{i/2}\left(\begin{smallmatrix} I_i & 0 \\ 0 & 1+2\gamma_i \end{smallmatrix}\right)$. Then we can see that

$$h_i \cdot x_{i,i} = \epsilon^{i/2} \begin{pmatrix} I_i & 0 \\ 0 & 1+2\gamma_i \end{pmatrix} \cdot \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & \pi z_i \end{pmatrix} = \epsilon^{i/2} \begin{pmatrix} I_i s_i & \pi I_i y_i \\ \pi(1+2\gamma_i) v_i & \pi(1+2\gamma_i) z_i \end{pmatrix}.$$

Thus, $h_i \cdot x_{i,i}$ satisfies the congruence condition given in (b) of the description of $T_3(R)$.

If L_i is of type I^e, then we express $x_{i,i}$ as a matrix

$$\begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & \pi x_i & \pi z_i \\ v_i & u_i & \pi w_i \end{pmatrix}.$$

The matrix form of h_i is given as in Remark 3.3(1) and again, in order to simplify our notation, write

$$h_i = \epsilon^{i/2} \begin{pmatrix} I_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2\gamma_i \end{pmatrix}.$$

Then we can see that

$$\begin{aligned} h_i \cdot x_{i,i} &= \epsilon^{i/2} \begin{pmatrix} I_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2\gamma_i \end{pmatrix} \cdot \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & \pi x_i & \pi z_i \\ v_i & u_i & \pi w_i \end{pmatrix} \\ &= \epsilon^{i/2} \begin{pmatrix} I_i s_i & I_i r_i & \pi I_i t_i \\ \pi y_i + v_i & \pi x_i + u_i & \pi(z_i + w_i) \\ \pi y_i + 2\gamma_i v_i & \pi x_i + 2\gamma_i u_i & \pi z_i + 2\gamma_i \pi w_i \end{pmatrix}. \end{aligned}$$

Thus, $h_i \cdot x_{i,i}$ satisfies the congruence condition given in c) of the description of $T_3(R)$ and our functor is well-defined.

For the second functor, we write $Y=\left(\pi^{\max(i,j)}y_{i,j}\right)$ and $h^{-1}=\left(\pi^{-i}\cdot h_i^{-1}\right)$. Then we have the following:

$$h^{-1}\cdot Y=\left(\pi^{\max\{0,j-i\}}x_{i,j}\right).$$

Here, $x_{i,i}=h_i^{-1}\cdot y_{i,i}$.

Then it suffices to show that $h^{-1}\cdot Y=\left(\pi^{\max\{0,j-i\}}x_{i,j}\right)$ satisfies the conditions defining $T_1(R)$ for a flat A-algebra R. Indeed, we do not describe the conditions defining $T_1(R)$ explicitly in this paper. However, these conditions can be read off from the conditions defining $\underline{M}(R)$ because of the definition of the functor T_1. The matrix form of an element of $\underline{M}(R)$ is described in Section 3B and based on this, it suffices to observe the diagonal blocks $x_{i,i}=h_i^{-1}\cdot y_{i,i}$ when L_i is of type I.

If L_i is of type I^o, then we express $y_{i,i}$ as a matrix $\begin{pmatrix} s_i & \pi y_i \\ \pi v_i & \pi z_i\end{pmatrix}$. The matrix form of h_i^{-1} is $h_i^{-1}=\epsilon^{-i/2}\begin{pmatrix} I_i & 0 \\ 0 & 1+2\gamma_i'\end{pmatrix}$ for a certain $\gamma_i'\in A$. Then we can see that

$$h_i^{-1}\cdot y_{i,i}=\epsilon^{-i/2}\begin{pmatrix} I_i & 0 \\ 0 & 1+2\gamma_i'\end{pmatrix}\cdot\begin{pmatrix} s_i & \pi y_i \\ \pi v_i & \pi z_i\end{pmatrix}=\epsilon^{i/2}\begin{pmatrix} I_i s_i & \pi I_i y_i \\ \pi(1+2\gamma_i')v_i & \pi(1+2\gamma_i')z_i\end{pmatrix}.$$

Thus, $h_i^{-1}\cdot y_{i,i}$ satisfies the relevant congruence condition in the definition of $T_1(R)$.

If L_i is of type I^e, then we express $y_{i,i}$ as a matrix

$$\begin{pmatrix} s_i & r_i & \pi t_i \\ y_i & x_i & \pi z_i \\ \pi v_i & \pi u_i & \pi w_i\end{pmatrix}.$$

The matrix form of h_i^{-1} is

$$h_i^{-1}=\epsilon^{-i/2}\begin{pmatrix} I_i & 0 & 0 \\ 0 & 2\epsilon'\gamma_i & -\epsilon' \\ 0 & -\epsilon' & \epsilon'\end{pmatrix}.$$

Here, $\epsilon'=(2\gamma_i-1)^{-1}$ is a unit in A. Then we can see that $h_i^{-1}\cdot y_{i,i}$ is

$$\begin{aligned}\epsilon^{-i/2}&\begin{pmatrix} I_i & 0 & 0 \\ 0 & 2\epsilon'\gamma_i & -\epsilon' \\ 0 & -\epsilon' & \epsilon'\end{pmatrix}\cdot\begin{pmatrix} s_i & r_i & \pi t_i \\ y_i & x_i & \pi z_i \\ \pi v_i & \pi u_i & \pi w_i\end{pmatrix}\\ &=\epsilon^{-i/2}\begin{pmatrix} I_i s_i & I_i r_i & \pi I_i t_i \\ \epsilon'(2\gamma_i y_i-\pi v_i) & \epsilon'(2\gamma_i x_i-\pi u_i) & \pi\epsilon'(2\gamma_i z_i-w_i) \\ \epsilon'(-y_i+\pi v_i) & \epsilon'(-x_i+\pi u_i) & \pi\epsilon'(-z_i+w_i)\end{pmatrix}.\end{aligned}$$

Thus, $h_i^{-1}\cdot y_{i,i}$ satisfies the relevant congruence condition in the definition of $T_1(R)$ and our functor is well-defined.

For (2), suppose that the functor

$$\underline{M}^*(R)\times T_3(R)\longrightarrow T_3(R),\ (m,Y)\mapsto\sigma({}^tm)\cdot Y,$$

for a flat A-algebra R, is well-defined. In other words, we suppose that $\sigma({}^tm)\cdot Y\in T_3(R)$. This functor is then represented by a morphism of schemes, a fact whose proof is similar to the argument used in the proof of Theorem 3.4, so we skip it. Thus it gives the map at the level of $\bar{\kappa}$-points

$$\underline{M}^*(\bar{\kappa}) \times T_3(\bar{\kappa}) \longrightarrow T_3(\bar{\kappa}),\ (m, Y) \mapsto \sigma({}^t m) \cdot Y.$$

This map implies that our map in (2) is well-defined. On the other hand, the inverse of our map in (2) is $Y \mapsto \sigma({}^t m)^{-1} \cdot Y$ and this map is well-defined as well since m^{-1} is also an element of $\underline{M}^*(\bar{\kappa})$. Therefore, the map in (2) is a bijection.

We now show that the above functor is well-defined. For a flat A-algebra, we choose an element $m \in \underline{M}^*(R)$ and $Y \in T_3(R)$ and we again express $m = \left(\pi^{\max\{0, j-i\}} m_{i,j}\right)$ and $Y = \left(\pi^{\max(i,j)} y_{i,j}\right)$. Then $\sigma({}^t m) \cdot Y$ obviously satisfies condition (a) in the definition of $T_3(R)$ and it suffices to show that $\sigma({}^t m_{i,i}) \cdot y_{i,i}$ satisfies conditions (b) and (c) when L_i is of type I.

If L_i is of type I^o, then we express $m_{i,i}$ as a matrix $\left(\begin{smallmatrix} s_i & \pi y_i \\ \pi v_i & 1+\pi z_i \end{smallmatrix}\right)$ and $y_{i,i}$ as a matrix $\left(\begin{smallmatrix} a_i & \pi b_i \\ \pi c_i & \pi d_i \end{smallmatrix}\right)$. Then

$$\sigma({}^t m_{i,i}) \cdot y_{i,i} = \begin{pmatrix} \sigma({}^t s_i) & \sigma(\pi \cdot {}^t v_i) \\ \sigma(\pi \cdot {}^t y_i) & 1+\sigma(\pi z_i) \end{pmatrix} \cdot \begin{pmatrix} a_i & \pi b_i \\ \pi c_i & \pi d_i \end{pmatrix}.$$

Then we can easily see that this matrix satisfies congruence condition (b) in the definition of $T_3(R)$.

If L_i is of type I^e, then we express $m_{i,i}$ and $y_{i,i}$ as matrices:

$$m_{i,i} = \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1+\pi x_i & \pi z_i \\ v_i & u_i & 1+\pi w_i \end{pmatrix} \quad \text{and} \quad y_{i,i} = \begin{pmatrix} a_i & b_i & \pi c_i \\ d_i & e_i & \pi f_i \\ \pi g_i & \pi h_i & \pi k_i \end{pmatrix}.$$

Then

$$\sigma({}^t m_{i,i}) \cdot y_{i,i} = \begin{pmatrix} \sigma({}^t s_i) & \sigma(\pi \cdot {}^t y_i) & \sigma({}^t v_i) \\ \sigma({}^t r_i) & 1+\sigma(\pi x_i) & \sigma(u_i) \\ \sigma(\pi \cdot {}^t t_i) & \sigma(\pi z_i) & 1+\sigma(\pi w_i) \end{pmatrix} \cdot \begin{pmatrix} a_i & b_i & \pi c_i \\ d_i & e_i & \pi f_i \\ \pi g_i & \pi h_i & \pi k_i \end{pmatrix}.$$

Then we can easily see that this matrix satisfies congruence condition (c) in the definition of $T_3(R)$. □

Let $\underline{G}$ be the stabilizer of h in $\underline{M}^*$. It is an affine group subscheme of $\underline{M}^*$, defined over A. Thus we have the following theorem.

Theorem 3.8. *The group scheme $\underline{G}$ is smooth, and $\underline{G}(R) = \mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h \otimes_A R)$ for any étale A-algebra R.*

Proof. Since $\underline{G}$ is the fiber of h along the smooth morphism $\rho : \underline{M}^* \to \underline{H}$, $\rho(m) = h \circ m$, the scheme $\underline{G}$ is smooth. Here, we use the fact that smoothness is stable under base change.

For the identity, we recall that each element of $\mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h \otimes_A R)$, for an étale A-algebra R, satisfies all congruence conditions defining $\underline{M}$, which is explained in Section 3A. Since $\underline{G}(R)$ is the group of R-points of $\underline{M}^*$ stabilizing the given hermitian form h, we have the identity $\underline{G}(R) = \mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h \otimes_A R)$ for any étale A-algebra R. □

Note that in the theorem, the equality holds only for an étale A-algebra R since we obtain conditions defining $\underline{M}$ by observing properties of elements of $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ for an étale A-algebra R (cf. Section 3A). For example, let (L, h) be the hermitian lattice of rank 1 as given in Appendix B. For simplicity, let $\pi+\sigma(\pi)=\pi^2=2$. As a set, $\mathrm{Aut}_{B\otimes_A R}(L\otimes_A R, h\otimes_A R)$ is the same as $\{(a,b): a, b\in R \text{ and } a^2+2ab+2b^2=1\}$ for a flat A-algebra R. Thus we cannot guarantee that $a-1$ is contained in the ideal (2), which should be necessary in order that (a,b) is an element of $\underline{G}(R)$.

4. The special fiber of the smooth integral model

In this section, we will determine the structure of the special fiber $\widetilde{G}$ of $\underline{G}$ by determining the maximal reductive quotient and the component group when E/F satisfies *Case 1*, by adapting the approach of Section 4 of [Cho 2015a]. From this section to the end, the identity matrix is denoted by id.

4A. ***The reductive quotient of the special fiber.*** Recall that Y_i is the sublattice of B_i such that $Y_i/\pi A_i$ is the radical of the alternating bilinear form $\xi^{-i/2}h \bmod \pi$ on $B_i/\pi A_i$ (when i is even) and that Z_i is the sublattice of A_i such that $Z_i/\pi A_i$ is the radical of the quadratic form $\frac{1}{2^m}q \bmod 2$ on $A_i/\pi A_i$, where $\frac{1}{2^m}q(x)=\frac{1}{2^m}h(x,x)$ (when $i=2m-1$ is odd).

Lemma 4.1. *Let i be odd. Consider the lattice $\pi A_{i-1}+A_{i+1}=\{x+y : x\in\pi A_{i-1},\ y\in A_{i+1}\}$. Then $\pi A_{i-1}+A_{i+1}=X_i$.*

Proof. Let $L=\bigoplus_i L_i$ be a Jordan splitting. We describe πA_{i-1}, A_{i+1}, X_i below:

$$\begin{aligned}\pi A_{i-1} &= \pi^i L_0\oplus\pi^{i-1}L_1\oplus\cdots\oplus\pi L_{i-1}\oplus\pi L_i\oplus\pi L_{i+1}\oplus\cdots;\\ A_{i+1} &= \pi^{i+1}L_0\oplus\pi^i L_1\oplus\cdots\oplus\pi^2 L_{i-1}\oplus\pi L_i\oplus L_{i+1}\oplus\cdots;\\ X_i &= \pi^i L_0\oplus\pi^{i-1}L_1\oplus\cdots\oplus\pi L_{i-1}\oplus\pi L_i\oplus L_{i+1}\oplus\cdots.\end{aligned}$$

Our claim follows directly from the above descriptions. □

Lemma 4.2. *Each element of $\underline{M}(R)$, for a flat A-algebra R, preserves $Y_i\otimes_A R$ (for i even) and $Z_i\otimes_A R$ (for i odd).*

Proof. The claim for Y_i follows from the fact that $Y_i=X_i$ or $Y_i=W_i$ according to the type of L_i as described in Remark 2.11.

To prove the claim for Z_i, use Lemma 4.1 to express a given arbitrary element of $Z_i\otimes_A R$ as $x+y$, where $x\in\pi A_{i-1}\otimes_A R$ and $y\in A_{i+1}\otimes_A R$. Let $g\in\underline{M}(R)$. Then $g(x+y)=g(x)+g(y)=(x+x')+(y+y')$, where $x'\in\pi B_{i-1}\otimes_A R$, $y'\in B_{i+1}\otimes_A R$ since g induces the identity on $(A_{i-1}\otimes_A R)/(B_{i-1}\otimes_A R)$ and on $(A_{i+1}\otimes_A R)/(B_{i+1}\otimes_A R)$. Since $\pi A_{i-1}\otimes_A R$ and $A_{i+1}\otimes_A R$ are contained in $W_i\otimes_A R$ and hence $\pi B_{i-1}\otimes_A R$ and $B_{i+1}\otimes_A R$ are contained in $Z_i\otimes_A R$, we have that $g(x+y)=(x+y)+x'+y'\in Z_i\otimes_A R$. □

Theorem 4.3. *Assume that i is even. Let h_i denote the nonsingular alternating bilinear form $\xi^{-i/2}h \bmod \pi$ on B_i/Y_i. Then there exists a unique morphism of algebraic groups*

$$\varphi_i : \widetilde{G}\longrightarrow \mathrm{Sp}(B_i/Y_i, h_i)$$

defined over κ such that for all étale local A-algebras R with residue field κ_R and every

$\tilde{m} \in \underline{G}(R)$ with reduction $m \in \widetilde{G}(\kappa_R)$, $\varphi_i(m) \in \mathrm{GL}(B_i \otimes_A R/Y_i \otimes_A R)$ is induced by the action of $\tilde{m}$ on $L \otimes_A R$ (which preserves $B_i \otimes_A R$ and $Y_i \otimes_A R$ by Lemma 4.2). Note that the dimension of B_i/Y_i, as a κ-vector space, is as follows:

$$\begin{cases} n_i & \text{if } L_i \text{ is of type } II; \\ n_i - 1 & \text{if } L_i \text{ is of type } I^o; \\ n_i - 2 & \text{if } L_i \text{ is of type } I^e. \end{cases}$$

Proof. Let R be an étale local A-algebra with κ_R as its residue field. Note that such an R is finite over A since any étale local algebra R over a henselian local ring is finite by Proposition 4 of Section 2.3 in [Bosch et al. 1990] and since A is henselian. For such a finite field extension κ_R of κ, R is uniquely determined up to isomorphism. Since $\underline{G}$ is smooth over A, the map $\underline{G}(R) \to \widetilde{G}(\kappa_R)$ is surjective by Hensel's lemma.

Now, we choose an element $m \in \widetilde{G}(\kappa_R)$ and a lift $\tilde{m} \in \underline{G}(R)$. Since the action of $\tilde{m}$ on $L \otimes_A R$ preserves $B_i \otimes_A R$ and $Y_i \otimes_A R$, $\tilde{m}$ determines an element of $\mathrm{GL}(B_i \otimes_A R/Y_i \otimes_A R)$. It is also easy to show that this element determined by $\tilde{m}$ fixes $h_i \otimes \kappa_R$ on $B_i/Y_i \otimes_\kappa \kappa_R$ $(= B_i \otimes_A R/Y_i \otimes_A R)$. Thus $\tilde{m}$ determines an element of $\mathrm{Sp}(B_i/Y_i, h_i)(\kappa_R)$ and so we have a map from $\widetilde{G}(\kappa_R)$ to $\mathrm{Sp}(B_i/Y_i, h_i)(\kappa_R)$. Indeed, this map is well-defined, i.e., independent of a lift $\tilde{m}$ of m as will be explained later after describing a matrix interpretation of this map. In order to show that this map is well-defined and representable, we interpret it in terms of matrices. Recall that $\underline{G}$ is a closed subgroup scheme of $\underline{M}^*$ and $\widetilde{G}$ is a closed subgroup scheme of $\widetilde{M}$, where $\widetilde{M}$ is the special fiber of $\underline{M}^*$. Thus we may consider an element of $\widetilde{G}(\kappa_R)$ as an element of $\widetilde{M}(\kappa_R)$. Based on Section 3B, an element m of $\widetilde{G}(\kappa_R)$ may be written as, say, $(m_{i,j}, s_i \cdots w_i)$ and it has the following formal matrix description:

$$m = \left(\pi^{\max\{0, j-i\}} m_{i,j}\right).$$

Here, if i is even and L_i is of type I^o or of type I^e, then

$$m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix},$$

respectively, where $s_i \in M_{(n_i-1)\times(n_i-1)}(B \otimes_A \kappa_R)$ (resp. $s_i \in M_{(n_i-2)\times(n_i-2)}(B \otimes_A \kappa_R)$), etc., and s_i is invertible. For the remaining $m_{i,j}$'s except for the cases explained above, $m_{i,j} \in M_{n_i \times n_j}(B \otimes_A \kappa_R)$ and $m_{i,i}$ is invertible. Note that the description of the multiplication in $\widetilde{M}(\kappa_R)$ given in Section 3B forces s_i and $m_{i,i}$ to be invertible.

We can write $m_{i,i} = m_{i,i}^1 + \pi \cdot m_{i,i}^2$ when L_i is of type II and for each block of $m_{i,i}$ when L_i is of type I, $s_i = s_i^1 + \pi \cdot s_i^2$ and so on. Here, $m_{i,i}^1, m_{i,i}^2 \in M_{n_i \times n_i}(\kappa_R) \subset M_{n_i \times n_i}(B \otimes_A \kappa_R)$ when L_i is of type II and so on, and π stands for $\pi \otimes 1 \in B \otimes_A \kappa_R$. Then m maps to $m_{i,i}^1$ if L_i is of type II and s_i^1 if L_i is of type I. Since this map is independent of the choice of $m_{i,i}^2$, s_i^2 and so on, it is independent of the choice of $\tilde{m}$, i.e., this map is well-defined.

We note that this map is given by polynomials over A of degree at most 1 as well as a group homomorphism. Thus the above matrix interpretation induces a Hopf algebra homomorphism over A from the coordinate ring of $\mathrm{Sp}(B_i/Y_i, h_i)$ to the coordinate ring of $\widetilde{G}$, which accordingly induces an algebraic group homomorphism $\varphi_i : \widetilde{G} \to \mathrm{Sp}(B_i/Y_i, h_i)$

such that the group homomorphism induced by φ_i at the level of κ_R-points is the same as the map explained above.

Since $\widetilde{G}$ is smooth over κ and κ is perfect, the set of κ_R-points of $\widetilde{G}$ for all finite field extensions κ_R/κ is dense in $\widetilde{G}$ by [Bosch et al. 1990, Corollary 13 of Section 2.2]. Therefore, φ_i is uniquely determined by the map constructed above at the level of κ_R-points. □

Theorem 4.4. *We next assume that $i = 2m-1$ is odd. Let $\bar{q}_i$ denote the nonsingular quadratic form $\frac{1}{2^m}q \bmod 2$ on A_i/Z_i. Then there exists a unique morphism of algebraic groups*

$$\varphi_i : \widetilde{G} \longrightarrow O(A_i/Z_i, \bar{q}_i)_{\text{red}}$$

defined over κ, where $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ is the reduced subgroup scheme of $O(A_i/Z_i, \bar{q}_i)$, such that for all étale local A-algebras R with residue field κ_R and every $\tilde{m} \in \underline{G}(R)$ with reduction $m \in \widetilde{G}(\kappa_R)$, $\varphi_i(m) \in \mathrm{GL}(A_i \otimes_A R/Z_i \otimes_A R)$ is induced by the action of $\tilde{m}$ on $L \otimes_A R$ (which preserves $A_i \otimes_A R$ and $Z_i \otimes_A R$ by Lemma 4.2).

Proof. The proof of this theorem is similar to that of Theorem 4.3 which deals with the case of even i. Thus we only provide the image of an element m of $\widetilde{G}(\kappa_R)$ in $O(A_i/Z_i, \bar{q}_i)_{\text{red}}(\kappa_R)$, where R is an étale local A-algebra with κ_R as its residue field. In this case, an element m of $\widetilde{G}(\kappa_R)$ maps to $m^1_{i,i}$ (if L_i is *free*) or to $\begin{pmatrix} m^1_{i,i} & 0 \\ \delta_{i-1}e_{i-1}\cdot m^1_{i-1,i}+\delta_{i+1}e_{i+1}\cdot m^1_{i+1,i} & 1 \end{pmatrix}$ (if L_i is *bound*). Here, $\delta_j = 1$ if L_j is of type I and $\delta_j = 0$ if L_j is of type II. Also, $e_j = (0, \cdots, 0, 1)$ (resp. $e_j = (0, \cdots, 0, 1, 0)$) of size $1 \times n_j$ if L_j is of type I^o (resp. of type I^e). □

Notice that if the dimension of A_i/Z_i is even and positive, then $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ ($= O(A_i/Z_i, \bar{q}_i)$) is disconnected. If the dimension of A_i/Z_i is odd, then $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ ($= \mathrm{SO}(A_i/Z_i, \bar{q}_i)$) is connected. The dimension of A_i/Z_i, as a κ-vector space, is as follows:

$$\begin{cases} n_i & \text{if } L_i \text{ is free;} \\ n_i + 1 & \text{if } L_i \text{ is bound.} \end{cases}$$

Note that the integer n_i, with i odd, is always even.

Theorem 4.5. *The morphism φ defined by*

$$\varphi = \prod_i \varphi_i : \widetilde{G} \longrightarrow \prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\text{red}}$$

is surjective.

Proof. Let us first prove the theorem under the assumption that

$$\dim \widetilde{G} = \dim \operatorname{Ker} \varphi + \sum_{i \text{ even}} (\dim \mathrm{Sp}(B_i/Y_i, h_i)) + \sum_{i \text{ odd}} (\dim O(A_i/Z_i, \bar{q}_i)_{\text{red}}). \tag{4-1}$$

This equation will be proved in Appendix A. Thus $\operatorname{Im} \varphi$ contains the identity component of $\prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\text{red}}$. Here $\operatorname{Ker} \varphi$ denotes the kernel of φ and $\operatorname{Im} \varphi$ denotes the image of φ. Note that it is well known that the image of a homomorphism of algebraic groups is a closed subgroup.

Recall from Section 3B that a matrix form of an element of $\widetilde{G}(R)$ for a κ-algebra R is written $(m_{i,j}, s_i \cdots w_i)$ with the formal matrix interpretation

$$m = \left(\pi^{\max\{0, j-i\}} m_{i,j}\right).$$

We represent the given hermitian form h by a hermitian matrix $(\pi^i \cdot h_i)$ with $\pi^i \cdot h_i$ for the (i, i)-block and 0 for the remaining blocks, as in Remark 3.3(1).

Let $\mathcal{H}$ be the set of odd integers i such that $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ is disconnected. Notice that $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ is disconnected exactly when L_i with i odd is *free*. We first prove that φ_i, for such an odd integer i, is surjective. We prove this by a series of reductions, after which we will be able to assume that L is of rank two.

For such an odd integer i with a free lattice L_i, we define the closed scheme H_i of $\widetilde{G}$ by the equations $m_{j,k} = 0$ if $j \neq k$, and $m_{j,j} = \text{id}$ if $j \neq i$. An element of $H_i(R)$ for a κ-algebra R can be represented by a matrix of the form

$$\begin{pmatrix} \text{id} & 0 & & \dots & & & 0 \\ 0 & \ddots & & & & & \\ & & \text{id} & & & & \\ \vdots & & & m_{i,i} & & & \vdots \\ & & & & \text{id} & & \\ & & & & & \ddots & 0 \\ 0 & & & \dots & & 0 & \text{id} \end{pmatrix}.$$

Obviously, H_i has a group scheme structure. We claim that φ_i is surjective from H_i to $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ (recall that $Z_i = X_i$ since L_i is free). Note that equations defining H_i are induced by the formal matrix equation

$$\sigma({}^t m_{i,i})(\pi^i \cdot h_i) m_{i,i} = \pi^i \cdot h_i$$

which is interpreted as in Remark 3.5. We emphasize that, in this formal matrix equation, we work with $m_{i,i}$, not m, because of the description of H_i. Note that none of the congruence conditions mentioned in Section 3A involve any entry from $m_{i,i}$.

On the other hand, let us consider the hermitian lattice L_i independently as a π^i-modular lattice. Since there is only one nontrivial Jordan component for this lattice and i is odd, the smooth integral model associated to L_i is determined by the following formal matrix equation which is interpreted as in Remark 3.5:

$$\sigma({}^t m)(\pi^i \cdot h_i) m = \pi^i \cdot h_i,$$

where m is an $(n_i \times n_i)$-matrix and is not subject to any congruence condition.

We consider the map from H_i to the special fiber of the smooth integral model associated to the hermitian lattice L_i such that $m_{i,i}$ maps to m. Since $m_{i,i}$ and m are subject to the same set of equations, this map is an isomorphism as algebraic groups. In addition, this map induces compatibility between the morphism φ_i from H_i to $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ and the morphism from the special fiber of the smooth integral model associated to L_i to $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$. Thus, in order to show that φ_i is surjective from H_i to $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$, we may and do assume that $L = L_i$ and in this case $Z_i = X_i = \pi L_i$. For simplicity, we can also assume that $i = 1$.

Because of Equation (4-1) stated at the beginning of the proof, the dimension of the image of φ_i, as a κ-algebraic group, is the same as that of $O(A_i/Z_i, \bar{q}_i)_{\text{red}}$ $(= O(L_i/\pi L_i, \bar{q}_i))$. Therefore, the image of φ_i contains the identity component of $O(L_i/\pi L_i, \bar{q}_i)$, namely $\text{SO}(L_i/\pi L_i, \bar{q}_i)$. Since $O(L_i/\pi L_i, \bar{q}_i)$ has two connected components, we only need to show the surjectivity of φ_i at the level of κ-points and it suffices to show that the image of $\varphi_i(\kappa)$ contains at least one element which is not contained in $\mathrm{SO}(L_i/\pi L_i, \bar{q}_i)(\kappa)$, where $\mathrm{SO}(L_i/\pi L_i, \bar{q}_i)(\kappa)$ is the group of κ-points of the algebraic group $\mathrm{SO}(L_i/\pi L_i, \bar{q}_i)$.

Recall that $L_i = \bigoplus_\lambda H_\lambda \oplus A(2, 2b, \pi)$ for a certain $b \in A$, cf. Theorem 2.10. We consider the orthogonal group associated to the quadratic κ-space $A(2, 2b, \pi)/\pi A(2, 2b, \pi)$ of dimension 2. Then this group is embedded into $O(L_i/\pi L_i, \bar{q}_i)(\kappa)$ as a closed subgroup and we denote the embedded group by $O(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$.

We express an element $m_{i,i} \in H_i(R)$, for a κ-algebra R, as $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ such that $x = x^1 + \pi x^2$ and so on, where $x^1, x^2 \in M_{(n_i-2)\times(n_i-2)}(R) \subset M_{(n_i-2)\times(n_i-2)}(R \otimes_A B)$ and π stands for $1 \otimes \pi \in R \otimes_A B$. Consider the closed subscheme of H_i defined by the equations $x = \mathrm{id}$, $y = 0$, and $z = 0$. An argument similar to one used above to reduce to the case where $L = L_i$ shows that this subscheme is isomorphic to the special fiber of the smooth integral model associated to the hermitian lattice $A(2, 2b, \pi)$ of rank 2. Then under the map $\varphi_i(\kappa)$, an element of this subgroup maps to an element of $O(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$ of the form $\begin{pmatrix} \mathrm{id} & 0 \\ 0 & w^1 \end{pmatrix}$. Note that $O(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$ is not contained in $\mathrm{SO}(L_i/\pi L_i, \bar{q}_i)(\kappa)$. Thus it suffices to show that the restriction of $\varphi_i(\kappa)$ to the above subgroup of $H_i(\kappa)$, which is given by letting $x = \mathrm{id}$, $y = 0$, $z = 0$, is surjective onto $O(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$ and we may and do assume that $L = L_i = A(2, 2b, \pi)$ is of rank 2.

Let $m_{i,i} = \begin{pmatrix} r & s \\ t & v \end{pmatrix}$ be an element of $H_i(\kappa)$ such that $r = r_1 + \pi r_2$ and so on, where $r_1, r_2 \in R \subset R \otimes_A B$ and π stands for $1 \otimes \pi \in R \otimes_A B$. Recall that $\pi = 1 + \sqrt{1+2u}$ for a certain unit $u \in A$ so that $\pi + \sigma(\pi) = 2$, $\sigma(\pi) = \epsilon\pi$ with $\epsilon \equiv 1 \bmod \pi$, and $\pi^2 \equiv (\sigma(\pi))^2 \equiv \xi^- \equiv 2u \bmod 2\pi$ as mentioned in Section 2A. Let $\bar{u} \in \kappa$ be the reduction of u modulo π. Then the equations defining $H_i(\kappa)$ are

$$r_1^2 + r_1 t_1 + b t_1^2 = 1, \quad r_1 v_1 + t_1 s_1 = 1,$$
$$r_1 s_1 + b t_1 v_1 + \bar{u}(r_2 v_1 + r_1 v_2 + t_2 s_1 + t_1 s_2) = 0, \quad s_1^2 + s_1 v_1 + b v_1^2 = b.$$

Under the map $\varphi_i(\kappa)$, $m_{i,i}$ maps to $\begin{pmatrix} r_1 & s_1 \\ t_1 & v_1 \end{pmatrix}$. Note that the quadratic form $\bar{q}_i$ restricted to $A(2, 2b, \pi)/\pi A(2, 2b, \pi)$ is given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix}$.

We now choose an element of $H_i(\kappa)$ by setting

$$r_1 = s_1 = v_1 = 1, \quad t_1 = 0, \quad 1 + \bar{u}(r_2 + v_2 + t_2) = 0.$$

Under the morphism $\varphi_i(\kappa)$, this element maps to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in O(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$. The Dickson invariant of this element is nontrivial so that it is not contained in $\mathrm{SO}(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$.

Therefore, $\varphi_i(\kappa)$ induces a surjection from $H_i(\kappa)$ to $O(A(2, 2b, \pi)/\pi A(2, 2b, \pi), \bar{q}_i)(\kappa)$ for $i \in \mathcal{H}$.

We now prove that $\varphi = \prod_i \varphi_i$ is surjective. We consider the morphism

$$\prod_{i \in \mathcal{H}} H_i \to \widetilde{G}$$
$$(h_i)_{i \in \mathcal{H}} \mapsto \prod_{i \in \mathcal{H}} h_i.$$

By considering a formal matrix form of an element of $H_i(R)$ for a κ-algebra R as given above, it is easy to see the following two facts. Firstly, H_i and H_j commute with each other in the sense that $h_i \cdot h_j = h_j \cdot h_i$ for all $i \neq j$, where $h_i \in H_i(R)$ and $h_j \in H_j(R)$ for a κ-algebra R. Based on this, the above morphism becomes a group homomorphism. Secondly, $H_i \cap H_j = 0$ for all $i \neq j$. This fact implies that the morphism $H_i \times H_j \longrightarrow \widetilde{G}$, $(h_i, h_j) \mapsto h_i \cdot h_j$ is injective and so $H_i \times H_j$ is a closed subgroup scheme of $\widetilde{G}$. A matrix form of an element of $H_i(R)$ also implies that $(H_i \times H_j) \cap H_k = 0$ for all pairwise different three integers i, j, k and so the morphism $(H_i \times H_j) \times H_k \longrightarrow \widetilde{G}$, $(h_i, h_j, h_k) \mapsto h_i \cdot h_j \cdot h_k$ is injective. Thus $H_i \times H_j \times H_k$ is a closed subgroup scheme of $\widetilde{G}$. Therefore, by repeating this argument, the product $\prod_{i \in \mathcal{H}} H_i$ is embedded into $\widetilde{G}$ as a closed subgroup scheme. Since $\varphi_i|_{H_j}$ is trivial for $i \neq j$ with $i, j \in \mathcal{H}$, the morphism

$$\prod_{i \in \mathcal{H}} \varphi_i : \prod_{i \in \mathcal{H}} H_i \to \prod_{i \in \mathcal{H}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}}$$

is surjective. Therefore, φ is surjective. Now it suffices to prove Equation (4-1) made at the beginning of the proof, which is the next lemma. □

Lemma 4.6. Ker φ *is smooth and unipotent of dimension* l. *In addition, the number of connected components of* Ker φ *is* 2^{β}. *Here,*

- *l is such that*

$$l + \sum_{i \text{ even}} (\dim \mathrm{Sp}(B_i/Y_i, h_i)) + \sum_{i \text{ odd}} (\dim O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}}) = \dim \widetilde{G}.$$

- *β is the number of even integers j such that L_j is of type I and L_{j+2} is of type II.*

Recall that the zero lattice is of type II. The proof is postponed to Appendix A.

Remark 4.7. We summarize the description of Im φ_i as follows.

type of lattice L_i	i	Im φ_i
II	even	$\mathrm{Sp}(n_i, h_i)$
I^o	even	$\mathrm{Sp}(n_i - 1, h_i)$
I^e	even	$\mathrm{Sp}(n_i - 2, h_i)$
free	odd	$O(n_i, \bar{q}_i)$
bound	odd	$\mathrm{SO}(n_i + 1, \bar{q}_i)$

Let i be odd and L_i be *free*. Then $A_i/Z_i = L_i/\pi L_i$ is a κ-vector space with even dimension. We now consider the question of whether the orthogonal group $O(A_i/Z_i, \bar{q}_i) = O(n_i, \bar{q}_i)$ is split or nonsplit.

By Theorem 2.10, we have that $L_i = \bigoplus_{\lambda} H_{\lambda} \oplus A(2, 2b_i, \pi)$ for certain $b_i \in A$. Thus the orthogonal group $O(A_i/Z_i, \bar{q}_i)$ $(= O(n_i, \bar{q}_i))$ is split if and only if the quadratic space $A(2, 2b_i, \pi)/\pi A(2, 2b_i, \pi)$ is isotropic. Recall that $\pi + \sigma(\pi) = 2$ and $\pi = 1 + \sqrt{1 + 2u}$ for a certain unit $u \in A$. Using this, the quadratic form on $A(2, 2b_i, \pi)/\pi A(2, 2b_i, \pi)$ is $q(x, y) = x^2 + xy + \bar{b}_i y^2$, where $\bar{b}_i$ is the reduction of b_i in κ.

We consider the identity $q(x, y) = x^2 + xy + \bar{b}_i y^2 = 0$. If $y = 0$, then $x = 0$. Assume that $y \neq 0$. Then we have that $\bar{b}_i = (x/y)^2 + x/y$.

Thus we can see that there exists a solution of the equation $z^2 + z = \bar{b}_i$ over κ if and only if $q(x, y)$ is isotropic if and only if $O(A_i/Z_i, \bar{q}_i)$ $(= O(n_i, \bar{q}_i))$ is split.

4B. ***The construction of component groups.*** The purpose of this subsection is to define a surjective morphism from $\widetilde{G}$ to $(\mathbb{Z}/2\mathbb{Z})^{\beta}$, where β is the number of even integers j such that L_j is of type I and L_{j+2} is of type II as defined in Lemma 4.6.

Definition 4.8. We set $L^0 = L$ and inductively define, for positive integers i,

$$L^i := \{x \in L^{i-1} \mid h(x, L^{i-1}) \subset (\pi^i)\}.$$

When $i = 2m$ is even,

$$L^{2m} = \pi^m (L_0 \oplus L_1) \oplus \pi^{m-1}(L_2 \oplus L_3) \oplus \cdots \oplus \pi(L_{2m-2} \oplus L_{2m-1}) \oplus \bigoplus_{i \geq 2m} L_i.$$

We choose a Jordan splitting for the hermitian lattice $(L^{2m}, \xi^{-m} h)$ as follows:

$$L^{2m} = \bigoplus_{i \geq 0} M_i,$$

where

$$\begin{aligned} M_0 &= \pi^m L_0 \oplus \pi^{m-1} L_2 \oplus \cdots \oplus \pi L_{2m-2} \oplus L_{2m}, \\ M_1 &= \pi^m L_1 \oplus \pi^{m-1} L_3 \oplus \cdots \oplus \pi L_{2m-1} \oplus L_{2m+1}, \\ M_k &= L_{2m+k} \text{ if } k \geq 2. \end{aligned}$$

Here, M_i is π^i-modular. We caution that the hermitian form we use on L^{2m} is not h, but its rescaled version $\xi^{-m} h$. Thus M_i is π^i-modular, not π^{2m+i}-modular.

Definition 4.9. We define $C(L)$ to be the sublattice of L such that

$$C(L) = \{x \in L \mid h(x, y) \in (\pi) \text{ for all } y \in B(L)\}.$$

We choose any even integer j such that L_j is of type I and L_{j+2} is of type II (possibly zero, by our convention), and consider the Jordan splitting $\bigoplus_{i \geq 0} M_i$ of L^j defined above. We stress that M_0 is nonzero and of type I, since it contains L_j as a direct summand so that $n(M_0) = s(M_0)$ (cf. Definition 2.1(c)), and $M_2 = L_{j+2}$ is of type II. Choose a basis $(\langle e_i \rangle, e)$ (resp. $(\langle e_i \rangle, a, e)$) for M_0 so that $M_0 = \bigoplus_{\lambda} H_{\lambda} \oplus K$ when the rank of M_0 is odd (resp. even). Here, we follow the notation from Theorem 2.10. Then $B(L^j)$ is spanned by

$$(\langle e_i \rangle, \pi e) \text{ (resp. } (\langle e_i \rangle, \pi a, e)) \quad \text{and} \quad M_1 \oplus \bigoplus_{i \geq 2} M_i$$

and $C(L^j)$ is spanned by

$$(\langle \pi e_i \rangle, e) \text{ (resp. } (\langle \pi e_i \rangle, \pi a, e)) \quad \text{and} \quad M_1 \oplus \bigoplus_{i \geq 2} M_i.$$

We now construct a morphism $\psi_j : \widetilde{G} \to \mathbb{Z}/2\mathbb{Z}$ as follows. (There are 2 cases depending on whether M_0 is of type I^e or of type I^o.)

(1) Firstly, we assume that M_0 is of type I^e. We choose a Jordan splitting for the hermitian lattice $(C(L^j), \xi^{-m} h)$ as follows:

$$C(L^j) = \bigoplus_{i \geq 1} M_i',$$

where

$$M_1' = (\pi)a \oplus Be \oplus M_1, \quad M_2' = \Big(\bigoplus_i (\pi)e_i\Big) \oplus M_2, \quad \text{and} \quad M_k' = M_k \text{ if } k \geq 3.$$

Here, M_i' is π^i-modular and (π) is the ideal of B generated by a uniformizer π. Notice that M_2' is of type II, since both $\bigoplus_i (\pi)e_i$ and M_2 are of type II, so that M_1' is *free*.

If m is an element of the group of R-points of the naive integral model associated to the hermitian lattice L, for a flat A-algebra R, then m stabilizes the hermitian lattice $(C(L^j) \otimes_A R, \xi^{-m} h \otimes 1)$ as well. If we use this fact in the case of an F-algebra R, where F is the quotient field of A, then we obtain a morphism of algebraic groups from the unitary group associated to the hermitian space $L \otimes_A F$ to the unitary group associated to the hermitian space $(C(L^j) \otimes_A F, \xi^{-m} h)$ by Yoneda's lemma. Furthermore, if we use the above fact in the case of an étale A-algebra R, then the morphism between unitary groups is extended to give a map from the group of R-points of the naive integral model associated to the hermitian lattice L to that of the hermitian lattice $(C(L^j), \xi^{-m} h)$. Note that the naive integral model and the associated smooth integral model have the same generic fiber and are the same at the level of étale A-points. Thus by Proposition 2.3 of [Yu 2002], the morphism between unitary groups is uniquely extended to a morphism of group schemes from the smooth integral model associated to L to the smooth integral model associated to $(C(L^j), \xi^{-m} h)$ such that the map induced from it at the level of étale A-points is the same as that described above. Let G_j denote the special fiber of the latter smooth integral model. We now have a morphism from $\widetilde{G}$ to G_j. Moreover, since M_1' is *free* and nonzero, we have a morphism from G_j to the even orthogonal group associated to M_1' as explained in Section 4A. Thus, the Dickson invariant of this orthogonal group induces the morphism

$$\psi_j : \widetilde{G} \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

(2) We next assume that M_0 is of type I^o. We choose a Jordan splitting for the hermitian lattice $(C(L^j), \xi^{-m} h)$ as follows:

$$C(L^j) = \bigoplus_{i \geq 0} M_i',$$

where

$$M_0' = Be, \quad M_1' = M_1, \quad M_2' = \Big(\bigoplus_i (\pi)e_i\Big) \oplus M_2, \quad \text{and} \quad M_k' = M_k \text{ if } k \geq 3.$$

Here, M_i' is π^i-modular and (π) is the ideal of B generated by a uniformizer π. Notice that the rank of the π^0-modular lattice M_0' is 1 and the lattice M_2' is of type II. If G_j denotes the special fiber of the smooth integral model associated to the hermitian lattice $(C(L^j), \xi^{-m} h)$, then we have a morphism from $\widetilde{G}$ to G_j as in the above argument (1).

We now consider the new hermitian lattice $M_0' \oplus C(L^j)$. Then for a flat A-algebra R, there is a natural embedding from the group of R-points of the naive integral model associated to the hermitian lattice $(C(L^j), \xi^{-m} h)$ to that of the hermitian lattice $M_0' \oplus C(L^j)$ such that m maps to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & m \end{smallmatrix}\right)$, where m is an element of the former group. As in the previous argument (1), the above fact induces a closed immersion of algebraic groups from the unitary group associated to the hermitian space $(C(L^j)\otimes_A F, \xi^{-m}h)$ to the unitary group associated to the hermitian space $(M_0'\oplus C(L^j))\otimes_A F$ and its extension at the level of étale A-algebra points between the associated naive integral models. Thus by Proposition 2.3 of [Yu 2002], the morphism between unitary groups is uniquely extended to a morphism of group schemes from the smooth integral model associated to the hermitian lattice $(C(L^j), \xi^{-m}h)$ to the smooth integral model associated to the hermitian lattice $M_0'\oplus C(L^j)$ such that the map induced from it at the level of étale A-points is the same as that described above. In Remark 4.10, we describe this morphism explicitly in terms of matrices.

Thus we have a morphism from the special fiber G_j of the smooth integral model associated to $C(L^j)$ to the special fiber G_j' of the smooth integral model associated to $M_0'\oplus C(L^j)$. Note that $(M_0'\oplus M_0')\oplus\bigoplus_{i\geq 1}M_i'$ is a Jordan splitting of the hermitian lattice $M_0'\oplus C(L^j)$. Let G_j'' be the special fiber of the smooth integral model associated to $C\big((M_0'\oplus M_0')\oplus\bigoplus_{i\geq 1}M_i'\big)$. Since the π^0-modular lattice $M_0'\oplus M_0'$ is of type I^e, we have a morphism $G_j'\to\mathbb{Z}/2\mathbb{Z}$ obtained by factoring through G_j'' and the corresponding even orthogonal group with the Dickson invariant as constructed in argument (1). ψ_j is defined to be the composite

$$\psi_j:\widetilde{G}\to G_j\to G_j'\to\mathbb{Z}/2\mathbb{Z}.$$

Remark 4.10. In this remark, we describe the morphism from the smooth integral model $\underline{G}_j$ associated to the hermitian lattice $(C(L^j), \xi^{-m}h)$ to the smooth integral model $\underline{G}_j'$ associated to the hermitian lattice $M_0'\oplus C(L^j)$ as given in argument (2) above, in terms of matrices. Let R be a flat A-algebra. We choose an element in $\underline{G}_j(R)$ and express it as a matrix $m=\big(\pi^{\max\{0,j-i\}}m_{i,j}\big)$. Then $m_{0,0}=\big(1+\pi z_0\big)$ since M_0' is of type I with rank 1 so that we may and do write m as $m=\begin{pmatrix}1+\pi z_0 & m_1\\ m_2 & m_3\end{pmatrix}$. We consider a morphism from $\underline{G}_j$ to $\mathrm{Aut}_B(M_0'\oplus C(L^j))$ such that m maps to

$$T=\begin{pmatrix}1&0\\0&m\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1+\pi z_0&m_1\\0&m_2&m_3\end{pmatrix},$$

where the set of R-points of the group scheme $\mathrm{Aut}_B(M_0'\oplus C(L^j))$ is the automorphism group of $(M_0'\oplus C(L^j))\otimes_A R$ by ignoring the hermitian form. Then the image of this morphism is represented by an affine group scheme which is isomorphic to $\underline{G}_j$. Note that T preserves the hermitian form attached to the lattice $M_0'\oplus C(L^j)$.

We claim that $\begin{pmatrix}1&0\\0&m\end{pmatrix}$ is contained in $\underline{G}_j'(R)$. If this is true, then the above matrix description defines a morphism from $\underline{G}_j$ to $\underline{G}_j'$ by Yoneda's lemma since $\underline{G}_j$ is flat. Furthermore, this matrix description is the same as that of naive integral models explained in the above argument (2) when R is an F-algebra or an étale A-algebra, since the naive integral model and the associated smooth integral model have the same generic fiber and are the same at the level of étale A-points. Since the desired morphism is completely determined at the level of F-algebra points and étale A-algebra points by Proposition 2.3 of [Yu 2002], the morphism from $\underline{G}_j$ to $\underline{G}_j'$ obtained by the above matrix description is the morphism we want to describe.

We rewrite the hermitian lattice $M_0'\oplus C(L^j)$ as $(M_0'\oplus M_0')\oplus\big(\bigoplus_{i\geq 1}M_i'\big)$. Let (e_1, e_2) be a basis for $(M_0'\oplus M_0')$ so that the corresponding Gram matrix of $(M_0'\oplus M_0')$ is $\left(\begin{smallmatrix} a & 0\\ 0 & a\end{smallmatrix}\right)$, where $a\equiv 1 \bmod 2$. Then the hermitian lattice $(M_0'\oplus M_0')$ has Gram matrix $\left(\begin{smallmatrix} a & a\\ a & 2a\end{smallmatrix}\right)$ with respect to the basis (e_1, e_1+e_2). $(M_0'\oplus M_0')$ is *unimodular of type* I^e with rank 2. With this basis, T becomes

$$\widetilde{T}=\begin{pmatrix}1 & -\pi z_0 & -m_1\\ 0 & 1+\pi z_0 & m_1\\ 0 & m_2 & m_3\end{pmatrix}.$$

On the other hand, an element of $\underline{G}'_j(R)$, with respect to a basis for $M_0'\oplus C(L^j)$ obtained by putting together the basis (e_1, e_1+e_2) for $(M_0'\oplus M_0')$ and a basis for $C(L^j)$, is given by an expression

$$\begin{pmatrix}1+\pi x_0' & -\pi z_0' & m_1'\\ u_0' & 1+\pi w_0' & m_1''\\ m_2' & m_2'' & m_3''\end{pmatrix},$$

cf. Section 3A. Then we can easily see that the congruence conditions on m_1, m_2, m_3 are the same as those of m_1'', m_2'', m_3'', respectively, and that the congruence conditions on m_1' are the same as those of m_1''. Thus $\widetilde{T}$ is an element of $\underline{M}^*_j(R)$, where $\underline{M}^*_j$ is the group scheme in Section 3B associated to $M_0'\oplus C(L^j)$ so that $\underline{G}'_j$ is defined as the closed subgroup scheme of $\underline{M}^*_j$ stabilizing the hermitian form on $M_0'\oplus C(L^j)$.

In conclusion, $\widetilde{T}$ preserves the hermitian form on $M_0'\oplus C(L^j)$. Therefore, it is an element of $\underline{G}'_j(R)$.

To summarize, if R is a nonflat A-algebra, then we can write an element of $\underline{G}_j(R)$ formally as $m=\left(\begin{smallmatrix} 1+\pi z_0 & m_1\\ m_2 & m_3\end{smallmatrix}\right)$. Then the image of m in $\underline{G}'_j(R)$ is $\widetilde{T}$ with respect to a basis as explained above.

(3) Combining all cases, the morphism

$$\psi=\prod_j \psi_j : \widetilde{G}\longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\beta},$$

where β is the number of even integers j such that L_j is of type I and L_{j+2} is of type II (possibly zero, by our convention).

Theorem 4.11. *The morphism*

$$\psi=\prod_j \psi_j : \widetilde{G}\longrightarrow (\mathbb{Z}/2\mathbb{Z})^{\beta}$$

is surjective. Moreover, the morphism

$$\varphi\times\psi : \widetilde{G}\longrightarrow \prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i)\times\prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}}\times(\mathbb{Z}/2\mathbb{Z})^{\beta}$$

is also surjective.

Proof. We first show that ψ_j is surjective. Recall that for such an even integer j, L_j is of type I and L_{j+2} is of type II (possibly zero by our convention). We define the closed subgroup scheme F_j of $\widetilde{G}$ defined by the following equations:

- $m_{i,k}=0$ *if* $i\neq k$;

- $m_{i,i} = \mathrm{id}$ *if* $i \neq j$;
- and for $m_{j,j}$,

$$\begin{cases} s_j = \mathrm{id},\, y_j = 0,\, v_j = 0 & \text{if } L_i \text{ is of type } I^o; \\ s_j = \mathrm{id},\, r_j = t_j = y_j = v_j = u_j = w_j = 0 & \text{if } L_i \text{ is of type } I^e. \end{cases}$$

A formal matrix form of an element of $F_j(R)$ for a κ-algebra R is then

$$\begin{pmatrix} \mathrm{id} & 0 & & \dots & & & 0 \\ 0 & \ddots & & & & & \\ & & \mathrm{id} & & & & \\ \vdots & & & m_{j,j} & & & \vdots \\ & & & & \mathrm{id} & & \\ & & & & & \ddots & 0 \\ 0 & & & \dots & & 0 & \mathrm{id} \end{pmatrix}$$

such that

$$m_{j,j} = \begin{cases} \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 1+\pi z_j \end{pmatrix} & \text{if } L_j \text{ is of type } I^o; \\ \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & 1+\pi x_j & \pi z_j \\ 0 & 0 & 1 \end{pmatrix} & \text{if } L_j \text{ is of type } I^e. \end{cases}$$

In Lemma A.9, we will show that F_j is isomorphic to $\mathbb{A}^1 \times \mathbb{Z}/2\mathbb{Z}$ as a κ-variety so that it has exactly two connected components, by enumerating equations defining F_j as a closed subvariety of an affine space of dimension 2 (resp. 4) if L_j is of type I^o (resp. of type I^e). Here, $\mathbb{A}^1$ is an affine space of dimension 1. These equations are necessary in this theorem and thus we state them in Equation (4-2) below. We refer to Lemma A.9 for the proof. Let α be the unit in B such that $\epsilon = 1 + \alpha\pi$ as explained in Section 2A, and $\bar{\alpha}$ be the image of α in κ. We write $x_j = x_j^1 + \pi x_j^2$ and $z_j = z_j^1 + \pi z_j^2$, where $x_j^1, x_j^2, z_j^1, z_j^2 \in R \subset R \otimes_A B$ and π stands for $1 \otimes \pi \in R \otimes_A B$. Then the equations defining F_j as a closed subvariety of an affine space of dimension 2 (resp. 4) are

$$\begin{cases} (z_j^1/\bar{\alpha}) + (z_j^1/\bar{\alpha})^2 = 0 & \text{if } L_j \text{ is of type } I^o; \\ x_j^1 = z_j^1,\, (z_j^1/\bar{\alpha}) + (z_j^1/\bar{\alpha})^2 = 0,\, z_j^2 + x_j^2 + x_j^1 z_j^1 = 0 & \text{if } L_j \text{ is of type } I^e. \end{cases} \tag{4-2}$$

The proof of the surjectivity of ψ_j is given below. The main idea is to show that $\psi_j|_{F_j}$ is surjective. There are 4 cases according to the types of M_0 and L_j. Recall that $\bigoplus_{i \geq 0} M_i$ is a Jordan splitting of a rescaled hermitian lattice $(L^j, \frac{1}{\xi^{j/2}} h)$ and that $M_0 = \pi^{j/2} L_0 \oplus \pi^{j/2-1} L_2 \oplus \cdots \oplus \pi L_{j-2} \oplus L_j$.

(1) Assume that both M_0 and L_j are of type I^e. In this case and the next case, we will describe $\psi_j|_{F_j} : F_j \to \mathbb{Z}/2\mathbb{Z}$ explicitly in terms of a formal matrix. To do that, we will first describe a morphism from F_j to the special fiber of the smooth integral model associated to L^j and then to G_j. Recall that G_j is the special fiber of the smooth integral model associated to $C(L^j) = \bigoplus_{i \geq 1} M_i'$. Then we will describe a morphism from F_j to the even

orthogonal group associated to M_1' and compute the Dickson invariant of the image of an element of F_j in this orthogonal group.

We write $M_0 = N_0 \oplus L_j$, where N_0 is unimodular with even rank. Thus N_0 is either of type II or of type I^e. First we assume that N_0 is of type I^e. Then we can write $N_0 = \left(\bigoplus_{\lambda'} H_{\lambda'}\right) \oplus A(1, 2b, 1)$ and $L_j = \left(\bigoplus_{\lambda''} H_{\lambda''}\right) \oplus A(1, 2b', 1)$ by Theorem 2.10, where $H_{\lambda'} = H(0) = H_{\lambda''}$ and $b, b' \in A$. Thus we write $M_0 = \left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus A(1, 2b, 1) \oplus A(1, 2b', 1)$, where $H_\lambda = H(0)$. For this choice of a basis of $L^j = \bigoplus_{i \geq 0} M_i$, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \begin{pmatrix} 1+\pi x_j & \pi z_j \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, id in the $(1, 1)$-block corresponds to the direct summand $\left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus A(1, 2b, 1)$ of M_0 and the diagonal block $\left(\begin{smallmatrix} 1+\pi x_j & \pi z_j \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to the direct summand $A(1, 2b', 1)$ of M_0.

Let (e_1, e_2, e_3, e_4) be a basis for the direct summand $A(1, 2b, 1) \oplus A(1, 2b', 1)$ of M_0. Since this is *unimodular of type* I^e, we can choose another basis based on Theorem 2.10. With the basis $(-2be_1 + e_2, (2b' - 1)e_1 + e_3 - e_4, e_3, e_2 + e_4)$, $A(1, 2b, 1) \oplus A(1, 2b', 1)$ becomes $A(2b(2b - 1), 2b'(2b' - 1), -(2b - 1)(2b' - 1)) \oplus A(1, 2(b + b'), 1)$. Since $A(2b(2b-1), 2b'(2b'-1), -(2b-1)(2b'-1))$ is *unimodular of type* II, it is isomorphic to $H(0)$ by Theorem 2.10. Thus we can write $M_0 = \left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus H(0) \oplus A(1, 2(b+b'), 1)$. For this basis, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} * & *' & 0 \\ *'' & \begin{pmatrix} 1+\pi x_j & \pi z_j \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, the diagonal block $\left(\begin{smallmatrix} 1+\pi x_j & \pi z_j \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to $A(1, 2(b+b'), 1)$ with basis $(e_3, e_2 + e_4)$ and the diagonal block $*$ corresponds to the direct summand $\left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus H(0)$ of M_0.

Then the direct summand M_1' of $C(L^j) = \bigoplus_{i \geq 1} M_i'$ is $(\pi)e_3 \oplus B(e_2 + e_4) \oplus M_1$. The image of a fixed element of F_j in the special fiber of the smooth integral model associated to $C(L^j)$ is then

$$\begin{pmatrix} \begin{pmatrix} 1+\pi x_j & z_j \\ 0 & 1 \end{pmatrix} & 0 & *' \\ 0 & \mathrm{id} & *'' \\ *''' & *'''' & * \end{pmatrix}.$$

Here, the diagonal block $\left(\begin{smallmatrix} 1+\pi x_j & z_j \\ 0 & 1 \end{smallmatrix}\right)$ corresponds to $(\pi)e_3 \oplus B(e_2 + e_4)$ and the diagonal block id corresponds to the direct summand M_1 of M_1'.

Now, the image of a fixed element of F_j in the orthogonal group associated to $M_1'/\pi M_1'$ is

$$T_1 = \begin{pmatrix} \begin{pmatrix} 1 & z_j^1 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Note that z_j^1 is in R such that $z_j = z_j^1 + \pi z_j^2$ as explained in the paragraph before Equation (4-2). The Dickson invariant of T_1 is the same as that of $\left(\begin{smallmatrix} 1 & z_j^1 \\ 0 & 1 \end{smallmatrix}\right)$. Here we consider $\left(\begin{smallmatrix} 1 & z_j^1 \\ 0 & 1 \end{smallmatrix}\right)$ as an element of the orthogonal group associated to $((\pi)e_3 \oplus B(e_2+e_4))/\pi((\pi)e_3 \oplus B(e_2+e_4))$. In order to compute the Dickson invariant, we use the scheme-theoretic description of the Dickson invariant explained in Remark 4.4 of [Cho 2015a]. The Dickson invariant of an orthogonal group of the quadratic space with dimension 2 is explicitly given at the end of the proof of Lemma 4.5 in [Cho 2015a]. Based on this, the Dickson invariant of $\left(\begin{smallmatrix} 1 & z_j^1 \\ 0 & 1 \end{smallmatrix}\right)$ is $z_j^1/\bar{\alpha}$. Note that $z_j^1/\bar{\alpha}$ is indeed an element of $\mathbb{Z}/2\mathbb{Z}$ by Equation (4-2).

In conclusion, $z_j^1/\bar{\alpha}$ is the image of a fixed element of F_j under the map ψ_j. Since $z_j^1/\bar{\alpha}$ can be either 0 or 1, $\psi_j|_{F_j}$ is surjective onto $\mathbb{Z}/2\mathbb{Z}$ and thus ψ_j is surjective.

If N_0 is of type II, then the proof of the surjectivity of ψ_j is similar to that of the above case and so we skip it.

(2) Assume that M_0 is of type I^e and L_j is of type I^o. We write $M_0 = N_0 \oplus L_j$, where N_0 is unimodular with odd rank so that it is of type I^o. Then we can write $N_0 = \left(\bigoplus_{\lambda'} H_{\lambda'}\right) \oplus (a)$ and $L_j = \left(\bigoplus_{\lambda''} H_{\lambda''}\right) \oplus (a')$ by Theorem 2.10, where $H_{\lambda'} = H(0) = H_{\lambda''}$ and $a, a' \in A$ such that $a, a' \equiv 1 \bmod 2$. Thus we write $M_0 = \left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus (a) \oplus (a')$, where $H_\lambda = H(0)$. For this choice of a basis of $L^j = \bigoplus_{i \geq 0} M_i$, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \left(1 + \pi z_j\right) & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, id in the (1, 1)-block corresponds to the direct summand $\left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus (a)$ of M_0 and the diagonal block $\left(1 + \pi z_j\right)$ corresponds to the direct summand (a') of M_0.

Let (e_1, e_2) be a basis for the direct summand $(a) \oplus (a')$ of M_0. Since this is *unimodular of type* I^e, we can choose another basis $(e_1, e_1 + e_2)$ such that the associated Gram matrix is $A(a, a + a', a)$, where $a + a' \in (2)$. For this basis, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \begin{pmatrix} 1 & -\pi z_j \\ 0 & 1 + \pi z_j \end{pmatrix} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, the diagonal block $\left(\begin{smallmatrix} 1 & -\pi z_j \\ 0 & 1+\pi z_j \end{smallmatrix}\right)$ corresponds to $A(a, a + a', a)$ with a basis $(e_1, e_1 + e_2)$ and id in the (1, 1)-block corresponds to the direct summand $\left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus (a)$ of M_0.

Then the direct summand M_1' of $C(L^j) = \bigoplus_{i \geq 1} M_i'$ is $(\pi)e_1 \oplus B(e_1 + e_2) \oplus M_1$. The image of a fixed element of F_j in the special fiber of the smooth integral model associated to $C(L^j)$ is then

$$\begin{pmatrix} \begin{pmatrix} 1 & -z_j \\ 0 & 1+\pi z_j \end{pmatrix} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, the diagonal block $\left(\begin{smallmatrix} 1 & -z_j \\ 0 & 1+\pi z_j \end{smallmatrix}\right)$ corresponds to $(\pi)e_1 \oplus B(e_1+e_2)$ and id in the (2×2)-block corresponds to the direct summand M_1 of M_1'.

Now, the image of a fixed element of F_j in the orthogonal group associated to $M_1'/\pi M_1'$ is

$$T_1 = \begin{pmatrix} \begin{pmatrix} 1 & z_j^1 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Note that $z_j^1 \in R$ is such that $z_j = z_j^1 + \pi z_j^2$, as explained in the paragraph before Equation (4-2). The Dickson invariant of T_1 is the same as that of $\left(\begin{smallmatrix} 1 & z_j^1 \\ 0 & 1 \end{smallmatrix}\right)$. Here, we consider $\left(\begin{smallmatrix} 1 & z_j^1 \\ 0 & 1 \end{smallmatrix}\right)$ as an element of the orthogonal group associated to $((\pi)e_1 \oplus B(e_1+e_2))/\pi((\pi)e_1 \oplus B(e_1+e_2))$. Then as explained in the above case (1), the Dickson invariant of $\left(\begin{smallmatrix} 1 & z_j^1 \\ 0 & 1 \end{smallmatrix}\right)$ is $z_j^1/\bar{\alpha}$. Note that $z_j^1/\bar{\alpha}$ is indeed an element of $\mathbb{Z}/2\mathbb{Z}$ by Equation (4-2).

In conclusion, $z_j^1/\bar{\alpha}$ is the image of a fixed element of F_j under the map ψ_j. Since $z_j^1/\bar{\alpha}$ can be either 0 or 1, $\psi_j|_{F_j}$ is surjective onto $\mathbb{Z}/2\mathbb{Z}$ and thus ψ_j is surjective.

(3) Assume that both M_0 and L_j are of type I^o. In this case, we will describe $\psi_j|_{F_j} : F_j \to \mathbb{Z}/2\mathbb{Z}$ explicitly in terms of a formal matrix. To do that, we will first describe a morphism from F_j to the special fiber of the smooth integral model associated to L^j and then to G_j. Recall that G_j is the special fiber of the smooth integral model associated to $C(L^j) = \bigoplus_{i\geq 0} M_i'$. Then we will describe a morphism from F_j to the special fiber of the smooth integral model associated to $M_0' \oplus C(L^j)$ and to the special fiber of the smooth integral model associated to $C(M_0' \oplus C(L^j))$. Finally, we will describe a morphism from F_j to a certain even orthogonal group associated to $C(M_0' \oplus C(L^j))$ and compute the Dickson invariant of the image of an element of F_j in this orthogonal group.

We write $M_0 = N_0 \oplus L_j$, where N_0 is unimodular with even rank. Thus N_0 is either of type *II* or of type I^e. First we assume that N_0 is of type I^e. Then we can write $N_0 = \left(\bigoplus_{\lambda'} H_{\lambda'}\right) \oplus A(1, 2b, 1)$ and $L_j = \left(\bigoplus_{\lambda''} H_{\lambda''}\right) \oplus (a)$ by Theorem 2.10, where $H_{\lambda'} = H(0) = H_{\lambda''}$, $b \in A$, and $a\ (\in A) \equiv 1 \bmod 2$. Thus we write $M_0 = \left(\bigoplus_{\lambda} H_{\lambda}\right) \oplus A(1, 2b, 1) \oplus (a)$, where $H_\lambda = H(0)$. For this choice of a basis of $L^j = \bigoplus_{i\geq 0} M_i$, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & (1+\pi z_j) & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, id in the $(1, 1)$-block corresponds to the direct summand $\left(\bigoplus_\lambda H_\lambda\right) \oplus A(1, 2b, 1)$ of M_0 and the diagonal block $(1+\pi z_j)$ corresponds to the direct summand (a) of M_0.

Let (e_1, e_2, e_3) be a basis for the direct summand $A(1, 2b, 1) \oplus (a)$ of M_0. Since this is *unimodular of type I^o*, we can choose another basis based on Theorem 2.10. Namely, if we choose $(-2be_1+e_2, -ae_1+e_3, e_2+e_3)$ as another basis, then $A(1,2b,1)\oplus(a)$ becomes $A(2b(2b-1), a(a+1), a(2b-1))\oplus(a+2b)$. Since $A(2b(2b-1), a(a+1), a(2b-1))$ is *unimodular of type II*, it is isomorphic to $H(0)$ by Theorem 2.10. Thus we can write $M_0=\left(\bigoplus_\lambda H_\lambda\right)\oplus H(0)\oplus(a+2b)$. For this basis, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} * & *' & 0 \\ *'' & \left(1+\frac{a}{a+2b}\pi z_j\right) & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, the diagonal block $\left(1+\frac{a}{a+2b}\pi z_j\right)$ corresponds to $(a+2b)$ with a basis e_2+e_3 and the diagonal block $*$ corresponds to the direct summand $\left(\bigoplus_\lambda H_\lambda\right)\oplus H(0)$ of M_0.

Then the direct summand M_0' of $C(L^j)=\bigoplus_{i\geq 0} M_i'$ is $B(e_2+e_3)$ of rank 1. The image of a fixed element of F_j in the special fiber of the smooth integral model associated to $C(L^j)$ is then

$$\begin{pmatrix} \left(1+\frac{a}{a+2b}\pi z_j\right) & 0 & *' \\ 0 & \mathrm{id} & *'' \\ *''' & *'''' & * \end{pmatrix}.$$

Here, the diagonal block $\left(1+\frac{a}{a+2b}\pi z_j\right)$ corresponds to $M_0'=B(e_2+e_3)$ with a Gram matrix $(a+2b)$ and the diagonal block id corresponds to $M_1'=M_1$.

We now describe the image of the above in the special fiber of the smooth integral model associated to $M_0'\oplus C(L^j)=(M_0'\oplus M_0')\oplus\left(\bigoplus_{i\geq 1} M_i'\right)$. If (e_1', e_2') is a basis for $(M_0'\oplus M_0')$, then we choose another basis $(e_1', e_1'+e_2')$ for $(M_0'\oplus M_0')$. For this basis, based on the description of the morphism from the smooth integral model associated to $C(L^j)$ to the smooth integral model associated to $M_0'\oplus C(L^j)$ explained in Remark 4.10, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to $M_0'\oplus C(L^j)$ is

$$\begin{pmatrix} 1 & -\frac{a}{a+2b}\pi z_j & 0 & *' \\ 0 & 1+\frac{a}{a+2b}\pi z_j & 0 & *' \\ 0 & 0 & \mathrm{id} & *'' \\ 0 & *''' & *'''' & * \end{pmatrix}.$$

Here, the diagonal block $\begin{pmatrix} 1 & -\frac{a}{a+2b}\pi z_j \\ 0 & 1+\frac{a}{a+2b}\pi z_j \end{pmatrix}$ corresponds to $(M_0'\oplus M_0')$ with a basis $(e_1', e_1'+e_2')$ and the diagonal block id corresponds to $M_1'=M_1$.

We now follow step (1) with $M_0'\oplus C(L^j)=(M_0'\oplus M_0')\oplus\left(\bigoplus_{i\geq 1} M_i'\right)$. Namely,

$$\begin{aligned} C(M_0'\oplus C(L^j)) &= (\pi)e_1'\oplus B(e_1'+e_2')\oplus\left(\bigoplus_{i\geq 1} M_i'\right) \\ &= \left((\pi)e_1'\oplus B(e_1'+e_2')\oplus M_1'\right)\oplus\left(\bigoplus_{i\geq 2} M_i'\right). \end{aligned}$$

Here, $((\pi)e_1'\oplus B(e_1'+e_2')\oplus M_1')$ is π^1-modular and M_i' is π^i-modular with $i\geq 2$. Then the image of a fixed element of F_j in the special fiber of the smooth integral model associated to $C(M_0'\oplus C(L^j))$ is

$$\begin{pmatrix} 1 & -\frac{a}{a+2b}z_j & 0 & *' \\ 0 & 1+\frac{a}{a+2b}\pi z_j & 0 & *' \\ 0 & 0 & \mathrm{id} & *'' \\ 0 & *''' & *'''' & * \end{pmatrix}.$$

Here, the top left 3×3-matrix corresponds to $(\pi e_1'\oplus B(e_1'+e_2')\oplus M_1')$.

Now, the image of a fixed element of F_j in the orthogonal group associated to $(\pi e_1\oplus B(e_1+e_2)\oplus M_1')/\pi(\pi e_1\oplus B(e_1+e_2)\oplus M_1')$ is

$$T_1=\begin{pmatrix} \begin{pmatrix} 1 & z_j^1 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

since mod 2 reduction of $\frac{a}{a+2b}$ is 1. Note that z_j^1 is in R such that $z_j=z_j^1+\pi z_j^2$ as explained in the paragraph before Equation (4-2). Then, as explained in step (1), the Dickson invariant of this is $z_j^1/\bar{\alpha}$. Note that $z_j^1/\bar{\alpha}$ is indeed an element of $\mathbb{Z}/2\mathbb{Z}$ by Equation (4-2).

In conclusion, $z_j^1/\bar{\alpha}$ is the image of a fixed element of F_j under the map ψ_j. Since $z_j^1/\bar{\alpha}$ can be either 0 or 1, $\psi_j|_{F_j}$ is surjective onto $\mathbb{Z}/2\mathbb{Z}$ and thus ψ_j is surjective.

If N_0 is of type II, then the proof of the surjectivity of ψ_j is similar to that of the above case and so we skip it.

(4) Assume that M_0 is of type I^o and L_j is of type I^e. We write $M_0=N_0\oplus L_j$, where N_0 is unimodular with odd rank so that it is of type I^o. Then we can write $N_0=\left(\bigoplus_{\lambda'}H_{\lambda'}\right)\oplus(a)$ and $L_j=\left(\bigoplus_{\lambda''}H_{\lambda''}\right)\oplus A(1,2b,1)$ by Theorem 2.10, where $H_{\lambda'}=H(0)=H_{\lambda''}$ and $a,b\in A$ such that $a\equiv 1 \mod 2$. We write $M_0=\left(\bigoplus_{\lambda}H_{\lambda}\right)\oplus(a)\oplus A(1,2b,1)$, where $H_\lambda=H(0)$. For this choice of a basis of $L^j=\bigoplus_{i\geq 0}M_i$, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \begin{pmatrix} 1+\pi x_j & \pi z_j \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, id in the $(1,1)$-block corresponds to the direct summand $\left(\bigoplus_{\lambda}H_{\lambda}\right)\oplus(a)$ of M_0 and the diagonal block $\begin{pmatrix} 1+\pi x_j & \pi z_j \\ 0 & 1 \end{pmatrix}$ corresponds to the direct summand $A(1,2b,1)$ of M_0.

Let (e_1,e_2,e_3) be a basis for the direct summand $(a)\oplus A(1,2b,1)$ of M_0. Since this is *unimodular of type* I^o, we can choose another basis based on Theorem 2.10. Namely, if we choose $(-2be_2+e_3, e_1-ae_2, e_1+e_3)$ as another basis, then $(a)\oplus A(1,2b,1)$ becomes $A(2b(2b-1), a(a+1), a(2b-1))\oplus(a+2b)$. Since $A(2b(2b-1), a(a+1), a(2b-1))$ is *unimodular of type* II, it is isomorphic to $H(0)$ by Theorem 2.10. Thus we can write $M_0=\left(\bigoplus_{\lambda}H_{\lambda}\right)\oplus H(0)\oplus(a+2b)$. For this basis, the image of a fixed element of F_j in the special fiber of the smooth integral model associated to L^j is

$$\begin{pmatrix} * & *' & 0 \\ *'' & \left(1+\frac{1}{a+2b}\pi z_j\right) & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Here, the diagonal block $\left(1+\frac{1}{a+2b}\pi z_j\right)$ corresponds to $(a+2b)$ with a basis (e_1+e_3) and the diagonal block $*$ corresponds to the direct summand $\left(\bigoplus_\lambda H_\lambda\right)\oplus H(0)$ of M_0.

Note that the reduction of $\frac{1}{a+2b}$ mod 2 is 1. The rest of the proof is similar to that of step (3) and so we skip it.

So far, we have proved that ψ_j is surjective. We now show that $\psi=\prod_j \psi_j$ is surjective. The proof is similar to the proof showing that $\prod_{i\in\mathcal{H}} H_i \to \widetilde{G}$ is a closed immersion in the last paragraph of the proof of Theorem 4.5.

We consider the morphism

$$F=\prod_j F_j \longrightarrow \widetilde{G}$$
$$(f_j)\mapsto \prod_j f_j.$$

By considering a matrix form of an element of $F_j(R)$ for a κ-algebra R as given at the beginning of the proof, it is easy to see the following two facts. Firstly, F_j and $F_{j'}$ commute with each other in the sense that $f_j\cdot f_{j'}=f_{j'}\cdot f_j$ for all even integers $j\neq j'$, where $f_j\in F_j(R)$ and $f_{j'}\in F_{j'}(R)$ for a κ-algebra R. Note that L_j and $L_{j'}$ (resp. L_{j+2} and $L_{j'+2}$) are of type I (resp. of type II). Based on this, the above morphism becomes a group homomorphism. Secondly, $F_j\cap F_{j'}=0$ for all $j\neq j'$. This fact implies that the morphism $F_j\times F_{j'}\to \widetilde{G}$ with $(f_j, f_{j'})\mapsto f_j\cdot f_{j'}$ is injective and so $F_j\times F_{j'}$ is a closed subgroup scheme of $\widetilde{G}$. A matrix form of an element of $F_j(R)$ also implies that $(F_j\times F_{j'})\cap F_{j''}=0$ for all pairwise different three integers j, j', j'' and so the morphism $(F_j\times F_{j'})\times F_{j''}\to\widetilde{G}$ with $(f_j, f_{j'}, f_{j''})\mapsto f_j\cdot f_{j'}\cdot f_{j''}$ is injective. Thus $F_j\times F_{j'}\times F_{j''}$ is a closed subgroup scheme of $\widetilde{G}$. Therefore, by repeating this argument, the product $F=\prod_j F_j$ is embedded into $\widetilde{G}$ as a closed subgroup scheme.

In addition, we claim that $\psi_j|_{F_{j'}}$ is trivial for all $j<j'$. The proof of our claim relies on the matrix interpretation of ψ_j. We first notice that $j'-j\geq 4$ since L_j is of type I and L_{j+2} is of type II. To obtain the morphism ψ_j, we observe that the lattice $C(L^j)=\bigoplus_{i\geq 1} M_i'$ (resp. $C(L^j)=\bigoplus_{i\geq 0} M_i'$) if M_0 is of type I^e (resp. of type I^o). In either case, $L_{j'}$ is a direct summand of $M_{j'-j}$ and the morphism ψ_j is attached to the Dickson invariant of the orthogonal group associated to M_1'. We should mention that if M_0 is of type I^o then we need a new hermitian lattice $M_0'\oplus C(L^j)$. In this case, the morphism ψ_j is also attached to the Dickson invariant of the orthogonal group associated to M_1' as a direct summand of $M_0'\oplus C(L^j)$. On the other hand, recall that G_j is the special fiber of the smooth integral model associated to $C(L^j)$. Then as a formal matrix, $F_{j'}$ maps to the block of G_j associated to $M_{j'-j}$. Therefore, since $j'-j$ is at least 4, the image of $F_{j'}$ under ψ_j is zero by observing the description of the orthogonal group associated to M_1' based on Section 4A.

We finally claim that the morphism ψ induces a surjective morphism from F to $(\mathbb{Z}/2\mathbb{Z})^\beta$ defined over κ. To show this, we express F as $F = F_{j_1} \times \cdots \times F_{j_\beta}$ and $(\mathbb{Z}/2\mathbb{Z})^\beta$ as $(\mathbb{Z}/2\mathbb{Z})^\beta = (\mathbb{Z}/2\mathbb{Z})_{j_1} \times \cdots \times (\mathbb{Z}/2\mathbb{Z})_{j_\beta}$, where $j_i < j_{i'}$ if $i < i'$. Choose an arbitrary element $(z_{j_1}, \cdots, z_{j_\beta})$ of $(\mathbb{Z}/2\mathbb{Z})_{j_1} \times \cdots \times (\mathbb{Z}/2\mathbb{Z})_{j_\beta}$ where each z_{j_i} is an element of $(\mathbb{Z}/2\mathbb{Z})_{j_i}$. We first choose $f_{j_1} \in F_{j_1}$ such that $\psi_{j_1}(f_{j_1}) = z_{j_1}$. Then choose $f_{j_2} \in F_{j_2}$ such that $\psi_{j_2}(f_{j_1} \cdot f_{j_2}) = z_{j_2}$. In this way, we choose $f_{j_t} \in F_{j_t}$ such that $\psi_{j_t}(f_{j_1} \cdot \cdots \cdot f_{j_t}) = z_{j_t}$. Note that $\psi_{j_t}(f_{j_{t'}}) = 0$ for all $t < t'$. Therefore, $\psi(f_{j_1} \cdots f_{j_\beta}) = \prod_t \psi_{j_t}(f_{j_1} \cdots f_{j_\beta}) = (z_{j_1}, \cdots, z_{j_\beta})$ and this shows the surjectivity of the morphism ψ.

For the surjectivity of $\varphi \times \psi$, we recall the following criterion ([Knus et al. 1998, Proposition 22.3]): the surjectivity of $\varphi \times \psi$ as algebraic groups is equivalent to the surjectivity of $\varphi \times \psi$ at the level of $\bar{\kappa}$-points since $\prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta$ is smooth.

Choose an element (x, y) in the group of $\bar{\kappa}$-points of

$$\prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta$$

such that $x \in \left(\prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}}\right)(\bar{\kappa})$ and $y \in (\mathbb{Z}/2\mathbb{Z})^\beta(\bar{\kappa})$. Then there is an element $a \in \widetilde{G}(\bar{\kappa})$ such that $\varphi(a) = x$ since φ is surjective by Theorem 4.5. We choose an element $b \in F(\bar{\kappa})$ such that $\psi(ab) = y$. On the other hand, φ vanishes on F since the morphism φ_i vanishes on F_j for all i, j. Thus $\varphi(b) = 0$ and $(\varphi \times \psi)(ab) = (x, y)$. This completes the proof. □

4C. *The maximal reductive quotient.* We finally have the structure theorem for the algebraic group $\widetilde{G}$.

Theorem 4.12. *The morphism*

$$\varphi \times \psi : \widetilde{G} \longrightarrow \prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta$$

is surjective and the kernel is unipotent and connected. Consequently,

$$\prod_{i \text{ even}} \mathrm{Sp}(B_i/Y_i, h_i) \times \prod_{i \text{ odd}} O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}} \times (\mathbb{Z}/2\mathbb{Z})^\beta$$

is the maximal reductive quotient. Here, $\mathrm{Sp}(B_i/Y_i, h_i)$ *and* $O(A_i/Z_i, \bar{q}_i)_{\mathrm{red}}$ *are explained in Section 4A (especially Remark 4.7) and* β *is defined in Lemma 4.6.*

Proof. We only need to prove that the kernel is unipotent and connected. The kernel of φ is a closed subgroup scheme of the unipotent group $\widetilde{M}^+$ which is defined in Lemma A.2 and so it suffices to show that the kernel of $\varphi \times \psi$ is connected. Equivalently, it suffices to show that the kernel of the restricted morphism $\psi|_{\mathrm{Ker}\,\varphi}$ is connected. From Lemma 4.6, the number of connected components of $\mathrm{Ker}\,\varphi$ is 2^β. Since $\varphi|_F = 0$ so that $F = \prod_j F_j \subset \mathrm{Ker}\,\varphi$, the restricted morphism $\psi|_{\mathrm{Ker}\,\varphi}$ is surjective onto $(\mathbb{Z}/2\mathbb{Z})^\beta$. We complete the proof by counting the number of connected components. □

5. Comparison of volume forms and final formulas

This section is based on Section 7 of [Gan and Yu 2000] and Section 5 of [Cho 2015a]. Let H be the F-vector space of hermitian forms on $V = L \otimes_A F$. Let $M' = \mathrm{End}_B(L)$ and let $H' = \{f : f \text{ is a hermitian form on } L\}$. Regarding $\mathrm{End}_E V$ and H as varieties over F, let ω_M and ω_H be nonzero, translation-invariant forms on $\mathrm{End}_E V$ and H, respectively, with normalization

$$\int_{M'} |\omega_M| = 1 \quad \text{and} \quad \int_{H'} |\omega_H| = 1.$$

Let $M^* = \mathrm{Res}_{E/F} \mathrm{GL}_E(V)$. Define a map $\rho : M^* \to H$ by $\rho(m) = h \circ m$. Here $h \circ m$ is the hermitian form $(v, w) \mapsto h(mv, mw)$. Then the inverse of h under ρ is G, which is the unitary group associated to the hermitian space (V, h). It is also the generic fiber of $\underline{G}'$. Put $\omega^{\mathrm{ld}} = \omega_M / \rho^* \omega_H$. For a detailed explanation of what $\omega_M / \rho^* \omega_H$ means, we refer to Section 3.2 of [Gan and Yu 2000].

We choose two forms ω'_M and ω'_H as generators for the spaces of the top degree forms on $\underline{M}'$, which is identified with the Lie algebra of $\underline{M}^*$, and $\underline{H}'$, which is identified with the tangent space to $\underline{H}$ at h, respectively. Here $\underline{M}'$ is defined in Remark 3.1 and $\underline{H}'$ is defined in the paragraph following the matrix description of an element of $\underline{H}(R)$ for a flat A-algebra R in Section 3C. They are nonzero translation-invariant forms on $\mathrm{End}_E V$ and H, respectively, with normalization

$$\int_{\underline{M}(A)} |\omega'_M| = 1 \quad \text{and} \quad \int_{\underline{H}(A)} |\omega'_H| = 1.$$

By Theorem 3.6, we have an exact sequence of locally free sheaves on $\underline{M}^*$:

$$0 \longrightarrow \rho^* \Omega_{\underline{H}/A} \longrightarrow \Omega_{\underline{M}^*/A} \longrightarrow \Omega_{\underline{M}^*/\underline{H}} \longrightarrow 0.$$

Put $\omega^{\mathrm{can}} = \omega'_M / \rho^* \omega'_H$. For a detailed explanation of what $\omega'_M / \rho^* \omega'_H$ means, we refer to Section 3.2 of [Gan and Yu 2000]. It follows that ω^{can} is a differential of top degree on $\underline{G}$, which is invariant under the generic fiber of $\underline{G}$, and which has nonzero reduction on the special fiber.

Lemma 5.1. *We have*:

$$|\omega_M| = |2|^{N_M} |\omega'_M|, \qquad N_M = \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I}} (2n_i - 1) + \sum_{i<j} (j - i) \cdot n_i \cdot n_j,$$

$$|\omega_H| = |2|^{N_H} |\omega'_H|, \qquad N_H = \sum_{\substack{i \text{ even} \\ L_i \text{ of type I}}} (n_i - 1) + \sum_{i<j} j \cdot n_i \cdot n_j + \sum_{i \text{ even}} \frac{i+2}{2} \cdot n_i + \sum_{i \text{ odd}} \frac{i+1}{2} \cdot n_i + \sum_i d_i,$$

$$|\omega^{\mathrm{ld}}| = |2|^{N_M - N_H} |\omega^{\mathrm{can}}|.$$

Here, $d_i = i \cdot n_i \cdot (n_i - 1)/2$.

Proof. Note that both ω_M and ω'_M are volume forms on $\mathrm{End}_E V$ with different normalizations, so that they differ by a scalar. The “difference” between the Haar measures associated to these volume forms can be detected at the level of F-points of $\mathrm{End}_E V$, since $\mathrm{End}_E V$ is an affine space.

Since $\underline{M}(A) = 1 + \underline{M}'(A)$, where $\underline{M}'$ is defined in Remark 3.1, we have the identity $\int_{\underline{M}'(A)} |\omega'_M| = 1$. Note that $\underline{M}'(A)$ is a finitely generated free A-submodule of M' whose rank is the same as that of M'. Thus N_M is the "difference" between these two modules M' and $\underline{M}'(A)$. More precisely, N_M is the length of the finitely generated torsion A-module $M'/\underline{M}'(A)$. Note that 2 is a uniformizer of A.

Similarly, N_H is the length of the finitely generated torsion A-module $H'/\underline{H}'(A)$. Here, $\underline{H}'$ is defined in the paragraph following the matrix description of an element of $\underline{H}(R)$ for a flat A-algebra R in Section 3C.

Then the above formula for N_M (resp. N_H) can be read off from the matrix interpretation for $\underline{M}(A)$ (resp. $\underline{H}(A)$) given in Sections 3A and 3B (resp. Section 3C). □

Let f be the cardinality of κ. The local density is defined as

$$\beta_L = \frac{1}{[G : G^\circ]} \cdot \lim_{N\to\infty} f^{-N \dim G} \# \underline{G}'(A/\pi^N A).$$

Here, $\underline{G}'$ is the naive integral model described at the beginning of Section 3 and G is the generic fiber of $\underline{G}'$ and G° is the identity component of G. In our case, G is the unitary group $U(V, h)$, where $V = L \otimes_A F$. Since $U(V, h)$ is connected, G° is the same as G so that $[G : G^\circ] = 1$.

Then based on Lemma 3.4 and Section 3.9 of [Gan and Yu 2000], we finally have the following local density formula.

Theorem 5.2. *Let f be the cardinality of κ. The local density of (L, h) is*

$$\beta_L = f^N \cdot f^{-\dim U(V,h)} \# \widetilde{G}(\kappa),$$

where

$$N = N_H - N_M = \sum_{i<j} i \cdot n_i \cdot n_j + \sum_{i \text{ even}} \frac{i+2}{2} \cdot n_i + \sum_{i \text{ odd}} \frac{i+1}{2} \cdot n_i + \sum_i d_i - \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I}} n_i.$$

Here, $\#\widetilde{G}(\kappa)$ can be computed explicitly based on Remark 5.3(1) below and Theorem 4.12.

For convenience, we repeat the following remark from Remark 5.3 in [Cho 2015a].

Remark 5.3 [Cho 2015a, Remark 5.3]. (1) In the above local density formula, $\#\widetilde{G}(\kappa)$ is computed as follows. We denote by $R_u\widetilde{G}$ the unipotent radical of $\widetilde{G}$ so that the maximal reductive quotient of $\widetilde{G}$ is $\widetilde{G}/R_u\widetilde{G}$. That is, there is the following exact sequence of group schemes over κ:

$$1 \longrightarrow R_u\widetilde{G} \longrightarrow \widetilde{G} \longrightarrow \widetilde{G}/R_u\widetilde{G} \longrightarrow 1.$$

Furthermore, the following sequence of groups

$$1 \longrightarrow R_u\widetilde{G}(\kappa) \longrightarrow \widetilde{G}(\kappa) \longrightarrow (\widetilde{G}/R_u\widetilde{G})(\kappa) \longrightarrow 1$$

is also exact by Lemma A.1. Using Lemma A.1, one can see that $\#R_u\widetilde{G}(\kappa) = f^m$, where m is the dimension of $R_u\widetilde{G}$. Notice that the dimension of $R_u\widetilde{G}$ can be computed explicitly based on Theorem 4.12, since the dimension of $\widetilde{G}$ is n^2 with $n = \mathrm{rank}_B L$. In addition, the orders of orthogonal and symplectic groups defined over a finite field are well known. Thus, one can compute $\#(\widetilde{G}/R_u\widetilde{G})(\kappa)$ explicitly based on Theorem 4.12. Finally, the order of the group $\widetilde{G}(\kappa)$ is identified as follows:

$$\#\widetilde{G}(\kappa) = \#R_u\widetilde{G}(\kappa)\cdot\#(\widetilde{G}/R_u\widetilde{G})(\kappa).$$

(2) As in Remark 7.4 of [Gan and Yu 2000], although we have assumed that $n_i = 0$ for $i < 0$, it is easy to check that the formula in the preceding theorem remains true without this assumption.

Appendix A: The proof of Lemma 4.6

The proof of Lemma 4.6 is based on Proposition 6.3.1 in [Gan and Yu 2000]. We first state a theorem of Lazard which is repeatedly used in this paper. Let U be a group scheme of finite type over κ which is isomorphic to an affine space as an algebraic variety. Then U is connected smooth unipotent group (cf. IV, § 4, Theorem 4.1 and IV, § 2, Corollary 3.9 in [Demazure and Gabriel 1970]).

For preparation, we state several lemmas.

Lemma A.1 [Gan and Yu 2000, Lemma 6.3.3]. *Let $1 \to X \to Y \to Z \to 1$ be an exact sequence of group schemes that are locally of finite type over κ, where κ is a perfect field. Suppose that X is smooth, connected, and unipotent. Then $1 \to X(R) \to Y(R) \to Z(R) \to 1$ is exact for any κ-algebra R.*

Let $\widetilde{M}$ be the special fiber of $\underline{M}^*$ and let R be a κ-algebra. Recall that we have described an element and the multiplication of elements of $\underline{M}(R)$ in Section 3B. Based on these, an element of $\widetilde{M}(R)$ is

$$m = \left(\pi^{\max\{0,j-i\}} m_{i,j}\right).$$

Here, if i is even and L_i is of type I^o (resp. of type I^e), then

$$m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1+\pi z_i \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1+\pi x_i & \pi z_i \\ v_i & u_i & 1+\pi w_i \end{pmatrix}\right),$$

where $s_i \in M_{(n_i-1)\times(n_i-1)}(B\otimes_A R)$ (resp. $s_i \in M_{(n_i-2)\times(n_i-2)}(B\otimes_A R)$), etc., and $s_i \mod \pi\otimes 1$ is invertible. For the remaining $m_{i,j}$'s except for the cases explained above, $m_{i,j}$ is contained in $M_{n_i\times n_j}(B\otimes_A R)$ and $m_{i,i} \mod \pi\otimes 1$ is invertible.

Let

$$\widetilde{M}_i = \begin{cases} \mathrm{GL}_\kappa(B_i/Y_i) & \text{if } i \text{ is even;} \\ \mathrm{GL}_\kappa(A_i/X_i) & \text{if } i \text{ is odd.} \end{cases}$$

Let $s_i = m_{i,i}$ if L_i is of type *II* in the above description of $\widetilde{M}(R)$. Then $s_i \mod \pi\otimes 1$ is an element of $\widetilde{M}_i(R)$. Therefore, we have a surjective morphism of algebraic groups

$$r : \widetilde{M} \longrightarrow \prod \widetilde{M}_i,$$

defined over κ. We now have the following easy lemma:

Lemma A.2. *The kernel of r is the unipotent radical $\widetilde{M}^+$ of $\widetilde{M}$, and $\prod \widetilde{M}_i$ is the maximal reductive quotient of $\widetilde{M}$.*

Proof. Since $\prod \widetilde{M}_i$ is a reductive group, we only have to show that the kernel of r is a connected smooth unipotent group. Let R be a κ-algebra. By the description of the morphism r in terms of matrices explained above, an element of the kernel of r is

$$m = \left(\pi^{\max\{0,j-i\}} m_{i,j}\right)$$

satisfying the following. If i is even and L_i is of type I^o (resp. of type I^e), then

$$m_{i,i} = \begin{pmatrix} \mathrm{id} + \pi s_i' & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} \mathrm{id} + \pi s_i' & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix}\right),$$

where $\mathrm{id} + \pi \otimes 1 \cdot s_i' \in M_{(n_i-1)\times(n_i-1)}(B \otimes_A R)$ (resp. $\mathrm{id} + \pi \otimes 1 \cdot s_i' \in M_{(n_i-2)\times(n_i-2)}(B \otimes_A R)$), etc., such that s_i' has entries in $R \subset B \otimes_A R$. For the remaining $m_{i,j}$'s except for the cases explained above, $m_{i,j} \in M_{n_i \times n_j}(B \otimes_A R)$ and $m_{i,i} = \mathrm{id} + \pi \otimes 1 \cdot m_{i,i}'$ such that $m_{i,i}'$ has entries in $R \subset B \otimes_A R$. Note that there are no equations among the variables given above. Thus the kernel of r is isomorphic to an affine space as an algebraic variety over κ. Therefore, it is a connected smooth unipotent group by a theorem of Lazard which is stated at the beginning of Appendix A. □

Recall that we have defined the morphism φ in Section 4A. The morphism φ extends to an obvious morphism

$$\tilde{\varphi} : \widetilde{M} \longrightarrow \prod_{i \text{ even}} \mathrm{GL}_\kappa(B_i/Y_i) \times \prod_{i \text{ odd}} \mathrm{GL}_\kappa(A_i/Z_i)$$

such that $\tilde{\varphi}|_{\tilde{G}} = \varphi$. Note that $Y_i \otimes_A R$ and $Z_i \otimes_A R$ are preserved by an element of $\underline{M}(R)$ for a flat A-algebra R (cf. Lemma 4.2). By using this, the construction of $\tilde{\varphi}$ is similar to Theorems 4.3 and 4.4 and thus we skip it. Let R be a κ-algebra. Based on the description of the morphism φ_i explained in Section 4A, $\mathrm{Ker}\,\tilde{\varphi}(R)$ is the subgroup of $\widetilde{M}(R)$ defined by the following conditions:

(a) If i is even and L_i is of type I, $s_i = \mathrm{id} \mod \pi \otimes 1$.

(b) If i is even and L_i is of type II, $m_{i,i} = \mathrm{id} \mod \pi \otimes 1$.

(c) If i is odd, $m_{i,i} = \mathrm{id} \mod \pi \otimes 1$ and $\delta_{i-1} e_{i-1} \cdot m_{i-1,i} + \delta_{i+1} e_{i+1} \cdot m_{i+1,i} = 0 \mod \pi \otimes 1$. Here, $\delta_j = 1$ if L_j is of type I and $\delta_j = 0$ if L_j is of type II, and $e_j = (0, \cdots, 0, 1)$ (resp. $e_j = (0, \cdots, 0, 1, 0)$) of size $1 \times n_j$ if L_j is of type I^o (resp. of type I^e).

It is obvious that $\mathrm{Ker}\,\tilde{\varphi}$ is a closed subgroup scheme of $\widetilde{M}^+$ and is smooth and unipotent since it is isomorphic to an affine space as an algebraic variety over κ.

Recall from Remark 3.1 that we defined the functor $\underline{M}'$ such that $(1 + \underline{M}')(R) = \underline{M}(R)$ inside $\mathrm{End}_{B \otimes_A R}(L \otimes_A R)$ for a flat A-algebra R. Thus there is an isomorphism of set valued functors

$$1+ : \underline{M}' \to \underline{M}$$

$$m \mapsto 1 + m,$$

where $m \in \underline{M}'(R)$ for a flat A-algebra R. We define a new operation $\star$ on $\underline{M}'(R)$ such that $x\star y = x + y + xy$ for a flat A-algebra R. Since $\underline{M}'(R)$ is closed under addition and multiplication, it is also closed under the new operation $\star$. Moreover, it has 0 as an identity element with respect to $\star$. Thus $\underline{M}'$ may and shall be considered as a scheme of monoids with $\star$. We claim that the above morphism $1+$ is an isomorphism of monoid schemes. Namely, we claim the following commutative diagram of schemes:

$$\begin{array}{ccc} \underline{M}' \times \underline{M}' & \xrightarrow{(1+)\times(1+)} & \underline{M} \times \underline{M} \\ \downarrow \star & & \downarrow \text{multiplication} \\ \underline{M}' & \xrightarrow{1+} & \underline{M} \end{array}$$

Since all schemes are irreducible and smooth, it suffices to check the commutativity of the diagram at the level of flat A-points as explained in the third paragraph from below in Remark 3.2, and this is obvious.

Since $\underline{M}^*$ is an open subscheme of $\underline{M}$, $(1+)^{-1}(\underline{M}^*)$ is an open subscheme of $\underline{M}'$. The composite of the following three morphisms

$$(1+)^{-1}(\underline{M}^*) \xrightarrow{(1+)} \underline{M}^* \xrightarrow{\text{inverse}} \underline{M}^* \xrightarrow{(1+)^{-1}} (1+)^{-1}(\underline{M}^*)$$

defines the inverse morphism on the scheme of monoids $(1+)^{-1}(\underline{M}^*)$ with respect to the operation $\star$. Thus we can see that $(1+)^{-1}(\underline{M}^*)$ is a group scheme with respect to $\star$ and the morphism $1+$ is an isomorphism of group schemes between $(1+)^{-1}(\underline{M}^*)$ and $\underline{M}^*$.

Let R be a κ-algebra. Since the morphism $1+$ is an isomorphism of monoid schemes between $\underline{M}'$ and $\underline{M}$, we can write each element of $\underline{M}(R)$ as $1 + x$ with $x \in \underline{M}'(R)$. Here, $1 + x$ means the image of x under the morphism $1+$ at the level of R-points. Note that $\underline{M}'(R)$ is a $B\otimes_A R$-algebra for any A-algebra R with respect to the original multiplication on it, not the operation $\star$. In particular, $\underline{M}'(R)$ is a $(B/2B)\otimes_A R$-algebra for any κ-algebra R. Therefore, we consider the subfunctor $\underline{\pi M'} : R \mapsto (\pi \otimes 1)\underline{M}'(R)$ of $\underline{M}' \otimes \kappa$ and the subfunctor $\widetilde{M}^1 : R \mapsto 1 + \underline{\pi M'}(R)$ of $\operatorname{Ker}\tilde{\varphi}$. Here, by $1 + \underline{\pi M'}(R)$, we mean the image of $\underline{\pi M'}(R)$ inside $\underline{M}(R)$ $(= \widetilde{M}(R))$ under the morphism $1+$ at the level of R-points. That $1 + \underline{\pi M'}(R)$ is contained in $\operatorname{Ker}\tilde{\varphi}(R)$ can easily be checked by observing the construction of $\tilde{\varphi}$. The multiplication on $\widetilde{M}^1$ is as follows: for two elements $1 + \pi x$ and $1 + \pi y$ in $\widetilde{M}^1(R)$, based on the above commutative diagram, the product of $1 + \pi x$ and $1 + \pi y$ is

$$(1 + \pi x) \cdot (1 + \pi y) = 1 + \pi x\star\pi y = 1 + (\pi(x + y) + \pi^2(xy)) = 1 + \pi(x + y).$$

Here, π stands for $\pi \otimes 1 \in B \otimes_A R$. Then we have the following lemma.

Lemma A.3. (i) *The functor $\widetilde{M}^1$ is representable by a smooth, connected, unipotent group scheme over κ. Moreover, $\widetilde{M}^1$ is a closed normal subgroup of* $\operatorname{Ker}\tilde{\varphi}$.

(ii) *The quotient group scheme* $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ *represents the functor*

$$R \mapsto \operatorname{Ker}\tilde{\varphi}(R)/\widetilde{M}^1(R)$$

by Lemma A.1 and is smooth, connected, and unipotent.

Proof. Let R be a κ-algebra. In the proof, π stands for $\pi \otimes 1 \in B \otimes_A R$. To show that $\widetilde{M}^1(R)$ is a subgroup of $\operatorname{Ker}\tilde{\varphi}(R)$, it suffices to show that the inverse $1+x'$ of $1+\pi x$ in $\operatorname{Ker}\tilde{\varphi}(R)$ is contained in $\widetilde{M}^1(R)$. From the identity

$$(1+x')(1+\pi x) = 1+x' \star \pi x = 1+(x'+\pi x+\pi x' x) = 1+0,$$

we see that x' is an element of $\pi \underline{M}'(R)$ so that $1+x'$ is an element of $\widetilde{M}^1(R)$, since $\underline{M}'(R)$ is closed under multiplication and addition which implies $x+x'x \in \underline{M}'(R)$.

Then the first sentence of (i) follows by a theorem of Lazard which is stated at the beginning of Appendix A since $\widetilde{M}^1$ is isomorphic to an affine space of dimension n^2 as an algebraic variety over κ.

To show that $\widetilde{M}^1(R)$ is a normal subgroup of $\operatorname{Ker}\tilde{\varphi}(R)$, we choose an element $1+\pi x \in \widetilde{M}^1(R)$ and $1+m \in \operatorname{Ker}\tilde{\varphi}(R)$ with $m \in \underline{M}'(R)$. Let $1+m'$ be the inverse of $1+m$ so that $(1+m')(1+m) = 1$. Then we have the following identity:

$$(1+m')(1+\pi x)(1+m) = 1+m' \star \pi x \star m = 1+\pi(x+m'x+xm+m'xm).$$

Since $\underline{M}'(R)$ is closed under multiplication and addition, $x+m'x+xm+m'xm \in \underline{M}'(R)$ so that $(1+m')(1+\pi x)(1+m) \in \widetilde{M}^1(R)$.

For (ii), smoothness and connectedness are stable under quotienting by algebraic groups (Proposition 22.4 in [Knus et al. 1998]) and a quotient of a unipotent group is also a unipotent group by part (a) of the first corollary in Section 8.3 in [Waterhouse 1979]. □

This paragraph is a reproduction of [Gan and Yu 2000, 6.3.6]. Recall that there is a closed immersion $\widetilde{G} \to \widetilde{M}$. Notice that $\operatorname{Ker}\varphi$ is the kernel of the composition $\widetilde{G} \to \widetilde{M} \to \widetilde{M}/\operatorname{Ker}\tilde{\varphi}$. We define $\widetilde{G}^1$ as the kernel of the composition

$$\widetilde{G} \to \widetilde{M} \to \widetilde{M}/\widetilde{M}^1.$$

Then $\widetilde{G}^1$ is the kernel of the morphism $\operatorname{Ker}\varphi \to \operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ and, hence, is a closed normal subgroup of $\operatorname{Ker}\varphi$. The induced morphism $\operatorname{Ker}\varphi/\widetilde{G}^1 \to \operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ is a monomorphism, and thus $\operatorname{Ker}\varphi/\widetilde{G}^1$ is a closed subgroup scheme of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ by (Exposé $\mathrm{VI_B}$, Corollary 1.4.2 in [SGA 3_{I} 1970]).

Theorem A.4. *$\widetilde{G}^1$ is connected, smooth, and unipotent. Furthermore, the underlying algebraic variety of $\widetilde{G}^1$ over κ is an affine space of dimension*

$$\sum_{i<j} n_i n_j + \sum_{i \text{ odd}} \frac{n_i^2+n_i}{2} + \sum_{i \text{ even}} \frac{n_i^2-n_i}{2} + \#\{i : i \text{ is even and } L_i \text{ is of type } I\}.$$

Proof. We prove this theorem by writing out a set of equations completely defining $\widetilde{G}^1$ (after all there are so many different sets of equations defining $\widetilde{G}^1$). Let R be a κ-algebra. As explained in Remark 3.3(2), we consider the given hermitian form h as an element of $\underline{H}(R)$ and write it as a formal matrix $h = \big(\pi^i \cdot h_i\big)$ with $(\pi^i \cdot h_i)$ for the (i,i)-block and 0 for the remaining blocks. We also write h as $(f_{i,j}, a_i \cdots f_i)$. Recall that the notation $(f_{i,j}, a_i \cdots f_i)$ is defined and explained in Section 3C and explicit values of $(f_{i,j}, a_i \cdots f_i)$ for the h are given in Remark 3.3(2).

We choose an element $m = (m_{i,j}, s_i \cdots w_i) \in (\mathrm{Ker}\,\tilde{\varphi})(R)$ with a formal matrix interpretation $m = \left(\pi^{\max\{0, j-i\}} m_{i,j}\right)$, where the notation $(m_{i,j}, s_i \cdots w_i)$ is explained in Section 3B. Then $h \circ m$ is an element of $\underline{H}(R)$ and $(\mathrm{Ker}\,\varphi)(R)$ is the set of m such that $h \circ m = (f_{i,j}, a_i \cdots f_i)$. The action $h \circ m$ is explicitly described in Remark 3.5. Based on this, we need to write the matrix product $h \circ m = \sigma({}^t m) \cdot h \cdot m$ formally. To do that, we write each block of $\sigma({}^t m) \cdot h \cdot m$ as follows:

The diagonal (i, i)-block of the formal matrix product $\sigma({}^t m) \cdot h \cdot m$ is the following:

$$\pi^i \left(\sigma({}^t m_{i,i}) h_i m_{i,i} + \sigma(\pi) \cdot \sigma({}^t m_{i-1,i}) h_{i-1} m_{i-1,i} + \pi \cdot \sigma({}^t m_{i+1,i}) h_{i+1} m_{i+1,i}\right) + \pi^i \left((\sigma\pi)^2 \cdot \sigma({}^t m_{i-2,i}) h_{i-2} m_{i-2,i} + \pi^2 \cdot \sigma({}^t m_{i+2,i}) h_{i+2} m_{i+2,i}\right), \quad \text{(A-1)}$$

where $0 \leq i < N$.

The (i, j)-block of the formal matrix product $\sigma({}^t m) \cdot h \cdot m$, where $i < j$, is the following:

$$\pi^j \left(\sum_{i \leq k \leq j} \sigma({}^t m_{k,i}) h_k m_{k,j} + \sigma(\pi) \cdot \sigma({}^t m_{i-1,i}) h_{i-1} m_{i-1,j} + \pi \cdot \sigma({}^t m_{j+1,i}) h_{j+1} m_{j+1,j} \right), \quad \text{(A-2)}$$

where $0 \leq i, j < N$.

Before studying $\widetilde{G}^1$, we describe the conditions for an element $m \in \widetilde{M}(R)$ as above to belong to the subgroup $\widetilde{M}^1(R)$.

(1) $m_{i,j} = \pi m'_{i,j}$ if $i \neq j$;

(2) $m_{i,i} = \mathrm{id} + \pi m'_{i,i}$ if L_i is of type II;

(3) $m_{i,i} = \begin{pmatrix} s_i & \pi y_i \\ \pi v_i & 1 + \pi z_i \end{pmatrix} = \begin{pmatrix} \mathrm{id} + \pi s'_i & \pi^2 y'_i \\ \pi^2 v'_i & 1 + \pi^2 z'_i \end{pmatrix}$ if i is even and L_i is of type I^o;

(4) $m_{i,i} = \begin{pmatrix} s_i & r_i & \pi t_i \\ \pi y_i & 1 + \pi x_i & \pi z_i \\ v_i & u_i & 1 + \pi w_i \end{pmatrix} = \begin{pmatrix} \mathrm{id} + \pi s'_i & \pi r'_i & \pi^2 t'_i \\ \pi^2 y'_i & 1 + \pi^2 x'_i & \pi^2 z'_i \\ \pi v'_i & \pi u'_i & 1 + \pi^2 w'_i \end{pmatrix}$ if i is even and L_i is of type I^e.

Here, all matrices having $'$ in the superscript are considered as matrices with entries in R. When i is even and L_i is of type I, we formally write $m_{i,i} = \mathrm{id} + \pi m'_{i,i}$. Then $\widetilde{G}^1(R)$ is the set of $m \in \widetilde{M}^1(R)$ such that $h \circ m = h = (f_{i,j}, a_i \cdots f_i)$. Since $h \circ m$ is an element of $\underline{H}(R)$, we can write $h \circ m$ as $(f'_{i,j}, a'_i \cdots f'_i)$. In what follows, we will write $(f'_{i,j}, a'_i \cdots f'_i)$ in terms of $h = (f_{i,j}, a_i \cdots f_i)$ and m, and will compare $(f'_{i,j}, a'_i \cdots f'_i)$ with $(f_{i,j}, a_i \cdots f_i)$, in order to obtain a set of equations defining $\widetilde{G}^1$.

If we put all these (1)–(4) into (A-2), then we obtain

$$\pi^j \left(\sigma(1 + \pi \cdot {}^t m'_{i,i}) h_i \pi m'_{i,j} + \sigma(\pi \cdot {}^t m'_{j,i}) h_j (1 + \pi m'_{j,j})\right).$$

Therefore,

$$f'_{i,j} = \left(\sigma(1 + \pi \cdot {}^t m'_{i,i}) h_i \pi m'_{i,j} + \sigma(\pi \cdot {}^t m'_{j,i}) h_j (1 + \pi m'_{j,j})\right),$$

where this equation is considered in $B\otimes_A R$ and π stands for $\pi\otimes 1\in B\otimes_A R$. Thus each term having π^2 as a factor is 0 and we have

$$f'_{i,j}=h_i\pi m'_{i,j}+\sigma(\pi\cdot {}^tm'_{j,i})h_j,\quad \text{where } i<j. \tag{A-3}$$

This equation is of the form $f'_{i,j}=X+\pi Y$ since it is an equation in $B\otimes_A R$. By letting $f'_{i,j}=f_{i,j}=0$, we obtain

$$\bar{h}_i m'_{i,j}+{}^tm'_{j,i}\bar{h}_j=0,\quad \text{where } i<j, \tag{A-4}$$

where $\bar{h}_i$ (resp. $\bar{h}_j$) is obtained by letting each term in h_i (resp. h_j) having π as a factor be zero so that this equation is considered in R. Note that $\bar{h}_i$ and $\bar{h}_j$ are invertible as matrices with entries in R by Remark 3.3. Thus $m'_{i,j}=\bar{h}_i^{-1}\cdot {}^tm'_{j,i}\cdot\bar{h}_j$. This induces that each entry of $m'_{i,j}$ is expressed as a linear combination of the entries of $m'_{j,i}$. Thus there are exactly n_in_j independent linear equations among the entries of $m'_{i,j}, m'_{j,i}$.

Next, we put (1)–(4) into (A-1). Then we obtain

$$\pi^i\big(\sigma(1+\pi\cdot {}^tm'_{i,i})h_i(1+\pi m'_{i,i})\big). \tag{A-5}$$

We interpret this so as to obtain equations defining $\widetilde{G}^1$. There are 4 cases, indexed by (i), (ii), (iii), (iv), according to types of L_i.

(i) Assume that i is odd. Then $\pi^ih_i=\xi^{(i-1)/2}\pi a_i$ as explained in Section 3C and thus we have

$$a'_i=\sigma(1+\pi\cdot {}^tm'_{i,i})a_i(1+\pi m'_{i,i}).$$

Here, the nondiagonal entries of this equation are considered in $B\otimes_A R$ and each diagonal entry of a'_i is of the form $\epsilon\pi x_i$ with $x_i\in R$.

Thus, we can cancel terms having π^2 as a factor and the above equation equals

$$a'_i=a_i+\sigma(\pi)\cdot {}^tm'_{i,i}a_i+\pi\cdot a_im'_{i,i}.$$

By letting $a'_i=a_i$, we have the following equation

$$\sigma(\pi)\cdot {}^tm'_{i,i}a_i+\pi\cdot a_im'_{i,i}=0.$$

Since this is an equation in $B\otimes_A R$, it is of the form $X+\pi Y=0$. Note that the reduction of $\epsilon \mod \pi$ is 1. We denote by $\bar{a}_i$ the reduction of $a_i \mod \pi$. Thus we have

$${}^tm'_{i,i}\bar{a}_i+\bar{a}_im'_{i,i}=0.$$

This is a matrix equation over R, in a usual sense, and $\bar{a}_i$ is symmetric and the diagonal entries of $\bar{a}_i$ are 0. More precisely,

$$\bar{a}_i=\begin{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}&&\\&\ddots&\\&&\begin{pmatrix}0&1\\1&0\end{pmatrix}\end{pmatrix}.$$

Then we can see that there is no contribution coming from the diagonal entries of ${}^t m'_{i,i}\bar{a}_i + \bar{a}_i m'_{i,i} = 0$ and that there are exactly $(n_i^2 - n_i)/2$ independent linear equations. Thus $(n_i^2 + n_i)/2$ entries of $m'_{i,i}$ determine all entries of $m'_{i,i}$. Note that the conditions on $m'_{i,i}$, viewed as a matrix with entries in κ, are tantamount to this matrix belonging to the Lie algebra of a symplectic group associated to an obvious alternating form given by $\bar{a}_i$. Then $(n_i^2 + n_i)/2$ is the dimension of this symplectic group.

For example, let $m'_{i,i} = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and $\bar{a}_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$ {}^t m'_{i,i}\bar{a}_i + \bar{a}_i m'_{i,i} = \begin{pmatrix} 2z & x+w \\ x+w & 2y \end{pmatrix} = \begin{pmatrix} 0 & x+w \\ x+w & 0 \end{pmatrix}. $$

Thus there is one linear equation $x + w = 0$ and x, y, z determine all entries of $m'_{i,i}$.

(ii) Assume that i is even and L_i is of type II. This case is parallel to the previous case. Then $\pi^i h_i = \xi^{i/2} a_i$ as explained in Section 3C and we have

$$ a'_i = \sigma(1 + \pi \cdot {}^t m'_{i,i}) a_i (1 + \pi m'_{i,i}). $$

Here, the nondiagonal entries of this equation are considered in $B \otimes_A R$ and each diagonal entry of a'_i is of the form $2x_i$ with $x_i \in R$. Now, the nondiagonal entries of $\sigma(\pi \cdot {}^t m'_{i,i}) a_i (\pi m'_{i,i})$ are all 0 since they contain π^2 as a factor. The diagonal entries of $\sigma(\pi \cdot {}^t m'_{i,i}) a_i (\pi m'_{i,i})$ are also 0 since they contain π^4 as a factor. Thus, the above equation equals

$$ a'_i = a_i + \sigma(\pi) \cdot {}^t m'_{i,i} a_i + \pi \cdot a_i m'_{i,i}. $$

By letting $a'_i = a_i$, we have the following equation

$$ \sigma(\pi) \cdot {}^t m'_{i,i} a_i + \pi \cdot a_i m'_{i,i} = 0. $$

Based on (2) of the description of $\underline{H}(R)$ for a κ-algebra R, which is explained in Section 3C, in order to investigate this equation, we need to consider the nondiagonal entries of $\sigma(\pi) \cdot {}^t m'_{i,i} a_i + \pi \cdot a_i m'_{i,i}$ as elements of $B \otimes_A R$ and the diagonal entries of $\sigma(\pi) \cdot {}^t m'_{i,i} a_i + \pi \cdot a_i m'_{i,i}$ as of the form $2x_i$ with $x_i \in R$. Recall from Remark 3.3 that

$$ a_i = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}. $$

Then we can see that each diagonal entry as well as each nondiagonal (upper triangular) entry of $\sigma(\pi) \cdot {}^t m'_{i,i} a_i + \pi \cdot a_i m'_{i,i}$ produces a linear equation. Thus there are exactly $(n_i^2 + n_i)/2$ independent linear equations and $(n_i^2 - n_i)/2$ entries of $m'_{i,i}$ determine all entries of $m'_{i,i}$.

For example, let $m'_{i,i} = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right)$ and $\bar{a}_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$ \sigma(\pi) \cdot {}^t m'_{i,i} a_i + \pi \cdot a_i m'_{i,i} = \sigma(\pi) \begin{pmatrix} z & x \\ w & y \end{pmatrix} + \pi \begin{pmatrix} z & w \\ x & y \end{pmatrix} = \begin{pmatrix} (\sigma(\pi)+\pi)z & \sigma(\pi)x + \pi w \\ \sigma(\pi)w + \pi x & (\sigma(\pi)+\pi)y \end{pmatrix}. $$

Recall that $\sigma(\pi) = \epsilon\pi$ with $\epsilon \equiv 1 \mod \pi$ and $\sigma(\pi) + \pi = 2$, as explained at the beginning of Section 2A. Thus there are three linear equations $z = 0$, $x + w = 0$, $y = 0$ and x determines every other entry of $m'_{i,i}$.

(iii) Assume that i is even and L_i is of type I^o. Then $\pi^i h_i = \xi^{i/2}\begin{pmatrix} a_i & \pi b_i \\ \sigma(\pi\cdot{}^t b_i) & 1+2c_i\end{pmatrix}$ as explained in Section 3C and we have

$$\begin{pmatrix} a_i' & \pi b_i' \\ \sigma(\pi\cdot{}^t b_i') & 1+2c_i'\end{pmatrix} = \sigma(1+\pi\cdot{}^t m_{i,i}')\cdot\begin{pmatrix} a_i & \pi b_i \\ \sigma(\pi\cdot{}^t b_i) & 1+2c_i\end{pmatrix}\cdot(1+\pi m_{i,i}'). \tag{A-6}$$

Here, the nondiagonal entries of a_i' as well as the entries of b_i' are considered in $B\otimes_A R$, each diagonal entry of a_i' is of the form $2x_i$ with $x_i\in R$, and c_i' is in R. In addition, $b_i=0$, $c_i=\bar{\gamma}_i$ as explained in Remark 3.3(2) and a_i is the diagonal matrix with $\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ on the diagonal.

Note that in this case, $m_{i,i}' = \begin{pmatrix} s_i' & \pi y_i' \\ \pi v_i' & \pi z_i'\end{pmatrix}$. Compute $\sigma(\pi\cdot{}^t m_{i,i}')\cdot\begin{pmatrix} a_i & 0 \\ 0 & 1+2c_i\end{pmatrix}\cdot(\pi m_{i,i}')$ formally and this equals $\sigma(\pi)\pi\begin{pmatrix} {}^t s_i' a_i s_i' + \pi^2 X_i & \pi Y_i \\ \sigma(\pi\cdot{}^t Y_i) & \pi^2 Z_i\end{pmatrix}$ for certain matrices X_i, Y_i, Z_i with suitable sizes. Thus we can ignore the contribution from $\sigma(\pi\cdot{}^t m_{i,i}')\begin{pmatrix} a_i & 0 \\ 0 & 1+2c_i\end{pmatrix}(\pi m_{i,i}')$ in Equation (A-6) and so Equation (A-6) equals

$$\begin{pmatrix} a_i' & \pi b_i' \\ \sigma(\pi\cdot{}^t b_i') & 1+2c_i'\end{pmatrix} = \begin{pmatrix} a_i & 0 \\ 0 & 1+2c_i\end{pmatrix} + \sigma(\pi)\begin{pmatrix} {}^t s_i' & \sigma(\pi)\cdot{}^t v_i' \\ \sigma(\pi)\cdot{}^t y_i' & \sigma(\pi) z_i'\end{pmatrix}\begin{pmatrix} a_i & 0 \\ 0 & 1+2c_i\end{pmatrix} + \pi\begin{pmatrix} a_i & 0 \\ 0 & 1+2c_i\end{pmatrix}\begin{pmatrix} s_i' & \pi y_i' \\ \pi v_i' & \pi z_i'\end{pmatrix}.$$

We interpret each block of the above equation below:

(a) Firstly, we consider the $(1,1)$-block. The computation associated to this block is similar to that for the above case (ii). Hence there are exactly $((n_i-1)^2+(n_i-1))/2$ independent linear equations and $((n_i-1)^2-(n_i-1))/2$ entries of s_i' determine all entries of s_i'.

(b) Secondly, we consider the $(1,2)$-block. We can ignore the contribution from ${}^t v_i' c_i$ since it contains π^3 as a factor. Then the $(1,2)$-block is

$$\pi b_i' = \sigma(\pi)\pi\cdot(\epsilon\cdot{}^t v_i' + 1/\epsilon\cdot a_i y_i'). \tag{A-7}$$

By letting $b_i' = b_i = 0$, we have

$$\sigma(\pi)\cdot(\epsilon\cdot{}^t v_i' + 1/\epsilon\cdot a_i y_i') = 0$$

as an equation in $B\otimes_A R$. Thus there are exactly (n_i-1) independent linear equations among the entries of v_i' and y_i' and the entries of v_i' determine all entries of y_i'.

(c) Finally, we consider the $(2,2)$-block. This is

$$1+2c_i' = 1+2c_i + (\pi^2+(\sigma(\pi))^2)z_i' + 2(\pi^2+(\sigma(\pi))^2)c_i z_i'. \tag{A-8}$$

Since $\pi^2+(\sigma(\pi))^2 = (\pi+\sigma(\pi))^2 - 2\sigma(\pi)\pi$, we see that $\pi^2+(\sigma(\pi))^2$ contains 4 as a factor. Thus by letting $c_i'=c_i$, this equation is trivial.

By combining the three cases (a)–(c), there are exactly $((n_i-1)^2+(n_i-1))/2+(n_i-1) = (n_i^2+n_i)/2-1$ independent linear equations and $(n_i^2-n_i)/2+1$ entries of $m_{i,i}'$ determine all entries of $m_{i,i}'$.

(iv) Assume that i is even and L_i is of type I^e. Then

$$\pi^i h_i = \xi^{i/2} \begin{pmatrix} a_i & b_i & \pi e_i \\ \sigma({}^t b_i) & 1+2f_i & 1+\pi d_i \\ \sigma(\pi\cdot{}^t e_i) & \sigma(1+\pi d_i) & 2c_i \end{pmatrix}$$

as explained in Section 3C and we have

$$\begin{pmatrix} a_i' & b_i' & \pi e_i' \\ \sigma({}^t b_i') & 1+2f_i' & 1+\pi d_i' \\ \sigma(\pi\cdot{}^t e_i') & \sigma(1+\pi d_i') & 2c_i' \end{pmatrix} = \sigma(1+\pi\cdot{}^t m_{i,i}')\cdot \begin{pmatrix} a_i & b_i & \pi e_i \\ \sigma({}^t b_i) & 1+2f_i & 1+\pi d_i \\ \sigma(\pi\cdot{}^t e_i) & \sigma(1+\pi d_i) & 2c_i \end{pmatrix}\cdot(1+\pi m_{i,i}'). \tag{A-9}$$

Here, the nondiagonal entries of a_i' as well as the entries of b_i', e_i', d_i' are considered in $B\otimes_A R$, each diagonal entry of a_i' is of the form $2x_i$ with $x_i \in R$, and c_i', f_i' are in R. In addition, $b_i=0$, $d_i=0$, $e_i=0$, $f_i=0$, $c_i=\bar{\gamma}_i$ as explained in Remark 3.3(2) and a_i is the diagonal matrix with $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the diagonal.

Notice that in this case,

$$m_{i,i}' = \begin{pmatrix} s_i' & r_i' & \pi t_i' \\ \pi y_i' & \pi x_i' & \pi z_i' \\ v_i' & u_i' & \pi w_i' \end{pmatrix}.$$

We compute

$$\sigma(\pi\cdot{}^t m_{i,i}')\cdot \begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i \end{pmatrix}\cdot(\pi m_{i,i}')$$

formally and this equals

$$\sigma(\pi)\pi \begin{pmatrix} {}^t s_i' a_i s_i' + \pi^2 X_i & Y_i & \pi Z_i \\ \sigma({}^t Y_i) & {}^t r_i' a_i r_i' + \pi^2 X_i' & \pi Y_i' \\ \sigma(\pi\cdot{}^t Z_i) & \sigma(\pi\cdot{}^t Y_i') & \pi^2 Z_i' \end{pmatrix}$$

for certain matrices $X_i, Y_i, Z_i, X_i', Y_i', Z_i'$ with suitable sizes. Thus we can ignore the contribution from this part in Equation (A-9) and so Equation (A-9) equals

$$\begin{pmatrix} a_i' & b_i' & \pi e_i' \\ \sigma({}^t b_i') & 1+2f_i' & 1+\pi d_i' \\ \sigma(\pi\cdot{}^t e_i') & \sigma(1+\pi d_i') & 2c_i' \end{pmatrix} = \begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i \end{pmatrix}$$
$$+\sigma(\pi) \begin{pmatrix} {}^t s_i' & \sigma(\pi)\cdot{}^t y_i' & {}^t v_i' \\ {}^t r_i' & \sigma(\pi) x_i' & u_i' \\ \sigma(\pi)\cdot{}^t t_i' & \sigma(\pi) z_i' & \sigma(\pi) w_i' \end{pmatrix} \begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i \end{pmatrix}$$
$$+\pi \begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i \end{pmatrix} \begin{pmatrix} s_i' & r_i' & \pi t_i' \\ \pi y_i' & \pi x_i' & \pi z_i' \\ v_i' & u_i' & \pi w_i' \end{pmatrix}.$$

We interpret each block of the above equation as follows:

(a) Let us consider the $(1, 1)$-block. The computation associated to this block is similar to that for the previous case (ii). Hence there are exactly $((n_i-2)^2+(n_i-2))/2$ independent linear equations and $((n_i-2)^2-(n_i-2))/2$ entries of s_i' determine all entries of s_i'.

(b) We consider the $(1, 2)$-block. This gives
$$b_i' = \pi(\epsilon^2\pi\cdot {}^ty_i' + \epsilon\cdot {}^tv_i' + a_ir_i'). \tag{A-10}$$
This is an equation in $B\otimes_A R$. By letting $b_i'=b_i=0$, there are exactly (n_i-2) independent linear equations among the entries of v_i', r_i'.

(c) The $(1, 3)$-block is
$$\pi e_i' = \pi^2(\epsilon^2\cdot {}^ty_i' + (2/\pi)\cdot\epsilon\cdot {}^tv_i'c_i + a_it_i').$$
By letting $e_i'=e_i=0$, we have
$$\begin{aligned} e_i' &= \pi(\epsilon^2\cdot {}^ty_i' + (2/\pi)\cdot\epsilon\cdot {}^tv_i'c_i + a_it_i') \\ &= \pi(\epsilon^2\cdot {}^ty_i' + a_it_i') = \pi({}^ty_i' + a_it_i') = 0. \end{aligned} \tag{A-11}$$
This is an equation in $B\otimes_A R$. Thus there are exactly (n_i-2) independent linear equations among the entries of y_i', t_i'.

(d) The $(2, 3)$-block is
$$1+\pi d_i' = 1+\sigma(\pi)(\sigma(\pi)x_i' + 2u_i'c_i) + \pi^2(z_i' + w_i').$$
By letting $d_i'=d_i=0$, we have
$$d_i' = \pi(\epsilon^2x_i' + z_i' + w_i') = \pi(x_i' + z_i' + w_i') = 0. \tag{A-12}$$
This is an equation in $B\otimes_A R$. Thus there is exactly one independent linear equation among the entries of x_i', z_i', w_i'.

(e) The $(2, 2)$-block is
$$\begin{aligned} 1+2f_i' &= 1+\sigma(\pi)(\sigma(\pi)x_i' + u_i') + \pi(\pi x_i' + u_i') \\ &= 1+2u_i' + ((\pi+\sigma(\pi))^2 - 2\pi\sigma(\pi))x_i'. \end{aligned}$$
By letting $f_i'=f_i=0$, we have
$$f_i' = u_i' + ((\pi+\sigma(\pi)) - \pi\sigma(\pi))x_i' = u_i' = 0.$$
This is an equation in R. Thus $u_i'=0$ is the only independent linear equation.

(f) The $(3, 3)$-block is
$$\begin{aligned} 2c_i' &= 2c_i + \sigma(\pi)(\sigma(\pi)z_i' + 2\sigma(\pi)w_i'c_i) + \pi(\pi z_i' + 2\pi w_i'c_i) \\ &= 2c_i + ((\pi+\sigma(\pi))^2 - 2\pi\sigma(\pi))(z_i' + 2w_i'c_i). \end{aligned} \tag{A-13}$$
Since $((\pi+\sigma(\pi))^2 - 2\pi\sigma(\pi))$ contains 4 as a factor, by letting $c_i'=c_i$, this equation is trivial.

By combining the six cases (a)–(f), there are exactly $((n_i-2)^2+(n_i-2))/2+2(n_i-2)+2 = (n_i^2+n_i)/2-1$ independent linear equations and $(n_i^2-n_i)/2+1$ entries of $m_{i,i}'$ determine all entries of $m_{i,i}'$.

We now combine all the work done in this proof. Namely, we collect the above (i), (ii), (iii), (iv) which are the interpretations of Equation (A-5), together with Equation (A-4). Then there are exactly

$$\sum_{i<j} n_i n_j + \sum_{i \text{ odd}} \frac{n_i^2 - n_i}{2} + \sum_{i \text{ even}} \frac{n_i^2 + n_i}{2} - \#\{i : i \text{ is even and } L_i \text{ is of type } I\}$$

independent linear equations among the entries of m. Furthermore, all coefficients of these equations are in κ. Therefore, we consider $\widetilde{G}^1$ as a subvariety of $\widetilde{M}^1$ determined by these linear equations. Since $\widetilde{M}^1$ is an affine space of dimension n^2, the underlying algebraic variety of $\widetilde{G}^1$ over κ is an affine space of dimension

$$\sum_{i<j} n_i n_j + \sum_{i \text{ odd}} \frac{n_i^2 + n_i}{2} + \sum_{i \text{ even}} \frac{n_i^2 - n_i}{2} + \#\{i : i \text{ is even and } L_i \text{ is of type } I\}.$$

This completes the proof by using a theorem of Lazard which is stated at the beginning of Appendix A. □

Let R be a κ-algebra. We describe the functor of points of the scheme $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ by using points of the scheme $(\underline{M}' \otimes \kappa)/\underline{\pi M}'$, based on Lemma A.3. Recall from two paragraphs before Lemma A.3 that $(1+)^{-1}(\underline{M}^*)$, which is an open subscheme of $\underline{M}'$, is a group scheme with the operation $\star$. Let $\widetilde{M}'$ be the special fiber of $(1+)^{-1}(\underline{M}^*)$. Since $\widetilde{M}^1$ is a closed normal subgroup of $\widetilde{M}(= \underline{M}^* \otimes \kappa)$ (cf. Lemma A.3(i)), $\underline{\pi M}'$, which is the inverse image of $\widetilde{M}^1$ under the isomorphism $1+$, is a closed normal subgroup of $\widetilde{M}'$. Therefore, the morphism $1+$ induces the following isomorphism of group schemes, which is also denoted by $1+$,

$$1+ : \widetilde{M}'/\underline{\pi M}' \longrightarrow \widetilde{M}/\widetilde{M}^1.$$

Note that $\widetilde{M}'/\underline{\pi M}'(R) = \widetilde{M}'(R)/\underline{\pi M}'(R)$ by Lemma A.1. Each element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ is therefore uniquely written as $1+\bar{x}$, where $\bar{x} \in \widetilde{M}'(R)/\underline{\pi M}'(R)$. Here, by $1+\bar{x}$, we mean the image of $\bar{x}$ under the morphism $1+$ at the level of R-points.

We still need a better description of an element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ by using a point of the scheme $(\underline{M}' \otimes \kappa)/\underline{\pi M}'$. Note that $(\underline{M}' \otimes \kappa)/\underline{\pi M}'$ is a quotient of group schemes with respect to the addition, whereas $\widetilde{M}'/\underline{\pi M}'$ is a quotient of group schemes with respect to the operation $\star$.

We claim that the open immersion $\iota : \widetilde{M}' \to \underline{M}' \otimes \kappa$ with $x \mapsto x$ induces a monomorphism of schemes

$$\bar{\iota} : \widetilde{M}'/\underline{\pi M}' \to (\underline{M}' \otimes \kappa)/\underline{\pi M}'.$$

Choose $x \in \widetilde{M}'(R)$ and $\pi y \in \underline{\pi M}'(R)$ for a κ-algebra R. Since $x \star \pi y = x + \pi(y + xy)$, both x and $x \star \pi y$ give the same element in $((\underline{M}' \otimes \kappa)/\underline{\pi M}')(R)$. Thus the morphism $\bar{\iota}$ is well-defined.

In order to show that $\bar{\iota}$ is a monomorphism, choose $x, y \in \widetilde{M}'(R)$ such that $x = y + \pi z$ with $\pi z \in \underline{\pi M}'(R)$. Let y' $(\in \widetilde{M}'(R))$ be the inverse of y so that $y \star y' = y + y' + yy' = 0$. Then $\pi(z + y'z)$ is an element of $\underline{\pi M}'(R)$. We have the following identity:

$$x \star \pi(z + y'z) = (y + \pi z) \star \pi(z + y'z) = y + \pi(y + y' + yy')z = y.$$

Therefore, x and y give the same element in $(\widetilde{M}'/\pi\underline{M'})(R)$, which shows the injectivity of the above morphism.

Note that the operation $\star$ is closed in $\underline{M}'\otimes\kappa$ as mentioned in the third paragraph following Lemma A.2. We can also easily check that the operation $\star$ is well-defined on $(\underline{M}'\otimes\kappa)/\pi\underline{M}'$, which turns to be a scheme of monoids with respect to $\star$, and that the morphism $\bar{\iota}$ is a monomorphism of monoid schemes.

To summarize, the morphism $1+ : \widetilde{M}'/\pi\underline{M'} \longrightarrow \widetilde{M}/\widetilde{M}^1$ is an isomorphism of group schemes and the morphism $\bar{\iota} : \widetilde{M}'/\pi\underline{M}' \to (\underline{M}' \otimes \kappa)/\pi\underline{M}'$ is a monomorphism preserving the operation $\star$. Therefore, each element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ is uniquely written as $1+\bar{x}$, where $\bar{x} \in (\underline{M}' \otimes \kappa)(R)/\pi\underline{M}'(R)$. Here, by $1+\bar{x}$, we mean $(1+)\circ\bar{\iota}^{-1}(\bar{x})$. From now on to the end of this paper, we keep the notation $1+\bar{x}$ to express an element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ such that $\bar{x}$ is an element of $(\underline{M}' \otimes \kappa)(R)/\pi\underline{M}'(R)$ which is a quotient of R-valued points of group schemes with respect to addition. Then the product of two elements $1+\bar{x}$ and $1+\bar{y}$ is the same as $1+\bar{x}\star\bar{y}$ $(= 1+(\bar{x}+\bar{y}+\bar{x}\bar{y}))$.

Remark A.5. By the above argument, we write an element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ formally as $m = \left(\pi^{\max\{0,j-i\}}m_{i,j}\right)$ with $s_i, \cdots, w_i$ as in Section 3B such that each entry of each of the matrices $(m_{i,j})_{i\neq j}, s_i, \cdots, w_i$ is in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R) \cong R$. In particular, based on the description of $\operatorname{Ker}\tilde{\varphi}(R)$ given at the paragraph following Lemma A.2, we have the following conditions on m:

(1) Assume that i is even and L_i is of type I. Then $s_i = \mathrm{id}$.

(2) $m_{i,i} = \mathrm{id}$ if L_i is of type II.

(3) Assume that i is odd. Then $\delta_{i-1}e_{i-1}\cdot m_{i-1,i} + \delta_{i+1}e_{i+1}\cdot m_{i+1,i} = 0$. Here, δ_j, e_j are as explained in the description of $\operatorname{Ker}\tilde{\varphi}(R)$.

Theorem A.6. $\operatorname{Ker}\varphi/\widetilde{G}^1$ *is isomorphic to* $\mathbb{A}^{l'}\times(\mathbb{Z}/2\mathbb{Z})^{\beta}$ *as a* κ*-variety, where* $\mathbb{A}^{l'}$ *is an affine space of dimension* l'*. Here,*

- l' *is such that* $l'+\dim\widetilde{G}^1 = l$*. Notice that* l *is defined in Lemma 4.6 and that the dimension of* $\widetilde{G}^1$ *is given in Theorem A.4.*
- β *is the number of even integers* j *such that* L_j *is of type I and* L_{j+2} *is of type II.*

Proof. Lemma A.1 and Theorem A.4 imply that $\operatorname{Ker}\varphi/\widetilde{G}^1$ represents the functor $R\mapsto \operatorname{Ker}\varphi(R)/\widetilde{G}^1(R)$. Recall that $\operatorname{Ker}\varphi/\widetilde{G}^1$ is a closed subgroup scheme of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ as explained at the paragraph just before Theorem A.4. Let $m = \left(\pi^{\max\{0,j-i\}}m_{i,j}\right)$ be an element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ such that m belongs to $(\operatorname{Ker}\varphi/\widetilde{G}^1)(R)$. We want to find equations which m satisfies. Note that the entries of m involve $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ as explained in Remark A.5.

Recall that h is the fixed hermitian form and we consider it as an element in $\underline{H}(R)$ as explained in Remark 3.3(2). We write it as a formal matrix $h = \left(\pi^i\cdot h_i\right)$ with $(\pi^i\cdot h_i)$ for the (i,i)-block and 0 for the remaining blocks. We choose a representative $1+x \in \operatorname{Ker}\varphi(R)$ of m so that $h\circ(1+x) = h$. Any other representative of m in $\operatorname{Ker}\tilde{\varphi}(R)$ is of the form $(1+x)(1+\pi y)$ with $y\in\underline{M}'(R)$ and we have $h\circ(1+x)(1+\pi y) = h\circ(1+\pi y)$. Notice that $h\circ(1+\pi y)$ is an element of $\underline{H}(R)$ so we express it as $(f'_{i,j}, a'_i\cdots f'_i)$. We also let

$h = (f_{i,j}, a_i \cdots f_i)$. Here, we follow notation from Section 3C, the paragraph just before Remark 3.3. Recall that $h = (f_{i,j}, a_i \cdots f_i)$ is described explicitly in Remark 3.3(2). Now, $1+\pi y$ is an element of $\widetilde{M}^1(R)$ and so we can use our result (Equations (A-3), (A-7), (A-8), (A-10), (A-11), (A-12), (A-13)) stated in the proof of Theorem A.4 in order to compute $h \circ (1+\pi y)$. Based on this, we enumerate equations which m satisfies as follows:

(1) Assume $i < j$. By Equation (A-3) which involves an element of $\widetilde{M}^1(R)$, each entry of $f'_{i,j}$ has π as a factor so that $f'_{i,j} \equiv f_{i,j} (=0) \mod (\pi \otimes 1)(B \otimes_A R)$. In other words, the (i, j)-block of $h \circ (1+x)(1+\pi y)$ divided by $\pi^{\max\{i,j\}}$ is $f_{i,j} (=0)$ modulo $(\pi \otimes 1)(B \otimes_A R)$, which is independent of the choice of $1+\pi y$. Let $\tilde{m} \in \operatorname{Ker} \tilde{\varphi}(R)$ be a lift of m. Therefore, if we write the (i, j)-block of $\sigma({}^t\tilde{m}) \cdot h \cdot \tilde{m}$ as $\pi^{\max\{i,j\}} \mathcal{X}_{i,j}(\tilde{m})$, where $\mathcal{X}_{i,j}(\tilde{m}) \in M_{n_i \times n_j}(B \otimes_A R)$, then the image of $\mathcal{X}_{i,j}(\tilde{m})$ in $M_{n_i \times n_j}(B \otimes_A R)/(\pi \otimes 1) M_{n_i \times n_j}(B \otimes_A R) \cong M_{n_i \times n_j}(R)$ is independent of the choice of the lift $\tilde{m}$ of m. Therefore, we may denote this image by $\mathcal{X}_{i,j}(m)$. On the other hand, by Equation (A-2), we have the following identity:

$$\mathcal{X}_{i,j}(m) = \sum_{i \le k \le j} \sigma({}^t m_{k,i}) \bar{h}_k m_{k,j} \quad \text{if } i < j. \tag{A-14}$$

We explain how to interpret the above equation. We know that $\mathcal{X}_{i,j}(m)$ and $m_{k,k'}$ (with $k \neq k'$) are matrices with entries in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$, whereas $m_{i,i}$ and $m_{j,j}$ are formal matrices as explained in Remark A.5. Thus we consider $\bar{h}_k$, $m_{i,i}$, and $m_{j,j}$ as matrices with entries in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$ by letting π be zero in each entry of the formal matrices h_k, $m_{i,i}$, and $m_{j,j}$. Here we keep using $m_{i,i}$ and $m_{j,j}$ for matrices with entries in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$ in the above equation in order to simplify notation. Later in Equation (A-23), they are denoted by $\bar{m}_{i,i}$ and $\bar{m}_{j,j}$. Then the right hand side is computed as a sum of products of matrices (involving the usual matrix addition and multiplication) with entries in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$. Thus, the assignment $m \mapsto \mathcal{X}_{i,j}(m)$ is polynomial in m. Furthermore, since m actually belongs to $\operatorname{Ker} \varphi(R)/\widetilde{G}^1(R)$, we have the following equation by the argument made at the beginning of this paragraph:

$$\mathcal{X}_{i,j}(m) = f_{i,j} \mod (\pi \otimes 1)(B \otimes_A R) = 0.$$

Thus we get an $n_i \times n_j$ matrix $\mathcal{X}_{i,j}$ of polynomials on $\operatorname{Ker} \tilde{\varphi}/\widetilde{M}^1$ defined by Equation (A-14), vanishing on the subscheme $\operatorname{Ker} \varphi/\widetilde{G}^1$.

Before moving to the following steps, we fix notation. Let m be an element in $(\operatorname{Ker} \tilde{\varphi}/\widetilde{M}^1)(R)$ and $\tilde{m} \in \operatorname{Ker} \tilde{\varphi}(R)$ be its lift. For any block x_i of m, $\tilde{x}_i$ is denoted by the corresponding block of $\tilde{m}$ whose reduction is x_i. Since x_i is a block of an element of $(\operatorname{Ker} \tilde{\varphi}/\widetilde{M}^1)(R)$, it involves $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$ as explained in Remark A.5, whereas $\tilde{x}_i$ involves $B \otimes_A R$. In addition, for a block a_i of h, $\bar{a}_i$ is denoted by the image of a_i in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$.

(2) Assume that i is even and L_i is of type I^o. By Equation (A-7) which involves an element of $\widetilde{M}^1(R)$, each entry of b'_i has π as a factor so that $b'_i \equiv b_i = 0 \mod (\pi \otimes 1)(B \otimes_A R)$. Let $\tilde{m} \in \operatorname{Ker} \tilde{\varphi}(R)$ be a lift of m. By using an argument similar to the paragraph just before Equation (A-14) of step (1), if we write the $(1, 2)$-block of the (i, i)-block of the formal matrix product $\sigma({}^t\tilde{m}) \cdot h \cdot \tilde{m}$ as $\xi^{i/2} \cdot \pi \mathcal{X}_{i,1,2}(\tilde{m})$, where $\mathcal{X}_{i,1,2}(\tilde{m}) \in M_{(n_i-1)\times 1}(B \otimes_A R)$, then the image of $\mathcal{X}_{i,1,2}(\tilde{m})$ in $M_{(n_i-1)\times 1}(B \otimes_A R)/(\pi \otimes 1) M_{(n_i-1)\times 1}(B \otimes_A R)$ is independent

of the choice of the lift $\tilde{m}$ of m. Therefore, we may denote this image by $\mathcal{X}_{i,1,2}(m)$. As for Equation (A-14) of step (1), we need to express $\mathcal{X}_{i,1,2}(m)$ as matrices. Recall that $\pi^i h_i = \xi^{i/2}\left(\begin{smallmatrix} a_i & 0 \\ 0 & 1+2c_i\end{smallmatrix}\right) = \pi^i \cdot \epsilon^{i/2}\left(\begin{smallmatrix} a_i & 0 \\ 0 & 1+2c_i\end{smallmatrix}\right)$ and $\epsilon \equiv 1 \mod \pi\otimes 1$. We write $m_{i,i}$ as $\left(\begin{smallmatrix} \mathrm{id} & \pi y_i \\ \pi v_i & 1+\pi z_i\end{smallmatrix}\right)$ and $\tilde{m}_{i,i}$ as $\left(\begin{smallmatrix} \tilde{s}_i & \pi \tilde{y}_i \\ \pi \tilde{v}_i & 1+\pi \tilde{z}_i\end{smallmatrix}\right)$ such that $\tilde{s}_i = \mathrm{id} \mod \pi\otimes 1$. Then

$$\sigma({}^t\tilde{m}_{i,i})h_i\tilde{m}_{i,i} = \epsilon^{i/2}\begin{pmatrix} \sigma({}^t\tilde{s}_i) & \sigma(\pi\cdot {}^t\tilde{v}_i) \\ \sigma(\pi\cdot {}^t\tilde{y}_i) & 1+\sigma(\pi\tilde{z}_i)\end{pmatrix}\begin{pmatrix} a_i & 0 \\ 0 & 1+2c_i\end{pmatrix}\begin{pmatrix} \tilde{s}_i & \pi\tilde{y}_i \\ \pi\tilde{v}_i & 1+\pi\tilde{z}_i\end{pmatrix}. \quad \text{(A-15)}$$

Then the $(1,2)$-block of $\sigma({}^t\tilde{m}_{i,i})h_i\tilde{m}_{i,i}$ is $\epsilon^{i/2}\pi(a_i\tilde{y}_i + \epsilon\sigma({}^t\tilde{v}_i)) + \pi^2(*)$ for a certain polynomial $(*)$. Therefore, by observing the $(1,2)$-block of Equation (A-1), we have

$$\mathcal{X}_{i,1,2}(m) = \bar{a}_i y_i + {}^t v_i + \mathcal{P}^i_{1,2}.$$

Here, $\mathcal{P}^i_{1,2}$ is a polynomial with variables in the entries of $m_{i-1,i}, m_{i+1,i}$. Note that this is an equation in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Thus ϵ, which is appeared in the $(1,2)$-block of $\sigma({}^t\tilde{m}_{i,i})h_i\tilde{m}_{i,i}$, has been ignored since $\epsilon \equiv 1 \mod \pi\otimes 1$. Furthermore, since m actually belongs to $\mathrm{Ker}\,\varphi(R)/\widetilde{G}^1(R)$, we have the following equation by the argument made at the beginning of this paragraph:

$$\mathcal{X}_{i,1,2}(m) = \bar{a}_i y_i + {}^t v_i + \mathcal{P}^i_{1,2} = \bar{b}_i = 0. \quad \text{(A-16)}$$

Thus we get polynomials $\mathcal{X}_{i,1,2}$ on $\mathrm{Ker}\,\tilde{\varphi}/\widetilde{M}^1$, vanishing on the subscheme $\mathrm{Ker}\,\varphi/\widetilde{G}^1$.

(3) Assume that i is even and L_i is of type I^e. The argument used in this step is similar to that of step (2) above. By Equations (A-10), (A-11) and (A-12), which involve an element of $\widetilde{M}^1(R)$, each entry of b'_i, e'_i, d'_i has π as a factor so that $b'_i \equiv b_i = 0$, $e'_i \equiv e_i = 0$, $d'_i \equiv d_i = 0 \mod (\pi\otimes 1)(B\otimes_A R)$. Let $\tilde{m} \in \mathrm{Ker}\,\tilde{\varphi}(R)$ be a lift of m. By using an argument similar to the paragraph just before Equation (A-14) of step (1), if we write the $(1,2)$, $(1,3)$, $(2,3)$-blocks of the (i,i)-block of the formal matrix product $\sigma({}^t\tilde{m})\cdot h\cdot\tilde{m}$ as $\xi^{i/2}\cdot\pi\mathcal{X}_{i,1,2}(\tilde{m})$, $\xi^{i/2}\cdot\pi\mathcal{X}_{i,1,3}(\tilde{m})$, $\xi^{i/2}\cdot\pi\mathcal{X}_{i,2,3}(\tilde{m})$, respectively, where $\mathcal{X}_{i,1,2}(\tilde{m})$ and $\mathcal{X}_{i,1,3}(\tilde{m}) \in M_{(n_i-2)\times 1}(B\otimes_A R)$ and $\mathcal{X}_{i,2,3}(\tilde{m}) \in B\otimes_A R$, then the images of $\mathcal{X}_{i,1,2}(\tilde{m})$ and $\mathcal{X}_{i,1,3}(\tilde{m})$ in $M_{(n_i-2)\times 1}(B\otimes_A R)/(\pi\otimes 1)M_{(n_i-2)\times 1}(B\otimes_A R)$ and the image of $\mathcal{X}_{i,2,3}(\tilde{m})$ in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ are independent of the choice of the lift $\tilde{m}$ of m. Therefore, we may denote these images by $\mathcal{X}_{i,1,2}(m)$, $\mathcal{X}_{i,1,3}(m)$, and $\mathcal{X}_{i,2,3}(m)$, respectively. As for Equation (A-14) of step (1), we need to express $\mathcal{X}_{i,1,2}(m)$, $\mathcal{X}_{i,1,3}(m)$, and $\mathcal{X}_{i,2,3}(m)$ as matrices. Recall that

$$\pi^i h_i = \xi^{i/2}\begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i\end{pmatrix} = \pi^i\cdot\epsilon^{i/2}\begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i\end{pmatrix}$$

and $\epsilon \equiv 1 \mod \pi\otimes 1$. We write

$$m_{i,i} = \begin{pmatrix} id & r_i & \pi t_i \\ \pi y_i & 1+\pi x_i & \pi z_i \\ v_i & u_i & 1+\pi w_i\end{pmatrix} \quad \text{and} \quad \tilde{m}_{i,i} = \begin{pmatrix} \tilde{s}_i & \tilde{r}_i & \pi\tilde{t}_i \\ \pi\tilde{y}_i & 1+\pi\tilde{x}_i & \pi\tilde{z}_i \\ \tilde{v}_i & \tilde{u}_i & 1+\pi\tilde{w}_i\end{pmatrix}$$

such that $\tilde{s}_i = \mathrm{id} \mod \pi \otimes 1$. Then

$$\sigma({}^t\tilde{m}_{i,i})h_i\tilde{m}_{i,i} = \epsilon^{i/2}\begin{pmatrix} \sigma({}^t\tilde{s}_i) & \sigma(\pi\cdot{}^t\tilde{y}_i) & \sigma({}^t\tilde{v}_i) \\ \sigma({}^t\tilde{r}_i) & 1+\sigma(\pi\tilde{x}_i) & \sigma(\tilde{u}_i) \\ \sigma(\pi\cdot{}^t\tilde{t}_i) & \sigma(\pi\cdot{}^t\tilde{z}_i) & 1+\sigma(\pi\tilde{w}_i)\end{pmatrix} \begin{pmatrix} a_i & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2c_i\end{pmatrix}\begin{pmatrix} \tilde{s}_i & \tilde{r}_i & \pi\tilde{t}_i \\ \pi\tilde{y}_i & 1+\pi\tilde{x}_i & \pi\tilde{z}_i \\ \tilde{v}_i & \tilde{u}_i & 1+\pi\tilde{w}_i\end{pmatrix}. \quad \text{(A-17)}$$

Then the $(1,2)$-block of $\sigma({}^t\tilde{m}_{i,i})h_i\tilde{m}_{i,i}$ is $\epsilon^{i/2}(a_i\tilde{r}_i + \sigma({}^t\tilde{v}_i)) + \pi(*)$, the $(1,3)$-block is $\epsilon^{i/2}\pi(a_i\tilde{t}_i + \epsilon\sigma({}^t\tilde{y}_i) + \sigma({}^t\tilde{v}_i)\tilde{z}_i) + \pi^2(**)$, and the $(2,3)$-block is $\epsilon^{i/2}(1+\pi(\sigma({}^t\tilde{r}_i)a_i\tilde{t}_i + \epsilon\sigma(\tilde{x}_i)+\tilde{z}_i+\tilde{w}_i+\sigma(\tilde{u}_i)\tilde{z}_i)+\pi^2(***))$ for certain polynomials $(*)$, $(**)$, $(***)$. Therefore, by considering the $(1,2)$, $(1,3)$, $(2,3)$-blocks of Equation (A-1) again, we have

$$\begin{cases} \mathcal{X}_{i,1,2}(m) = \bar{a}_i r_i + {}^t v_i; \\ \mathcal{X}_{i,1,3}(m) = \bar{a}_i t_i + {}^t y_i + {}^t v_i z_i + \mathcal{P}^i_{1,3}; \\ \mathcal{X}_{i,2,3}(m) = {}^t r_i \bar{a}_i t_i + x_i + z_i + w_i + u_i z_i + \mathcal{P}^i_{2,3}. \end{cases}$$

Here, $\mathcal{P}^i_{1,3}$, $\mathcal{P}^i_{2,3}$ are suitable polynomials with variables in the entries of $m_{i-1,i}$, $m_{i+1,i}$. These equations are considered in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Since m actually belongs to $\mathrm{Ker}\,\varphi(R)/\widetilde{G}^1(R)$, we have the following equation by the argument made at the beginning of this paragraph:

$$\begin{cases} \mathcal{X}_{i,1,2}(m) = \bar{a}_i r_i + {}^t v_i = \bar{b}_i = 0; \\ \mathcal{X}_{i,1,3}(m) = \bar{a}_i t_i + {}^t y_i + {}^t v_i z_i + \mathcal{P}^i_{1,3} = \bar{e}_i = 0; \\ \mathcal{X}_{i,2,3}(m) = {}^t r_i \bar{a}_i t_i + x_i + z_i + w_i + u_i z_i + \mathcal{P}^i_{2,3} = \bar{d}_i = 0. \end{cases} \quad \text{(A-18)}$$

Thus we get polynomials $\mathcal{X}_{i,1,2}$, $\mathcal{X}_{i,1,3}$, $\mathcal{X}_{i,2,3}$ on $\mathrm{Ker}\,\tilde{\varphi}/\widetilde{M}^1$, vanishing on the subscheme $\mathrm{Ker}\,\varphi/\widetilde{G}^1$.

(4) Assume that i is even and L_i is of type I. By Equations (A-8) and (A-13) which involve an element of $\widetilde{M}^1(R)$, $c'_i \equiv c_i = 0 \mod (\pi\otimes 1)(B\otimes_A R)$. Let $\tilde{m} \in \mathrm{Ker}\,\tilde{\varphi}(R)$ be a lift of m. By using an argument similar to the paragraph just before Equation (A-14) of step (1), if we write the $(2,2)$-block (when L_i is of type I^o) or the $(3,3)$-block (when L_i is of type I^e) of the (i,i)-block of $h\circ\tilde{m} = \sigma({}^t\tilde{m})\cdot h\cdot\tilde{m}$ as $\xi^{i/2}\cdot(1+2\mathcal{X}_{i,i}(\tilde{m}))$ or $\xi^{i/2}\cdot(2\mathcal{X}_{i,i}(\tilde{m}))$ respectively, where $\mathcal{X}_{i,i}(\tilde{m}) \in B\otimes_A R$, then the image of $\mathcal{X}_{i,i}(\tilde{m})$ in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ is independent of the choice of the lift $\tilde{m}$ of m. Therefore, we may denote this image by $\mathcal{X}_{i,i}(m)$. As in Equation (A-14) of step (1), we need to express $\mathcal{X}_{i,i}(m)$ as matrices. By considering Equations (A-15) and (A-17), the $(2,2)$-block (when L_i is of type I^o) or the $(3,3)$-block (when L_i is of type I^e) of the formal matrix product $\sigma({}^t\tilde{m}_{i,i})h_i\tilde{m}_{i,i}$ is ($\epsilon^{i/2}$ if L_i is of type I^o) $+\epsilon^{i/2}(2c_i + (\pi+\sigma(\pi))\tilde{z}_i + \pi\sigma(\pi)\tilde{z}_i^2) + 4(*)$ for a certain polynomial $(*)$. Therefore, by considering the $(2,2)$-block (when L_i is of type I^o) or the $(3,3)$-block (when L_i is of type I^e) of Equation (A-1) again, we have

$$\begin{aligned}\mathcal{X}_{i,i}(\tilde{m}) = \frac{1}{\pi^2}\big((\pi+\sigma(\pi))\tilde{z}_i + \pi\sigma(\pi)\tilde{z}_i^2 + \sigma({}^t\tilde{m}'_{i-1,i})\cdot\sigma(\pi)h_{i-1}\cdot\tilde{m}'_{i-1,i} \\ + \sigma({}^t\tilde{m}'_{i+1,i})\cdot\pi h_{i+1}\cdot\tilde{m}'_{i+1,i} + \sigma({}^t\tilde{m}'_{i-2,i})\cdot\sigma(\pi)^2 h_{i-2}\cdot\tilde{m}'_{i-2,i} \\ + \sigma({}^t\tilde{m}'_{i+2,i})\cdot\pi^2 h_{i+2}\cdot\tilde{m}'_{i+2,i}\big).\end{aligned}$$

Here, $\tilde{m}'_{j,i}$ is the last column vector of the matrix $\tilde{m}_{j,i}$. Note that the right hand side is a formal polynomial with entries in $\tilde{m}$. This equation should be interpreted as follows. We formally compute the right hand side and then it is of the form $1/\pi^2(\pi^2 X)$. The left hand side $\mathcal{X}_{i,i}(\tilde{m})$ is defined as the modified X by letting each term having π^2 as a factor in X be zero. It is a polynomial with entries in $B\otimes_A R$. Furthermore, $\mathcal{X}_{i,i}(m)$ is the image of $\mathcal{X}_{i,i}(\tilde{m})$ in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Let α be the unit in B such that $\epsilon=1+\alpha\pi$, as explained in Section 2A. Then $(\pi+\sigma(\pi))z_i+\pi\sigma(\pi)z_i^2=(2+\alpha\pi)\pi z_i+(1+\alpha\pi)\pi^2 z_i^2$ and so $\mathcal{X}_{i,i}(\tilde{m})$ is written as follows:

$$\begin{aligned}\mathcal{X}_{i,i}(\tilde{m})=\frac{1}{\pi^2}\big(&\alpha\pi^2\tilde{z}_i+\pi^2\tilde{z}_i^2+\sigma({}^t\tilde{m}'_{i-1,i})\cdot\sigma(\pi)h_{i-1}\cdot\tilde{m}'_{i-1,i}+\sigma({}^t\tilde{m}'_{i+1,i})\cdot\pi h_{i+1}\cdot\tilde{m}'_{i+1,i}\\ &+\sigma({}^t\tilde{m}'_{i-2,i})\cdot\sigma(\pi)^2h_{i-2}\cdot\tilde{m}'_{i-2,i}+\sigma({}^t\tilde{m}'_{i+2,i})\cdot\pi^2h_{i+2}\cdot\tilde{m}'_{i+2,i}\big).\end{aligned}$$

We can then write $\mathcal{X}_{i,i}(m)$ by using m and $\tilde{m}$ as follows:

$$\begin{aligned}\mathcal{X}_{i,i}(m)=\ &(\bar{\alpha}z_i+z_i^2+{}^tm'_{i-2,i}\cdot\bar{h}_{i-2}\cdot m'_{i-2,i}+m'_{i+2,i}\cdot\bar{h}_{i+2}\cdot m'_{i+2,i})\\ &+\frac{1}{\pi^2}\big(\sigma({}^t\tilde{m}'_{i-1,i})\cdot\sigma(\pi)h_{i-1}\cdot\tilde{m}'_{i-1,i}+\sigma({}^t\tilde{m}'_{i+1,i})\cdot\pi h_{i+1}\cdot\tilde{m}'_{i+1,i}\big).\end{aligned}\tag{A-19}$$

Here, $\bar{\alpha}$ is the image of α in κ and $m'_{j,i}$ is the last column vector of the matrix $m_{j,i}$. Note that the σ-action on $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ is trivial and so we remove σ in the first line of the above equation. Here, the reason we do not express $\mathcal{X}_{i,i}(m)$ based only on the entries in m as in steps (1)–(3) is that two terms involving h_{i-1} and h_{i+1} have only π as a factor which makes the expression with m complicated notation wise. Thus, in the above expression of $\mathcal{X}_{i,i}(m)$, the first line is just a polynomial in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ and the second line is interpreted as explained above as a formal expression. Note that the second line is independent of the choice of lifts $\tilde{m}'_{i-1,i}$ and $\tilde{m}'_{i+1,i}$ of $m'_{i-1,i}$ and $m'_{i+1,i}$, respectively, as explained in the first paragraph of step (4). For example, let $\pi h_{i+1}=\begin{pmatrix}2&\pi\\ \sigma(\pi)&2b\end{pmatrix}$ with $b\in A$ and let $\tilde{m}'_{i+1,i}=\begin{pmatrix}x_1+\pi x_2\\ y_1+\pi y_2\end{pmatrix}$ such that $m'_{i+1,i}=\begin{pmatrix}x_1\\ y_1\end{pmatrix}$. By Section 2A, we may assume that $\pi+\sigma(\pi)=2$ and $\pi\cdot\sigma(\pi)=\epsilon\pi^2=2u$ with $\epsilon\equiv 1 \bmod \pi$ and a unit $u\in A$. Then as a part of $\mathcal{X}_{i,i}(m)$, we can see that

$$\frac{1}{\pi^2}\sigma({}^t\tilde{m}'_{i+1,i})\cdot\pi h_{i+1}\cdot\tilde{m}'_{i+1,i}=\frac{1}{u}(x_1^2+x_1y_1+by_1^2).$$

Since m actually belongs to $\operatorname{Ker}\varphi(R)/\widetilde{G}^1(R)$, we have the following equation by the argument made at the beginning of this paragraph:

$$\begin{aligned}\mathcal{F}_i:\mathcal{X}_{i,i}(m)=\ &(\bar{\alpha}z_i+z_i^2+{}^tm'_{i-2,i}\cdot\bar{h}_{i-2}\cdot m'_{i-2,i}+m'_{i+2,i}\cdot\bar{h}_{i+2}\cdot m'_{i+2,i})\\ &\frac{1}{\pi^2}\big(\sigma({}^t\tilde{m}'_{i-1,i})\cdot\sigma(\pi)h_{i-1}\cdot\tilde{m}'_{i-1,i}+\sigma({}^t\tilde{m}'_{i+1,i})\cdot\pi h_{i+1}\cdot\tilde{m}'_{i+1,i}\big)\\ =\ &\bar{c}_i=0.\end{aligned}\tag{A-20}$$

Thus we get polynomials $\mathcal{X}_{i,i}$ on $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$, vanishing on the subscheme $\operatorname{Ker}\varphi/\widetilde{G}^1$.

(5) We now choose an even integer j such that L_j is of type I and L_{j+2} is of type II (possibly zero, by our convention). For each such j, there is a nonnegative integer m_j such that L_{j-2l} is of type I for every l with $0\le l\le m_j$ and $L_{j-2(m_j+1)}$ is of type II. Then we

claim that the sum of equations

$$\sum_{l=0}^{m_j} \frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l}$$

is the same as

$$\sum_{l=0}^{m_j}\Big(\frac{z_{j-2l}}{\bar{\alpha}}+\Big(\frac{z_{j-2l}}{\bar{\alpha}}\Big)^2\Big)=\Big(\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\bar{\alpha}}\Big)\Big(\sum_{l=0}^{m_j}(\frac{z_{j-2l}}{\bar{\alpha}})+1\Big)=0. \tag{A-21}$$

Here, $\bar{\alpha}$ is the image of α in κ and we consider this equation in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. We postpone the proof of this claim to Lemma A.7.

Let $G^\ddagger$ be the subfunctor of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ consisting of those m satisfying Equations (A-14), (A-16), (A-18) and (A-20). Note that such m also satisfy Equation (A-21). In Lemma A.8 below, we will prove that $G^\ddagger$ is represented by a smooth closed subscheme of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ and is isomorphic to $\mathbb{A}^{l'}\times(\mathbb{Z}/2\mathbb{Z})^\beta$ as a κ-variety, where $\mathbb{A}^{l'}$ is an affine space of dimension

$$l'=\sum_{i<j}n_in_j-\sum_{\substack{i\text{ odd}\\ L_i\text{ bound}}}n_i+\sum_{\substack{i\text{ even}\\ L_i\text{ of type }I^o}}(n_i-1)+\sum_{\substack{i\text{ even}\\ L_i\text{ of type }I^e}}(2n_i-2).$$

For ease of notation, let $G^\dagger=\operatorname{Ker}\varphi/\widetilde{G}^1$. Since $G^\dagger$ and $G^\ddagger$ are both closed subschemes of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ and $G^\dagger(\bar{\kappa})\subset G^\ddagger(\bar{\kappa})$, $(G^\dagger)_{\text{red}}$ is a closed subscheme of $(G^\ddagger)_{\text{red}}=G^\ddagger$. It is easy to check that $\dim G^\dagger=\dim G^\ddagger$ since $\dim G^\dagger=\dim\operatorname{Ker}\varphi-\dim\widetilde{G}^1=l-\dim\widetilde{G}^1$ and $\dim G^\ddagger=l'=l-\dim\widetilde{G}^1$. Here, $\dim\operatorname{Ker}\varphi=l$ is given in Lemma 4.6 and $\dim\widetilde{G}^1$ is given in Theorem A.4.

We claim that $(G^\dagger)_{\text{red}}$ contains at least one (closed) point of each connected component of $G^\ddagger$. Choose an even integer j such that L_j is of type I and L_{j+2} is of type II (possibly zero, by our convention). Consider the closed subgroup scheme F_j of $\widetilde{G}$ defined by the following equations:

- $m_{i,k}=0$ if $i\neq k$;
- $m_{i,i}=\text{id}$ if $i\neq j$;
- and for $m_{j,j}$,

$$\begin{cases} s_j=\text{id},\ y_j=0,\ v_j=0 & \text{if } L_i \text{ is of type } I^o;\\ s_j=\text{id},\ r_j=t_j=y_j=v_j=u_j=w_j=0 & \text{if } L_i \text{ is of type } I^e.\end{cases}$$

We will prove in Lemma A.9 below that each element of $F_j(R)$ for a κ-algebra R satisfies $(z_j^1/\bar{\alpha})+(z_j^1/\bar{\alpha})^2=0$, where $z_j=z_j^1+\pi z_j^2$, and that F_j is isomorphic to $\mathbb{A}^1\times\mathbb{Z}/2\mathbb{Z}$ as a κ-variety, where $\mathbb{A}^1$ is an affine space of dimension 1.

Notice that F_j and $F_{j'}$ commute with each other for all even integers $j\neq j'$, in the sense that $f_j\cdot f_{j'}=f_{j'}\cdot f_j$, where $f_j\in F_j$ and $f_{j'}\in F_{j'}$. Let $F=\prod_j F_j$. Then F is smooth and is a closed subgroup scheme of $\operatorname{Ker}\varphi$ as mentioned in the proof of Theorem 4.11. If $F^\dagger$ is the image of F in $G^\dagger$, then it is smooth and thus a closed subscheme of $(G^\dagger)_{\text{red}}$. By observing Equation (A-21) and $(z_j^1/\bar{\alpha})+(z_j^1/\bar{\alpha})^2=0$ above, we can easily see that $F^\dagger$ contains at least one (closed) point of each connected component of $G^\ddagger$ and this proves our claim.

Combining this fact with $\dim G^\dagger=\dim G^\ddagger$, we conclude that $(G^\dagger)_{\text{red}}\simeq G^\ddagger$, and hence, $G^\dagger=G^\ddagger$ because $G^\dagger$ is a subfunctor of $G^\ddagger$. This completes the proof. □

Lemma A.7. *Choose an even integer j such that L_j is of type I and L_{j+2} is of type II (possibly zero, by our convention). For such j, there is a nonnegative integer m_j such that L_{j-2l} is of type I for every l with $0 \leq l \leq m_j$ and $L_{j-2(m_j+1)}$ is of type II. Then the sum of the equations*

$$\sum_{l=0}^{m_j} \frac{1}{\bar{\alpha}^2} \mathcal{F}_{j-2l}$$

equals

$$\sum_{l=0}^{m_j} \Big(\frac{z_{j-2l}}{\bar{\alpha}} + \Big(\frac{z_{j-2l}}{\bar{\alpha}}\Big)^2 \Big) = \Big(\sum_{l=0}^{m_j} \frac{z_{j-2l}}{\bar{\alpha}} \Big) \Big(\sum_{l=0}^{m_j} \big(\frac{z_{j-2l}}{\bar{\alpha}}\big) + 1 \Big) = 0.$$

Proof. Our strategy to prove this lemma is the following. We will first prove that for each odd integer i, the terms containing an h_i add to zero in the sum $\sum_{l=0}^{m_j} \frac{1}{\alpha^2} \mathcal{F}_{j-2l}$. Then we will show that for each even integer i, the terms containing an $\bar{h}_i$ add to zero in the sum $\sum_{l=0}^{m_j} \frac{1}{\alpha^2} \mathcal{F}_{j-2l}$, so that only the terms containing the z_i remain.

We recall the notations used in the theorem. Let m be an element in $(\operatorname{Ker} \tilde{\varphi}/\widetilde{M}^1)(R)$ and $\tilde{m} \in \operatorname{Ker} \tilde{\varphi}(R)$ be its lift. For any block x_i of m, $\tilde{x}_i$ denotes the corresponding block of $\tilde{m}$ whose reduction is x_i. Since x_i is a block of an element of $(\operatorname{Ker} \tilde{\varphi}/\widetilde{M}^1)(R)$, its entries are elements of $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R) \cong R$ as explained in Remark A.5, whereas entries of $\tilde{x}_i$ are elements of $B \otimes_A R$. In addition, for a block a_i of h, where we consider h as an element of $\underline{H}(R)$ as explained in Remark 3.3(2), $\bar{a}_i$ denotes the image of a_i in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$, which is mentioned in step (i) of the proof of Theorem A.4. If we write h as a formal matrix $h = \big(\pi^i \cdot h_i\big)$ with $(\pi^i \cdot h_i)$ for the (i, i)-block and 0 for the remaining blocks, then recall from the paragraph following Equation (A-14) that $\bar{h}_k$ is the matrix with entries in $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R) \cong R$ by letting π be zero in each entry of the formal matrix h_k. To help our computation, we write $\bar{h}_i$. Note that $\epsilon\ (\in B) \equiv 1 \bmod \pi$.

$$\bar{h}_i = \begin{cases} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & & 1 \end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^o; \\ \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \ddots & & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^e; \\ \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} & \text{if } i \text{ is odd or if } i \text{ is even and } L_i \text{ is of type } II. \end{cases} \tag{A-22}$$

We recall that $m_{i,i}$ is a formal matrix as described in Remark A.5, not a matrix in $M_{n_i\times n_i}((B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R))$, whereas $m_{i,j}$ for $i\neq j$ is a matrix with entries in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Thus we need to modify $m_{i,i}$ into a matrix with entries in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ in order to use Equation (A-14) as explained in the paragraph following Equation (A-14). We define $\bar{m}_{i,i}$ $(\in M_{n_i\times n_i}((B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)))$ to be obtained from $m_{i,i}$ by letting π be zero in each entry of the formal matrix $m_{i,i}$. The matrix $\bar{m}_{i,i}$ is described as follows.

$$\bar{m}_{i,i}=\begin{cases}\begin{pmatrix}\mathrm{id}&0\\0&1\end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^o;\\ \begin{pmatrix}\mathrm{id}&r_i&0\\0&1&0\\v_i&u_i&1\end{pmatrix} & \text{if } i \text{ is even and } L_i \text{ is of type } I^e;\\ \mathrm{id} & \text{if } i \text{ is even and } L_i \text{ is of type } \mathit{II};\\ \mathrm{id} & \text{if } i \text{ is odd.}\end{cases} \tag{A-23}$$

In addition, if i is odd, then we have

$$\delta_{i-1}e_{i-1}\cdot m_{i-1,i}+\delta_{i+1}e_{i+1}\cdot m_{i+1,i}=0. \tag{A-24}$$

Here, δ_j, e_j are as explained in the description of $\mathrm{Ker}\,\tilde{\varphi}(R)$, the paragraph following Lemma A.2.

We choose an even integer k (assuming $m_j>0$) such that $j-2(m_j-1)\leq k\leq j$ so that both L_k and L_{k-2} are of type I. We observe $\sigma({}^t\tilde{m}'_{k-1,k})\cdot\sigma(\pi)h_{k-1}\cdot\tilde{m}'_{k-1,k}$ in $\mathcal{F}_k$ and $\sigma({}^t\tilde{m}'_{k-1,k-2})\cdot\sigma(\pi)h_{k-1}\cdot\tilde{m}'_{k-1,k-2}$ in $\mathcal{F}_{k-2}$ (cf. Equation (A-20)). We claim that

$$\frac{1}{\pi^2}\big(\sigma({}^t\tilde{m}'_{k-1,k})\cdot\sigma(\pi)h_{k-1}\cdot\tilde{m}'_{k-1,k}+\sigma({}^t\tilde{m}'_{k-1,k-2})\cdot\sigma(\pi)h_{k-1}\cdot\tilde{m}'_{k-1,k-2}\big)=0. \tag{A-25}$$

Note that this equation is interpreted as explained in the paragraph following Equation (A-19).

We use Equation (A-14) for $i=k-1$ and $j=k$ so that we have

$${}^t\bar{m}_{k-1,k-1}\bar{h}_{k-1}m_{k-1,k}={}^tm_{k,k-1}\bar{h}_k\bar{m}_{k,k}. \tag{A-26}$$

Note that this equation is over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Indeed, there is a σ-action in Equation (A-14) but it is trivial over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Recall that $m'_{k-1,k}$ is the last column vector of $m_{k-1,k}$. Let $e_{k-1}=(0,\cdots,0,1)$ be of size $1\times n_k$. Then $m'_{k-1,k}=m_{k-1,k}\cdot{}^te_{k-1}$. We multiply both sides of the above equation by ${}^te_{k-1}$ on the right. Then the left hand side is ${}^t\bar{m}_{k-1,k-1}\bar{h}_{k-1}m_{k-1,k}\cdot{}^te_{k-1}=\bar{h}_{k-1}m'_{k-1,k}$ since ${}^t\bar{m}_{k-1,k-1}=\mathrm{id}$. The right hand side is ${}^tm_{k,k-1}\bar{h}_k\bar{m}_{k,k}\cdot{}^te_{k-1}$. Since $\bar{m}_{k,k}\cdot{}^te_{k-1}$ is the last column vector of $\bar{m}_{k,k}$, $\bar{m}_{k,k}\cdot{}^te_{k-1}={}^te_{k-1}$ by Equation (A-23) so that ${}^tm_{k,k-1}\bar{h}_k\bar{m}_{k,k}\cdot{}^te_{k-1}={}^tm_{k,k-1}\bar{h}_k\cdot{}^te_{k-1}$. Furthermore, $\bar{h}_k$ is symmetric over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ and so ${}^tm_{k,k-1}\bar{h}_k\cdot{}^te_{k-1}={}^t(e_{k-1}\cdot\bar{h}_km_{k,k-1})$. Then based on the matrix form of $\bar{h}_k$ in Equation (A-22), we have that $e_{k-1}\cdot\bar{h}_k$ is the same as e_k, where e_k is defined in the paragraph following Lemma A.2. (There, e_j is defined when j is even and L_j is of type I.) In conclusion, Equation (A-26) induces the equation

$$\bar{h}_{k-1}m'_{k-1,k}={}^t(e_k\cdot m_{k,k-1}) \tag{A-27}$$

over $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$.

We again use Equation (A-14) for $i = k-2$ and $j = k-1$ so that we have

$$ {}^t\bar{m}_{k-2,k-2}\bar{h}_{k-2}m_{k-2,k-1} = {}^t m_{k-1,k-2}\bar{h}_{k-1}\bar{m}_{k-1,k-1}. \tag{A-28}$$

Note that this equation is over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Since $k-1$ is odd, $\bar{m}_{k-1,k-1} = \mathrm{id}$ by Equation (A-23). Recall that $m'_{k-1,k-2}$ is the last column vector of $m_{k-1,k-2}$. Let $e'_{k-1} = (0, \cdots, 0, 1)$ of size $1 \times n_{k-2}$. Then $m'_{k-1,k-2} = m_{k-1,k-2} \cdot {}^t e'_{k-1}$. We multiply both sides of the above equation by e'_{k-1} on the left. Then the right hand side is $e'_{k-1} \cdot {}^t m_{k-1,k-2}\bar{h}_{k-1}\bar{m}_{k-1,k-1} = {}^t m'_{k-1,k-2}\bar{h}_{k-1}$. Note that $\bar{m}_{k-2,k-2} \cdot {}^t e'_{k-1} = {}^t e'_{k-1}$ by Equation (A-23) since this is the last column vector of $\bar{m}_{k-2,k-2}$. Thus in the left hand side, $e'_{k-1} \cdot {}^t\bar{m}_{k-2,k-2}\bar{h}_{k-2}m_{k-2,k-1} = e'_{k-1} \cdot \bar{h}_{k-2}m_{k-2,k-1}$. Based on the matrix form of $\bar{h}_k$ for an even integer k in Equation (A-22), $e'_{k-1} \cdot \bar{h}_{k-2}$ is the same as e_{k-2}, where e_k is defined in the paragraph following Lemma A.2. In conclusion, Equation (A-28) induces the equation

$$ {}^t m'_{k-1,k-2}\bar{h}_{k-1} = e_{k-2} \cdot m_{k-2,k-1} \tag{A-29}$$

over $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$.

Now we use Equations (A-27) and (A-29). Based on the matrix form of $\bar{h}_{k-1}$ for an odd integer $k-1$ in Equation (A-22), we have that $\bar{h}_{k-1} \cdot \bar{h}_{k-1} = \mathrm{id}$ and $\bar{h}_{k-1}$ is symmetric. Thus, by multiplying Equations (A-27) and (A-29) by $\bar{h}_{k-1}$, we obtain $m'_{k-1,k} = {}^t(e_k \cdot m_{k,k-1} \cdot \bar{h}_{k-1})$ and ${}^t m'_{k-1,k-2} = e_{k-2} \cdot m_{k-2,k-1} \cdot \bar{h}_{k-1}$, respectively, as equations over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$.

On the other hand, we observe that $k-1$ is odd and both L_{k-2} and L_k are of type I. Thus $e_{k-2} \cdot m_{k-2,k-1} = e_k \cdot m_{k,k-1}$ by Equation (A-24). We multiply this equation by $\bar{h}_{k-1}$ and so obtain

$$ e_{k-2} \cdot m_{k-2,k-1} \cdot \bar{h}_{k-1} = e_k \cdot m_{k,k-1} \cdot \bar{h}_{k-1} $$

as an equation over $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$. Therefore, we have the equation

$$ m'_{k-1,k} = m'_{k-1,k-2}. $$

As mentioned at the paragraph following Equation (A-19), Equation (A-25) is independent of the choice of a lift of $m'_{k-1,k}$ and $m'_{k-1,k-2}$. Therefore, two terms in Equation (A-25) are same and this verifies our claim.

In the case of $\mathcal{F}_j$, following the proof of Equation (A-29), we have ${}^t m'_{j+1,j} \cdot \bar{h}_{j+1} = e_j \cdot m_{j,j+1}$. Since L_{j+2} is of type II (possibly zero, by our convention), $e_j \cdot m_{j,j+1} = 0$ by Equation (A-24). Thus, the term involving h_{j+1} in $\mathcal{F}_j$ is zero. In the case of $j-2m_j$, where $m_j \geq 0$, the term involving h_{j-2m_j-1} in $\mathcal{F}_{j-2m_j}$ is zero in a manner similar to that of the above case of $\mathcal{F}_j$.

To summarize, for each odd integer i, the terms containing an h_i add to zero in $\sum_{l=0}^{m_j} \frac{1}{\alpha^2}\mathcal{F}_{j-2l}$.

We now prove that for each even integer i, the terms containing an $\bar{h}_i$ add to zero in $\sum_{l=0}^{m_j} \frac{1}{\alpha^2}\mathcal{F}_{j-2l}$. We again choose an even integer k (assuming $m_j > 0$) such that $j-2(m_j-1) \leq k \leq j$ so that both L_k and L_{k-2} are of type I. We observe ${}^t m'_{k-2,k} \cdot \bar{h}_{k-2} \cdot m'_{k-2,k}$

in $\mathcal{F}_k$ and ${}^t m'_{k,k-2} \cdot \bar{h}_k \cdot m'_{k,k-2}$ in $\mathcal{F}_{k-2}$, and we claim that

$$ {}^t m'_{k-2,k} \cdot \bar{h}_{k-2} \cdot m'_{k-2,k} + {}^t m'_{k,k-2} \cdot \bar{h}_k \cdot m'_{k,k-2} = 0, \tag{A-30}$$

as an equation over $(B \otimes_A R)/(\pi \otimes 1)(B \otimes_A R)$. Let $\widehat{m}'_{k-2,k}$ be the $(n_{k-2} \times n_k)$-th entry (resp. $((n_{k-2}-1) \times n_k)$-th entry) of $m_{k-2,k}$ when L_{k-2} is of type I^o (resp. I^e). We can also define $\widehat{m}'_{k,k-2}$ as the $(n_k \times n_{k-2})$-th entry (resp. $((n_k - 1) \times n_{k-2})$-th entry) of $m_{k,k-2}$ when L_k is of type I^o (resp. I^e). Then the above Equation (A-30) is the same as

$$(\widehat{m}'_{k-2,k})^2 + (\widehat{m}'_{k,k-2})^2 = 0. \tag{A-31}$$

We use Equation (A-14) for $i = k-2$ and $j = k$ so that we have

$$ {}^t \bar{m}_{k-2,k-2} \bar{h}_{k-2} m_{k-2,k} + {}^t m_{k-1,k-2} \bar{h}_{k-1} m_{k-1,k} + {}^t m_{k,k-2} \bar{h}_k \bar{m}_{k,k} = 0. \tag{A-32}$$

Let $\widetilde{e}_k = (0, \cdots, 0, 1)$ of size $1 \times n_k$ and $\widetilde{e}_{k-2} = (0, \cdots, 0, 1)$ of size $1 \times n_{k-2}$. Then we have

$$\widetilde{e}_{k-2} \cdot {}^t \bar{m}_{k-2,k-2} \bar{h}_{k-2} m_{k-2,k} \cdot {}^t \widetilde{e}_k = \widehat{m}'_{k-2,k} \tag{A-33}$$

since $m_{k-2,k} \cdot {}^t \widetilde{e}_k = m'_{k-2,k}$ and $\bar{m}_{k-2,k-2} \cdot {}^t \widetilde{e}_{k-2} = {}^t \widetilde{e}_{k-2}$. We also have

$$\widetilde{e}_{k-2} \cdot {}^t m_{k,k-2} \bar{h}_k \bar{m}_{k,k} \cdot {}^t \widetilde{e}_k = \widehat{m}'_{k,k-2} \tag{A-34}$$

since $\bar{m}_{k,k} \cdot {}^t \widetilde{e}_k = {}^t \widetilde{e}_k$ and $m_{k,k-2} \cdot {}^t \widetilde{e}_{k-2} = m'_{k,k-2}$. Note that we use Equations (A-22) and (A-23) for our matrix computation. On the other hand, due to the fact that $\bar{h}_{k-1} \cdot \bar{h}_{k-1} = \mathrm{id}$ and $\bar{h}_{k-1}$ is symmetric, we have

$$\widetilde{e}_{k-2} \cdot {}^t m_{k-1,k-2} \bar{h}_{k-1} m_{k-1,k} \cdot {}^t \widetilde{e}_k = (\widetilde{e}_{k-2} \cdot {}^t m_{k-1,k-2} \bar{h}_{k-1}) \cdot \bar{h}_{k-1} \cdot (\bar{h}_{k-1} m_{k-1,k} \cdot {}^t \widetilde{e}_k). \tag{A-35}$$

Now, $\widetilde{e}_{k-2} \cdot {}^t m_{k-1,k-2} \bar{h}_{k-1} = {}^t m'_{k-1,k-2} \bar{h}_{k-1} = e_{k-2} \cdot m_{k-2,k-1}$ by Equation (A-29) and $\bar{h}_{k-1} m_{k-1,k} \cdot {}^t \widetilde{e}_k = \bar{h}_{k-1} m'_{k-1,k} = {}^t (e_k \cdot m_{k,k-1})$ by Equation (A-27). Since $e_{k-2} \cdot m_{k-2,k-1} = e_k \cdot m_{k,k-1}$ by Equation (A-24), Equation (A-35) equals

$$(\widetilde{e}_{k-2} \cdot {}^t m_{k-1,k-2} \bar{h}_{k-1}) \cdot \bar{h}_{k-1} \cdot (\bar{h}_{k-1} m_{k-1,k} \cdot \widetilde{e}_k) = (e_k \cdot m_{k,k-1}) \cdot \bar{h}_{k-1} \cdot {}^t (e_k \cdot m_{k,k-1}) = 0. \tag{A-36}$$

We now combine Equations (A-33), (A-34), and (A-36). Namely, if we multiply $\widetilde{e}_{k-2}$ to the left of each side in Equation (A-32) and we multiply ${}^t \widetilde{e}_k$ to the right of each side in Equation (A-32), then we have

$$\widehat{m}'_{k-2,k} + 0 + \widehat{m}'_{k,k-2} = 0 \tag{A-37}$$

and so Equations (A-31) and (A-30) are proved.

In the case of $\mathcal{F}_j$, the term ${}^t m'_{j+2,j} \cdot \bar{h}_{j+2} \cdot m'_{j+2,j} = 0$ since L_{j+2} is of type $I\!I$ (possibly zero, by our convention). Similarly, the term ${}^t m'_{j-2m_j-2,j-2m_j} \cdot \bar{h}_{j-2m_j-2} \cdot m'_{j-2m_j-2,j-2m_j}$ of $\mathcal{F}_{j-2m_j}$, where $m_j \geq 0$, is 0 since L_{j-2m_j-2} is of type $I\!I$. Here, we use Equation (A-22) for our matrix multiplication.

To summarize, for each even integer i, the terms containing an h_i add to zero in $\sum_{l=0}^{m_j} \frac{1}{\alpha^2} \mathcal{F}_{j-2l}$.

Therefore, the sum of equations $\sum_{l=0}^{m_j} \frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l}$ equals

$$\sum_{l=0}^{m_j} \frac{1}{\bar{\alpha}^2}(\bar{\alpha}\bar{z}_{j-2l} + \bar{z}_{j-2l}^2) = 0.$$

This is the same as

$$\sum_{l=0}^{m_j}\Big(\frac{\bar{z}_{j-2l}}{\bar{\alpha}} + \Big(\frac{\bar{z}_{j-2l}}{\bar{\alpha}}\Big)^2\Big) = \Big(\sum_{l=0}^{m_j}\frac{\bar{z}_{j-2l}}{\bar{\alpha}}\Big)\Big(\sum_{l=0}^{m_j}\Big(\frac{\bar{z}_{j-2l}}{\bar{\alpha}}\Big)+1\Big) = 0. \tag{A-38}$$

This completes the proof of the lemma. □

Lemma A.8. *Let $G^\ddagger$ be the subfunctor of* $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ *consisting of those m satisfying Equations* (A-14), (A-16), (A-18), *and* (A-20). *Note that such m then satisfies Equation* (A-21) *as well. Then $G^\ddagger$ is represented by a smooth closed subscheme of* $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ *and is isomorphic to $\mathbb{A}^{l'} \times (\mathbb{Z}/2\mathbb{Z})^\beta$ as a κ-variety, where $\mathbb{A}^{l'}$ is an affine space of dimension l'. Here,*

$$l' = \sum_{i<j} n_i n_j - \sum_{\substack{i \text{ odd} \\ L_i \text{ bound}}} n_i + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (n_i - 1) + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^e}} (2n_i - 2).$$

Proof. Let $\mathcal{J}$ be the set of even integers j such that L_j is of type I and L_{j+2} is of type *II* (possibly empty, by our convention). Note that Equation (A-20) implies Equation (A-21) by Lemma A.7. Equation (A-21) implies that $G^\ddagger$ is disconnected with at least 2^β connected components (Exercise 2.19 of [Hartshorne 1977]). Here, $\beta = \#\mathcal{J}$. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be a pair of two (possibly empty) subsets of $\mathcal{J}$ such that $\mathcal{J}$ is the disjoint union of $\mathcal{J}_1$ and $\mathcal{J}_2$. Let $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$ be the subfunctor of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ consisting of those m satisfying Equations (A-14), (A-16), (A-18), and (A-20), the equations $\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\bar{\alpha}} = 0$ for any $j \in \mathcal{J}_1$, and the equations $\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\bar{\alpha}} = 1$ for any $j \in \mathcal{J}_2$. Here m_j is the integer associated to j defined in Lemma A.7. We claim that $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$ is represented by a smooth closed subscheme of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ and is isomorphic to $\mathbb{A}^{l'}$. Since the scheme $G^\ddagger$ is a direct product of $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$'s for any such pair of $\mathcal{J}_1$, $\mathcal{J}_2$ by Exercise 2.19 of [Hartshorne 1977], the lemma follows from this claim.

It is obvious that $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$ is represented by a closed subscheme of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ since the equations defining $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$ as a subfunctor of $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ are all polynomials. Thus it suffices to show that $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$ is isomorphic to an affine space $\mathbb{A}^{l'}$. Our strategy to show this is that the coordinate ring of $\widetilde{G}^\ddagger_{\mathcal{J}_1,\mathcal{J}_2}$ is isomorphic to a polynomial ring. To do that, we use the following trick over and over. We consider the polynomial ring $\kappa[x_1, \cdots, x_n]$ and it quotient ring $\kappa[x_1, \cdots, x_n]/(x_1 + P(x_2, \cdots, x_n))$. Then the quotient ring $\kappa[x_1, \cdots, x_n]/(x_1 + P(x_2, \cdots, x_n))$ is isomorphic to $\kappa[x_2, \cdots, x_n]$ and in this case we say that x_1 *can be eliminated by* $x_2, \cdots, x_n$.

By the description of an element of $(\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1)(R)$ in Remark A.5, we see that $\operatorname{Ker}\tilde{\varphi}/\widetilde{M}^1$ is isomorphic to an affine space of dimension

$$2\sum_{i<j} n_i n_j - \sum_{\substack{i \text{ odd} \\ L_i \text{ bound}}} n_i + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (2n_i - 1) + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^e}} (4n_i - 4)$$

with variables

$$(m_{i,j})_{i\neq j},\quad (y_i, v_i, z_i)_{\substack{i \text{ even}\\ L_i \text{ of type } I^o}},\quad (r_i, t_i, y_i, v_i, x_i, z_i, u_i, w_i)_{\substack{i \text{ even}\\ L_i \text{ of type } I^e}}$$

such that $\delta_{i-1}e_{i-1}\cdot m_{i-1,i}+\delta_{i+1}e_{i+1}\cdot m_{i+1,i}=0$ with i odd. Here, δ_j, e_j are as explained in the description of $\mathrm{Ker}\,\tilde{\varphi}(R)$, the paragraph right after Lemma A.2.

From now on, we eliminate suitable variables based on Equations (A-14), (A-16), (A-18), and (A-20), the equations $\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\alpha}=0$ for all $j\in\mathcal{J}_1$, and the equations $\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\alpha}=1$ for all $j\in\mathcal{J}_2$.

(1) We first consider Equation (A-14). For two integers i, j with $i<j$, we have

$${}^t m_{j,i}\bar{h}_j\bar{m}_{j,j}=\sum_{i\leq k\leq j-1}{}^t m_{k,i}\bar{h}_k m_{k,j}\ \text{ over }\ (B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R).$$

By Equation (A-23), $\bar{m}_{j,j}=\mathrm{id}$ if L_j is not of type I^e. Thus the above equation equals ${}^t m_{j,i}\bar{h}_j=\sum_{i\leq k\leq j-1}{}^t m_{k,i}\bar{h}_k m_{k,j}$ over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$ if L_j is not of type I^e. Since $\bar{h}_j$ is a nonsingular matrix by Equation (A-22), $m_{j,i}$ can be eliminated by the right hand side. If L_j is of type I^e, we have

$$\bar{m}_{j,j}=\begin{pmatrix}\mathrm{id} & r_j & 0\\ 0 & 1 & 0\\ v_j & u_j & 1\end{pmatrix}\quad\text{and}\quad \bar{h}_j=\begin{pmatrix}a_j & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 0\end{pmatrix}$$

by Equations (A-23) and (A-22), respectively. Then

$$\bar{h}_j\bar{m}_{j,j}=\begin{pmatrix}a_j & a_j r_j & 0\\ v_j & 1+u_j & 1\\ 0 & 1 & 0\end{pmatrix}.$$

To compute ${}^t m_{j,i}\bar{h}_j\bar{m}_{j,j}$, we write ${}^t m_{j,i}=\begin{pmatrix}A_j & B_j & C_j\end{pmatrix}$ so that

$${}^t m_{j,i}\bar{h}_j\bar{m}_{j,j}=\begin{pmatrix}A_j a_j+B_j v_j & A_j a_j r_j+B_j(1+u_j)+C_j & B_j\end{pmatrix}.$$

By first considering the $(1,3)$-block of the matrix ${}^t m_{j,i}\bar{h}_j\bar{m}_{j,j}$, B_j can be eliminated by $\sum_{i\leq k\leq j-1}{}^t m_{k,i}\bar{h}_k m_{k,j}$. Then we consider the $(1,1)$-block of ${}^t m_{j,i}\bar{h}_j\bar{m}_{j,j}$. Since a_j is a nonsingular matrix, we see that A_j can be eliminated by $\sum_{i\leq k\leq j-1}{}^t m_{k,i}\bar{h}_k m_{k,j}$ with $B_j v_j$. By considering the $(1,2)$-block of ${}^t m_{j,i}\bar{h}_j\bar{m}_{j,j}$, C_j can be eliminated by $\sum_{i\leq k\leq j-1}{}^t m_{k,i}\bar{h}_k m_{k,j}$ with $A_j a_j r_j+B_j(1+u_j)$. Therefore, all lower triangular blocks $m_{j,i}$ (with $j>i$) can be eliminated by upper triangular blocks $m_{i,j}$ together with r_j, v_j, u_j (resp. r_i, v_i, u_i) if L_j (resp. L_i) is of type I^e. Here r_i, v_i, u_i are nontrivial blocks of $\bar{m}_{i,i}$, if L_i is of type I^e, which appeared in the right hand side of the above equation.

On the other hand, the equation $\delta_{i-1}e_{i-1}\cdot m_{i-1,i}+\delta_{i+1}e_{i+1}\cdot m_{i+1,i}=0$ for an odd integer i, which is one equation defining $\mathrm{Ker}\,\tilde{\varphi}/\widetilde{M}^1$ (cf. Remark A.5(3)), should be rewritten in terms of upper triangular blocks. To do that, we use Equation (A-27) with $i=k-1$. Note that the only assumption needed in Equation (A-27) is that L_k is of type I. Thus the above equation is the same as

$$\delta_{i-1}e_{i-1}\cdot m_{i-1,i}+\delta_{i+1}{}^t(\bar{h}_i m'_{i,i+1})=0.$$

(2) We secondly consider Equation (A-16). If L_i is of type I^o, then v_i can be eliminated by y_i and $m_{i-1,i}, m_{i,i+1}$.

(3) Next, we consider Equation (A-18). By $\mathcal{X}_{i,1,2}$, v_i can be eliminated by r_i. By $\mathcal{X}_{i,1,3}$, y_i can be eliminated by t_i, v_i, z_i and entries from $m_{i-1,i}, m_{i,i+1}$. By $\mathcal{X}_{i,2,3}$, x_i can be eliminated by r_i, t_i, z_i, w_i, u_i and entries from $m_{i-1,i}, m_{i,i+1}$.

(4) Finally, we consider $\frac{1}{\bar{\alpha}^2}\mathcal{F}_i$, instead of $\mathcal{F}_i$ (Equation (A-20)), together with equations $\sum_{l=0}^{m_j} \frac{z_{j-2l}}{\alpha} = 0$ with $j \in \mathcal{J}_1$ and equations $\sum_{l=0}^{m_j} \frac{z_{j-2l}}{\alpha} = 1$ with $j \in \mathcal{J}_2$. Note that $\frac{1}{\bar{\alpha}^2}\mathcal{F}_i$ is equivalent to $\mathcal{F}_i$ since α is a unit in B. For each $j \in \mathcal{J}$, there is a nonnegative integer m_j such that L_{j-2l} is of type I for every l with $0 \le l \le m_j$ and $L_{j-2(m_j+1)}$ is of type II (cf. Lemma A.7).

To analyze these equations, we investigate $\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l}$ for a fixed $j \in \mathcal{J}$. First assume that $m_j \ge 1$. Since we have eliminated all lower triangular blocks in step (1), we need to replace lower triangular blocks appeared in $\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l}$ by suitable upper triangular blocks. If $m_j \ge 2$, then we choose an integer l such that $0 < l < m_j$. By definition, $\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l}$ is

$$\begin{aligned}
&\frac{1}{\bar{\alpha}^2\cdot\pi^2}\big(\sigma({}^t\tilde{m}'_{j-2l-1,j-2l})\cdot\sigma(\pi)h_{j-2l-1}\cdot\tilde{m}'_{j-2l-1,j-2l}\\
&\qquad\qquad +\sigma({}^t\tilde{m}'_{j-2l+1,j-2l})\cdot\pi h_{j-2l+1}\cdot\tilde{m}'_{j-2l+1,j-2l}\big)\\
&\qquad\qquad +\frac{z_{j-2l}}{\bar{\alpha}}+\Big(\frac{z_{j-2l}}{\bar{\alpha}}\Big)^2+\frac{{}^tm'_{j-2l-2,j-2l}\cdot\bar{h}_{j-2l-2}\cdot m'_{j-2l-2,j-2l}}{\bar{\alpha}^2}\\
&\qquad\qquad +\frac{{}^tm'_{j-2l+2,j-2l}\cdot\bar{h}_{j-2l+2}\cdot m'_{j-2l+2,j-2l}}{\bar{\alpha}^2}\\
&=0.
\end{aligned}$$

The first two lines are interpreted as explained in the paragraph following Equation (A-19) and the third and fourth line is a polynomial in $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. We claim that the equation $\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l}$ is the same as the following:

$$\begin{aligned}
&\frac{1}{\bar{\alpha}^2\cdot\pi^2}\big(\sigma({}^t\tilde{m}'_{j-2l-1,j-2l})\cdot\sigma(\pi)h_{j-2l-1}\cdot\tilde{m}'_{j-2l-1,j-2l}\\
&\qquad +(e_{j-2l}\cdot\sigma(\tilde{m}_{j-2l,j-2l+1}))\cdot\pi h^3_{j-2l+1}\cdot{}^t(e_{j-2l}\cdot\tilde{m}_{j-2l,j-2l+1})\big)\\
&\qquad\qquad +\frac{z_{j-2l}}{\bar{\alpha}}+\Big(\frac{z_{j-2l}}{\bar{\alpha}}\Big)^2+\Big(\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}\Big)^2+\Big(\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}\Big)^2\\
&=0.
\end{aligned}\tag{A-39}$$

The third line easily follows from the definition of $\widehat{m}'_{k-2,k}$ and $\widehat{m}'_{k,k-2}$ (given in the paragraph following Equation (A-30)) combined with Equation (A-37). For the first two lines, we consider Equation (A-29) with $k-2=j-2l$ which gives the identity $m'_{j-2l+1,j-2l} = \bar{h}_{j-2l+1}\cdot{}^t(e_{j-2l}\cdot m_{j-2l,j-2l+1})$ over $(B\otimes_A R)/(\pi\otimes 1)(B\otimes_A R)$. Note that the only assumption needed in Equation (A-29) is that L_{k-2} is of type I. Then $h_{j-2l+1}\cdot{}^t(e_{j-2l}\cdot\tilde{m}_{j-2l,j-2l+1})$ is a lift of $\bar{h}_{j-2l+1}\cdot{}^t(e_{j-2l}\cdot m_{j-2l,j-2l+1})$. The first line is independent of the choice of a lift $\tilde{m}'_{j-2l+1,j-2l}$ of $m'_{j-2l+1,j-2l}$ as explained at the paragraph following Equation (A-19). This fact completes our claim. The above equation is equivalent to

$$\frac{1}{\bar{\alpha}^2\cdot\pi^2}\Big(\sigma({}^t\tilde{m}'_{j-2l-1,j-2l})\cdot\sigma(\pi)h_{j-2l-1}\cdot\tilde{m}'_{j-2l-1,j-2l} + (e_{j-2l}\cdot\sigma(\tilde{m}_{j-2l,j-2l+1}))\cdot\pi h^3_{j-2l+1}\cdot{}^t(e_{j-2l}\cdot\tilde{m}_{j-2l,j-2l+1})\Big) + \Big(\frac{z_{j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}\Big)+\Big(\frac{z_{j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}\Big)^2 = \Big(\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}\Big) \tag{A-40}$$

by adding $\big(\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}\big)$ to both sides.

For $\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2m_j}$, we observe that L_{j-2m_j-2} is of type *II*. By Equation (A-27) with $k = j-2m_j$, we have ${}^t m'_{j-2m_j-1,j-2m_j} = e_{j-2m_j}\cdot m_{j-2m_j,j-2m_j-1}\bar{h}_{j-2m_j-1}$. Here we use the fact that $\bar{h}^2_{j-2m_j-1} = \mathrm{id}$ (cf. Equation (A-22)). Note that the only assumption needed in Equation (A-27) is that L_k is of type I. On the other hand, the equation in Remark A.5(3), when $i = j-2m_j-1$, is $e_{j-2m_j}\cdot m_{j-2m_j,j-2m_j-1} = 0$ since L_{j-2m_j-2} is of type *II*. Thus ${}^t m'_{j-2m_j-1,j-2m_j} = 0$. Therefore, $\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2m_j}$ is

$$\frac{1}{\bar{\alpha}^2\cdot\pi^2}\big((e_{j-2m_j}\cdot\sigma(\tilde{m}_{j-2m_j,j-2m_j+1}))\cdot\pi h^3_{j-2m_j+1}\cdot{}^t(e_{j-2m_j}\cdot\tilde{m}_{j-2m_j,j-2m_j+1})\big) + \frac{z_{j-2m_j}}{\bar{\alpha}}+\Big(\frac{z_{j-2m_j}}{\bar{\alpha}}\Big)^2+\Big(\frac{\widehat{m}'_{j-2m_j,j-2m_j+2}}{\bar{\alpha}}\Big)^2 = 0. \tag{A-41}$$

This equation is equivalent to

$$\frac{1}{\bar{\alpha}^2\cdot\pi^2}\Big((e_{j-2m_j}\cdot\sigma(\tilde{m}_{j-2m_j,j-2m_j+1}))\cdot\pi h^3_{j-2m_j+1}\cdot{}^t(e_{j-2m_j}\cdot\tilde{m}_{j-2m_j,j-2m_j+1})\Big) + \Big(\frac{z_{j-2m_j}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2m_j,j-2m_j+2}}{\bar{\alpha}}\Big)+\Big(\frac{z_{j-2m_j}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2m_j,j-2m_j+2}}{\bar{\alpha}}\Big)^2 = \frac{\widehat{m}'_{j-2m_j,j-2m_j+2}}{\bar{\alpha}} \tag{A-42}$$

by adding $\frac{\widehat{m}'_{j-2m_j,j-2m_j+2}}{\bar{\alpha}}$ to both sides.

We emphasize that it is unnecessary to investigate $\mathcal{F}_j$ since the equation $\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\bar{\alpha}} = 0$ (resp. $\sum_{l=0}^{m_j}\frac{z_{j-2l}}{\bar{\alpha}} = 1$) if $j\in\mathcal{J}_1$ (resp. if $j\in\mathcal{J}_2$) already implies Equation (A-21) so that $\sum_{l=0}^{m_j}\frac{1}{\bar{\alpha}^2}\mathcal{F}_{j-2l} = 0$.

We now observe Equations (A-40) and (A-42). We introduce a new variable

$$z'_{j-2l} = \begin{cases} \dfrac{z_{j-2l}}{\bar{\alpha}}+\dfrac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}+\dfrac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}} & \text{if } 0<l<m_j; \\ \dfrac{z_{j-2m_j}}{\bar{\alpha}}+\dfrac{\widehat{m}'_{j-2m_j,j-2m_j+2}}{\bar{\alpha}} & \text{if } l=m_j. \end{cases}$$

Then z_{j-2l} can be eliminated by z'_{j-2l}, $\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}$, $\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}$. In addition, by using Equations (A-40) and (A-42), the term $\frac{\widehat{m}'_{j-2l-2,j-2l}}{\bar{\alpha}}+\frac{\widehat{m}'_{j-2l,j-2l+2}}{\bar{\alpha}}$ can be eliminated by

z'_{j-2l} and $m'_{j-2l-1,j-2l}, m_{j-2l,j-2l+1}$. Furthermore, the equation $\sum_{l=0}^{m_j} \frac{z_{j-2l}}{\bar{\alpha}} = 0$ (resp. $\sum_{l=0}^{m_j} \frac{z_{j-2l}}{\bar{\alpha}} = 1$) if $j \in \mathcal{J}_1$ (resp. if $j \in \mathcal{J}_2$) implies that z_j can be eliminated by z'_{j-2l}, $m'_{j-2l-1,j-2l}, m_{j-2l,j-2l+1}$ with $0 < l \le m_j$.

If $m_j = 0$, then we can show that the equation $\frac{1}{\bar{\alpha}^2}\mathcal{F}_j$ is the same as

$$\frac{z_j}{\bar{\alpha}} + \left(\frac{z_j}{\bar{\alpha}}\right)^2 = 0$$

by using an argument similar to that used in the proof of Equation (A-41). Then the equation $\frac{z_j}{\bar{\alpha}} = 0$ (resp. $\frac{z_j}{\bar{\alpha}} = 1$) if $j \in \mathcal{J}_1$ (resp. if $j \in \mathcal{J}_2$) implies that z_j can be eliminated.

We now combine all cases (1)–(4) observed above.

(a) By (1), we eliminate $\sum_{i<j} n_i n_j$ variables.

(b) By (2), we eliminate $\sum_{i \text{ even and } L_i \text{ of type } I^o} (n_i - 1)$ variables.

(c) By (3), we eliminate $\sum_{i \text{ even and } L_i \text{ of type } I^e} (2(n_i - 2) + 1)$ variables.

(d) By (4), we eliminate $\#\{i : i \text{ is even and } L_i \text{ is of type } I\}$ variables.

Recall from the third paragraph of the proof that $\operatorname{Ker} \tilde{\varphi}/\widetilde{M}^1$ is isomorphic to an affine space of dimension

$$2\sum_{i<j} n_i n_j - \sum_{\substack{i \text{ odd} \\ L_i \text{ bound}}} n_i + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (2n_i - 1) + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^e}} (4n_i - 4).$$

Thus, $\widetilde{G}^{\ddagger}_{\mathcal{J}_1,\mathcal{J}_2}$ is isomorphic to an affine space of dimension

$$\Bigg(2\sum_{i<j} n_i n_j - \sum_{\substack{i \text{ odd} \\ L_i \text{ bound}}} n_i + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (2n_i - 1) + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^e}} (4n_i - 4)\Bigg) \tag{A-43}$$

$$-\Bigg(\sum_{i<j} n_i n_j + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (n_i - 1) + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^e}} (2(n_i - 2) + 1) + \#\{i : i \text{ is even and } L_i \text{ is of type } I\}\Bigg).$$

Therefore, the dimension of $\widetilde{G}^{\ddagger}_{\mathcal{J}_1,\mathcal{J}_2}$ is

$$\sum_{i<j} n_i n_j - \sum_{\substack{i \text{ odd} \\ L_i \text{ bound}}} n_i + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (n_i - 1) + \sum_{\substack{i \text{ even} \\ L_i \text{ of type } I^o}} (2n_i - 2), \tag{A-44}$$

which finishes the proof. □

Lemma A.9. *Let F_j be the closed subgroup scheme of $\widetilde{G}$ defined by the following equations*:

- $m_{i,k} = 0$ *if* $i \neq k$;
- $m_{i,i} = \mathrm{id}$ *if* $i \neq j$;
- *and for* $m_{j,j}$,

$$\begin{cases} s_j = \mathrm{id},\ y_j = 0,\ v_j = 0 & \text{if } L_i \text{ is of type } I^o; \\ s_j = \mathrm{id},\ r_j = t_j = y_j = v_j = u_j = w_j = 0 & \text{if } L_i \text{ is of type } I^e. \end{cases}$$

Then F_j is isomorphic to $\mathbb{A}^1 \times \mathbb{Z}/2\mathbb{Z}$ as a κ-variety, where $\mathbb{A}^1$ is an affine space of dimension 1, *and has exactly two connected components.*

Proof. A matrix form of an element m of $F_j(R)$ for a κ-algebra R is

$$\begin{pmatrix} \mathrm{id} & 0 & & \dots & & 0 \\ 0 & \ddots & & & & \\ & & \mathrm{id} & & & \\ \vdots & & & m_{j,j} & & \vdots \\ & & & & \mathrm{id} & \\ & & & & \ddots & 0 \\ 0 & & & \dots & 0 & \mathrm{id} \end{pmatrix}$$

such that

$$m_{j,j} = \begin{cases} \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 1+\pi z_j \end{pmatrix} & \text{if } L_j \text{ is of type } I^o; \\ \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & 1+\pi x_j & \pi z_j \\ 0 & 0 & 1 \end{pmatrix} & \text{if } L_j \text{ is of type } I^e. \end{cases}$$

To prove the lemma, we consider the matrix equation $\sigma({}^t m) \cdot h \cdot m = h$. Recall that h, as an element of $\underline{H}(R)$, is as explained in Remark 3.3(2). Based on Equations (A-1) and (A-2), the diagonal (i, i)-blocks of $\sigma({}^t m) \cdot h \cdot m = h$ with $i \neq j$ are trivial and the nondiagonal blocks of $\sigma({}^t m) \cdot h \cdot m = h$ are also trivial. The (j, j)-block of $\sigma({}^t m) \cdot h \cdot m$ is

$$\begin{cases} \pi^j \cdot \begin{pmatrix} a_j & 0 \\ 0 & (1+\sigma(\pi z_j)) \cdot (1+2\bar{\gamma}_j) \cdot (1+\pi z_j) \end{pmatrix} & \text{if } L_j \text{ is of type } I^o; \\ \pi^j \cdot \begin{pmatrix} a_j & 0 & 0 \\ 0 & (1+\sigma(\pi x_j))(1+\pi x_j) & (1+\sigma(\pi x_j))(1+\pi z_j) \\ 0 & (1+\sigma(\pi z_j))(1+\pi x_j) & (1+\pi z_j)\sigma(\pi z_j)+\pi z_j+2\bar{\gamma}_j \end{pmatrix} & \text{if } L_j \text{ is of type } I^e. \end{cases}$$

We write $x_j = x_j^1 + \pi x_j^2$ and $z_j = z_j^1 + \pi z_j^2$, where $x_j^1, x_j^2, z_j^1, z_j^2 \in R \subset R \otimes_A B$ and π stands for $1 \otimes \pi \in R \otimes_A B$. When L_j is of type I^o, by considering the (2, 2)-block of the matrix above, we obtain the equation

$$\bar{\alpha}(z_j^1) + (z_j^1)^2 = 0.$$

Recall that α is the unit in B such that $\epsilon = 1 + \alpha\pi$ as explained in Section 2A, and $\bar{\alpha}$ is the image of α in κ.

Then this equation is equivalent to

$$(z_j^1/\bar{\alpha}) + (z_j^1/\bar{\alpha})^2 = 0$$

by dividing by $\bar{\alpha}^2$ in both sides. Therefore, in this case, F_j is isomorphic to $\mathbb{A}^1 \times \mathbb{Z}/2\mathbb{Z}$ as a κ-variety.

When L_j is of type I^e, by considering the (2, 2)-block of the matrix above, we obtain the equation

$$\bar{\alpha}(x_j^1) + (x_j^1)^2 = 0.$$

We also consider the (2, 3)-block of the matrix above, and we obtain two equations

$$x_j^1 + z_j^1 = 0, \quad \bar{\alpha} x_j^1 + x_j^2 + z_j^2 + \bar{\alpha} x_j^1 z_j^1 = 0.$$

By considering the (3, 3)-block of the matrix above, we obtain the equation

$$\bar{\alpha}(z_j^1) + (z_j^1)^2 = 0.$$

By combining all these, we see that F_j is isomorphic to $\mathbb{A}^1 \times \mathbb{Z}/2\mathbb{Z}$ as a κ-variety. □

We introduce the final lemma in order to prove Lemma 4.6 below. This lemma is about the number of connected components in a short exact sequence of algebraic groups.

Lemma A.10. *Assume that there is a short exact sequence*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

of linear algebraic groups over κ. Let $\pi_0(B)$ be the component group of B which is defined as the spectrum of the largest separable subalgebra $\pi_0(\kappa[B])$ of $\kappa[B]$, where $\kappa[B]$ is the coordinate ring of B. Let $\#(\pi_0(B))$ be the order of $\pi_0(B)$, which is defined as the dimension of $\pi_0(\kappa[B])$ as a κ-vector space. Note that B is connected if and only if $\pi_0(B)$ is trivial if and only if $\#(\pi_0(B)) = 1$. Thus $\#(\pi_0(B))$ is the number of connected components of $B \otimes_\kappa \bar{\kappa}$. Then

$$\#(\pi_0(B)) \leq \#(\pi_0(A)) \cdot \#(\pi_0(C)).$$

Moreover, the equality holds if A is connected and in this case, $\pi_0(B) = \pi_0(C)$.

Proof. By definition of a component group, there exists a surjective morphism $\pi : B \longrightarrow \pi_0(B)$ whose kernel is connected. Let A' ($\subseteq \pi_0(B)$) be the image of A under the morphism π. Notice that A' is a normal subgroup of $\pi_0(B)$ and that $\#(A') \leq \#(\pi_0(A))$. Then the morphism π induces a surjective morphism from C to $\pi_0(B)/A'$ and so $\#(\pi_0(B)/A') \leq \#(\pi_0(C))$. Therefore, $\#(\pi_0(B)) \leq \#(\pi_0(A)) \cdot \#(\pi_0(C))$.

It is clear that $\#(\pi_0(C)) \leq \#(\pi_0(B))$. Thus, if A is connected, then $\#(\pi_0(C)) = \#(\pi_0(B))$. In this case, since there exists a surjective morphism from B to $\pi_0(C)$ (through C), there exists a surjective morphism from $\pi_0(B)$ to $\pi_0(C)$. Since $\#(\pi_0(C)) = \#(\pi_0(B))$, we can conclude that $\pi_0(B) = \pi_0(C)$. □

We finally prove Lemma 4.6.

Proof. We start with the following short exact sequence

$$1 \longrightarrow \widetilde{G}^1 \longrightarrow \operatorname{Ker}\varphi \longrightarrow \operatorname{Ker}\varphi/\widetilde{G}^1 \longrightarrow 1.$$

It is obvious that $\operatorname{Ker}\varphi$ is smooth by Theorems A.4 and A.6. $\operatorname{Ker}\varphi$ is also unipotent since it is a subgroup of a unipotent group $\widetilde{M}^+$. Since $\widetilde{G}^1$ is connected by Theorem A.4, the component group of $\operatorname{Ker}\varphi$ is the same as that of $\operatorname{Ker}\varphi/\widetilde{G}^1$ by Lemma A.10. Moreover, the dimension of $\operatorname{Ker}\varphi$ is the sum of the dimension of $\widetilde{G}^1$ and the dimension of $\operatorname{Ker}\varphi/\widetilde{G}^1$. This completes the proof. □

Appendix B: Examples

In this appendix, we provide an example with a unimodular lattice (L, h) of rank 1. Let L be Be, a rank 1 hermitian lattice with hermitian form $h(le, l'e) = \sigma(l)l'$. With this lattice, we construct the smooth integral model and its special fiber and compute the local density.

B.1: ***Naive construction (without using our technique).*** We first construct the smooth integral model and its special fiber, without using any techniques introduced in this paper. If we write an element of L as $x + \pi y$ where $x, y \in A$, then it is easy to see that a naive integral model $\underline{G}'$ is Spec $A[x, y]/(x^2 + (\pi + \sigma(\pi))xy + \pi\sigma(\pi)y^2 - 1)$. As mentioned in Section 2A, we may assume that $\pi + \sigma(\pi) = 2$ and $\pi\sigma(\pi) = 2u$ for a unit $u \in A$. We remark that $\underline{G}'$ is smooth if $p \neq 2$, and in this case its special fiber is Spec $\kappa[x, y]/(x^2 - 1) = \mathbb{A}^1 \times \mu_2$ as a κ-variety. However, if $p = 2$, then its special fiber is no longer smooth since $\kappa[x, y]/(x^2 - 1) = \kappa[x, y]/(x - 1)^2$ is nonreduced. Some of the difficulty in the case $p = 2$ arises from this. The associated smooth integral model is obtained by a finite sequence of *dilatations* (at least once) of $\underline{G}'$ (cf. [Bosch et al. 1990]).

On the other hand, the difficulty can also be explained in terms of quadratic forms. Namely, the smoothness of any scheme over A should be closely related to the smoothness of its special fiber. If we define a function $q : L \longrightarrow A$ by $l \mapsto h(l, l)$, then $q \bmod 2$ is a quadratic form over κ. Therefore, the associated smooth integral model should contain information about this quadratic form, which is more subtle than quadratic forms over a field of characteristic not equal 2.

To construct the smooth integral model, we observe the characterization of $\underline{G}$ that $\underline{G}(R) = \underline{G}'(R)$ for an étale A-algebra R. Thus any element of $\underline{G}(R)$ is of the form $x + \pi y$ such that $x^2 + 2xy + 2uy^2 = 1$. Therefore, $(x - 1)^2$ is contained in the ideal (2) of R so that we can rewrite $x = 1 + 2x'$ since R is étale over A. With this, any element of $\underline{G}(R)$ is of the form $1 + 2x' + \pi y$ such that $y + uy^2 + 2(x' + (x')^2 + x'y) = 0$. We consider the affine scheme Spec $A[x, y]/(y + uy^2 + 2(x + x^2 + xy))$. Its special fiber is then reduced and smooth. Thus, this affine scheme is the desired smooth integral model $\underline{G}$. Furthermore, its special fiber Spec $\kappa[x, y]/(y + uy^2)$ is isomorphic to $\mathbb{A}^1 \times \mathbb{Z}/2\mathbb{Z}$ as a κ-variety so that the number of rational points is $2f$, where f is the cardinality of κ.

B.2: ***Construction following our technique.*** Define the map $q : L \to A$ via

$$l \mapsto h(l, l).$$

If we write $l = x + \pi y$ such that $x, y \in A$, then $q(l) = h(x + \pi y, x + \pi y) = x^2 + (\pi + \sigma(\pi))xy + \pi \cdot \sigma(\pi)y^2$. Thus $q \bmod 2$ is an additive polynomial over κ. Let $B(L)$ be the sublattice of L such that $B(L)/\pi L$ is the kernel of the additive polynomial $q \bmod 2$ on $L/\pi L$. In this case, $B(L) = \pi L$.

For an étale A-algebra R with $g \in \mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h \otimes_A R)$, it is easy to see that g induces the identity on $L/B(L) = L/\pi L$. Based on this, we construct the following functor from the category of commutative flat A-algebras to the category of monoids as follows. For any commutative flat A-algebra R, set

$$\underline{M}(R) = \{m \in \mathrm{End}_{B\otimes_A R}(L \otimes_A R)\} \mid m \text{ induces the identity on } L \otimes_A R/B(L) \otimes_A R\}.$$

This functor $\underline{M}$ is then representable by a polynomial ring and has the structure of a scheme of monoids. Let $\underline{M}^*(R)$ be the set of invertible elements in $\underline{M}(R)$ for any commutative A-algebra R. Then $\underline{M}^*$ is representable by a group scheme which is an open subscheme of $\underline{M}$ (Section 3B). Thus $\underline{M}^*$ is smooth. As a matrix, each element of $\underline{M}^*(R)$ for a flat A-algebra R can be written as $\big(1+\pi z\big)$.

We define another functor from the category of commutative flat A-algebras to the category of sets as follows. For any commutative flat A-algebra R, let $\underline{H}(R)$ be the set of hermitian forms f on $L \otimes_A R$ (with values in $B \otimes_A R$) such that $f(a,a) \bmod 2 = h(a,a) \bmod 2$, where $a \in L \otimes_A R$. As a matrix, each element of $\underline{M}^*(R)$ for a flat A-algebra R is $\big(1+2c\big)$.

Then for any flat A-algebra R, the group $\underline{M}^*(R)$ acts on the right of $\underline{H}(R)$ by $f \circ m = \sigma({}^t m) \cdot f \cdot m$ and this action is represented by an action morphism (Theorem 3.4)

$$\underline{H} \times \underline{M}^* \longrightarrow \underline{H}.$$

Let ρ be the morphism $\underline{M}^* \to \underline{H}$ defined by $\rho(m) = h \circ m$, which is obtained from the above action morphism. As a matrix, for a flat A-algebra R,

$$\rho(m) = \rho\big(\big(1+\pi z\big)\big) = \big(1+\pi z + \sigma(\pi z) + \pi\sigma(\pi)\cdot z\sigma(z)\big).$$

Then ρ is smooth of relative dimension 1 (Theorem 3.6). Let $\underline{G}$ be the stabilizer of h in $\underline{M}^*$. The group scheme $\underline{G}$ is smooth, and $\underline{G}(R) = \mathrm{Aut}_{B\otimes_A R}(L \otimes_A R, h \otimes_A R)$ for any étale A-algebra R (Theorem 3.8).

We now describe the structure of the special fiber $\widetilde{G}$ of $\underline{G}$. For a κ-algebra R, each element of $\underline{M}(R)$ (resp. $\underline{H}(R)$) can be written as a formal matrix $m = \big(1+\pi z\big)$ (resp. $f = \big(1+2c\big)$). Firstly, it is easy to see that $B_0 = Y_0 = \pi L$ so that the morphism φ in Section 4A is trivial.

For the component groups, as explained in Theorem 4.11, there is a surjective morphism from $\widetilde{G}$ to $\mathbb{Z}/2\mathbb{Z}$. Let us describe this morphism explicitly below. It is easy to see that $L^0 = M_0 = L$ and $C(L^0) = M_0' = L$. Here, we follow notation of Section 4B. Since $M_0 = L$ is of type I^o, there exists a morphism from the special fiber $\widetilde{G}$ $(= G_0)$ to the special fiber of the smooth integral model associated to $M_0' \oplus C(L^0) = L \oplus L$ of type I^e as explained in the argument 2 just before Remark 4.10. Remark 4.10 tells us how to describe this morphism as formal matrices. Let (e_1, e_2) be a basis for $L \oplus L$ so that the associated Gram matrix of the hermitian lattice $L \oplus L$ with respect to this basis is $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Then we consider the basis (e_1, e_1+e_2), with respect to which the morphism described in Remark 4.10 is given as

$$\big(1+\pi z\big) \mapsto \begin{pmatrix}1 & -\pi z\\ 0 & 1+\pi z\end{pmatrix}.$$

We now construct a morphism from the special fiber of the smooth integral model associated to $M_0' \oplus C(L^0) = L \oplus L$ to $\mathbb{Z}/2\mathbb{Z}$ and describe the image of $\left(\begin{smallmatrix}1 & -\pi z\\ 0 & 1+\pi z\end{smallmatrix}\right)$ in $\mathbb{Z}/2\mathbb{Z}$.

Let R be a κ-algebra. The Gram matrix for the hermitian lattice $L \oplus L$ with respect to the basis (e_1, e_1+e_2) is $\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$. Since $L \oplus L$ is *unimodular of type* I^e, an R-point of the special fiber associated to $L \oplus L$ with respect to this basis is expressed as the formal matrix $\left(\begin{smallmatrix}1+\pi x' & \pi z'\\ u' & 1+\pi w'\end{smallmatrix}\right)$, as explained in Section 3B. Based on argument (1) following

Definition 4.9, the morphism mapping to $\mathbb{Z}/2\mathbb{Z}$ factors through the special fiber associated to $C(L\oplus L)$, composed with the Dickson invariant associated to the corresponding orthogonal group. $C(L\oplus L)$ is then generated by $(\pi e_1, e_1+e_2)$ and is π^1-modular. Thus there is no congruence condition on an element of the smooth integral model associated to $C(L\oplus L)$ as explained in Section 3B. Write $x'=x_1'+\pi x_2'$, $y'=y_1'+\pi y_2'$, and $z'=z_1'+\pi z_2'$. The image of $\begin{pmatrix} 1+\pi x' & \pi z' \\ u' & 1+\pi w' \end{pmatrix}$ in the special fiber associated to $C(L\oplus L)$ is $\begin{pmatrix} 1+\pi x_1' & z_1'+\pi z_2' \\ \pi u_1' & 1+\pi w_1' \end{pmatrix}$. Since $C(L\oplus L)$ is π^1-modular with rank 2, there is a morphism from the special fiber associated to $C(L\oplus L)$ to the orthogonal group associated to $C(L\oplus L)/\pi C(L\oplus L)$, as described in Theorem 4.4 or Remark 4.7. Then the image of $\begin{pmatrix} 1+\pi x_1' & z_1'+\pi z_2' \\ \pi u_1' & 1+\pi w_1' \end{pmatrix}$ in this orthogonal group is $\begin{pmatrix} 1 & z_1' \\ 0 & 1 \end{pmatrix}$. The Dickson invariant of $\begin{pmatrix} 1 & z_1' \\ 0 & 1 \end{pmatrix}$ is $z_1'/\bar{\alpha}$ as mentioned in step (1) of the proof of Theorem 4.11. Here, α is the unit in B such that $\epsilon=1+\alpha\pi$ as explained in Section 2A, and $\bar{\alpha}$ is the image of α in κ.

In conclusion, the image of $(1+\pi z)$, which is an element of $\widetilde{G}(R)$ for a κ-algebra R, in $\mathbb{Z}/2\mathbb{Z}$ is $z_1/\bar{\alpha}$, where we write $z=z_1+\pi z_2$. On the other hand, the equation defining $\widetilde{G}$ is $\bar{\alpha}z_1+z_1^2=0$ which is equivalent to $\frac{z_1}{\bar{\alpha}}+(\frac{z_1}{\bar{\alpha}})^2=0$. Thus, the morphism from $\widetilde{G}$ to $\mathbb{Z}/2\mathbb{Z}$ is surjective. Therefore the maximal reductive quotient of $\widetilde{G}$ is $\mathbb{Z}/2\mathbb{Z}$ and using Remark 5.3,

$$\#(\widetilde{G}(\kappa)) = \#(\mathbb{Z}/2\mathbb{Z})\cdot\#(\mathbb{A}^1) = 2f,$$

where f is the cardinality of κ. Based on Theorem 5.2, the local density is

$$\beta_L = f^0\cdot 2f = 2f.$$

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