GROUP VELOCITY OF DIELECTRIC WAVEGUIDE MODES

The group velocity and dispersion of surface-wave modes propagating along a circular dielectric rod are computed and presented graphically in normalised form. The group velocity as each mode approaches the cutoff frequency is calculated approximately.

Interest in circular dielectric surface-wave guides has been reawakened by recent proposals[1] to use the cladded glass fibre as a transmission line for laser communication systems. The cladded glass fibre can support nonradiative surface waves when the refractive index of the core, n, is greater than the refractive index of the cladding, n_1.

The solution of Maxwell's equations in the core and cladding permits the formulation of a boundary equation[2,3] when tangential field components are equated at the core–cladding boundary:

$$\frac{N^2\bar{\beta}^2}{\bar{k}_0^2}\left(\frac{1}{u^2}+\frac{1}{w^2}\right)^2 = (F_N + M_N)\left(F_N + \frac{M_N}{\epsilon'}\right) \quad . \quad (1)$$

where N = azimuthal mode number
r_1 = boundary radius
$\bar{\beta} = \beta r_1$ = normalised guide phase constant
$\bar{k}_0 = k_0 r_1 \sqrt{\epsilon}$ = normalised free-space phase constant
$\epsilon = n^2$ = relative permittivity of core
$\epsilon_1 = n_1^2$ = relative permittivity of cladding
$\epsilon' = \epsilon/\epsilon_1$
$u^2 = \bar{k}_0^2 - \bar{\beta}^2$
$w^2 = \bar{\beta}^2 - \bar{k}_0^2/\epsilon'$
$F_N = J_N'(u)/\{uJ_N(u)\}$
$M_N = K_N'(w)/\{wK_N(w)\}$

J_N and K_N are first kind and modified Bessel functions[4] of order N; J_N' and K_N' are differentiated with respect to the argument.

Eqn. 1 has been solved numerically for lossless dielectrics. For each azimuthal mode number N, there is a family of radial modes. When $N = 0$, the modes are circularly symmetrical, and occur in pairs of TE and TM modes. When $N > 0$, pairs of modes still occur, except for the lowest-order mode of each family; however, the modes are hybrid, with both H and E field components in the axial direction. The first mode in the family, $N = 1$, the HE_{11} mode, has a zero-frequency cutoff. One of a pair of modes is designated[2] $HE_{N(M+1)}$ or HE_{NM}, depending on whether the H or E axial field dominates. The numerical solution of eqn. 1 for $\bar{k}_0$ and $\bar{\beta}$ produces the mode plots of Fig. 1, where some low-order modes are shown. The normalised phase velocity $\bar{v}_p$ is given by

$$\bar{v}_p = v_p \frac{n}{c}, \text{ where } c \text{ is the velocity of light}$$
$$= \bar{k}_0/\bar{\beta}$$

Fig. 1 is essentially a graph of phase velocity against frequency, normalised for small refractive-index difference $\Delta n/n$, where $\Delta n = n - n_1$. The graph applies when $\Delta n/n < 10^{-2}$; for $\Delta n/n > 10^{-2}$, the boundary equation must be solved using the particular value of $\Delta n/n$ required.

The normalised group velocity, $\bar{v}_g$, is defined as

$$\bar{v}_g = v_g \frac{n}{c} \quad \text{where } v_g \text{ is the group velocity and } c \text{ is the velocity of light}$$
$$= \frac{d\omega}{d\beta}\frac{n}{c}$$
$$= \frac{d\bar{k}_0}{d\bar{\beta}} \quad \text{since } k_0 = \frac{\omega}{c}$$

By differentiation of eqn. 1 with respect to $\bar{k}_0$, an expression is obtained for the normalised group velocity of the surface-wave modes:

$$\frac{d\bar{k}_0}{d\bar{\beta}} = \frac{\dfrac{\bar{\beta}}{\bar{k}_0}\left\{X - Y + \dfrac{2N^2}{\bar{k}_0^2}\left(\dfrac{1}{u^2}+\dfrac{1}{w^2}\right)^2 + 4N^2\dfrac{\bar{\beta}^2}{\bar{k}_0^2}\left(\dfrac{1}{u^2}+\dfrac{1}{w^2}\right)\left(\dfrac{1}{u^4}-\dfrac{1}{w^4}\right)\right\}}{X - \dfrac{Y}{\epsilon'} + 2N^2\dfrac{\bar{\beta}^2}{\bar{k}_0^2}\left(\dfrac{1}{u^2}+\dfrac{1}{w^2}\right)^2 + 4N^2\dfrac{\bar{\beta}^2}{\bar{k}_0^2}\left(\dfrac{1}{u^2}+\dfrac{1}{w^2}\right)\left(\dfrac{1}{u^4}-\dfrac{1}{w^4\epsilon'}\right)} \quad . \ . \ . \ . \ . \quad (2)$$

where

$$X = \left\{\frac{N^2}{u^4} - \frac{1}{u^2}(1 + 2F_N) - F_N^2\right\}\left\{2F_N + \left(1 + \frac{1}{\epsilon'}\right)M_N\right\}$$

$$Y = \left\{\frac{N^2}{w^4} + \frac{1}{w^2}(1 - 2M_N) - M_N^2\right\}\left\{2M_N + \left(1 + \frac{1}{\epsilon'}\right)F_N\right\}$$

Insertion of solutions of eqn. 1 for $\bar{k}_0$ and $\bar{\beta}$ into eqn. 2 gives $d\bar{k}_0/d\bar{\beta}$ directly. Fig. 2 shows group velocity as a function of frequency, in a generalised form, for $\Delta n/n < 10^{-2}$.

Eqn. 2 may be differentiated with respect to $\bar{k}_0$ to obtain an expression for $d\bar{v}_g/d\bar{k}_0$, which is a normalised form of dispersion or rate of variation of group velocity with frequency. Fig. 3 shows generalised dispersion curves for low-order modes, for $\Delta n/n$ small.

Near the cutoff frequency of a particular mode, the group velocity cannot be determined by the computer program used to solve the boundary equation, since some quantities tend to infinity. For example, w is zero on the asymptote to which the phase velocity tends near cutoff, and $1/w$ is infinite. Consequently an approximate solution[5] of the boundary

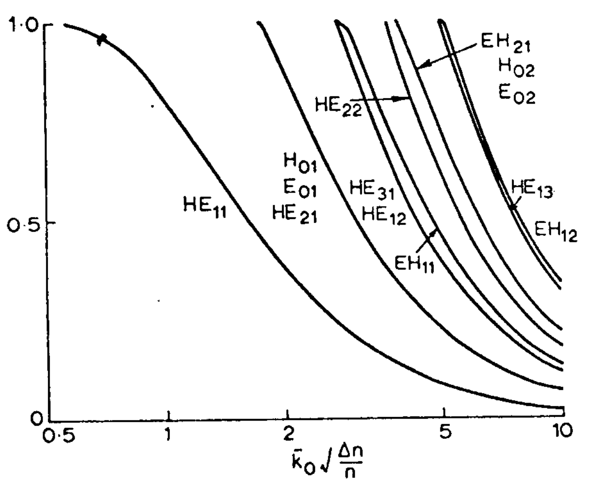
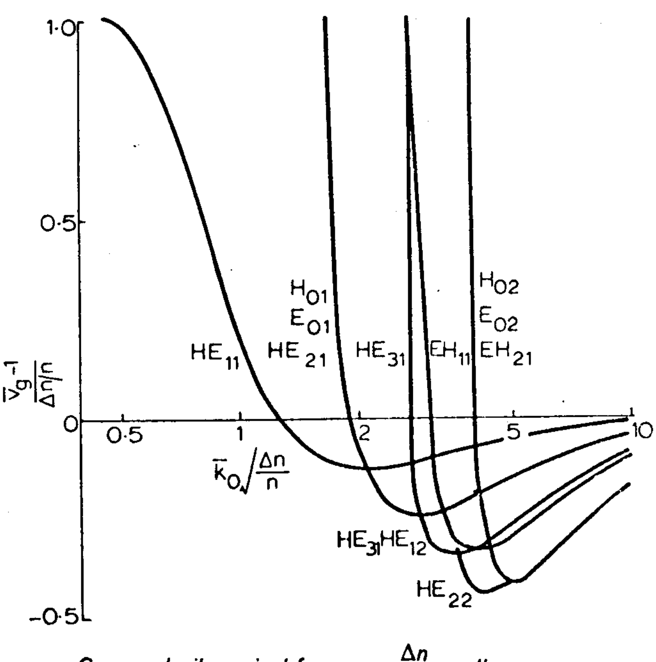


Fig. 1 *Phase velocity against frequency generalised for small refractive-index difference, for lowest-order modes*



Group velocity against frequency, $\frac{\Delta n}{n}$ small equation for the case when w is small has been applied to eqn. 2, the expression for normalised group velocity.

In this case,

$$\left(\frac{k_0}{\beta}\right)^2 \simeq \epsilon'$$

and $$M_N = \frac{K_N'(w)}{wK_N(w)} \simeq -\frac{1}{w^2} + \ln\frac{w}{2} \qquad N = 1$$

$$\simeq -\frac{N}{w^2} - \frac{1}{2(n-1)} \qquad N > 1$$

There are two sets of solutions for the boundary condition, giving the pairs of modes. For $N > 1$, they occur because $J_N(u)$ tends to a zero at cutoff for one of the pair, the HE mode, and does not for the other, the EH mode. The lone HE_{N1} modes behave like EH modes, since $J_N(u)$ does not tend to zero at cutoff.

For $J_N(u) \to 0$ near cutoff, eqn. 1 tends to

$$F_N = \frac{\epsilon' + 1}{\epsilon'} \frac{N}{w^2}$$

For $J_N(u)$ finite at cutoff,

$$F_N = \frac{-N}{u^2} + \frac{1}{(\epsilon' + 1)(N - 1)}$$

When $N = 1$, the cutoff frequencies occur for all modes at zeros of $J_1(u)$; the $HE_{1(M+1)}$ and EH_{1M} become degenerate.[2] However, there are two distinct orders of magnitude that $J_1(u)$ can have as w tends to zero.[5] If $J_1(u)$ is very small, of order w^2, eqn. 1 reduces to

$$F_1 = \frac{\epsilon' + 1}{\epsilon' w^2}$$ near cutoff, corresponding to the HE modes.

If $J_1(u)$ is of order $\ln\frac{w}{2}$, then near cutoff

$$F_1 = -\left(\frac{2\ln\frac{w}{2}}{\epsilon' + 1} + \frac{1}{u^2}\right)$$

The appropriate expressions for F_N and M_N can be inserted in eqn. 2 to obtain the group velocity near cutoff:

(i) HE modes:

$$F_N \simeq \frac{\epsilon' + 1}{\epsilon} \frac{N}{w^2}.$$

The numerator and denominator of eqn. 2 contain terms in $1/w^6$, $1/w^4$ and $1/w^2$. Since w is very small, terms of order less than $1/w^6$ may be ignored; hence

$$\bar{v}_g \simeq \frac{1}{\sqrt{\epsilon'}} \frac{N(\epsilon' + 1) + 2\epsilon'}{N(\epsilon' + 1) + 2} \qquad N \geqslant 1$$

If $\Delta n/n$ is small, i.e. $\epsilon' = 1 + \delta$,

$$\bar{v}_g \simeq 1 - \left(\frac{1}{2} - \frac{1}{N+1}\right)\delta$$

If $N = 1$, $\bar{v}_g \simeq 1$.

(ii) EH modes and HE_{N1} modes:

$$(a)\ N > 1: F_N \simeq -\frac{N}{u^2} + \frac{1}{(\epsilon' + 1)(N - 1)}$$

Terms in $1/w^6$ cancel in the numerator and denominator of $\bar{v}_g$. Ignoring terms of order less than $1/w^4$, the group velocity is given by

$$\bar{v}_g \simeq \sqrt{\epsilon'}$$

$= \bar{v}_p$, the phase velocity, at cutoff

$$(b)\ N = 1: F_1 = -\frac{2\ln\frac{w}{2}}{\epsilon' + 1} + \frac{1}{u^2}$$

Ignoring terms of order less than $1/w^4$, the group velocity is again

$$\bar{v}_g \simeq \sqrt{\epsilon'}$$

$= \bar{v}_p$ at cutoff.

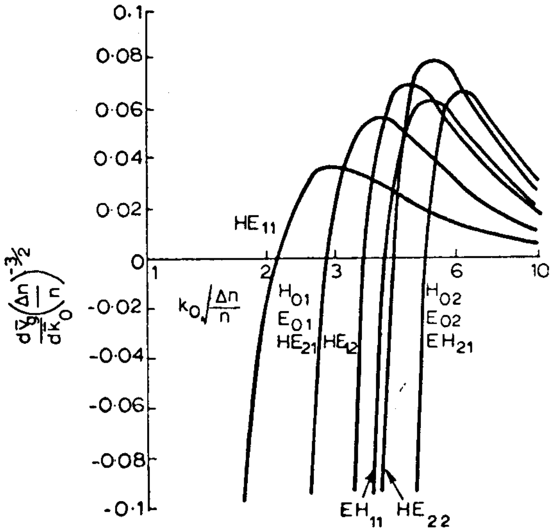


Fig. 3 *Variation of group velocity with frequency, $\frac{\Delta n}{n}$ small*

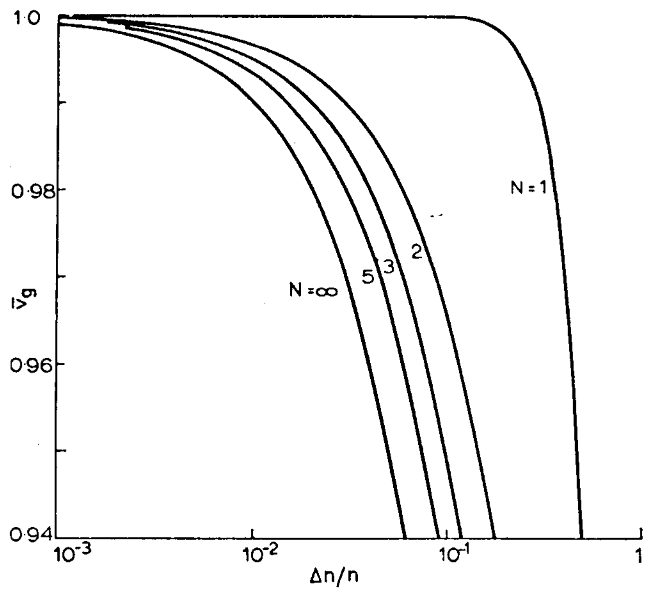


Fig. 4 *Group velocity at cutoff against refractive-index difference*

Thus the group velocity of all EH and HE_{N1} modes tends to the phase velocity as cutoff is approached. Removing the normalising factors gives

$$v_g = v_p = \frac{c}{n_1} \quad \text{at cutoff}$$

c/n_1 is the plane-wave velocity in the cladding material.

The group velocity at cutoff of the remaining HE modes is given by

$$\bar{v}_g = \frac{1}{\sqrt{\epsilon'}} \frac{N(\epsilon' + 1) + 2\epsilon'}{N(\epsilon' + 1) + 2}$$

Fig. 4 shows $\bar{v}_g$ plotted against $\Delta n/n$. When $N = 1$, $\bar{v}_g$ is approximately unity for $\Delta n/n < 10^{-1}$. As the azimuthal mode number N increases, the group velocity at cutoff tends to the curve for $N = \infty$, where

$$\bar{v}_g = \frac{1}{\sqrt{\epsilon'}}$$

The calculations that have been made of group velocity and dispersion do not take account of the variation of permittivity with frequency of the guide materials, since no general assumptions can be made about such variations. In a glass-fibre system, the bulk glass dispersion might typically be 10^{-8} m/cycle, which corresponds to a bandwidth of 5 GHz, allowing for a group delay of 30° over a 10 km path. To calculate the dispersion of a single mode, the normalised dispersion (Fig. 3) may be multiplied by the factor $\pi d_1 \left(\frac{\Delta n}{n}\right)^{3/2}$, where d_1 is the core diameter. For a fibre with $5\,\mu$m diameter core and refractive-index difference of 1%, this factor is $1 \cdot 6 \times 10^{-8}$ m/cycle. The reduction in bandwidth because of multimode propagation may be estimated from the normalised group-velocity curves of Fig. 2.

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