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Research Article

GROUPING BY PROXIMITY AND MULTISTABILITY IN DOT LATTICES:
A Quantitative Gestalt Theory

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Abstract—*Gestalt phenomena have long resisted quantification. In the spirit of Gestalt field theory, we propose a theory that predicts the probability of grouping by proximity in the six kinds of dot lattices (hexagonal, rhombic, square, rectangular, centered rectangular, and oblique). We claim that the unstable perceptual organization of dot lattices is caused by competing forces that attract each dot to other dots in its neighborhood. We model the decline of these forces as a function of distance with an exponential decay function. This* attraction function *has one parameter, the* attraction constant. *Simple assumptions allow us to predict the entropy of the perceptual organization of different dot lattices. We showed dot lattices tachistoscopically to 7 subjects, and from the probabilities of the perceived organizations, we calculated the entropy of each lattice for each subject. The model fit the data exceedingly well. The attraction constant did not vary much over subjects.*

The three dot patterns (or dot lattices) in Figure 1 are ambiguous. The pattern in Figure 1a is ambiguous because it is usually seen as a collection of columns, but it is sometimes seen as a collection of rows. The pattern in Figure 1b is more ambiguous: The chance that people will see it as a collection of columns is not very different from the chance that they will see it as a collection of rows (e.g., Schumann, 1900; Wertheimer, 1923). Figure 1c is most ambiguous; it can be seen with almost equal ease in three ways: as a collection of rows, a collection of lines sloping upward to the right, or a collection of lines sloping downward to the right. Starting from this observation, we present a new theory of grouping by proximity and of pattern ambiguity.

This theory requires us to take several steps. First, we enlarge the domain of stimuli. The Gestalt psychologists' demonstration of grouping by proximity is based on special cases. The dot lattices they used (rectangular, as in Fig. 1a; square, as in Fig. 1b) are only two of six classes of lattices (Kubovy, 1994). Our analysis of the geometry of lattices allows us to vary stimulus parameters systematically.

Second, we broaden the notion of ambiguity. Up to now, most psychologists have thought a figure to be either ambiguous or not. (Fisher, 1967, is a notable exception. His article is also a good source of references to the earlier literature.) We argue that ambiguity is quantifiable. Let us call ambiguous figures with two aspects *bistable*, figures with three aspects *tristable*, and figures with four aspects *quadristable*. Bistable figures (Necker, 1832; Rubin, 1921) are less ambiguous than tristable figures (Attneave, 1968, 1971). People spontaneously see each dot lattice as a collection of parallel strips of dots, most often aligned in the direction of the shortest interdot distance, but sometimes aligned in other directions. As we show, dot lattices are at least quadristable, and thus are candidates for the title of "most ambiguous stimuli." However, not all lattices are equally ambiguous (Kubovy, 1994).

Third, we give a specific expression to the Gestalt-theoretic idea that "like processes in the visual field attract one another" (Osgood, 1953, p. 202). From this idea flows an information-theoretic model of grouping and multistability, which we test against data.

We investigate the grouping and the multistability of dot lattices using a Gestalt-theoretic model. In doing so, we overcome three weaknesses in the Gestalt psychologists' treatment of grouping by proximity: First, whereas they studied only two types of lattices, we can study all lattices. Second, whereas they did not quantify their notion of attraction, we can propose a specific attraction function, which leads to a prediction of the probability of different perceptual organizations. Third, whereas they relied on phenomenology, we can compare the observed probability of the different organizations of dot lattices to the predictions of the theory.

A TWO-PARAMETER SPACE OF LATTICES

A dot lattice is a collection of dots in the plane; this collection is invariant under translation. A dot lattice possesses two important properties: It is discrete (i.e., dots are not too close together), and the dots are spread over the whole plane (i.e., dots are not too far apart). The latter property implies that a dot lattice is infinite. Consider two dots, *A* and *B*, of the lattice shown in Figure 1c, a fragment of which is shown in Figure 2. If you draw a line $\overleftrightarrow{AB}$ through *A* and *B*, the line will pass through infinitely many dots, all equally spaced. Now draw another line $\overleftrightarrow{AC}$, not parallel to $\overleftrightarrow{AB}$, through dot *A* and dot *C*. It too will pass through infinitely many dots, which also will be equally spaced. Call the distance between neighboring dots on $\overleftrightarrow{AB}$, *a*, and the distance between neighboring dots on $\overleftrightarrow{AC}$, *b*. Take a copy of the dot lattice and slide it without rotation (*translate* it) a distance *a* along the direction $\overleftrightarrow{AB}$. The copy and the original will coincide once more. This is why we say that the lattice is invariant under translation. Similarly, if you translated a copy of the lattice a distance *b* along the direction $\overleftrightarrow{AC}$, the copy and the original would also coincide. So all lattices are invariant under two independent translations.

Because no real lattice is infinite, no real system can be

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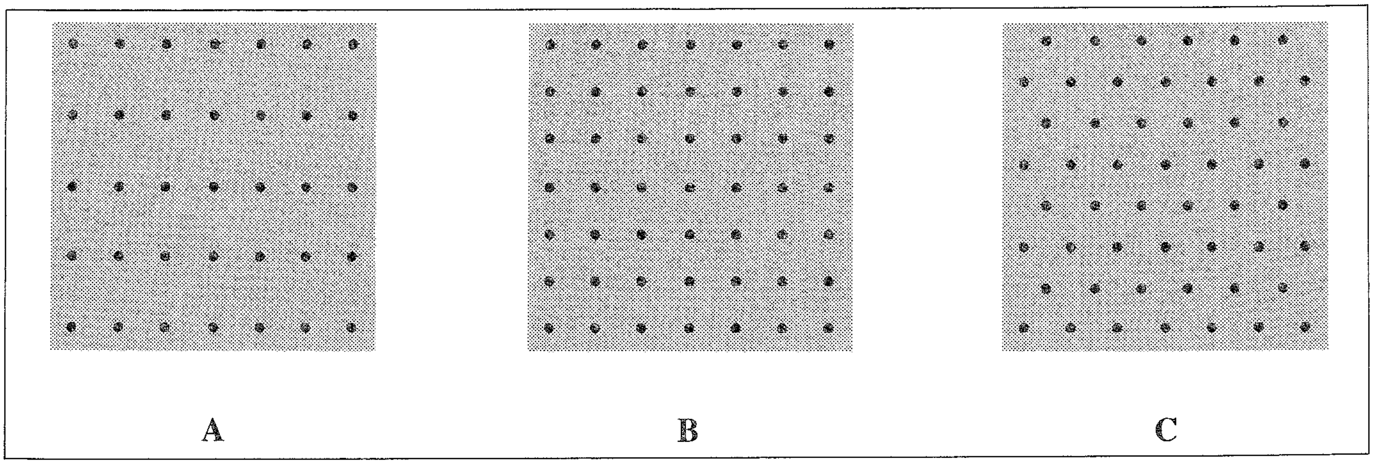

Fig. 1. Three ambiguous dot lattices: a rectangular dot lattice demonstrating grouping by proximity and low instability (a), a square dot lattice demonstrating bistable grouping by proximity (b), and a hexagonal dot lattice demonstrating tristable grouping by proximity (c).

exactly invariant under translation. So the lattices we deal with are only approximately invariant under translation.

Now suppose that *B* is *A*'s nearest neighbor, and *C* is *A*'s second nearest neighbor. A lattice is specified by its two shortest translations in the directions $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, that is, by a pair of translation vectors $\mathbf{a}$ and $\mathbf{b}$ (Fig. 3). The triangle ΔABC is called the lattice's basic or principal triangle. Bravais (1850/1949) showed that for all lattices, $60° \leq \angle BAC \leq 90°$; $45° \leq \angle CBA \leq 90°$; $0° \leq \angle ACB \leq 60°$ (see also Armstrong, 1988). As a consequence, the lattice's basic parallelogram, *ABDC,* whose two sides are the vectors $\mathbf{a}$ (*AB*) and $\mathbf{b}$ (*AC*), is limited by the following conditions: $\|\mathbf{a}\| \leq \|\mathbf{b}\| \leq \|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a} + \mathbf{b}\|$ $(AB \leq AC \leq BC \leq AD)$.[1]

The distance of any dot from its eight nearest neighbors is $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\|\mathbf{a} - \mathbf{b}\|$, and $\|\mathbf{a} + \mathbf{b}\|$. This is evident from Figure 4, in which we show dot *A*'s neighbors, and the four basic parallelograms that share this dot. The distances are: $AB = AB' = \|\mathbf{a}\|$, $AC = AC' = \|\mathbf{b}\|$, $AD = AD' = \|\mathbf{a} + \mathbf{b}\|$, and $AE = AE' = \|\mathbf{a} - \mathbf{b}\|$.

Bravais laid the foundation of crystallography when he classified the lattices into five classes (they are still called the Bravais lattices; e.g., Senechal, 1990): square (Fig. 1b), rectangular (Fig. 1a), centered rectangular, hexagonal (Fig. 1c), and oblique. These five classes are distinguished only by their symmetry groups. Consider once again the square lattice of Figure 1b. It is not only invariant under translation, it is also invariant under reflection about mirrors in four orientations (vertical, horizontal, and two diagonal) and under fourfold rotation (90°, 180°, 270°, and 360°). So translation, reflection, and fourfold rotation are elements of the symmetry group of square lattices. In contrast, the rectangular lattice of Figure 1a, although it is invariant under translation, is invariant under reflection about mirrors in two orientations only (vertical and horizontal) and under only twofold rotation (180° and 360°). Thus, translation, reflection, and twofold rotation are elements of the symmetry group of rectangular lattices. Hence, rectangular and square lattices are different classes of Bravais lattices.

1. The magnitude (or length or norm) of vector $\mathbf{x}$ is $\|\mathbf{x}\|$. The symbols + and − represent vector addition and subtraction.

The Bravais classification disregards the metric properties of lattices. For example, any lattice for which $\|\mathbf{a}\| \neq \|\mathbf{b}\|$ and $\gamma = 90°$ (Fig. 3) is a rectangular lattice. Recognizing that this disregard for the metric properties of lattices can only hamper our understanding of grouping by proximity, we must extend Bravais's classification of lattices (Kubovy, 1994). From the fact that a lattice's fundamental parallelogram (and hence the lattice itself) is specified by three parameters—$\|\mathbf{a}\|$, $\|\mathbf{b}\|$ and $\gamma = \angle(\mathbf{a},\mathbf{b})$—it follows that if $\|\mathbf{a}\|$ is held constant, any lattice can be located in a two-parameter space whose coordinates are $\|\mathbf{b}\|$ and

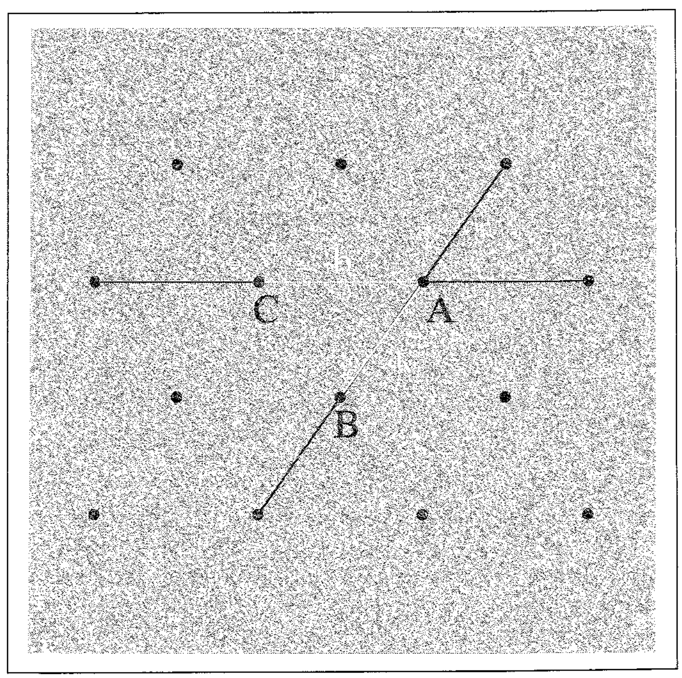

Fig. 2. Two independent translations in a dot lattice. Line $\overleftrightarrow{AB}$ passes through an infinite number of dots, a distance *a* apart. Line $\overleftrightarrow{AC}$ also passes through an infinite number of dots, a distance *b* apart.

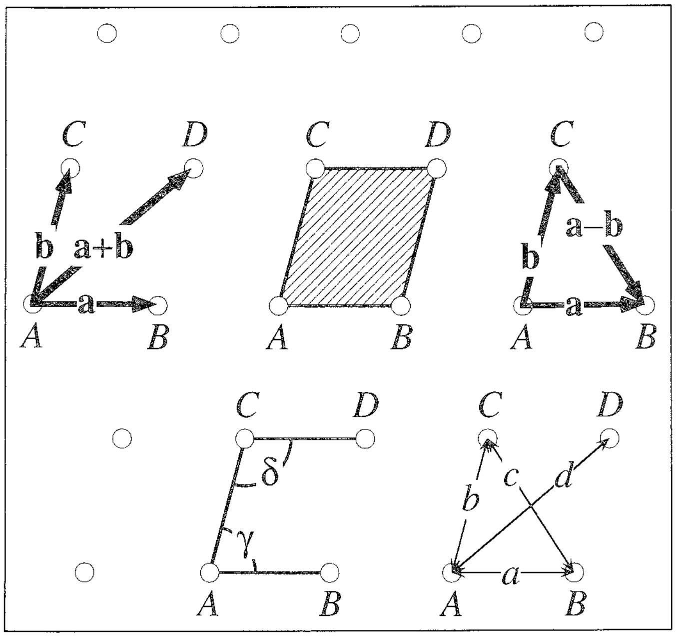

Fig. 3. Defining a lattice. In the lattice at the left of the top row, vectors **a** and **b** are the basis of (i.e., are the translation vectors that generate) the lattice. The triangle *ABC* is the lattice's basic (or principal) triangle. The parallelogram *ABDC* is the lattice's basic parallelogram. The long diagonal of the basic parallelogram is **a** + **b**. The other diagrams in the top row show the lattice's basic parallelogram (center) and the short diagonal of the basic parallelogram, **a** – **b**. The diagrams in the bottom row show definitions of the angles γ and δ (left) and the distances *a*, *b*, *c*, and *d* (right).

$\gamma = \angle(\mathbf{a},\mathbf{b})$ (Fig. 5). The boundaries of this space suggest that one of Bravais's classes—centered rectangular—should be divided into two: centered rectangular and rhombic. As a result, there are six different lattice types (Fig. 6).

Because the parameters $\|\mathbf{b}\|$ and γ suffice to describe any lattice, we can express $\|\mathbf{a} - \mathbf{b}\|$ and $\|\mathbf{a} + \mathbf{b}\|$ in their terms. We simplify our notation by letting $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{a} - \mathbf{b}\|$, and $d = \|\mathbf{a} + \mathbf{b}\|$. Without loss of generality, we set $a = 1$. Let us return to Figure 3. We can calculate c because we know the two other sides of the triangle (a and b) and the angle they enclose (γ). Using the second law of cosines for triangles—$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$—and substituting $a = 1$, we obtain (remembering that lengths must be positive): $c = \sqrt{1 + b^2 - 2b\cos(\gamma)}$. To calculate d, we note from Figure 3 that $\delta = \pi - \gamma$. Therefore, $\cos(\delta) = \cos(\pi - \gamma) = -\cos(\gamma)$, and $d = \sqrt{1 + b^2 + 2b\cos(\gamma)}$.

These results allow us to hold a and b constant while we vary c and d; a and c constant while we vary b and d; and a and d constant while we vary b and c.

A ONE-PARAMETER MODEL OF GROUPING BY PROXIMITY

Koffka (1935) had a prescient view of neural networks:

> Where the local processes are not *completely* insulated . . . what happens in one place will depend upon what happens in all the others. . . . there are innumerable cross connections which probably connect every nerve cell with every other. . . . But if it is so, then the events in this network . . . must [have] . . . a degree of interdependence varying inversely with the actual operative resistances. (p. 60)

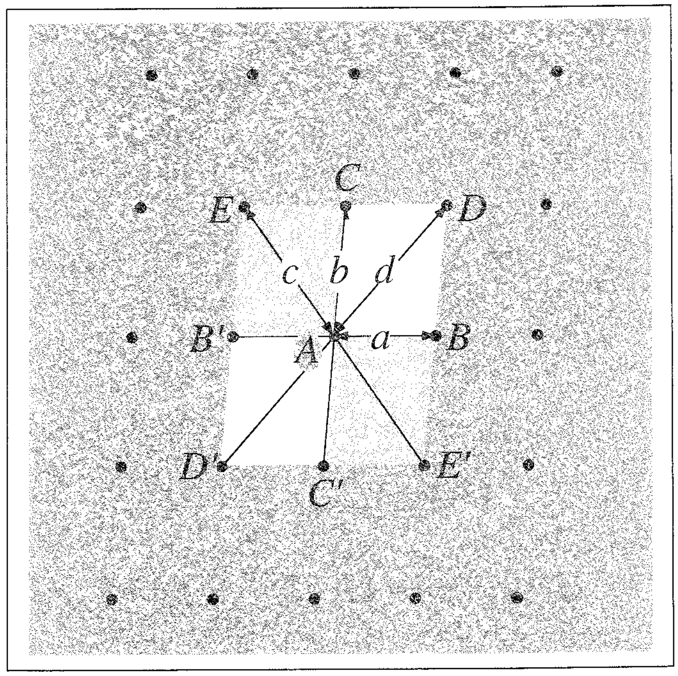

Fig. 4. The distance of dot *A* to its eight nearest neighbors. $AB = AB' = a$; $AC = AC' = b$; $AD = AD' = d$; $AE = AE' = c$. See Figure 3 for the definition of *a*, *b*, *c*, and *d*.

From this point of departure, Koffka developed his theory of grouping:

> We must think of . . . group formation as due to actual forces of attraction between the members of the group. (p. 165)

> Two parts in the field will attract each other according to their degree of proximity and equality. . . . the degree of proximity, or rather its opposite, distance, can easily be varied quantitatively. We need only remove two field parts sufficiently from each other, and the force of attraction will, at least for all practical purposes, vanish. (p. 166)

In this article, we use our space of dot lattices to develop and test Koffka's ideas. All other factors being equal, grouping by proximity implies that a dot lattice will be seen as a collection of parallel strips of dots along the direction of the shortest vector **a**. When b approaches or equals a, the perceived organization may change to strips parallel to **b**. Because each lattice has a different distribution of a, b, c, and d (Fig. 6), each will be multistable to a different degree. We now formalize this idea.

Let $\mathrm{V} = \{\mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}\}$ be the sides and the diagonals of a lattice's basic parallelogram, and let $V = \{a, b, c, d\}$ be the corresponding magnitudes of these vectors. Grouping by proximity implies that the probability of organizing the lattice in a direction $\mathbf{v} \in \mathrm{V}$, $p(\mathbf{v})$, is a decreasing function, $f(v)$, of the magnitude of the vector in that direction ($v \in V$). We call $f(v)$ the *attraction function* because it determines how the attraction between two dots diminishes as the distance between them

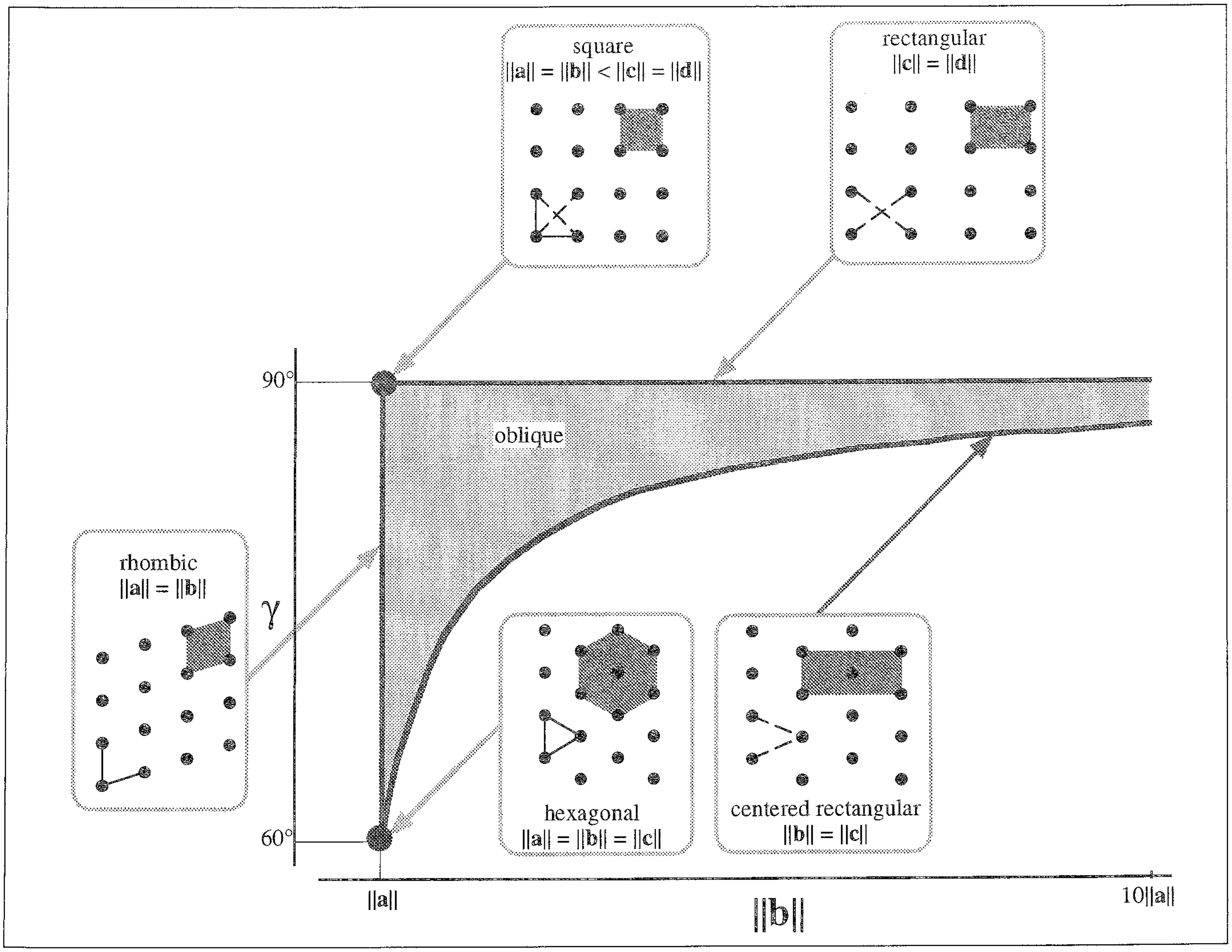

Fig. 5. The lattices organized in a two-dimensional space in which ||a|| is held constant and ||b|| and γ vary. ||b|| varies from ||a|| to 10 · ||a||; γ varies from 60° to 90°. From Kubovy (1994). Copyright 1994 by the Psychonomic Society, Inc. Reproduced by permission.

grows. In what follows, we choose the function $f(v) = e^{-\alpha((v/a)-1)}$ (which is similar to the function proposed by Shepard, 1987, as a universal law of generalization), where α (> 0) is the *attraction constant*. Figure 7 shows attraction functions for $\alpha = 5$ and $\alpha = 11$.

To obtain the distribution of the four probabilities $p(\mathbf{a})$, $p(\mathbf{b})$, $p(\mathbf{a} - \mathbf{b})$, $p(\mathbf{a} + \mathbf{b})$, we first assume that the probability of organizing the lattice in a direction $\mathbf{v} \notin \mathrm{V}$ is negligible. (A lattice can be organized in infinitely many directions. However, if we use the exponential attraction function we just described, with realistic values for α, the probabilities we calculate by neglecting all but the four primary directions will be overestimated by less than 10^{-11}.) We then let

$$p(\mathbf{v}) = \frac{f(v)}{f(a) + f(b) + f(c) + f(d)}, \quad (1)$$

where $\mathbf{v}$ takes on the values $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $\mathbf{a} + \mathbf{b}$ and v takes on the values a, b, c, and d. The denominator of Equation 1—which we henceforth denote by u—ensures that $\Sigma_{\mathbf{v} \in \mathrm{V}}\, p(\mathbf{v}) = 1$.

The role of the attraction constant may become clearer if we show what happens to the four probabilities as α decreases. Consider a hexagonal lattice for which the magnitudes of the four vectors are $(a, b, c, d) = (1, 1, 1, \sqrt{3})$. If $\alpha = 0.1$, the four probabilities are $(p(\mathbf{a}), p(\mathbf{b}), p(\mathbf{a} - \mathbf{b}), p(\mathbf{a} + \mathbf{b})) = (.254, .254, .254, .237)$. There is a small difference between the probability of grouping by proximity and the probability of grouping in a direction that does not minimize distance, $\mathbf{a} + \mathbf{b}$. If $\alpha = 4$, the probabilities are approximately (.327, .327, .327, .018). The difference has increased sharply in favor of grouping by proximity. Thus, α is directly related to the strength of the tendency to see grouping by proximity. At the limit, when α approaches ∞, $\lim_{\alpha \to \infty} f(\mathbf{v}) = 0$ for $v \neq a$, and the probabilities are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. In such a case, only grouping by proximity is possible.

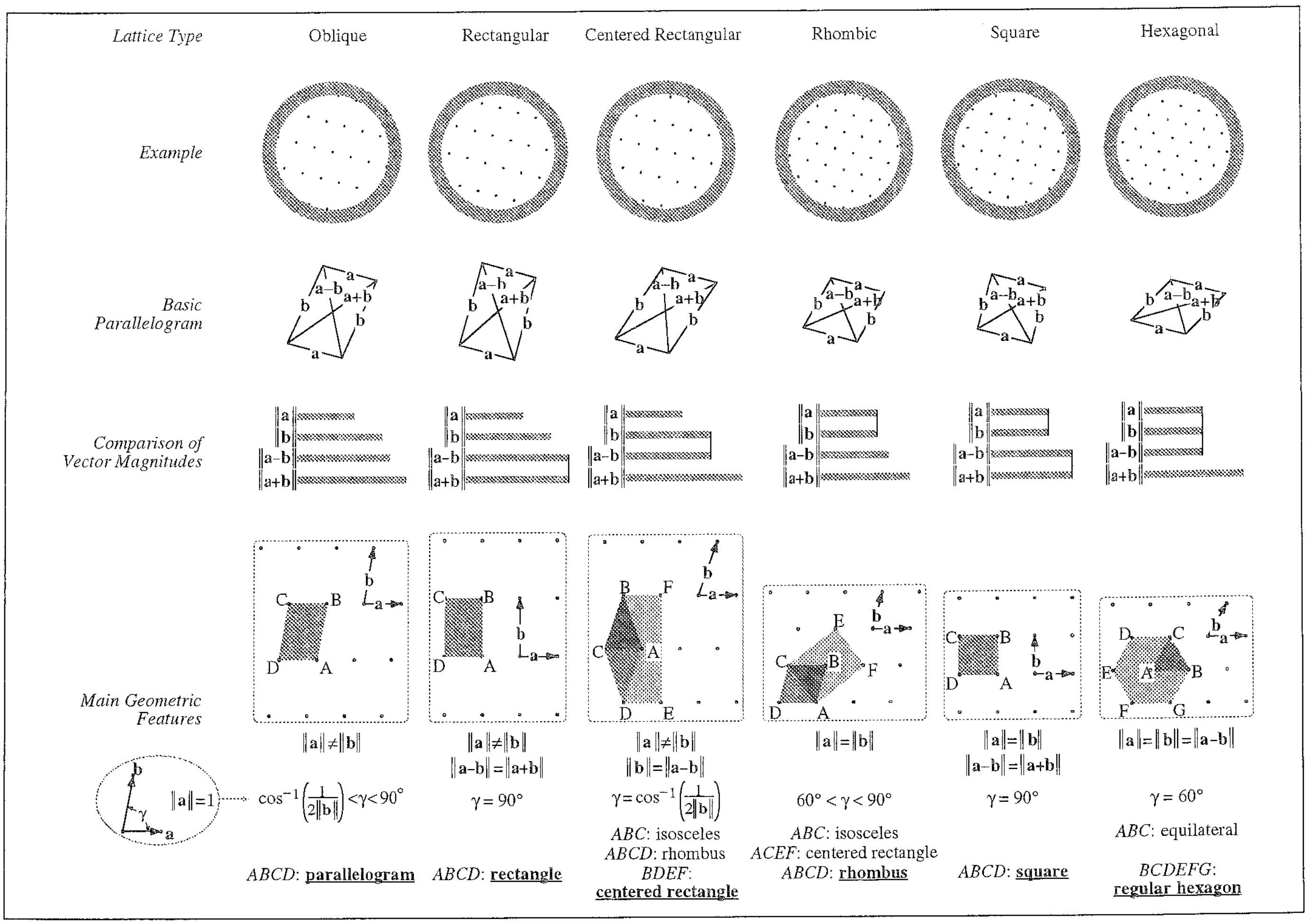

Fig. 6. The six types of lattices compared. The lower left-hand corner defines γ and expresses the assumption that $\|\mathbf{a}\| = 1$. From Kubovy (1994). Copyright 1994 by the Psychonomic Society, Inc. Reproduced by permission.

From these premises, we derive a measure of the instability of lattices. We use the Shannon (1948) and Wiener (1948) measure of entropy (or average uncertainty):

$$H = -\sum_{v \in V} p(\mathrm{v})\log_2[p(\mathrm{v})]. \qquad (2)$$

Substituting the attraction function, Equation 1, we obtain the predicted entropy:

$$\hat{H} = \frac{\sum_{v \in V} p(\mathrm{v})\log_2[p(\mathrm{v})]}{u \log_2 u}, \qquad (3)$$

where u is the denominator of the attraction function.

With this expression in hand, we can describe the instability of the different types of lattices. In Figure 8, we plot $\hat{H}$ as a function of the two parameters that specify the lattices, for $\alpha = 7.937$.

EXPERIMENT

Because we wanted to test a theory of grouping by proximity, we designed our stimuli either to minimize the effects of other principles of grouping or to control them. For example,

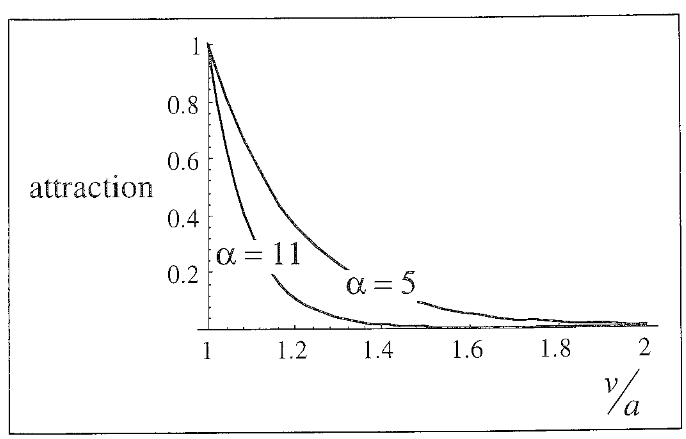

Fig. 7. Attraction functions for $\alpha = 5$ and $\alpha = 11$.

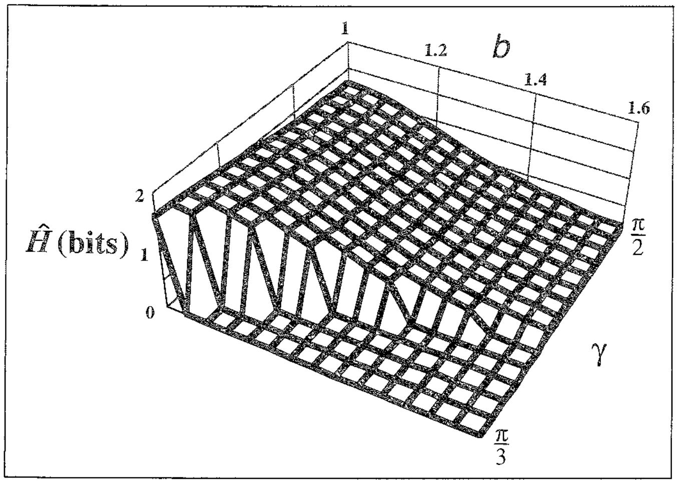

Fig. 8. Predicted entropy ($\hat{H}$) as a function of $\|\mathbf{b}\|$ and γ for the space shown in Figure 5, with the attraction constant, α, ≈ 8.

we made grouping by similarity impossible by using dot patterns in which all dots were identical. We minimized the effects of reference frames by presenting the dot lattices within a circular aperture in a darkened room. We controlled for orientation biases by presenting the lattices in random orientations. We repeatedly showed subjects brief exposures of lattices to obtain estimates of the probabilities of the lattices' four principal organizations (for other studies using this technique, see Price, 1969).

Method

Subjects

We recruited seven members of the University of Virginia community for this experiment: the second author, three graduate students, and three undergraduate students. The undergraduates did not know the purpose of the experiment and participated for research credit.

Apparatus and stimuli

We presented each lattice for 300 ms on a computer screen (1280 × 1024 spatially and 60 Hz temporally, controlled by an SGI Indigo). The screen was divided into two regions, a blue circle (radius of 505 pixels, 12.6° visual angle) in the center of the screen and a black region around it. The lattices, which consisted of large numbers of yellow dots (filled circles with a 5-pixel radius, separated by a minimal distance a, which was fixed at 60 pixels, 1.5° visual angle), were visible on the blue region of the screen only. We used this configuration to suggest that the lattices were viewed through a circular aperture beyond which they could continue infinitely in all directions. Figure 9a shows an example of a stimulus display.

After removing a lattice, we showed the subject a four-alternative response screen (Fig. 9b). Each alternative consisted of a circle and one of its diameters. The slope of the diameter corresponded to the slope of one of the four vectors of the lattice just presented.

We presented 16 different lattices that systematically sampled the space of lattices shown in Figure 5 in the following way (Fig. 10). With a fixed dot size and distance a, the hexagonal and square lattices are determined by the parameters b $(=a)$ and γ (60° and 90°, respectively). We selected two cases of rhombic lattices distributed evenly between the hexagonal and square with $\gamma = 70°$ and $\gamma = 80°$. To sample the second parameter, b, we selected three values different from a. Because multistability is more likely when b is close to a, we restricted b to the interval $[a, 2a]$. Because psychophysical comparisons are based on ratios (cf. Weber's law), we selected b_{i+1} such that $b_{i+1}/b_i = 1.26$, and $b_1 = a$. For the rectangular lattices, this gave us the following values of b to be multiplied by the length selected for a: $b_2 = 1.26$, $b_3 = 1.59$, and $b_4 = 2.00$. We computed the γ values for the centered rectangular lattices from these three values of b, because $\gamma = \cos^{-1}(1/2b)$ (see Kubovy, 1994): The values are shown in Figure 10. We derived the parameter values for the remaining six oblique lattices in a similar fashion.

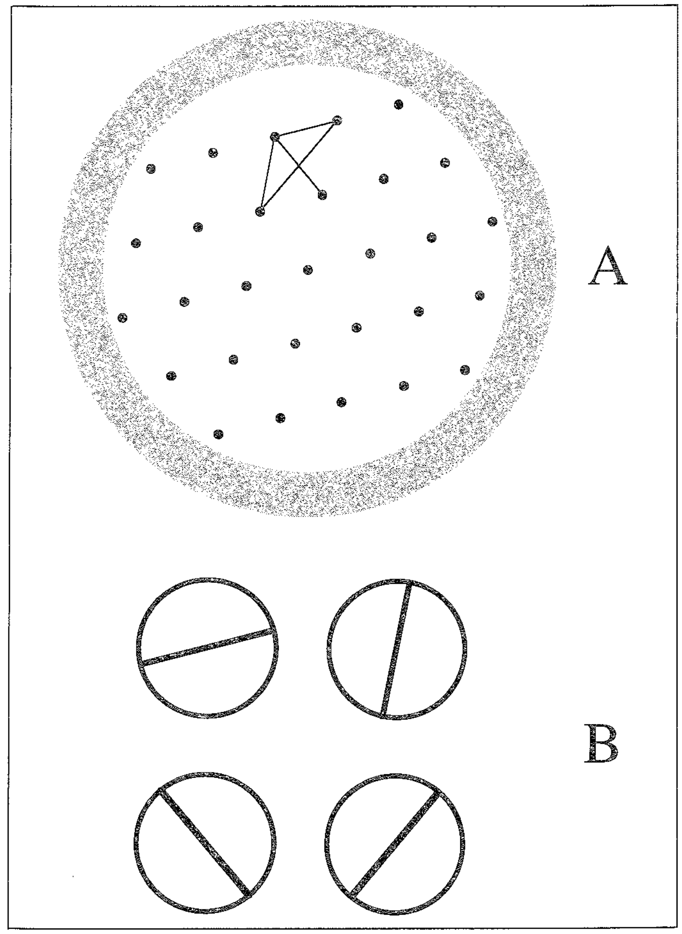

Fig. 9. The two parts of a typical trial: (a) stimulus display (with lines added to indicate the directions of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} - \mathbf{b}$, $\mathbf{a} + \mathbf{b}$) and (b) response screen.

Procedure

Each subject performed three blocks of 1,600 trials, consisting of 100 presentations of each of the 16 lattices defined in

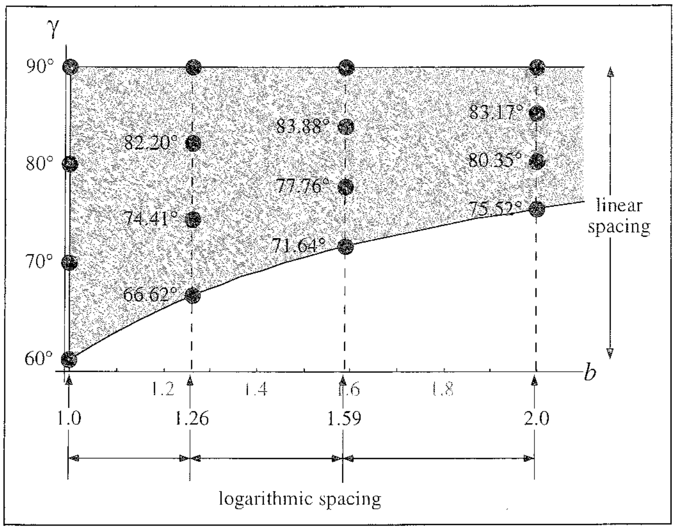

Fig. 10. The parameters of the 16 lattices sampled for the experiment.

Figure 10, each in an orientation selected randomly from a uniform distribution over the half-open interval [0°, 360°]. A block of 1,600 trials was run in a 1-hr session, with a short break after every 400 trials. The sessions were separated by at least 1 hr and often by a day. Each block consisted of 100 random permutations of the 16 different lattices.

Subjects were told that the lattices could be perceived as collections of strips of dots and that the same lattice could have alternative organizations. Subjects used a computer mouse to indicate the perceived organization of each lattice (i.e., the orientation of the strips) by selecting one of the four circles on the response screen. They received 16 practice trials.

Results

There are many different ways of analyzing the data we obtained in this experiment. In the present article, we focus on the entropy of the responses to the lattices. From the values of $p(\mathbf{a})$, $p(\mathbf{b})$, $p(\mathbf{a} - \mathbf{b})$, and $p(\mathbf{a} + \mathbf{b})$, we calculated the value of H for each lattice and each subject using Equation 2. We then regressed the 16 observed values of H on $\hat{H}$ for each subject. The value of $\hat{H}$—calculated from Equation 3—depends on the value of α, so we had to find α and the regression of H on $\hat{H}$ simultaneously. For each subject, we varied α until we found a value of α that maximized the coefficient of determination (R^2) for the regression of H on $\hat{H}$.

The values of α ranged from 6.72 to 11.57 (median: 7.65). The maximum adjusted coefficients of determination ranged from .962 to .987 (median: .983). The regressions are shown in Figure 11.

The model slightly, but systematically, underestimates the observed values of H. The intercepts of the regressions ranged from −0.025 to 0.209 (median: 0.076). The error resulting from the median intercept represents about 4% of the observed range of H. The slopes ranged from 1.00 to 1.21 (median: 1.09). The median slope would produce a deviation of 9% in the observed H from $\hat{H}$ at the highest value of $\hat{H}$ if there were no intercept.

DISCUSSION

The theory we have proposed has three components: (a) We postulate a negatively accelerated attraction function, (b) we estimate choice probabilities using Equation 1, and (c) we use entropy to quantify instability. We discuss each of these in turn.

The attraction function we have proposed is not unique. We have fit the data equally well with other similarly shaped attraction functions, such as power functions. Therefore, we cannot claim generality for the particular decaying exponential function we have chosen. Furthermore, we do not yet know under what conditions the attraction constant α is invariant.

The model proposed in Equation 1 implies Luce's (1959) choice axiom: namely, that a subject's chance of choosing direction $\mathbf{v}$ is the same regardless of whether the choice options were $\mathbf{V} = \{\mathbf{a}, \mathbf{b}, \mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}\}$ or a more restricted set. We can illustrate this point with the homely (but by now traditional) example of choices from a menu. Suppose we knew a person well enough to say that at a given restaurant she chooses chicken 20% of the time and fish 30% of the time. Now consider her choices on a day when only chicken and fish happen to be available. According to the choice axiom, the relative chances that she will choose chicken or fish will remain unchanged. Because she has only two choices, the chance that she will choose chicken is 40%, and the chance that she will choose fish is 60%. When some alternatives on the menu are made irrelevant because they are unavailable, the relative attraction of the remaining alternatives is unchanged. This is why the choice axiom is sometimes labeled *independence from irrelevant alternatives*.

The choice axiom can be violated. Consider the following example (Restle & Greeno, 1970, p. 213). Suppose that when given the choice between ice cream (i) and sausages (s), a young soldier tends to choose ice cream: $p(i|i, s) > p(s|i, s)$. Now suppose that on another occasion, sauerkraut (k) is also available. It is not unreasonable to suppose that in the presence of sauerkraut, the soldier may reverse his preference: $p(i|i, s, k) < p(s|i, s, k)$.

Similarly, the choice axiom may not hold in our data. Assume that a subject chooses direction $\mathbf{v}$ with a probability $p(\mathbf{v})$ when given a set of choices $\mathbf{V}$ to which $\mathbf{v}$ belongs. Suppose that on some trials, she sees the lattice organized in direction $\mathbf{v}$. If on such a trial we had given her a set of choices $\mathbf{V}'$ to which $\mathbf{v}$ did not belong, her ignorance about the choices in $\mathbf{V}'$ might lead her to choose one of them at random and violate the choice axiom. We cannot use the data collected in the present experiment to test the choice axiom directly.

We do have some evidence consistent with such a violation. We noted that our model underestimates the entropy of the lattices. It is possible that the impression of an organization in a lattice grows over time, and that this latency is variable. Suppose that on some trials, subjects do not perceive an orientation in a particular direction, and therefore choose among the four alternatives at random. Their choice probabilities would be diluted with random choices, and hence would be closer to being equal than if such trials had not occurred. Therefore, observed entropy would be greater than predicted entropy for all lattices, which would manifest itself as a positive intercept.

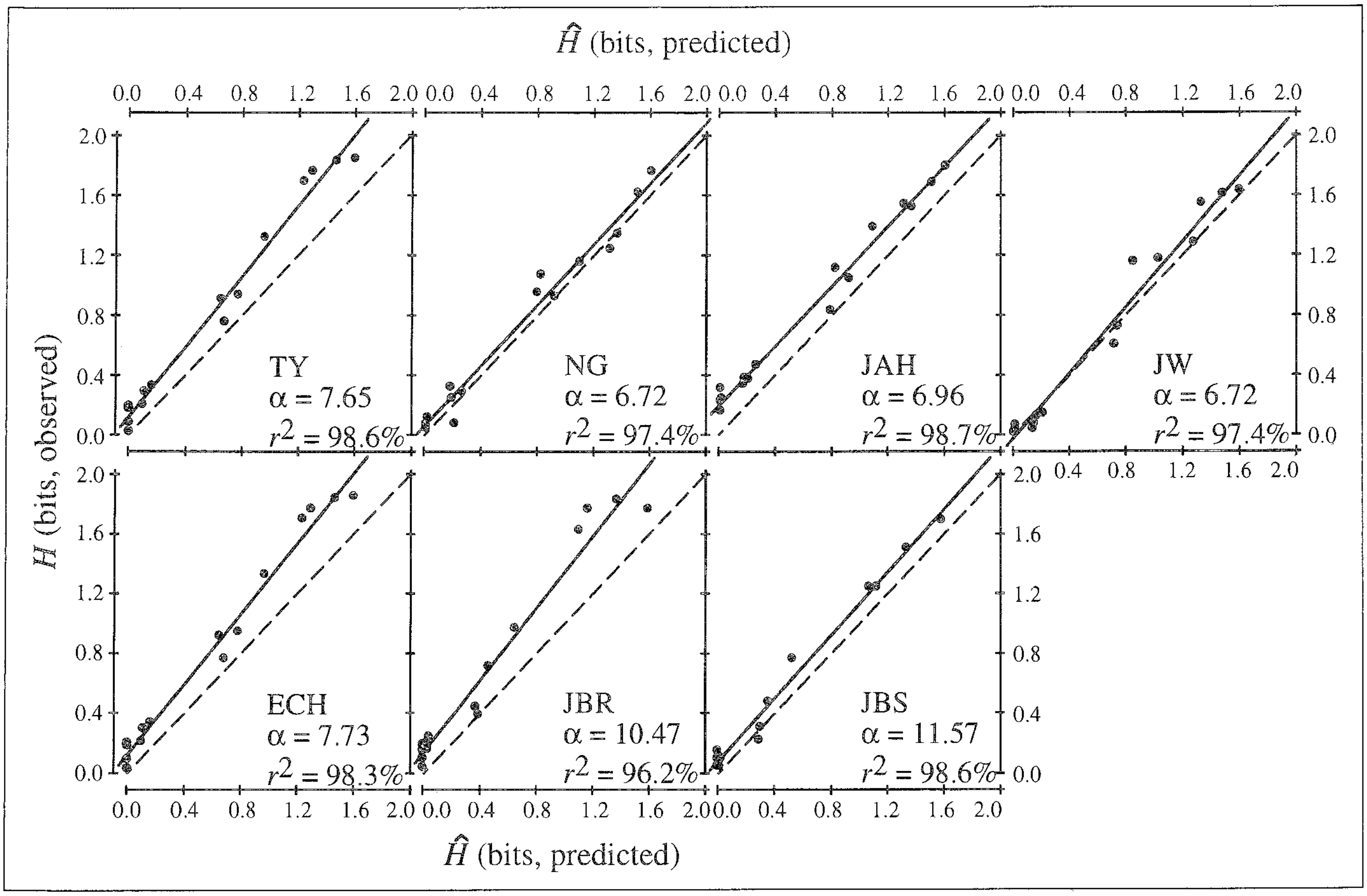

Fig. 11. Observed entropy (H) as a function of predicted entropy ($\hat{H}$) for the 7 participants (4,800 trials each). Each data point corresponds to a different dot lattice (300 trials). The least squares regression line is the solid line. The line of perfect prediction is the dashed line.

Entropy has not often been proposed as a measure of ambiguity (van Leeuwen & van de Hof, 1991, is an exception). Previous researchers have mostly recorded how long observers hold one organization of an ambiguous figure before switching to the other, or counted the number of reversals per unit time (Hochberg & Peterson, 1987; Lindauer & Baust, 1974; Peterson, Harvey, & Wiedenbacher, 1991; Peterson & Hochberg, 1983; Price, 1969). Some research programs may benefit from the use of entropy to quantify degrees of multistability. For instance, Palmer and his colleagues (Palmer, 1980, 1989; Palmer & Bucher, 1981, 1982) have studied the multistability of equilateral triangles. They have found that if a collection of equilateral triangles is drawn so that their bases are collinear, they are most likely to be seen all pointing in a direction perpendicular to the line on which their bases lie. If they are drawn so that their axes of symmetry are collinear, they are mostly likely to be seen pointing in a direction parallel to the line on which their axes lie. If neither alignment is present, the triangles still tend collectively to point in one direction, but the dominance of one direction over all others is sharply reduced. Thus, alignment reduces the entropy of a collection of equilateral triangles.

Our model has its limitations, however. It cannot describe the perceptual clustering of random dots in the plane. Other models have been designed to describe clustering of random dots (e.g., Compton & Logan, 1993; van Oeffelen & Vos, 1982, 1983), but such models can predict neither grouping in, nor the ambiguity of, dot lattices. Without describing these models in detail, we can explain why they must fail with dot lattices. Models of this type assert that a collection of dots will give rise to a surface, called the *strength gradient,* that has a peak at each dot and valleys between the dots. The left-hand column of Figure 12 shows two sets of dots: eight dots in two clusters and a nine-dot portion of a rectangular lattice. Figure 13 shows the strength gradients of these two dot patterns. Grouping or clustering is predicted by applying a threshold to the strength gradient, which amounts to decapitating its surface at a given altitude. As the three columns on the right of Figure 12 show, the lower the threshold, the larger the clusters. The model successfully detects the clusters in the eight-dot pattern. However, it cannot predict any grouping other than grouping by proximity. It therefore cannot deal with multistability. In the case of the hexagonal, the rhombic, or the square lattices, in which the distances between neighboring dots along **a** are equal to the distances between neighboring dots along **b**, the model cannot predict any grouping at all.

Our model (and all others) suffers from another limitation. It does not distinguish between small fragments of lattices (e.g., the nine points $A, B, \ldots, E, B', \ldots E'$ in Fig. 4) and extensive

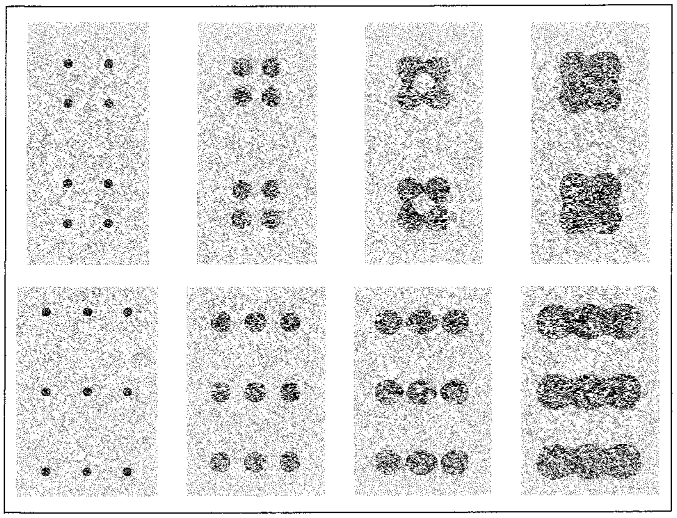

Fig. 12. Illustration of the outputs of a clustering algorithm discussed in the text. The left column shows two sets of dots used to test the algorithm, and the other columns represent the corresponding clusters the algorithm produces as the threshold for clustering becomes increasingly lax (from left to right).

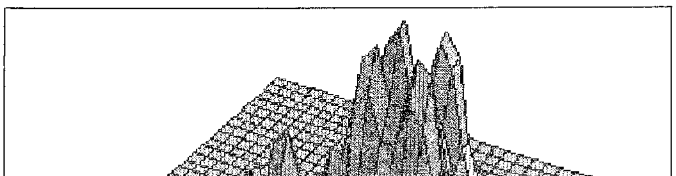

Fig. 13. The strength gradients of the two sets of dots shown in the left column of Figure 12.

lattices. At issue is the problem of cooperativity (Julesz, 1971). There are two aspects to this phenomenon. First, lattices do not undergo organization unless their size is considerable. People do not see small lattice fragments organized in strips. Second, lattices do not undergo piecemeal organization; this fact implies that the organization of local sets of dots in different parts of the lattice is linked.

To summarize, we have proposed a quantitative Gestalt model according to which dots in a lattice are attracted to each other as a decreasing exponential function of the distance between them, independently of the geometry of the lattice in which they are embedded. The model fits our data well. Thus, the ambiguity (or entropy) of lattices is well described by surfaces like the one shown in Figure 8.

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