

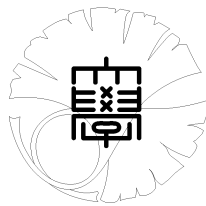
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**Groupoid-theoretical methods  
in the mapping class groups  
of surfaces**

by

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# Groupoid-theoretical methods in the mapping class groups of surfaces

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## Abstract

We provide some language for algebraic study of the mapping class groups for surfaces with non-connected boundary. As applications, we generalize our previous results on Dehn twists in [13] and [17] to any compact connected oriented surfaces with non-empty boundary. Moreover we embed the ‘smallest’ Torelli group in the sense of Putman [23] into a pro-nilpotent group coming from the Goldman Lie algebra. The graded quotients of the embedding equal the Johnson homomorphisms of all degrees in the case the boundary is connected.

## Introduction

In algebraic study of the mapping class group of a surface, its action on the fundamental group of the surface plays an essential role. Let  $\Sigma_{g,r}$  be a compact connected oriented surface of genus  $g$  with  $r$  boundary components,  $r \geq 1$ ,  $\mathcal{M}_{g,r}$  the mapping class group of  $\Sigma_{g,r}$ . Choose a basepoint  $*$   $\in \partial\Sigma_{g,r}$ . Then the action induces a group homomorphism

$$\text{DN}: \mathcal{M}_{g,r} \rightarrow \text{Aut}\pi_1(\Sigma_{g,r}, *).$$

The Dehn-Nielsen theorem says DN is injective if  $r = 1$ . If  $r \geq 2$ , it is *not* injective. In fact, the right handed Dehn twist along a boundary component without the basepoint  $*$  is a non-trivial element in the kernel of the homomorphism DN. To study the case  $r \geq 2$ , we have to consider the action of  $\mathcal{M}_{g,r}$  on a groupoid  $\mathcal{C}$ , more precisely, a full subcategory of the fundamental groupoid of the surface  $\Sigma_{g,r}$ ,  $\Pi\Sigma_{g,r}$ , whose object set has at least one point in each boundary component. Then, as will be shown in §3.1, the natural homomorphism  $\text{DN}: \mathcal{M}_{g,r} \rightarrow \text{Aut}\mathcal{C}$  is injective.

The purpose of this paper is to provide some language for studying the action of the mapping class group on such groupoids. In our previous paper [13], we found the Goldman Lie algebra of a surface acts on the group ring of the fundamental group of the surface as derivations. This action plays a central role in our language. As a consequence, we come to the notion of the completed Goldman Lie algebra for any oriented surface. From the results in [13], in the case for  $\Sigma_{g,1}$ , this Lie algebra includes Kontsevich’s ‘associative’ and ‘Lie’ as Lie subalgebras.

In [13] the authors gave some description of Dehn twists on the surface  $\Sigma_{g,1}$ , which led us to the definition of a generalized Dehn twist along a non-simple closed curve as an automorphism of the completed group ring of the fundamental group of  $\Sigma_{g,1}$ . In [17], the second-named author proved the Dehn twist along a figure eight is *not* realized by any element of  $\mathcal{M}_{g,1}$ . Our practical goal is to generalize all these results to the case  $r \geq 2$ .

In fact, Theorem 5.2.1 is a generalization of our description of Dehn twists ([13] Theorem 1.1.1) to any oriented surfaces, and Theorem 5.4.2 is a generalization of the non-realizability as a diffeomorphism of the generalized Dehn twist along a figure eight ([17] Theorem 5.1.1) to any oriented surfaces of finite type with non-empty boundary.

As was shown in [17] Theorem 3.3.2, the generalized Dehn twist along a closed curve  $C$  is localized inside a regular neighborhood of the curve  $C$ . In almost all cases, the regular neighborhood has a non-connected boundary. This also leads us to studying groupoids. Our groupoid-theoretical methods make all the arguments on (generalized) Dehn twists much shorter than those in [13] and [17]. In our forthcoming paper [14], we prove the generalized Dehn twists along non-simple closed curves in wider classes are not realized by any diffeomorphisms.

In our previous papers [13] [17], the notions as Magnus expansions or symplectic expansions played a crucial role. The main theorems and constructions of this paper are basically free from them, although we have used them in several arguments. Recently Massuyeau and Turaev [19] developed a theory on generalized Dehn twists without use of Magnus expansions. Comparing their approach and ours seems very interesting.

The higher Johnson homomorphism of the Torelli group is an important tool to study the algebraic structure of the mapping class group  $\mathcal{M}_{g,1}$ . If  $r \geq 2$ , theory of higher Johnson homomorphisms for  $\mathcal{M}_{g,r}$  has not been established, since the map DN is not injective. Moreover it has not been clarified what Lie algebra should be an appropriate target of the higher Johnson homomorphism in these cases. In §6, we discuss the Johnson homomorphisms via our groupoid-theoretical methods. The completed Goldman Lie algebra we introduce here gives a geometric interpretation of a completion of a Lie algebra introduced by Morita [20] [21] as an appropriate target of the higher Johnson homomorphism for  $r = 1$ , which is also Kontsevich’s ‘Lie’ [15]. By generalizing this construction and using Putman’s result on generators of the Torelli groups [23], we embed the ‘smallest’ Torelli group in the sense of Putman [23] into a pro-nilpotent group coming from the Goldman Lie algebra in the case  $g$  and  $r$  are positive. If  $r = 1$ , the graded quotients of the embedding equal the Johnson homomorphisms of all degrees. Moreover, our construction has a compatibility with respect to an inclusion of surfaces (Proposition 6.3.4). Recently Church [5] introduced the first Johnson homomorphism for all kinds of Putman’s partitioned Torelli groups. It would be very interesting to describe an explicit relation between Church’s homomorphisms and ours.

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## 1 Special additive categories (SAC's)

In this and the next sections we develop some algebraic machineries which do not need any surface topology. What we want to consider is a *groupoid version* of several constructions attached to a group, such as the group ring and derivations on the group ring (§1.1), the completed group ring with respect to the augmentation ideal and its powers (§1.2 and §2.1), the Hopf algebra structure on the group ring (§2.1), and the abelianization (§2.2).

Let  $\mathcal{R}$  be a small category. Throughout this paper we denote by  $\text{Ob}\mathcal{R}$  the set of objects in  $\mathcal{R}$  and write  $\mathcal{R}(p_0, p_1) = \text{Hom}_{\mathcal{R}}(p_0, p_1)$  for any objects  $p_0$  and  $p_1 \in \text{Ob}\mathcal{R}$ . The composition  $\circ$  in  $\mathcal{R}$  gives a multiplication  $\cdot$  in  $\mathcal{R}$  by

$$\cdot : \mathcal{R}(p_0, p_1) \times \mathcal{R}(p_1, p_2) \rightarrow \mathcal{R}(p_0, p_2), \quad \gamma_1 \gamma_2 = \gamma_1 \cdot \gamma_2 := \gamma_2 \circ \gamma_1.$$

### 1.1 Definition of special additive categories

Let  $K$  be a commutative ring with unit.

**Definition 1.1.1.** *A small additive category  $\mathcal{R}$  is called a  $K$ -special additive category ( $K$ -SAC) if it satisfies*

- (i) *For any  $p_0, p_1$  and  $p_2 \in \text{Ob}\mathcal{R}$ , the additive group  $\mathcal{R}(p_0, p_1)$  is a  $K$ -module, and the multiplication  $\cdot : \mathcal{R}(p_0, p_1) \times \mathcal{R}(p_1, p_2) \rightarrow \mathcal{R}(p_0, p_2)$  is  $K$ -bilinear.*

- (ii) For any  $p_0$  and  $p_1 \in \text{Ob}\mathcal{R}$  with  $\mathcal{R}(p_0, p_1) \neq 0$ , there exists an isomorphism in  $\mathcal{R}(p_0, p_1)$ .

We denote by  $\pi_0\mathcal{R}$  the set of isomorphism classes in  $\text{Ob}\mathcal{R}$ . For any  $q \in \text{Ob}\mathcal{R}$  the additive group  $\mathcal{R}(q, q)$  is an associative  $K$ -algebra with unit by the condition (i). Further,  $\mathcal{R}(p_0, p_1)$  is a left  $\mathcal{R}(p_0, p_0)$ - and right  $\mathcal{R}(p_1, p_1)$ - module.

A typical example of a  $K$ -SAC is the free  $K$ -module over a groupoid  $\mathcal{G}$ . For any  $p_0$  and  $p_1 \in \text{Ob}\mathcal{G}$ , we define  $(K\mathcal{G})(p_0, p_1)$  to be the free  $K$ -module over the set  $\mathcal{G}(p_0, p_1) = \text{Hom}_{\mathcal{G}}(p_0, p_1)$ . Clearly  $K\mathcal{G}$  is a  $K$ -SAC, and  $\pi_0 K\mathcal{G} = \pi_0\mathcal{G}$ .

A family of  $K$ -endomorphisms  $D = D^{(p_0, p_1)}: \mathcal{R}(p_0, p_1) \rightarrow \mathcal{R}(p_0, p_1)$ ,  $p_0, p_1 \in \text{Ob}\mathcal{R}$ , is called a *derivation* of  $\mathcal{R}$ , if it satisfies Leibniz' rule

$$D(uv) = (Du)v + u(Dv)$$

for any  $p_0, p_1, p_2 \in \text{Ob}\mathcal{R}$ ,  $u \in \mathcal{R}(p_0, p_1)$  and  $v \in \mathcal{R}(p_1, p_2)$ . Then, for any  $q \in \text{Ob}\mathcal{R}$ ,  $D = D^{(q, q)}$  is a derivation of the associative  $K$ -algebra  $\mathcal{R}(q, q)$  in a usual sense. It should be remarked a derivation is not a covariant functor. We denote by  $\text{Der}\mathcal{R}$  the  $K$ -Lie algebra consisting of all derivations of  $\mathcal{R}$ .

Let  $\mathcal{R}$  and  $\mathcal{R}'$  be  $K$ -SAC's with the same object set. We denote  $(\mathcal{R} \otimes \mathcal{R}')(p_0, p_1) := \mathcal{R}(p_0, p_1) \otimes_K \mathcal{R}'(p_0, p_1)$  for  $p_0, p_1 \in \text{Ob}\mathcal{R} = \text{Ob}\mathcal{R}'$ . The tensor product of multiplications in  $\mathcal{R}$  and  $\mathcal{R}'$  makes  $\mathcal{R} \otimes \mathcal{R}'$  a  $K$ -SAC whose object set is  $\text{Ob}\mathcal{R} = \text{Ob}\mathcal{R}'$ , which we call the *tensor product* of  $K$ -SAC's  $\mathcal{R}$  and  $\mathcal{R}'$ .

We call a covariant functor  $\mathcal{F}: \mathcal{R} \rightarrow \mathcal{R}'$  a *homomorphism* of  $K$ -SAC's, if  $\mathcal{F}(p) = p$  for any  $p \in \text{Ob}\mathcal{R} = \text{Ob}\mathcal{R}'$ , and  $\mathcal{F}: \mathcal{R}(p_0, p_1) \rightarrow \mathcal{R}'(p_0, p_1)$  is  $K$ -linear for any  $p_0$  and  $p_1 \in \text{Ob}\mathcal{R}$ . We denote by  $\text{Hom}(\mathcal{R}, \mathcal{R}')$  the  $K$ -vector space consisting of all homomorphisms of  $K$ -SAC's from  $\mathcal{R}$  to  $\mathcal{R}'$ .

A homomorphism  $U \in \text{Hom}(\mathcal{R}, \mathcal{R})$  is called an *automorphism* of  $\mathcal{R}$ , if  $U: \mathcal{R}(p_0, p_1) \rightarrow \mathcal{R}(p_0, p_1)$  is a  $K$ -linear isomorphism for any  $p_0$  and  $p_1 \in \text{Ob}\mathcal{R}$ . We denote by  $\text{Aut}\mathcal{R}$  the group consisting of all automorphisms of  $\mathcal{R}$ .

## 1.2 Filtered SAC's

**Definition 1.2.1.** A  $K$ -SAC  $\mathcal{R}$  is called *filtered*, if each  $\mathcal{R}(p_0, p_1)$ ,  $p_0, p_1 \in \text{Ob}\mathcal{R}$ , has a sequence of  $K$ -submodules  $\{F_n\mathcal{R}(p_0, p_1)\}_{n \geq 0}$  such that

- (i)  $F_0\mathcal{R}(p_0, p_1) = \mathcal{R}(p_0, p_1)$  and  $F_n\mathcal{R}(p_0, p_1) \supset F_{n+1}\mathcal{R}(p_0, p_1)$  for any  $n \geq 0$ .  
(ii)  $F_{n_1}\mathcal{R}(p_0, p_1) \cdot F_{n_2}\mathcal{R}(p_1, p_2) \subset F_{n_1+n_2}\mathcal{R}(p_0, p_2)$  for any  $p_0, p_1, p_2 \in \text{Ob}\mathcal{R}$  and  $n_1, n_2 \geq 0$ .

Suppose  $p_0, p_1, q_0$  and  $q_1 \in \text{Ob}\mathcal{R}$  satisfy  $[p_0] = [q_0]$  and  $[p_1] = [q_1] \in \pi_0\mathcal{R}$ . Let  $\gamma \in \mathcal{R}(p_0, q_0)$  and  $\delta \in \mathcal{R}(q_1, p_1)$  be isomorphisms. Then, from the condition (ii) with  $n_1$  or  $n_2 = 0$ , we have

$$F_n\mathcal{R}(p_0, p_1) = \gamma(F_n\mathcal{R}(q_0, q_1))\delta \quad (1.2.1)$$

for any  $n \geq 0$ . If  $n < 0$ , we define  $F_n\mathcal{R}(p_0, p_1) = \mathcal{R}(p_0, p_1)$ .

If we define

$$\text{gr}\mathcal{R}(p_0, p_1) := \bigoplus_{n=0}^{\infty} \text{gr}_n\mathcal{R}(p_0, p_1), \quad \text{gr}_n\mathcal{R}(p_0, p_1) = F_n\mathcal{R}(p_0, p_1)/F_{n+1}\mathcal{R}(p_0, p_1)$$

for  $p_0$  and  $p_1 \in \text{Ob}\mathcal{R}$ , then  $\text{gr}\mathcal{R}$  is a  $K$ -SAC with  $\text{Obgr}\mathcal{R} = \text{Ob}\mathcal{R}$ .

Now, in view of [24] p.265, we consider the following four conditions about a filtered  $K$ -SAC  $\mathcal{R}$ .

(C1)  $\mathcal{R}(q, q)/F_1\mathcal{R}(q, q) \cong K$  for any  $q \in \text{Ob}\mathcal{R}$ .

(C2) The algebra  $\text{gr}\mathcal{R}(q, q)$  is generated by  $\text{gr}_1\mathcal{R}(q, q)$  for any  $q \in \text{Ob}\mathcal{R}$ . In other words, the sum of the multiplication and the inclusion  $F_1\mathcal{R}(q, q)^{\otimes n} \oplus F_{n+1}\mathcal{R}(q, q) \rightarrow F_n\mathcal{R}(q, q)$  is surjective for any  $q \in \text{Ob}\mathcal{R}$  and  $n \geq 1$ .

(C3)  $\mathcal{R}(q, q) \cong \varprojlim_{n \rightarrow \infty} \mathcal{R}(q, q)/F_n\mathcal{R}(q, q)$  for any  $q \in \text{Ob}\mathcal{R}$ .

(C4) The multiplication  $F_1\mathcal{R}(q, q)^{\otimes n} \rightarrow F_n\mathcal{R}(q, q)$  is surjective for any  $q \in \text{Ob}\mathcal{R}$  and  $n \geq 1$ .

**Lemma 1.2.2.** *These conditions are equivalent to the followings, respectively.*

(C'1)  $\mathcal{R}(p_0, p_1)/F_1\mathcal{R}(p_0, p_1) \cong K$  for any  $p_0, p_1 \in \text{Ob}\mathcal{R}$  with  $[p_0] = [p_1] \in \pi_0\mathcal{R}$ .

(C'2) *If  $[p_0] = [p_1] \in \pi_0\mathcal{R}$ , then the sum of the multiplication and the inclusion  $\bigotimes_{i=1}^n F_1\mathcal{R}(q_{i-1}, q_i) \oplus F_{n+1}\mathcal{R}(p_0, p_1) \rightarrow F_n\mathcal{R}(p_0, p_1)$  is surjective for any  $q_1, \dots, q_{n-1} \in [p_0] \subset \text{Ob}\mathcal{R}$  and  $n \geq 1$  with  $q_0 = p_0$  and  $q_n = p_1$ .*

(C'3)  $\mathcal{R}(p_0, p_1) \cong \varprojlim_{n \rightarrow \infty} \mathcal{R}(p_0, p_1)/F_n\mathcal{R}(p_0, p_1)$  for any  $p_0, p_1 \in \text{Ob}\mathcal{R}$ .

(C'4) *If  $[p_0] = [p_1] \in \pi_0\mathcal{R}$ , then the multiplication  $\bigotimes_{i=1}^n F_1\mathcal{R}(q_{i-1}, q_i) \rightarrow F_n\mathcal{R}(p_0, p_1)$  is surjective for any  $q_1, \dots, q_{n-1} \in [p_0] \subset \text{Ob}\mathcal{R}$  and  $n \geq 1$  with  $q_0 = p_0$  and  $q_n = p_1$ .*

*Proof.* (C1)  $\Leftrightarrow$  (C'1) and (C3)  $\Leftrightarrow$  (C'3) follow immediately from (1.2.1). (C'2)  $\Rightarrow$  (C2) and (C'4)  $\Rightarrow$  (C4) are clear.

Choose an object  $q \in [p_0] \subset \text{Ob}\mathcal{R}$ , and isomorphisms  $\gamma_i \in \mathcal{R}(q, q_i)$ ,  $0 \leq i \leq n$ . By (1.2.1), we have  $F_n\mathcal{R}(p_0, p_1) = \gamma_0^{-1}F_n\mathcal{R}(q, q)\gamma_n$ . On the other hand, we have

$$\begin{aligned} & \gamma_0^{-1} \overbrace{F_1\mathcal{R}(q, q) \cdots \cdots F_1\mathcal{R}(q, q)}^n \gamma_n \\ &= \gamma_0^{-1}F_1\mathcal{R}(q, q)\gamma_1 \cdot \gamma_1^{-1}F_1\mathcal{R}(q, q)\gamma_2 \cdots \cdots \gamma_{n-1}^{-1}F_1\mathcal{R}(q, q)\gamma_n \\ &= F_1\mathcal{R}(q_0, q_1) \cdot F_1\mathcal{R}(q_1, q_2) \cdots \cdots F_1\mathcal{R}(q_{n-1}, q_n). \end{aligned}$$

Hence we obtain (C2)  $\Rightarrow$  (C'2) and (C4)  $\Rightarrow$  (C'4).  $\square$

Let  $\mathcal{R}$  be a filtered  $K$ -SAC. We define

$$\begin{aligned} \widehat{\mathcal{R}}(p_0, p_1) &:= \varprojlim_{m \rightarrow \infty} \mathcal{R}(p_0, p_1)/F_m\mathcal{R}(p_0, p_1), \quad \text{and} \\ F_n\widehat{\mathcal{R}}(p_0, p_1) &:= \varprojlim_{m \rightarrow \infty} F_n\mathcal{R}(p_0, p_1)/F_m\mathcal{R}(p_0, p_1) \subset \widehat{\mathcal{R}}(p_0, p_1) \end{aligned}$$

for  $p_0, p_1 \in \text{Ob}\mathcal{R}$  and  $n \geq 0$ . Then  $\widehat{\mathcal{R}}$  is a filtered  $K$ -SAC satisfying the condition (C3). We call it the *completion* of  $\mathcal{R}$ . There is a natural homomorphism  $\mathcal{R} \rightarrow \widehat{\mathcal{R}}$  of  $K$ -SAC's.

Let  $\mathcal{R}$  and  $\mathcal{R}'$  be filtered  $K$ -SAC's with the same object set. We define

$$(\mathcal{R} \widehat{\otimes} \mathcal{R}') (p_0, p_1) := \varprojlim_{n \rightarrow \infty} (\mathcal{R} \otimes \mathcal{R}') (p_0, p_1) / \sum_{n_1+n_2=n} F_{n_1}\mathcal{R}(p_0, p_1) \otimes F_{n_2}\mathcal{R}'(p_0, p_1)$$

for  $p_0, p_1 \in \text{Ob}\mathcal{R} = \text{Ob}\mathcal{R}'$ . Then  $\mathcal{R} \widehat{\otimes} \mathcal{R}'$  is a filtered  $K$ -SAC whose object set is  $\text{Ob}\mathcal{R} = \text{Ob}\mathcal{R}'$  in an obvious way, which we call the *completed tensor product* of filtered  $K$ -SAC's  $\mathcal{R}$  and  $\mathcal{R}'$ . If  $a \otimes b \in (\mathcal{R} \otimes \mathcal{R}') (p_0, p_1)$ , then its image under the homomorphism  $\mathcal{R} \otimes \mathcal{R}' \rightarrow \mathcal{R} \widehat{\otimes} \mathcal{R}'$  is denoted by  $a \widehat{\otimes} b$ .

### 1.3 Derivations of a filtered SAC

Let  $\mathcal{R}$  be a filtered  $K$ -SAC. We define

$$F_n \text{Der} \mathcal{R} := \{D \in \text{Der} \mathcal{R}; \forall p_0, \forall p_1 \in \text{Ob} \mathcal{R}, \forall l \geq 0, D(F_l \mathcal{R}(p_0, p_1)) \subset F_{l+n} \mathcal{R}(p_0, p_1)\}$$

for any  $n \in \mathbb{Z}$ . It is clear that

$$[F_{n_1} \text{Der} \mathcal{R}, F_{n_2} \text{Der} \mathcal{R}] \subset F_{n_1+n_2} \text{Der} \mathcal{R} \quad (1.3.1)$$

for any  $n_1$  and  $n_2 \in \mathbb{Z}$ .

**Lemma 1.3.1.** *Let  $\mathcal{R}$  be a filtered  $K$ -SAC with the condition (C4) and  $n \in \mathbb{Z}$ . If a derivation  $D \in \text{Der} \mathcal{R}$  satisfies*

$$D(\mathcal{R}(p_0, p_1)) \subset F_n \mathcal{R}(p_0, p_1)$$

for any  $p_0$  and  $p_1 \in \text{Ob} \mathcal{R}$ , then we have  $D \in F_{n-1} \text{Der} \mathcal{R}$ . In particular, we obtain

$$\text{Der} \mathcal{R} = F_{-1} \text{Der} \mathcal{R}.$$

*Proof.* Let  $p_0$  and  $p_1$  be objects in  $\mathcal{R}$ . We may assume  $[p_0] = [p_1] \in \pi_0 \mathcal{R}$ , since  $\mathcal{R}(p_0, p_1) = 0$  if  $[p_0] \neq [p_1]$ . Let  $l$  be a positive integer. Choose  $q_1, \dots, q_{l-1} \in [p_0] = [p_1] \subset \text{Ob} \mathcal{R}$ , and denote  $q_0 = p_0$  and  $q_l = p_1$ . From the condition (C4), the multiplication  $\bigotimes_{i=1}^l F_1 \mathcal{R}(q_{i-1}, q_i) \rightarrow F_l \mathcal{R}(p_0, p_1)$  is surjective. Hence it suffices to show  $D(u_1 u_2 \dots u_l) \in F_{n+l-1} \mathcal{R}(p_0, p_1)$  for any  $u_i \in F_1 \mathcal{R}(q_{i-1}, q_i)$ . Now we have

$$D(u_1 u_2 \dots u_l) = \sum_{i=1}^l u_1 \dots u_{i-1} (D u_i) u_{i+1} \dots u_l,$$

and  $u_1 \dots u_{i-1} (D u_i) u_{i+1} \dots u_l \in F_{n+l-1} \mathcal{R}(p_0, p_1)$  from the assumption. This proves the lemma.  $\square$

Now we study an analytic function of a derivation  $D \in \text{Der} \mathcal{R}$ .

**Lemma 1.3.2.** *Let  $\mathcal{R}$  be a filtered  $K$ -SAC with the conditions (C2) and (C3), and  $f(t) = \sum_{k=0}^{\infty} a_k t^k \in K[[t]]$  a formal power series. Suppose a derivation  $D \in \text{Der} \mathcal{R}$  satisfies the following three conditions.*

- (i)  $D \in F_0 \text{Der} \mathcal{R}$ .
- (ii) For any  $p_0$  and  $p_1 \in \text{Ob} \mathcal{R}$ , there exists a positive integer  $\nu$  such that  $D^\nu = 0$  on  $\text{gr}_1 \mathcal{R}(p_0, p_1)$ .
- (iii) For any  $p_0$  and  $p_1 \in \text{Ob} \mathcal{R}$ ,  $D(\mathcal{R}(p_0, p_1)) \subset F_1 \mathcal{R}(p_0, p_1)$ .

Then the series  $f(D) = \sum_{k=0}^{\infty} a_k D^k \in \text{End}(\mathcal{R}(p_0, p_1))$  converges for any  $p_0$  and  $p_1 \in \text{Ob} \mathcal{R}$ .

*Proof.* We have

$$D^{n(\nu-1)+1}(F_n\mathcal{R}(p_0, p_1)) \subset F_{n+1}\mathcal{R}(p_0, p_1) \quad (1.3.2)$$

for any  $n \geq 1$ . In fact, we may assume  $[p_0] = [p_1] \in \pi_0\mathcal{R}$ , since it is trivial in the case  $[p_0] \neq [p_1]$ . Then choose  $q_1, \dots, q_{n-1} \in [p_0] = [p_1] \subset \text{Ob}\mathcal{R}$ , and denote  $q_0 = p_0$  and  $q_n = p_1$ . From the condition (C2), the multiplication induces a surjection  $\bigotimes_{i=1}^n F_1\mathcal{R}(q_{i-1}, q_i) \oplus F_{n+1}\mathcal{R}(p_0, p_1) \rightarrow F_n\mathcal{R}(p_0, p_1)$ . By the condition (ii) we have  $D^\nu u_i \in F_2\mathcal{R}(q_{i-1}, q_i)$  for any  $u_i \in F_1\mathcal{R}(q_{i-1}, q_i)$ . Hence  $D^{n(\nu-1)+1}(u_1 u_2 \cdots u_n) \in F_{n+1}\mathcal{R}(p_0, p_1)$ , while  $DF_{n+1}\mathcal{R}(p_0, p_1) \subset F_{n+1}\mathcal{R}(p_0, p_1)$  from the condition (i). This proves (1.3.2).

For any  $u \in \mathcal{R}(p_0, p_1)$ , we have  $Du \in F_1\mathcal{R}(p_0, p_1)$  by the condition (iii). Hence, by (1.3.2), we have  $D^m u \in F_n\mathcal{R}(p_0, p_1)$  if  $m \geq 1 + \sum_{k=1}^{n-1} k(\nu-1) + 1 = n + \frac{1}{2}n(n-1)(\nu-1)$ . By the condition (C3) the series  $f(D)$  converges as an element of  $\text{End}(\mathcal{R}(p_0, p_1))$ . This completes the proof.  $\square$

If  $K$  includes the rationals  $\mathbb{Q}$ , then we may consider  $\exp(t)$  and  $\frac{1}{t}(\exp(t) - 1) \in K[[t]]$ .

**Proposition 1.3.3.** *Suppose  $K$  includes  $\mathbb{Q}$  and let  $\mathcal{R}$  be a filtered  $K$ -SAC with the conditions (C2) and (C3), and  $D$  and  $D'$  derivations of  $\mathcal{R}$  satisfying the three conditions (i)-(iii) in Lemma 1.3.2. Then*

- (1) *If  $[D, D'] = 0$ , then the sum  $D+D'$  satisfies the three conditions (i)-(iii), and  $\exp(D+D') = (\exp D)(\exp D')$ .*
- (2)  *$\exp D \in \text{Aut}\mathcal{R}$ .*
- (3) *If  $\exp(D) = \exp(D') \in \text{Aut}\mathcal{R}$ , then we have  $D = D' \in \text{Der}\mathcal{R}$ .*

*Proof.* (1) The conditions (i) and (iii) are clear. By  $[D, D'] = 0$ , we have

$$(D + D')^m = \sum_{k=0}^m \binom{m}{k} D^k D'^{m-k}. \quad (1.3.3)$$

Hence, if  $D^\nu = D'^{\nu'} = 0$  on  $\text{gr}_1\mathcal{R}(p_0, p_1)$ , then  $(D+D')^{\nu+\nu'} = 0$  on  $\text{gr}_1\mathcal{R}(p_0, p_1)$ . This implies  $D+D'$  satisfies the condition (ii). By (1.3.3) we compute  $\exp(D+D') = (\exp D)(\exp D')$ .

(2) It is clear that  $(\exp D)(1_q) = 1_q$  for any  $q \in \text{Ob}\mathcal{R}$ . Leibniz' formula implies  $D^k(uv) = \sum_{j=0}^k \binom{k}{j} (D^j u)(D^{k-j} v)$  for any  $p_0, p_1, p_2 \in \text{Ob}\mathcal{R}$ ,  $u \in \mathcal{R}(p_0, p_1)$  and  $v \in \mathcal{R}(p_1, p_2)$ . Hence  $(\exp D)(uv) = (\exp D)(u)(\exp D)(v)$ , which means  $\exp D$  is a covariant functor from  $\mathcal{R}$  to  $\mathcal{R}$  itself. By (1),  $\exp(-D)$  is the inverse of  $\exp D$ . Hence we obtain  $\exp D \in \text{Aut}\mathcal{R}$ .

(3) We denote  $f(t) := \frac{1}{t}(\exp t - 1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} t^k \in K[[t]]$ . We have  $(\exp(D) - 1)^n = D^n f(D)^n$ . Hence, as was proved in the proof of Lemma 1.3.2, we have  $(\exp(D) - 1)^m u \in F_n\mathcal{R}(p_0, p_1)$  for any  $p_0, p_1 \in \text{Ob}\mathcal{R}$  and  $u \in \mathcal{R}(p_0, p_1)$ , if  $m \geq n + \frac{1}{2}n(n-1)(\nu-1)$ . This implies  $\log(\exp D) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\exp(D) - 1)^n$  converges as an element of  $\text{End}\mathcal{R}(p_0, p_1)$ . On the other hand,  $\log(\exp D) \equiv D \pmod{D^m}$  for any  $m \geq 1$ . Hence, from the condition (C3), we have  $\log(\exp D) = D$ . In particular, if  $\exp D = \exp D'$ , then we have  $D = D'$ .  $\square$



## 2 Groupoids

Let  $K$  be a commutative ring with unit, and  $\mathcal{G}$  a groupoid. As was stated in §1.1,  $K\mathcal{G}$ , the free  $K$ -module over  $\mathcal{G}$ , is a  $K$ -SAC with  $\text{Ob}K\mathcal{G} = \text{Ob}\mathcal{G}$ . A homomorphism of  $K$ -SAC's  $\Delta: K\mathcal{G} \rightarrow K\mathcal{G} \otimes K\mathcal{G}$  is defined by  $\Delta\gamma := \gamma \otimes \gamma$  for any  $\gamma \in \mathcal{G}(p_0, p_1)$ ,  $p_0, p_1 \in \text{Ob}\mathcal{G}$ , which we call the *coproduct* of  $K\mathcal{G}$ . If  $\text{Ob}\mathcal{G}$  is a singleton, namely,  $\mathcal{G}$  is a group, then  $\Delta$  defines the standard Hopf algebra structure on the group ring  $K\mathcal{G}$ .

### 2.1 Filtration on $K\mathcal{G}$

We have the augmentation  $\varepsilon: K\mathcal{G}(q, q) \rightarrow K$  and the augmentation ideal  $I\mathcal{G}(q, q) := \text{Ker}\varepsilon \subset K\mathcal{G}(q, q)$  for any  $q \in \text{Ob}\mathcal{G}$ . We remark the power  $I\mathcal{G}(q, q)^n$  is a two-sided ideal of the group ring  $K\mathcal{G}(q, q)$  for any  $n \geq 0$ .

**Proposition 2.1.1.** *The  $K$ -SAC  $K\mathcal{G}$  has a filtration  $\{F_n K\mathcal{G}(p_0, p_1)\}_{n \geq 0}$ ,  $p_0, p_1 \in \text{Ob}\mathcal{G}$ , such that  $F_n K\mathcal{G}(q, q) = I\mathcal{G}(q, q)^n$  for any  $n \geq 0$ . The filtered  $K$ -SAC  $K\mathcal{G}$  satisfies the conditions (C1) and (C4).*

*Proof.* Let  $p_0$  and  $p_1$  be objects in  $\mathcal{G}$  with  $[p_0] = [p_1] \in \pi_0\mathcal{G} = \pi_0 K\mathcal{G}$ . Then, for any  $q, q_1 \in [p_0]$ ,  $\gamma \in \mathcal{G}(p_0, q)$ ,  $\gamma_1 \in \mathcal{G}(p_0, q_1)$ ,  $\delta \in \mathcal{G}(q, p_1)$ ,  $\delta_1 \in \mathcal{G}(q_1, p_1)$  and  $n \geq 0$ , we have

$$\gamma I\mathcal{G}(q, q)^n \delta = \gamma_1 I\mathcal{G}(q_1, q_1)^n \delta_1 \subset K\mathcal{G}(p_0, p_1) \quad (2.1.1)$$

In fact, since the map  $\mathcal{G}(q, q) \rightarrow \mathcal{G}(q_1, q_1)$ ,  $x \mapsto \gamma_1^{-1} \gamma x \gamma^{-1} \gamma_1$ , is an isomorphism of groups, we have  $\gamma_1^{-1} \gamma I\mathcal{G}(q, q)^n \gamma^{-1} \gamma_1 = I\mathcal{G}(q_1, q_1)^n$ . Since  $I\mathcal{G}(q_1, q_1)^n$  is a two-sided ideal of  $K\mathcal{G}(q_1, q_1)$  and  $\delta_1 \delta^{-1} \gamma^{-1} \gamma_1 \in \mathcal{G}(q_1, q_1)$  is an invertible element of  $K\mathcal{G}(q_1, q_1)$ , we have  $I\mathcal{G}(q_1, q_1)^n = I\mathcal{G}(q_1, q_1)^n \delta_1 \delta^{-1} \gamma^{-1} \gamma_1$ . Hence  $\gamma_1^{-1} \gamma I\mathcal{G}(q, q)^n \gamma^{-1} \gamma_1 = I\mathcal{G}(q_1, q_1)^n \delta_1 \delta^{-1} \gamma^{-1} \gamma_1$ , and so  $\gamma I\mathcal{G}(q, q)^n \delta = \gamma_1 I\mathcal{G}(q_1, q_1)^n \delta_1$ .

Hence we may define

$$F_n K\mathcal{G}(p_0, p_1) := \gamma I\mathcal{G}(q, q)^n \delta, \quad n \geq 0,$$

if  $[p_0] = [p_1]$ . In the case  $[p_0] \neq [p_1]$ , we define  $F_n K\mathcal{G}(p_0, p_1) := 0 \subset K\mathcal{G}(p_0, p_1) = 0$ .

Next we prove the condition (ii) in Definition 1.2.1

$$F_{n_1} K\mathcal{G}(p_0, p_1) \cdot F_{n_2} K\mathcal{G}(p_1, p_2) \subset F_{n_1+n_2} K\mathcal{G}(p_0, p_2)$$

for any  $p_0, p_1, p_2 \in \text{Ob}\mathcal{G}$  and  $n_1, n_2 \geq 0$ . Choose  $\gamma \in \mathcal{G}(p_0, p_1)$  and  $\delta \in \mathcal{G}(p_1, p_2)$ . Then  $\gamma^{-1} F_{n_1} K\mathcal{G}(p_0, p_1) = I\mathcal{G}(p_1, p_1)^{n_1}$  and  $F_{n_2} K\mathcal{G}(p_1, p_2) \delta^{-1} = I\mathcal{G}(p_1, p_1)^{n_2}$ , so that  $\gamma^{-1} F_{n_1} K\mathcal{G}(p_0, p_1) \cdot F_{n_2} K\mathcal{G}(p_1, p_2) \delta^{-1} \subset I\mathcal{G}(p_1, p_1)^{n_1+n_2}$ . This proves  $K\mathcal{G}$  is a filtered  $K$ -SAC.

The conditions (C1) and (C4) are clear from the definition of the power of the augmentation ideal  $I\mathcal{G}(q, q)^n$ ,  $n \geq 0$ ,  $q \in \text{Ob}\mathcal{G}$ .  $\square$

Following 1.2 we can define the completion of the filtered  $K$ -SAC  $K\mathcal{G}$ ,  $\widehat{K\mathcal{G}}$ , which satisfies the conditions (C1), (C2) and (C3). Since  $K\mathcal{G}$  satisfies the condition (C4), we have  $\text{Der}K\mathcal{G} = F_{-1}\text{Der}K\mathcal{G}$  from Lemma 1.3.1. In particular, any derivation  $D \in \text{Der}K\mathcal{G}$  induces a derivation of the completion  $\widehat{K\mathcal{G}}$  in a natural way. In other words, we have a natural homomorphism of  $K$ -Lie algebras,  $\text{Der}K\mathcal{G} \rightarrow \text{Der}\widehat{K\mathcal{G}}$ .

The coproduct  $\Delta$  on  $K\mathcal{G}$  satisfies

$$\Delta F_n K\mathcal{G}(p_0, p_1) \subset \sum_{n_1+n_2=n} F_{n_1} K\mathcal{G}(p_0, p_1) \otimes F_{n_2} K\mathcal{G}(p_0, p_1) \quad (2.1.2)$$

for any  $p_0, p_1 \in \text{Ob}\mathcal{G}$  and  $n \geq 0$ . In fact, it is easy to see  $\Delta(IG(q, q)^n) \subset \sum_{n_1+n_2=n} IG(q, q)^{n_1} \otimes IG(q, q)^{n_2}$  for any  $q \in \text{Ob}\mathcal{G}$ . For  $\gamma \in \mathcal{G}(p_0, q)$  and  $\delta \in \mathcal{G}(q, p_1)$ ,  $p_0, p_1 \in [q]$ , we have  $\Delta F_n K\mathcal{G}(p_0, p_1) = \Delta(\gamma IG(q, q)^n \delta) = (\gamma \otimes \gamma) \Delta(IG(q, q)^n) (\delta \otimes \delta) \subset \sum_{n_1+n_2=n} \gamma IG(q, q)^{n_1} \delta \otimes \gamma IG(q, q)^{n_2} \delta = \sum_{n_1+n_2=n} F_{n_1} K\mathcal{G}(p_0, p_1) \otimes F_{n_2} K\mathcal{G}(p_0, p_1)$ , as was to be shown.

Hence  $\Delta$  induces a homomorphism of  $K$ -SAC's

$$\Delta: \widehat{K\mathcal{G}} \rightarrow \widehat{K\mathcal{G}} \widehat{\otimes} \widehat{K\mathcal{G}},$$

which we call the coproduct of  $\widehat{K\mathcal{G}}$ . We denote by  $\text{Der}_\Delta \widehat{K\mathcal{G}}$  the Lie subalgebra of  $\text{Der} \widehat{K\mathcal{G}}$  consisting of all the continuous derivations  $D$  stabilizing the coproduct  $\Delta$ , namely, satisfying

$$\Delta D = (D \widehat{\otimes} 1 + 1 \widehat{\otimes} D) \Delta: \widehat{K\mathcal{G}}(p_0, p_1) \rightarrow (\widehat{K\mathcal{G}} \widehat{\otimes} \widehat{K\mathcal{G}})(p_0, p_1)$$

for any  $p_0, p_1 \in \text{Ob}\mathcal{G}$ .

## 2.2 Abelianization $\mathcal{G}^{\text{abel}}$

In this subsection we introduce the abelianization of a groupoid  $\mathcal{G}$ ,  $\mathcal{G}^{\text{abel}}$ . Before defining the abelianization, we remark that, for any filtered  $K$ -SAC  $\mathcal{R}$ , the quotient  $\mathcal{R}/F_n \mathcal{R}$  given by  $\text{Ob}\mathcal{R}/F_n \mathcal{R} := \text{Ob}\mathcal{R}$  and  $(\mathcal{R}/F_n \mathcal{R})(p_0, p_1) := \mathcal{R}(p_0, p_1)/F_n \mathcal{R}(p_0, p_1)$ ,  $p_0, p_1 \in \text{Ob}\mathcal{R}$ , is also a filtered  $K$ -SAC. Any derivation  $D \in F_0 \text{Der}\mathcal{R}$  defines a derivation of the quotient  $\mathcal{R}/F_n \mathcal{R}$  in a natural way. In other words, one can define a natural homomorphism of  $K$ -Lie algebras  $F_0 \text{Der}\mathcal{R} \rightarrow F_0 \text{Der}(\mathcal{R}/F_n \mathcal{R})$ . Moreover we remark that, for any group  $G$ , we have a natural isomorphism

$$IG/(IG)^2 \cong G^{\text{abel}}, \quad x - 1 \mapsto x \text{ mod } [G, G].$$

Here  $IG$  is the augmentation ideal of the integral group ring  $\mathbb{Z}G$ . The unit  $1 \in \mathbb{Z}G$  gives a decomposition  $\mathbb{Z}G/(IG)^2 = \mathbb{Z} \oplus IG/(IG)^2 = \mathbb{Z} \oplus G^{\text{abel}}$ . The multiplication of any two elements in  $G^{\text{abel}}$  vanishes in the ring  $\mathbb{Z}G/(IG)^2$ . In particular,  $\mathbb{Z}G/(IG)^2$  is a commutative ring. The conjugate action of  $G$  on the ring  $\mathbb{Z}G/(IG)^2$  is trivial.

For any groupoid  $\mathcal{G}$ , we call the  $\mathbb{Z}$ -SAC

$$\mathcal{G}^{\text{abel}} := \mathbb{Z}\mathcal{G}/F_2 \mathbb{Z}\mathcal{G}$$

the *abelianization* of the groupoid  $\mathcal{G}$ . The conjugate action of  $\mathcal{G}(q, q)$  on the ring  $\mathcal{G}^{\text{abel}}(q, q)$  is trivial. Hence, for any  $\lambda \in \pi_0 \mathcal{G}$  and  $q_1, q_2 \in \lambda$ , the isomorphism

$$\mathcal{G}^{\text{abel}}(q_1, q_1) \xrightarrow{\cong} \mathcal{G}^{\text{abel}}(q_2, q_2), \quad x \mapsto \gamma x \gamma^{-1}$$

does not depend on the choice of  $\gamma \in \mathcal{G}(q_2, q_1)$ . Under this identification we define

$$H\mathcal{G}(\lambda) := \mathcal{G}^{\text{abel}}(q, q), \quad q \in \lambda.$$

$H\mathcal{G}(\lambda)$  is a commutative ring. If  $p_0, p_1 \in \lambda$ , then  $\mathcal{G}^{\text{abel}}(p_0, p_1)$  is a left and right  $H\mathcal{G}(\lambda)$ -module. We remark  $\gamma x = x \gamma \in \mathcal{G}^{\text{abel}}(p_0, p_1)$  for any  $\gamma \in \mathcal{G}(p_0, p_1)$  and  $x \in H\mathcal{G}(\lambda)$ .

### 3 Oriented surfaces

Let  $S$  be an oriented surface, or equivalently, an oriented 2-dimensional  $C^\infty$ -manifold. Throughout this paper, a *simple closed curve* (SCC) on  $S$  means a smooth submanifold of the interior  $S \setminus \partial S$  which is diffeomorphic to the circle  $S^1$ . It can be regarded as an unoriented free loop in  $S$ .

Let  $E$  be a non-empty closed subset of the surface  $S$ . What we actually have in mind is the case  $E$  is a disjoint union of finitely many simple closed curves and finitely many points. Then we introduce a groupoid  $\mathcal{C} = \mathcal{C}(S, E)$  by setting  $\text{Ob}\mathcal{C} := E$  and

$$\mathcal{C}(p_0, p_1) = \text{Hom}_{\mathcal{C}}(p_0, p_1) := \Pi S(p_0, p_1) = [([0, 1], 0, 1), (S, p_0, p_1)],$$

the homotopy set of continuous paths on  $S$  connecting  $p_0$  to  $p_1$ , for  $p_0, p_1 \in E = \text{Ob}\mathcal{C}$ . As usual a path and its homotopy class will be denoted by the same letter, if there is no fear of confusion. The multiplication  $\gamma_1\gamma_2 = \gamma_1 \cdot \gamma_2$ , where  $\gamma_1 \in \mathcal{C}(p_0, p_1)$ ,  $\gamma_2 \in \mathcal{C}(p_1, p_2)$  and  $p_0, p_1, p_2 \in E$ , means the (homotopy class of) conjunction, which traverses  $\gamma_1$  first. The natural map  $\pi_0\mathcal{C}(S, E) \rightarrow \pi_0 S$  is injective.

Let  $K$  be a commutative ring including the rationals  $\mathbb{Q}$ . Then, if  $G$  is a finitely generated free group or a surface group, the completion map  $KG \rightarrow \widehat{KG} = \varprojlim_{n \rightarrow \infty} KG/(IG)^n$  is injective [3] [4]. Hence, if the fundamental group of any connected component of  $S$  is finitely generated, then the completion map  $K\mathcal{C}(S, E) \rightarrow \widehat{K\mathcal{C}(S, E)}$  is injective.

#### 3.1 Dehn-Nielsen homomorphism

We define *the mapping class group of the pair*  $(S, E)$ , which is denoted by  $\mathcal{M}(S, E)$ , to be the set of orientation preserving diffeomorphisms of  $S$  that fix  $E \cup \partial S$  pointwise, modulo isotopies relative to  $E \cup \partial S$ . A diffeomorphism and its class in  $\mathcal{M}(S, E)$  will be denoted by the same letter, if no confusion might occur. Adopting the functional notation, the product  $\varphi_1\varphi_2$  for  $\varphi_1, \varphi_2 \in \mathcal{M}(S, E)$  means the (class of) composition  $\varphi_1 \circ \varphi_2$ . Then we can define a group homomorphism of Dehn-Nielsen type

$$\text{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut}(K\mathcal{C}(S, E)),$$

by  $\text{DN}(\varphi)(\gamma) := \varphi(\gamma)$ , for  $\gamma \in \mathcal{C}(p_0, p_1)$ ,  $p_0, p_1 \in E$ . For any groupoid  $\mathcal{G}$  we denote by  $\text{Aut}\mathcal{G}$  the group consisting of all covariant functors  $\mathcal{F}: \mathcal{G} \rightarrow \mathcal{G}$  satisfying the conditions  $\mathcal{F}(p_0) = p_0$  and  $\mathcal{F}: \mathcal{G}(p_0, p_1) \rightarrow \mathcal{G}(p_0, p_1)$  is an isomorphism for any  $p_0$  and  $p_1 \in \text{Ob}\mathcal{G}$ . By definition the Dehn-Nielsen homomorphism  $\text{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut}(K\mathcal{C}(S, E))$  factors through the group  $\text{Aut}\mathcal{C}(S, E)$ .

We say  $S$  is of *finite type*, if  $S$  is a compact connected oriented surface, or a surface obtained from a compact connected oriented surface by removing finitely many points in the interior.

**Theorem 3.1.1.** *Suppose  $S$  is of finite type with non-empty boundary,  $E \subset \partial S$ , and any connected component of  $\partial S$  has an element of  $E$ . Then the homomorphism  $\text{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut}(K\mathcal{C}(S, E))$  is injective.*

*Proof.* Let  $\varphi \in \mathcal{M}(S, E)$  and suppose  $\text{DN}(\varphi) = 1$ . Since  $\varphi$  is identity on  $\partial S$ , for any  $p, q \in \partial S$  and  $\gamma \in \Pi S(p, q)$ , we have  $\varphi(\gamma) = \gamma$ . Moreover, by [7] Theorem 3.1,  $\varphi(\gamma)$  is isotopic to  $\gamma$ . Now we take a system of proper arcs  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_{r-1}$ , and

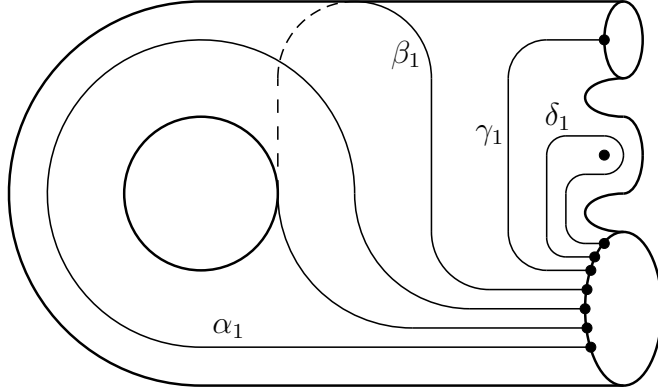


Figure 1: the case  $g = 1$ ,  $r = 2$ , and  $n = 1$

$\delta_1, \dots, \delta_n$  in  $S$  ( $g$  is the genus of  $S$ ,  $r$  is the number of components of  $\partial S$ , and  $n$  is the number of punctures), such that the surface obtained from  $S$  by cutting along these arcs is the union of a disk and  $n$  punctured disks. See Figure 1.

Applying [8] Proposition 2.8, we may assume  $\varphi$  is identity on these arcs. Finally, by [25], we conclude that  $\varphi$  is isotopic to the identity.  $\square$

If  $K$  includes  $\mathbb{Q}$  and  $S$  is of finite type, the completion map  $K\mathcal{C}(S, E) \rightarrow \widehat{K\mathcal{C}(S, E)}$  is injective, as was stated in the beginning of this section. Hence the natural homomorphism  $\widehat{\text{DN}}: \mathcal{M}(S, E) \rightarrow \text{Aut}(\widehat{K\mathcal{C}(S, E)})$  is also injective under the assumption of Theorem 3.1.1.

It would be very interesting if one could find a characterization of the image of the homomorphism  $\text{DN}$  for any  $(S, E)$ .

Next we consider the case  $E \not\subset \partial S$  and  $E \setminus \partial S$  is finite. We number the elements of the set  $E \setminus \partial S$ , as  $E \setminus \partial S = \{q_1^0, q_2^0, \dots, q_s^0\}$ , where  $s = \sharp(E \setminus \partial S) \geq 1$ .

**Lemma 3.1.2.** *Assume  $S$  is connected and  $E \cap \partial S \neq \emptyset$ . Then the kernel of the forgetful homomorphism  $\text{Aut}\mathcal{C}(S, E) \rightarrow \text{Aut}\mathcal{C}(S, E \cap \partial S)$  is isomorphic to the fundamental group  $\pi_1(S^s, (q_1^0, \dots, q_s^0)) = \prod_{i=1}^s \pi_1(S, q_i^0)$ .*

*Proof.* For any  $x_i \in \pi_1(S, q_i^0)$ ,  $1 \leq i \leq s$ , we define  $\mathcal{F} = \mathcal{F}(x_1, \dots, x_s) \in \text{Aut}\mathcal{C}(S, E)$  by

$$\mathcal{F}\gamma := \begin{cases} \gamma, & \text{if } p_0, p_1 \in E \cap \partial S, \\ x_{i_0}\gamma, & \text{if } p_0 = q_{i_0}^0 \text{ and } p_1 \in E \cap \partial S, \\ \gamma x_{i_1}^{-1}, & \text{if } p_0 \in E \cap \partial S \text{ and } p_1 = q_{i_1}^0, \\ x_{i_0}\gamma x_{i_1}^{-1}, & \text{if } p_0 = q_{i_0}^0 \text{ and } p_1 = q_{i_1}^0, \end{cases}$$

for  $\gamma \in \mathcal{C}(p_0, p_1)$ . It is clear that the map

$$\prod_{i=1}^s \pi_1(S, q_i^0) \rightarrow \text{Aut}\mathcal{C}(S, E), \quad (x_1, \dots, x_s) \mapsto \mathcal{F}(x_1, \dots, x_s)$$

is an injective group homomorphism, and its image is in the kernel of the forgetful homomorphism. Hence it suffices to show the kernel is included in the image.

Let  $U$  be an element of the kernel of the forgetful homomorphism. Choose a point  $*$   $\in E \cap \partial S$  and paths  $\gamma_i \in \Pi S(p_i, *)$ ,  $1 \leq i \leq s$ . Define  $x_i := (U\gamma_i)\gamma_i^{-1} \in \Pi S(p_i, p_i) = \pi_1(S, p_i)$ ,  $1 \leq i \leq s$ . Then  $x_i$  does not depend on the choice of  $*$  and  $\gamma_i$ . In fact, for another  $*' \in E \cap \partial S$  and  $\gamma'_i \in \Pi S(p_i, *')$ , take some  $\delta \in \Pi S(*', *)$ . Then we have  $(U\gamma_i)^{-1}(U\gamma'_i)\delta = U(\gamma_i^{-1}\gamma'_i\delta) = \gamma_i^{-1}\gamma'_i\delta \in \pi_1(S, *)$  since  $U$  is an element of the kernel of the forgetful homomorphism. Hence we have  $(U\gamma'_i)\gamma_i^{-1} = x_i$ . This means  $U = \mathcal{F}(x_1, \dots, x_s)$  and proves the lemma.  $\square$

Let  $F_s(S \setminus \partial S)$  be the configuration space of ordered distinct  $s$  points

$$F_s(S \setminus \partial S) := \{(q_1, q_2, \dots, q_s) \in (S \setminus \partial S)^s; \forall i \neq \forall j, q_i \neq q_j\}.$$

Then we have a natural exact sequence

$$1 \rightarrow \pi_1(F_s(S \setminus \partial S), (q_1^0, \dots, q_s^0)) \rightarrow \mathcal{M}(S, E) \rightarrow \mathcal{M}(S, E \cap \partial S) \rightarrow 1$$

[2] Theorem 4.3. See also [8] Theorem 4.6.

**Theorem 3.1.3.** *Assume  $S$  is of finite type with non-empty boundary, any component of  $\partial S$  has an element of  $E$ , and  $E \setminus \partial S$  is a non-empty finite set. Then the kernel of the Dehn-Nielsen homomorphism  $\text{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut}KC(S, E)$  is isomorphic to the kernel of the inclusion homomorphism  $\pi_1(F_s(S \setminus \partial S), (q_1^0, \dots, q_s^0)) \rightarrow \pi_1(S^s, (q_1^0, \dots, q_s^0))$ . In particular, the homomorphism  $\text{DN}$  is injective if and only if  $\sharp(E \setminus \partial S) = 1$ .*

*Proof.* Consider the morphism of exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(F_s(S \setminus \partial S)) & \longrightarrow & \mathcal{M}(S, E) & \longrightarrow & \mathcal{M}(S, E \cap \partial S) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(S^s) & \longrightarrow & \text{Aut}\mathcal{C}(S, E) & \longrightarrow & \text{Aut}\mathcal{C}(S, E \cap \partial S). \end{array}$$

The right vertical arrow is injective from Theorem 3.1.1. The theorem follows from chasing the diagram.  $\square$

## 3.2 van Kampen theorem

In this subsection we prove the easier half of the van Kampen theorem for the groupoid  $\mathcal{C}(S, E)$ . Let  $S_1$  and  $S_2$  be oriented surfaces,  $\partial' S_1$  and  $\partial' S_2$  sums of some connected components of the boundary  $\partial S_1$  and  $\partial S_2$ , respectively, and  $\varphi: \partial' S_1 \xrightarrow{\cong} \partial' S_2$  an orientation-reversing diffeomorphism. Moreover let  $E_1 \subset S_1$  and  $E_2 \subset S_2$  be non-empty closed subsets. We assume the condition

Any connected component of  $\partial' S_i$  has some point in  $E_i$  for  $i = 1, 2$ , and  $\varphi$  maps the set  $E_1 \cap \partial' S_1$  onto the set  $E_2 \cap \partial' S_2$ .

Then we define  $S_3 := S_1 \cup_\varphi S_2$ ,  $E_3 := E_1 \cup_\varphi E_2$  and  $E_3^\partial \subset E_3$  the image of  $E_1 \cap \partial' S_1$  and  $E_2 \cap \partial' S_2$ . We write simply  $\mathcal{C}_i := \mathcal{C}(S_i, E_i)$  for  $i = 1, 2, 3$ . We have the inclusion map  $\iota_i: \mathcal{C}_i \rightarrow \mathcal{C}_3$  for  $i = 1, 2$ .

The van Kampen theorem says  $\mathcal{C}_3$  is “generated” by  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . In order to formulate it in a rigorous way, we prepare some notations. Let  $p_0$  and  $p_1$  be points in  $E_3$ . Then we denote by  $\bar{\mathcal{E}}(p_0, p_1)$  the set of finite sequences of points in  $E_3$ ,  $\lambda = (q_0, q_1, \dots, q_n) \in E_3^{n+1}$ ,  $n \geq 0$ , satisfying the conditions

(i)  $q_0 = p_0$  and  $q_n = p_1$ .

(ii) For  $1 \leq j \leq n$ , either  $\{q_{j-1}, q_j\} \subset S_1$  or  $\{q_{j-1}, q_j\} \subset S_2$ .

Further we denote by  $\mathcal{E}(p_0, p_1)$  the set of pairs  $(\lambda, \mu)$ ,  $\lambda = (q_0, q_1, \dots, q_n) \in \overline{\mathcal{E}}(p_0, p_1)$ ,  $\mu = (\mu_1, \dots, \mu_n) \in \{1, 2\}^n$  such that  $\{q_{j-1}, q_j\} \subset S_{\mu_j}$  for any  $1 \leq j \leq n$ . For  $(\lambda, \mu) \in \mathcal{E}(p_0, p_1)$ , we denote  $K\mathcal{C}(\lambda, \mu) := \bigotimes_{j=1}^n K\mathcal{C}_{\mu_j}(q_{j-1}, q_j)$ . One can define the multiplication map  $K\mathcal{C}(\lambda, \mu) \rightarrow K\mathcal{C}_3(p_0, p_1)$  in an obvious way.

**Proposition 3.2.1.** *The multiplication map*

$$\bigoplus_{(\lambda, \mu) \in \mathcal{E}(p_0, p_1)} K\mathcal{C}(\lambda, \mu) \rightarrow K\mathcal{C}_3(p_0, p_1)$$

is surjective for any  $p_0$  and  $p_1 \in E_3$ .

*Proof.* Take an open tubular neighborhood  $U$  of  $\partial' S_1 \cup_{\varphi} \partial' S_2$  in  $S_3$ . Then  $\{S_1 \cup U, S_2 \cup U\}$  is an open covering of  $S_3$ , and  $S_i$  is a deformation retract of  $S_i \cup U$  for  $i = 1, 2$ . For any path  $\ell: ([0, 1], 0, 1) \rightarrow (S, p_0, p_1)$ , there exists a positive integer  $n \geq 1$  such that  $\ell([\frac{j-1}{n}, \frac{j}{n}])$  is included in  $S_1 \cup U$  or  $S_2 \cup U$  for any  $1 \leq j \leq n$ . Then we may assume  $\ell(\frac{j}{n}) \in E_3$  for  $1 \leq j \leq n-1$ . In fact, if  $\ell(\frac{j}{n}) \in U$ , the connected component of  $\ell(\frac{j}{n})$  in  $U$  contains a point in  $E_3^\partial$  by the assumption on  $E_i$  and  $\partial' S_i$ . We insert a path connecting  $\ell(\frac{j}{n})$  to a point in  $E_3^\partial$  inside  $U$  into the path  $\ell$ , and deform it to obtain a new path homotopic to the original path  $\ell$  with  $\ell(\frac{j}{n}) \in E_3^\partial$ . On the other hand, if  $\ell(\frac{j}{n}) \in S_i \setminus U$ , where  $i = 1$  or  $2$ , then we can deform the path  $\ell$  inside  $S_i \cup U$  such that  $\ell(\frac{j}{n}) \in \{p_0, p_1\} \cup E_3^\partial \subset E_3$ . Moreover, using the deformation retraction of  $S_i \cup U$  onto  $S_i$ , we deform the path  $\ell([\frac{j-1}{n}, \frac{j}{n}])$  inside  $S_1$  or  $S_2$ . Hence we obtain a new path  $\ell$  such that

- For  $1 \leq j \leq n$ , either  $\ell([\frac{j-1}{n}, \frac{j}{n}]) \subset S_1$  or  $\ell([\frac{j-1}{n}, \frac{j}{n}]) \subset S_2$ ,
- $\ell(\frac{j}{n})$  is in  $E_3$  for  $1 \leq j \leq n-1$ , and
- $\ell$  is homotopic to the original path relative to  $\{0, 1\}$ .

Then we have  $\ell|_{[\frac{0}{n}, \frac{1}{n}]} \otimes \ell|_{[\frac{1}{n}, \frac{2}{n}]} \otimes \dots \otimes \ell|_{[\frac{n-1}{n}, \frac{n}{n}]} \in K\mathcal{C}(\lambda, \mu)$  for some  $(\lambda, \mu) \in \mathcal{E}(p_0, p_1)$ . This proves the proposition.  $\square$

### 3.3 Automorphisms

From now until the end of §5 we suppose  $K$  is a commutative ring including the rationals  $\mathbb{Q}$ .

Let  $S$  be an oriented surface, and  $E$  and  $E'$  closed subsets, as before. If  $E \subset E'$ , the inclusion  $E \hookrightarrow E'$  induces a homomorphism of filtered SAC's  $\widehat{K\mathcal{C}(S, E)} \rightarrow \widehat{K\mathcal{C}(S, E')}$  and the forgetful homomorphism  $\phi: \text{Aut}\widehat{K\mathcal{C}(S, E')} \rightarrow \text{Aut}\widehat{K\mathcal{C}(S, E)}$ . In this subsection we study this forgetful homomorphism. For topological study of surfaces, the group  $\text{Aut}\widehat{K\mathcal{C}(S, E)}$  is too large, so that we begin with introducing an appropriate subgroup of  $\text{Aut}\widehat{K\mathcal{C}(S, E)}$ ,  $A(S, E)$ .

**Definition 3.3.1.** *We define  $A(S, E)$  to be the subgroup consisting of all the automorphisms  $U \in \text{Aut}\widehat{K\mathcal{C}(S, E)}$  satisfying the following four conditions.*

- (i)  $U$  is a homeomorphism with respect to the filtration  $\{F_p \widehat{K\mathcal{C}(S, E)}\}_{p \geq 0}$ .
- (ii) If  $\gamma \in \Pi S(p_0, p_1)$ ,  $p_0, p_1 \in E$ , is represented by a path included in  $E$ , then  $U\gamma = \gamma$ .
- (iii)  $\varepsilon U = \varepsilon: \widehat{K\mathcal{C}(p_0, p_1)} \rightarrow K$  for any  $p_0, p_1 \in E$ . Here  $\varepsilon$  is the augmentation, which is induced by the  $K$ -linear map  $K\mathcal{C}(p_0, p_1) \rightarrow K$ ,  $\mathcal{C}(p_0, p_1) \ni \gamma \mapsto 1$ .
- (iv)  $\Delta U = (U \widehat{\otimes} U) \Delta: \widehat{K\mathcal{C}(p_0, p_1)} \rightarrow (\widehat{K\mathcal{C}} \widehat{\otimes} \widehat{K\mathcal{C}})(p_0, p_1)$  for any  $p_0, p_1 \in E$ . Here  $\Delta$  is the coproduct of  $\widehat{K\mathcal{C}}$  as in §2.1.

Clearly the image of the Dehn-Nielsen homomorphism  $\widehat{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut} \widehat{K\mathcal{C}(S, E)}$  is included in  $A(S, E)$ , so that we obtain  $\widehat{DN}: \mathcal{M}(S, E) \rightarrow A(S, E)$ . We have the forgetful homomorphism  $\phi: A(S, E) \rightarrow A(S, E)$ .

For the rest of this subsection we suppose each component of  $S$  is a surface of finite type and not closed (hence its fundamental group is a finitely generated free group). Let  $C_i \subset S \setminus (E \cup \partial S)$ ,  $1 \leq i \leq n$ , be disjoint simple closed curves which are *not* null-homotopic in  $S$ . Choose a basepoint  $*_i \in C_i$  and a simple loop  $\eta_i: ([0, 1], \{0, 1\}) \rightarrow (C_i, *_i)$  going around  $C_i$ . We also denote its homotopy class by  $\eta_i \in \pi_1(S, *_i)$ . We can define  $\eta_i^a := \exp(a \log \eta_i) \in K \widehat{\pi_1(S, *_i)}$  for  $a \in K$ . For any  $p \in C_i$ , we have some  $t \in [0, 1]$  such that  $\eta_i(t) = p$ . We define  $\eta_{i,p}^a := (\eta_i|_{[0,t]})^{-1} \eta_i^a (\eta_i|_{[0,t]}) \in K \widehat{\pi_1(S, p)}$ , which is independent of the choice of the path  $\eta_i|_{[0,t]}$ . We denote  $E_1 := \bigcup_{i=1}^n C_i$ .

**Proposition 3.3.2.** *Suppose  $U \in A(S, E \cup E_1)$  is in the kernel of the forgetful homomorphism  $\phi: A(S, E \cup E_1) \rightarrow A(S, E)$ . Then there exist some  $a_i = a_i^U \in K$ ,  $1 \leq i \leq n$ , such that*

$$Uv = \begin{cases} v, & \text{if } p_0, p_1 \in E, \\ \eta_{i_0, p_0}^{a_{i_0}} v, & \text{if } p_0 \in C_{i_0}, p_1 \in E, \\ v(\eta_{i_1, p_1}^{a_{i_1}})^{-1}, & \text{if } p_0 \in E, p_1 \in C_{i_1}, \\ \eta_{i_0, p_0}^{a_{i_0}} v(\eta_{i_1, p_1}^{a_{i_1}})^{-1}, & \text{if } p_0 \in C_{i_0}, p_1 \in C_{i_1}, \end{cases}$$

for any  $v \in \widehat{K\mathcal{C}(p_0, p_1)}$ ,  $p_0, p_1 \in E \cup E_1$ .

*Proof.* For each  $1 \leq i \leq n$ , choose a point  $* \in E$  which is in the connected component containing  $*_i$ , and a path  $\gamma_i \in \Pi S(*, *_i)$ . Consider  $u_i := \gamma_i^{-1}(U\gamma_i) \in K \widehat{\pi_1(S, *_i)}$ . We have  $(\gamma_i \widehat{\otimes} \gamma_i)(u_i \widehat{\otimes} u_i) = (U\gamma_i) \widehat{\otimes} (U\gamma_i) = (U \widehat{\otimes} U) \Delta \gamma_i = \Delta(\gamma_i u_i) = (\gamma_i \widehat{\otimes} \gamma_i) \Delta u_i$ , and so  $u_i$  is group-like. Moreover  $u_i$  does not depend on the choice of  $*$  and  $\gamma_i$ . In fact, for another  $*' \in E$  and  $\gamma'_i \in \Pi S(*', *_i)$ , take some  $\delta \in \Pi S(*, *')$ . Then  $\gamma'_i \gamma_i^{-1} \delta = (U\gamma'_i)(U\gamma_i)^{-1} \delta$  since  $\gamma'_i \gamma_i^{-1} \delta \in \pi_1(S, *')$  and  $\phi(U) = 1$ . Hence we have  $\gamma'_i{}^{-1}(U\gamma'_i) = \gamma_i^{-1}(U\gamma_i) = u_i$ . In particular, if  $\gamma'_i = \gamma_i \eta_i$ , we obtain  $\eta_i^{-1} u_i \eta_i = u_i$ , since  $U\eta_i = \eta_i$  from the condition (ii) in Definition 3.3.1. Now we have the following.

**Proposition 3.3.3.** *Let  $S$  be a surface of finite type and not closed, and  $C$  a simple closed curve in  $S$  which is not null-homotopic in  $S$ . Choose a basepoint  $* \in C$  and a simple loop  $\eta: ([0, 1], \{0, 1\}) \rightarrow (C, *)$  going around  $C$ . We also denote its homotopy class by  $\eta \in \pi_1(S, *)$ . Then the subalgebra*

$$Z(\eta) := \{u \in K \widehat{\pi_1(S, *)}; \eta u = u \eta\}$$

of  $K \widehat{\pi_1(S, *)}$  equals the ring of formal power series in  $\eta - 1$ ,  $K[[\eta - 1]] = K[[\log \eta]]$ .

The proof will be given in the second half of this subsection. From this proposition we obtain  $u_i \in K[[\eta_i - 1]]$ . Since the inclusion homomorphism  $K[[\eta_i - 1]] \rightarrow K\widehat{\pi_1(S, *_i)}$  is injective,  $u_i$  is group-like also in  $K[[\eta_i - 1]]$ . Hence we have  $u_i = \eta_i^{-a_i}$  for some  $a_i \in K$ . We have  $U\gamma_i = \gamma_i u_i$  and  $U\gamma_i^{-1} = (u_i)^{-1}\gamma_i^{-1}$ . Since  $u_i$  does not depend on the choice of  $\gamma_i$ , we have

$$\forall v \in \widehat{KC}(*, *_i), Uv = vu_i, \quad \forall v' \in \widehat{KC}(*_i, *), Uv' = (u_i)^{-1}v'. \quad (3.3.1)$$

Now let  $p_0, p_1 \in E \cup E_1$  and  $v \in \widehat{KC}(p_0, p_1)$ .

(i) If  $p_0, p_1 \in E$ , then  $Uv = v$  since  $\phi(U) = 1$ .

(ii) Suppose  $p_0 = \eta_{i_0}(t_0) \in C_{i_0}$  and  $p_1 \in E$ . Choose  $\delta_1 \in \Pi S(p_1, *)$ . Then, since  $U\delta_1 = \delta_1$  and  $U(\eta_{i_0}|_{[0, t_0]}) = \eta_{i_0}|_{[0, t_0]}$ , we have  $(\eta_{i_0}|_{[0, t_0]})(Uv)\delta_1 = U((\eta_{i_0}|_{[0, t_0]})v\delta_1) = (u_{i_0})^{-1}(\eta_{i_0}|_{[0, t_0]})v\delta_1$ , and so  $Uv = \eta_{i_0, p_0}^{a_{i_0}}v$ .

(iii) Suppose  $p_0 \in E$  and  $p_1 = \eta_{i_1}(t_1) \in C_{i_1}$ . Choose  $\delta_0 \in \Pi S(*, p_0)$ . Then, since  $U\delta_0 = \delta_0$  and  $U(\eta_{i_1}|_{[0, t_1]})^{-1} = (\eta_{i_1}|_{[0, t_1]})^{-1}$ , we have  $\delta_0(Uv)(\eta_{i_1}|_{[0, t_1]})^{-1} = U(\delta_0v(\eta_{i_1}|_{[0, t_1]})^{-1}) = \delta_0v(\eta_{i_1}|_{[0, t_1]})^{-1}u_{i_1}$ , and so  $Uv = v(\eta_{i_1, p_1}^{a_{i_1}})^{-1}$ .

(iv) Suppose  $p_0 = \eta_{i_0}(t_0) \in C_{i_0}$  and  $p_1 = \eta_{i_1}(t_1) \in C_{i_1}$ . Then  $\gamma_{i_0}(\eta_{i_0}|_{[0, t_0]})v(\eta_{i_1}|_{[0, t_1]})^{-1} \in \widehat{KC}(*, *_i)$ . Hence we have  $\gamma_{i_0}u_{i_0}(\eta_{i_0}|_{[0, t_0]})(Uv)(\eta_{i_1}|_{[0, t_1]})^{-1} = U(\gamma_{i_0}(\eta_{i_0}|_{[0, t_0]})v(\eta_{i_1}|_{[0, t_1]})^{-1}) = \gamma_{i_0}(\eta_{i_0}|_{[0, t_0]})v(\eta_{i_1}|_{[0, t_1]})^{-1}u_{i_1}$ , and so  $Uv = \eta_{i_0, p_0}^{a_{i_0}}v(\eta_{i_1, p_1}^{a_{i_1}})^{-1}$ .

This completes the proof of Proposition 3.3.2.  $\square$

As a corollary, we have

**Proposition 3.3.4.** *Let  $N \subset S \setminus (E \cup \partial S)$  be a connected compact subsurface with non-empty boundary, which is not diffeomorphic to the disk  $D^2$ . Assume the inclusion homomorphism of fundamental groups  $\pi_1(N) \rightarrow \pi_1(S)$  is injective. Let  $i: KC(N, \partial N) \rightarrow KC(S, E \cup \partial N)$  be the inclusion homomorphism. Numbering the boundary components of  $\partial N$  as  $\partial N = \coprod_{i=1}^n \partial_i N$ , we choose  $*_i \in \partial_i N$  and  $\eta_i \in \pi_1(N, *_i)$  as in Proposition 3.3.2.*

*Suppose  $U \in A(N, \partial N)$  and  $\tilde{U} \in A(S, E \cup \partial N)$  satisfy  $\tilde{U} \circ i = i \circ U: KC(N, \partial N) \rightarrow KC(S, E \cup \partial N)$  and  $\tilde{U}$  is in the kernel of the forgetful homomorphism  $\phi: A(S, E \cup \partial N) \rightarrow A(S, E)$ . Then there exist some  $a_i = a_i^U \in K$ ,  $1 \leq i \leq n$ , such that*

$$Uv = \eta_{i_0, p_0}^{a_{i_0}}v(\eta_{i_1, p_1}^{a_{i_1}})^{-1} \quad (3.3.2)$$

*for any  $v \in KC(S, E \cup \partial N)(p_0, p_1)$ ,  $p_0 \in \partial_{i_0} N$ , and  $p_1 \in \partial_{i_1} N$ .*

*Proof.* Since  $N \neq D^2$ , the inclusion homomorphism  $\pi_1(\partial_i N) \rightarrow \pi_1(N)$  is injective. It follows from the assumption the simple closed curve  $\partial_i N$  is *not* null-homotopic in  $S$ . Hence we can apply Proposition 3.3.2 to  $\partial N$ , from which it follows that there exist some  $a_i = a_i^U \in K$ ,  $1 \leq i \leq n$ , such that

$$\tilde{U}v = \eta_{i_0, p_0}^{a_{i_0}}v(\eta_{i_1, p_1}^{a_{i_1}})^{-1}$$

*for any  $v \in KC(S, E \cup \partial N)(p_0, p_1)$ ,  $p_0 \in \partial_{i_0} N$ , and  $p_1 \in \partial_{i_1} N$ .*

From the assumption the inclusion homomorphism  $K\Pi N(p_0, p_1) \rightarrow K\Pi S(p_0, p_1)$  is injective for any  $p_0, p_1 \in \partial N$ . Since  $\pi_1(S)$  is a finitely generated free group, the completion map  $K\Pi S(p_0, p_1) \rightarrow \widehat{K\Pi S}(p_0, p_1)$  is also injective. Hence the equation (3.3.2) holds for any  $p_0, p_1 \in \partial N$  and  $v \in K\Pi N(p_0, p_1)$ , while  $K\Pi N(p_0, p_1)$  is dense in  $\widehat{K\Pi N}(p_0, p_1)$  and  $U$  is continuous. Hence the equation (3.3.2) holds for any  $v \in KC(N, \partial N)(p_0, p_1)$ ,  $p_0 \in \partial_{i_0} N$ , and  $p_1 \in \partial_{i_1} N$ . This proves the proposition.  $\square$



The rest of this subsection is devoted to the proof of Proposition 3.3.3.

First of all we need some algebraic facts. Let  $H_{\mathbb{Z}}$  be a  $\mathbb{Z}$ -free module of finite rank. We denote  $H := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$  and  $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ , the completed tensor algebra generated by  $H$ . Throughout this paper we omit the symbol  $\otimes$  as a multiplication in the algebra  $\widehat{T}$ .

**Lemma 3.3.5.** *Let  $X' \in H_{\mathbb{Z}}$  be a primitive element, and  $X := X' \otimes 1 \in H$ . Then we have*

$$\{u \in \widehat{T}; Xu = uX\} = K[[X]].$$

*Proof.* It suffices to prove  $\{u \in H^{\otimes n}; Xu = uX\} \subset K[[X]] \cap H^{\otimes n}$  by induction on  $n \geq 0$ . It is clear for the case  $n = 0$ .

Choose a  $\mathbb{Z}$ -free basis  $\{X'_i\}_{i=1}^{\text{rank}H_{\mathbb{Z}}}$  of  $H_{\mathbb{Z}}$  with  $X'_1 = X'$ . The set  $\{X_i\}_{i=1}^{\text{rank}H_{\mathbb{Z}}}$  defined by  $X_i := X'_i \otimes 1 \in H$  is a  $K$ -free basis of  $H$ . Assume  $n \geq 1$ . For any  $u \in H^{\otimes n}$ , there exist unique elements  $u_i \in H^{\otimes(n-1)}$ ,  $1 \leq i \leq \text{rank}H_{\mathbb{Z}}$ , such that  $u = Xu_1 + \sum_{i \geq 2} X_i u_i$ . If  $Xu = uX$ , then  $X^2u_1 + \sum_{i \geq 2} XX_i u_i = Xu_1X + \sum_{i \geq 2} X_i u_i X$ . Since  $\{X_i\}_{i=1}^{\text{rank}H_{\mathbb{Z}}}$  is linearly independent, we have  $u_i = 0$  for  $i \geq 2$ , and  $Xu_1 = u_1X$ . Hence, by the inductive assumption,  $u_1 \in K[[X]]$ , and so  $u = Xu_1 \in K[[X]]$ . This completes the induction.  $\square$

We remark Lemma 3.3.5 holds for any commutative ring with unit. We identify  $\Lambda^2 H_{\mathbb{Z}}$  with the  $\mathbb{Z}$ -submodule of  $H_{\mathbb{Z}}^{\otimes 2}$  generated by the set  $\{X'Y' - Y'X'; X', Y' \in H_{\mathbb{Z}}\}$ .

**Lemma 3.3.6.** *Let  $v'_0 \in \Lambda^2 H_{\mathbb{Z}}$  be primitive as an element of  $H_{\mathbb{Z}}^{\otimes 2}$ , and  $v_0 := v'_0 \otimes 1 \in \Lambda^2 H = \Lambda^2 H_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ . Then we have*

$$\{u \in \widehat{T}; v_0 u = uv_0\} = K[[v_0]].$$

*Proof.* It is clear  $K[[v_0]]$  is in the LHS. We prove  $\{u \in H^{\otimes n}; v_0 u = uv_0\} \subset K[[v_0]] \cap H^{\otimes n}$  by induction on  $n \geq 1$ .

To prove the case  $n = 1$ , we consider the adjoint map  $\text{adv}'_0 : H_{\mathbb{Z}} \rightarrow H_{\mathbb{Z}}^{\otimes 3}$ ,  $Y' \mapsto v'_0 Y' - Y' v'_0$ . Since the image  $(\text{adv}'_0)(H_{\mathbb{Z}})$  is  $\mathbb{Z}$ -free,  $\text{Ker}(\text{adv}'_0)$  is a direct summand of  $H_{\mathbb{Z}}$ , and we have  $\{Y \in H; v_0 Y = Y v_0\} = \text{Ker}(\text{adv}'_0) \otimes_{\mathbb{Z}} K$ . Assume  $\{Y \in H; v_0 Y = Y v_0\} \neq 0$ . Then  $\text{Ker}(\text{adv}'_0) \neq 0$ . In particular, we have some primitive element  $Z' \in H_{\mathbb{Z}}$  such that  $v'_0 Z' = Z' v'_0$ . From Lemma 3.3.5 for  $K = \mathbb{Z}$ , we have  $v'_0 = \lambda Z'^2$  for some  $\lambda \in \mathbb{Z}$ . Since  $v'_0 \in \Lambda^2 H_{\mathbb{Z}}$ , this implies  $v'_0 = 0$ , which contradicts the assumption  $v'_0$  is primitive, and proves the case  $n = 1$ .

Choose a  $\mathbb{Z}$ -free basis  $\{v'_i\}_{i=1}^{(\text{rank}H_{\mathbb{Z}})^2}$  of  $H_{\mathbb{Z}}^{\otimes 2}$  with  $v'_1 = v'_0$ . The subset  $\{v_i\}_{i=1}^{(\text{rank}H_{\mathbb{Z}})^2}$  defined by  $v_i := v'_i \otimes 1 \in \Lambda^2 H$  is a  $K$ -free basis of  $H^{\otimes 2}$ . Assume  $n \geq 2$ . For any  $u \in H^{\otimes n}$ , there exist unique elements  $u_i \in H^{\otimes(n-2)}$ ,  $1 \leq i \leq (\text{rank}H_{\mathbb{Z}})^2$ , such that  $u = v_0 u_1 + \sum_{i \geq 2} v_i u_i$ . If  $v_0 u = uv_0$ , then  $v_0^2 u_1 + \sum_{i \geq 2} v_0 v_i u_i = v_0 u_1 v_0 + \sum_{i \geq 2} v_i u_i v_0$ . Since  $\{v_i\}_{i=1}^{(\text{rank}H_{\mathbb{Z}})^2}$  is linearly independent, we have  $u_i = 0$  for  $i \geq 2$ , and  $v_0 u_1 = u_1 v_0$ . Hence, by the inductive assumption,  $u_1 \in K[[v_0]]$ , and so  $u = v_0 u_1 \in K[[v_0]]$ . This completes the induction.  $\square$

Further we need some general result on a filtered  $\mathbb{Q}$ -vector space. Let  $M = F_0 M \supset F_1 M \supset \cdots$  be a filtered  $\mathbb{Q}$ -vector space. Assume the filtration  $\{F_p M\}_{p \geq 0}$  is separated  $\bigcap_{p=0}^{\infty} F_p M = 0$ , and complete  $M = \varprojlim_{p \rightarrow \infty} M/F_p M$ . Let  $\{p_q\}_{q=0}^{\infty}$  be a sequence of natural numbers with  $\lim_{q \rightarrow \infty} p_q = +\infty$ . If a sequence  $\{a_q\}_{q=0}^{\infty} \subset M$  satisfies  $a_q \in F_{p_q} M$  for each  $q \geq 0$ , then the series  $\sum_{q=0}^{\infty} a_q x^q$  converges as an element of  $M = \varprojlim_{p \rightarrow \infty} M/F_p M$  for any  $x \in \mathbb{Q}$ .

**Lemma 3.3.7.** *If there exists an infinite subset  $X \subset \mathbb{Q}$  such that*

$$\forall x \in X, \quad \sum_{q=0}^{\infty} a_q x^q = 0,$$

*then we have  $a_q = 0$  for any  $q \geq 0$ .*

*Proof.* It suffices to show  $a_q \in F_r M$  for any  $r \geq 0$ . There exists some  $n = n(r)$  such that  $a_q \in F_r M$  for any  $q > n$ . This implies  $\sum_{q=0}^n a_q x^q = 0 \in M/F_r M$  for any  $x \in X$ . Since  $X$  is infinite, we have some distinct  $n+1$  elements  $x_0, x_1, \dots, x_n$  in  $X$ . Because of the Vandermonde determinants, the  $(n+1) \times (n+1)$ -matrix  $(x_i^j)_{0 \leq i, j \leq n}$  has an inverse matrix. Hence we obtain  $a_q \in F_r M$ ,  $0 \leq \forall q \leq n$ . This proves the lemma.  $\square$

In order to deduce Proposition 3.3.3 from these algebraic facts, we need the notion of a *group-like expansion* of a free group [18]. This notion will be also used in §4.3 and §6.1. Let  $\pi$  be a finitely generated free group, and  $H$  the  $K$ -first homology group of  $\pi$ ,

$$H := H_1(\pi; K) = \pi^{\text{abel}} \otimes_{\mathbb{Z}} K.$$

We denote by  $[x] \in H$  the homology class of  $x \in \pi$ . Let  $\widehat{\mathcal{L}}$  be the space of all Lie-like elements in the completed tensor algebra  $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ . The image of the exponential

$$\exp: \widehat{\mathcal{L}} \rightarrow \widehat{T}, \quad u \mapsto \exp(u) := \sum_{k=0}^{\infty} \frac{1}{k!} u^k$$

is a subgroup of the multiplicative group of the algebra  $\widehat{T}$ . We denote by  $u * v \in \widehat{\mathcal{L}}$  the Hausdorff series of  $u$  and  $v \in \widehat{\mathcal{L}}$ . By definition, we have  $\exp(u * v) = (\exp u)(\exp v)$ .

**Definition 3.3.8** (Massuyeau [18]). *A map  $\theta: \pi \rightarrow \widehat{T}$  is called a group-like expansion, if  $\theta$  is a group homomorphism of  $\pi$  into the multiplicative group  $\exp \widehat{\mathcal{L}}$ , and  $\theta(x) \equiv 1 + [x] \pmod{\prod_{m=2}^{\infty} H^{\otimes m}}$  for any  $x \in \pi$ .*

Any group-like expansion  $\theta$  induces a filter-preserving isomorphism of Hopf algebras [12] [18]

$$\theta: \widehat{K\pi} \xrightarrow{\cong} \widehat{T}. \quad (3.3.3)$$

Here the algebra  $\widehat{T}$  is filtered by the ideals  $\widehat{T}_p := \prod_{m=p}^{\infty} H^{\otimes m}$ ,  $p \geq 1$ .

**Proposition 3.3.9.** *Let  $S$  be a surface of finite type and not closed, and  $C$  a simple closed curve in  $S$ . Choose a basepoint  $*$  in  $C$  and let  $\eta \in \pi_1(S, *)$  be a simple loop around  $C$ . Then there exists a group-like expansion  $\theta$  of the free group  $\pi_1(S, *)$  such that*

- (i)  $\theta(\eta) = \exp([\eta])$ , if  $[\eta] \neq 0 \in H = H_1(S; K)$ ,
- (ii)  $\theta(\eta) = \exp(\eta'_0 \otimes 1)$  for some  $\eta'_0 \in \Lambda^2 H_{\mathbb{Z}}$ , if  $[\eta] = 0 \in H$ . Here  $\eta'_0$  is primitive as an element of  $H_{\mathbb{Z}}^{\otimes 2}$ .

*Proof.* From the assumption on  $S$ , the fundamental group  $\pi_1(S, *)$  has a presentation

$$\langle \alpha_i, \beta_i (1 \leq i \leq g), \gamma_j (1 \leq j \leq r); [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_r = 1 \rangle \quad (3.3.4)$$

for some  $g \geq 0$  and  $r \geq 1$ . It suffices to consider the following three cases.

- (1)  $C$  is non-separating.
- (2)  $C$  is separating, and  $[\eta] \neq 0 \in H$ .
- (3)  $C$  is separating, and  $[\eta] = 0 \in H$ .

(1) By the classification theorem of surfaces, we may take  $\eta = \alpha_1$ . Since  $r \geq 1$ ,  $\pi_1(S, *)$  is freely generated by  $\{\alpha_i, \beta_i(1 \leq i \leq g), \gamma_j(2 \leq j \leq r)\}$ . We can define  $\theta$  by  $\theta(\alpha_i) := \exp([\alpha_i])$ ,  $\theta(\beta_i) := \exp([\beta_i])$ ,  $1 \leq i \leq g$ , and  $\theta(\gamma_j) := \exp([\gamma_j])$ ,  $2 \leq j \leq r$ . Clearly we have  $\theta(\eta) = \exp([\eta])$ .

(2) The complement  $S \setminus C$  has two connected components, so that we have  $r \geq 2$ . By the classification theorem of surfaces, we may take  $\eta^{\pm 1} = \gamma_{k+1} \cdots \gamma_r [\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]$  for some  $1 \leq k \leq r-1$  and  $0 \leq h \leq g$ . The fundamental group  $\pi_1(S, *)$  is freely generated by  $\{\alpha_i, \beta_i(1 \leq i \leq g), \gamma_j(2 \leq j \leq r)\}$ . We define  $\theta(\alpha_i) := \exp([\alpha_i])$ ,  $\theta(\beta_i) := \exp([\beta_i])$ ,  $1 \leq i \leq g$ , and  $\theta(\gamma_j) := \exp([\gamma_j])$ ,  $2 \leq j \leq r-1$ . We denote  $\log \theta(x) = \sum_{p=1}^{\infty} \ell_p(x)$ ,  $\ell_p(x) \in \widehat{\mathcal{L}} \cap H^{\otimes p}$ . We define  $\ell_p(\gamma_r)$  by induction on  $p \geq 1$ .  $\ell_1(\gamma_r)$  must be  $[\gamma_r]$ . Assume  $p \geq 2$  and  $\ell_q(\gamma_r)$  is defined for  $q \leq p-1$ . Then we define  $\ell_p(\gamma_r)$  to be minus the  $H^{\otimes p}$ -component of  $[\gamma_{k+1}] * \cdots * [\gamma_{r-1}] * \left( \sum_{q=1}^{p-1} \ell_q(\gamma_r) \right) * [\alpha_1] * [\beta_1] * (-[\alpha_1]) * (-[\beta_1]) * \cdots * [\alpha_h] * [\beta_h] * (-[\alpha_h]) * (-[\beta_h])$ . From this definition we have  $\ell_p(\eta^{\pm 1}) = \pm \ell_p(\eta) = 0$  for  $p \geq 2$ , namely  $\theta(\eta) = \exp([\eta])$ .

(3) By the classification theorem of surfaces, we may take  $\eta^{\pm 1} = [\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]$  for some  $0 \leq h \leq g$ . There exists a group-like expansion  $\theta'$  of the free group  $\langle \alpha_i, \beta_i(1 \leq i \leq h) \rangle$  satisfying  $\theta'([\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h]) = \exp(\sum_{i=1}^h [\alpha_i] \wedge [\beta_i])$ , c.f., Definition 6.1.1 which is originally due to [18]. See also [16]. The fundamental group  $\pi_1(S, *)$  is freely generated by  $\{\alpha_i, \beta_i(1 \leq i \leq g), \gamma_j(2 \leq j \leq r)\}$ . We can define  $\theta$  by  $\theta(\alpha_i) := \theta'(\alpha_i)$ ,  $\theta(\beta_i) := \theta'(\beta_i)$ , if  $1 \leq i \leq h$ ,  $\theta(\alpha_i) := \exp([\alpha_i])$ ,  $\theta(\beta_i) := \exp([\beta_i])$ , if  $h+1 \leq i \leq g$ , and  $\theta(\gamma_j) := \exp([\gamma_j])$ ,  $2 \leq j \leq r$ . Then we have  $\theta(\eta) = \exp(\pm \sum_{i=1}^h [\alpha_i] \wedge [\beta_i])$ . It is easy to show that  $\eta'_0 := \pm \sum_{i=1}^h [\alpha_i] \wedge [\beta_i] \in \Lambda^2 H_1(S; \mathbb{Z})$  is primitive as an element of  $H_1(S; \mathbb{Z})^{\otimes 2}$ .  $\square$

We remark  $[\eta] \in H_1(S; \mathbb{Z})$  is primitive in the case (i), and  $\eta'_0 \in \Lambda^2 H_1(S; \mathbb{Z})$  is also primitive as an element of  $H_1(S; \mathbb{Z})^{\otimes 2}$  in the case (ii). Hence one can apply Lemmas 3.3.5 and 3.3.6 to these primitive elements, respectively.

*Proof of Proposition 3.3.3.* It is clear  $K[[\eta - 1]] \subset Z(\eta)$ . Let  $u \in Z(\eta)$ . For any  $n \in \mathbb{Z}_{\geq 0}$  we have  $0 = u\eta^n - \eta^n u = u \exp(n \log \eta) - \exp(n \log \eta)u$ . It follows from Lemma 3.3.7 that  $u(\log \eta) = (\log \eta)u$ . Apply the group-like expansion  $\theta$  in Proposition 3.3.9 to this equation, we obtain  $\theta(u)\theta(\log \eta) = \theta(\log \eta)\theta(u) \in \widehat{T}$ . Here we remark  $\theta(\log \eta) \in H$  or  $\theta(\log \eta) \in \Lambda^2 H$ . Since  $C$  is not null-homotopic in  $S$ , we have  $\widehat{\theta(\log \eta)} \neq 0$ . Hence, from Lemmas 3.3.5 and 3.3.6,  $\theta(u) \in K[[\theta(\log \eta)]]$ . Since  $\theta: K\widehat{\pi_1(S, *)} \rightarrow \widehat{T}$  is an isomorphism of algebras, we have  $u \in K[[\log \eta]] = K[[\eta - 1]]$ . This completes the proof.  $\square$

## 4 Completion of the Goldman Lie algebra

Let  $K$  be a commutative ring including the rationals  $\mathbb{Q}$ . Let  $S$  be an oriented surface,  $E$  a non-empty closed subset with the property  $E \setminus \partial S$  is closed in  $S$ . In this section we recall the Goldman Lie algebra of an oriented surface, and look at its action as derivations on the  $K$ -SAC  $K\mathcal{C}(S, E)$ . This action turns out to be compatible with the filtrations, and we

are naturally lead to the definition of the completed Goldman Lie algebra and its action on the completion  $\widehat{K\mathcal{C}(S, E)}$ .

We denote by  $\hat{\pi}(S) = [S^1, S]$  the homotopy set of free loops on  $S$ . The free  $K$ -module over the set  $\hat{\pi}(S)$ ,  $K\hat{\pi}(S)$ , has a natural structure of a  $K$ -Lie algebra, called *the Goldman Lie algebra of  $S$* , as follows [9]. For any  $q \in S$  we denote by  $|\cdot|: \pi_1(S, q) \rightarrow \hat{\pi}(S)$  the natural map forgetting the basepoint  $q$ . For a loop  $\alpha: S^1 \rightarrow S$  and a simple point  $p \in \alpha$ , let  $\alpha_p$  be the oriented loop  $\alpha$  based at  $p$ . Let  $\alpha$  and  $\beta$  be immersed loops in  $S$  such that  $\alpha \cup \beta: S^1 \cup S^1 \rightarrow S$  is an immersion with at worst transverse double points. For each intersection  $p \in \alpha \cap \beta$ , the conjunction  $\alpha_p \beta_p \in \pi_1(S, p)$  is defined. Let  $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$  be the local intersection number of  $\alpha$  and  $\beta$  at  $p$  and set

$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) |\alpha_p \beta_p| \in K\hat{\pi}(S).$$

This bracket makes the vector space  $K\hat{\pi}(S)$  a  $K$ -Lie algebra.

Moreover we denote  $S^* := S \setminus (E \setminus \partial S)$ . Let  $\alpha: S^1 \rightarrow S^*$  be an immersed loop and  $\beta: ([0, 1], 0, 1) \rightarrow (S, *_0, *_1)$  an immersed path connecting  $*_0 \in E$  to  $*_1 \in E$ , and suppose  $\alpha \cup \beta$  has at worst transverse double points. For each intersection  $p \in \alpha \cap \beta$ , let  $\alpha_p$  and  $\varepsilon(p; \alpha, \beta)$  be the same as before and let  $\beta_{*_0 p}$  (resp.  $\beta_{p*_1}$ ) be the path along  $\beta$  from  $*_0$  to  $p$  (resp.  $p$  to  $*_1$ ). Then the conjunction  $\beta_{*_0 p} \alpha_p \beta_{p*_1} \in \Pi S(*_0, *_1)$  is defined. For such  $\alpha$  and  $\beta$ , define

$$\sigma(\alpha)\beta := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \beta_{*_0 p} \alpha_p \beta_{p*_1} \in K\Pi S(*_0, *_1) = K\mathcal{C}(S, E)(*_0, *_1).$$

Then, by a similar way to Proposition 3.2.2 [13], we obtain a well-defined homomorphism of  $K$ -Lie algebras

$$\sigma: K\hat{\pi}(S^*) \rightarrow \text{Der}K\mathcal{C}(S, E).$$

On the other hand, in §2.1 we introduced a natural homomorphism  $\text{Der}K\mathcal{C}(S, E) \rightarrow \text{Der}\widehat{K\mathcal{C}(S, E)}$ . Hence we have a natural homomorphism of  $K$ -Lie algebras

$$\sigma: K\hat{\pi}(S^*) \rightarrow \text{Der}\widehat{K\mathcal{C}(S, E)}.$$

## 4.1 Filtration on the Goldman Lie algebra

We denote by  $S_\lambda$  the connected component corresponding to  $\lambda \in \pi_0 S$ . Then we have a direct sum decomposition of  $K$ -Lie algebras

$$K\hat{\pi}(S) = \bigoplus_{\lambda \in \pi_0 S} K\hat{\pi}(S_\lambda). \quad (4.1.1)$$

Now we introduce a  $K$ -submodule  $K\hat{\pi}(S)(n) \subset K\hat{\pi}(S)$  for any  $n \geq 1$ . We begin by considering the case  $S$  is connected. Then  $\hat{\pi}(S)$  is the set of all conjugacy classes in the fundamental group  $\pi_1(S, q)$  for any  $q \in S$ , namely, the forgetful map  $|\cdot|: \pi_1(S, q) \rightarrow \hat{\pi}(S)$  is surjective. We denote by  $1_q \in \pi_1(S, q)$  the constant loop based at  $q$ . We have

$$|K1_{q_1} + (I\pi_1(S, q_1))^n| = |K1_q + (I\pi_1(S, q))^n|$$

for any  $n \geq 1$  and any other  $q_1 \in S$ . In fact, if we choose  $\gamma \in \Pi S(q_1, q)$ , then  $\gamma(K1_q + (I\pi_1(S, q))^n)\gamma^{-1} = K1_{q_1} + (I\pi_1(S, q_1))^n$ . Hence we may define

$$K\hat{\pi}(S)(n) := |K1_q + (I\pi_1(S, q))^n|.$$

In the general case, we define

$$K\hat{\pi}(S)(n) := \bigoplus_{\lambda \in \pi_0 S} K\hat{\pi}(S_\lambda)(n)$$

for any  $n \geq 1$ . In any cases, we have  $K\hat{\pi}(S)(1) = K\hat{\pi}(S)$ .

**Theorem 4.1.1.** *In the situation stated at the beginning of this section, we have*

$$\sigma(K\hat{\pi}(S^*)(n)) \subset F_{n-2}\text{Der}(K\mathcal{C}(S, E))$$

for any  $n \geq 1$ .

*Proof.* We may assume  $S$  is connected. Choose  $q \in S \setminus (E \cup \partial S)$ . By Lemma 1.3.1, it suffices to show  $\sigma(|u|)(\gamma) \subset F_{n-1}K\mathcal{C}(p_0, p_1)$  for any  $n \geq 1$ ,  $u \in I\pi_1(S, q)^n$ ,  $p_0, p_1 \in E$ , and  $\gamma \in \mathcal{C}(p_0, p_1)$ . The  $K$ -module  $I\pi_1(S, q)^n$  is generated by the set  $\{(x_1 - y_1)(x_2 - y_2) \cdots (x_n - y_n); x_i, y_i \in \pi_1(S, q)\}$ . Putting  $x_1, x_2, \dots, x_n$  and  $\gamma$  in general position, we may assume  $(x_i \cap \gamma) \cap (x_j \cap \gamma) = \emptyset$  if  $i \neq j$ . Then we have

$$\sigma(|x_1 x_2 \cdots x_n|)(\gamma) = \sum_{i=1}^n \sum_{p \in x_i \cap \gamma} \varepsilon(p; x_i, \gamma) \gamma_{p_0 p}(x_i)_{p q} x_{i+1} \cdots x_n x_1 \cdots x_{i-1} (x_i)_{q p} \gamma_{p p_1},$$

and so

$$\begin{aligned} & \sigma(|(x_1 - y_1)(x_2 - y_2) \cdots (x_n - y_n)|)(\gamma) \\ &= \sum_{i=1}^n \sum_{p \in x_i \cap \gamma} \varepsilon(p; x_i, \gamma) \gamma_{p_0 p}(x_i)_{p q} (x_{i+1} - y_{i+1}) \cdots (x_{i-1} - y_{i-1}) (x_i)_{q p} \gamma_{p p_1} \\ & \quad - \sum_{i=1}^n \sum_{p \in y_i \cap \gamma} \varepsilon(p; y_i, \gamma) \gamma_{p_0 p}(y_i)_{p q} (x_{i+1} - y_{i+1}) \cdots (x_{i-1} - y_{i-1}) (y_i)_{q p} \gamma_{p p_1}, \end{aligned}$$

which is in  $F_{n-1}K\mathcal{C}(p_0, p_1)$ . This completes the proof.  $\square$

As a by-product we see the filtration of  $K\hat{\pi}(S)$  is compatible with the bracket.

**Theorem 4.1.2.** *Let  $S$  be an oriented surface. Then we have*

$$[K\hat{\pi}(S)(n_1), K\hat{\pi}(S)(n_2)] \subset K\hat{\pi}(S)(n_1 + n_2 - 2)$$

for any  $n_1, n_2 \geq 1$ .

*Proof.* We may assume  $S$  is connected. Choose a point  $p \in S \setminus \partial S$ , and set  $E = \{p\}$ . By Theorem 4.1.1,  $\sigma(u)(v) \in F_{n_1+n_2-2}K\mathcal{C}(p, p) = (I\pi_1(S, p))^{n_1+n_2-2}$  for any  $u \in K\hat{\pi}(S^*)(n_1)$  and  $v \in (I\pi_1(S, p))^{n_2}$ . Hence  $[u, v] = |\sigma(u)(v)| \in K\hat{\pi}(S)(n_1 + n_2 - 2)$ . On the other hand, the inclusion homomorphism  $\pi_1(S^*, q) \rightarrow \pi_1(S, q)$  is surjective for any  $q \in S^* = S \setminus \{p\}$ . Hence the inclusion homomorphism  $K\hat{\pi}(S^*)(n) \rightarrow K\hat{\pi}(S)(n)$  is also surjective for any  $n \geq 1$ . This proves the theorem.  $\square$

In particular, if  $n \geq 2$ ,  $K\hat{\pi}(S)(n)$  is a Lie subalgebra of  $K\hat{\pi}(S)$  and an ideal of  $K\hat{\pi}(S)(2)$ . We define

$$\widehat{K\hat{\pi}(S)} := \varprojlim_{n \rightarrow \infty} K\hat{\pi}(S)/K\hat{\pi}(S)(n),$$

and call it *the completed Goldman Lie algebra* of the surface  $S$ . It is a  $K$ -Lie algebra from Theorem 4.1.2. We define

$$\widehat{K\hat{\pi}(S)}(n) := \varprojlim_{m \rightarrow \infty} K\hat{\pi}(S)(n)/K\hat{\pi}(S)(m)$$

for  $n \geq 1$ . From Theorem 4.1.2 we have

$$[\widehat{K\hat{\pi}(S)}(n_1), \widehat{K\hat{\pi}(S)}(n_2)] \subset \widehat{K\hat{\pi}(S)}(n_1 + n_2 - 2)$$

for any  $n_1, n_2 \geq 1$ . For any  $q \in S$  the quotient map  $||: \pi_1(S, q) \rightarrow \hat{\pi}(S)$  induces a natural map

$$||: K\widehat{\pi_1(S, q)} \rightarrow \widehat{K\hat{\pi}(S)}.$$

If  $E \subset S$  is a non-empty closed subset with the property  $E \setminus \partial S$  is closed in  $S$ , then the homomorphism  $\sigma$  induces a natural homomorphism of  $K$ -Lie algebras

$$\sigma: \widehat{K\hat{\pi}(S^*)} \rightarrow \text{Der}(\widehat{K\mathcal{C}(S, E)})$$

by Theorem 4.1.1. Since  $\sigma(K\hat{\pi}(S^*)) \subset F_{-1}\text{Der}K\mathcal{C}$ , we have  $\sigma(\widehat{K\hat{\pi}(S^*)}) \subset F_{-1}\text{Der}\widehat{K\mathcal{C}}$ . This implies the action of any element of  $\widehat{K\hat{\pi}(S^*)}$  on  $\widehat{K\mathcal{C}(S, E)}$  is continuous with respect to the topology induced by the filtration  $\{F_n K\mathcal{C}(S, E)\}_{n \geq 0}$ .

As will be shown in §6.1, if  $S = \Sigma_{g,1}$  and  $E \subset \partial\Sigma_{g,1}$ , then the Lie algebra  $\widehat{K\hat{\pi}(\Sigma_{g,1})}$  is isomorphic to the Lie algebra of symplectic derivations of the completed tensor algebra  $\prod_{m=0}^{\infty} H_1(\Sigma_{g,1}; K)^{\otimes m}$  and  $\widehat{K\hat{\pi}(\Sigma_{g,1})}(2)$  a completion of Kontsevich's ‘associative’  $a_g$  [15]. These isomorphisms are essentially due to our previous work [13].

## 4.2 Action on $\mathcal{C}(S, E)^{\text{abel}}$

Let  $S$  be an oriented surface, and  $E$  a non-empty closed subset of  $S$  with the property  $E \setminus \partial S$  is closed in  $S$ . We consider the abelianization of the groupoid  $\mathcal{C} = \mathcal{C}(S, E)$  introduced in §2.2. From Theorem 4.1.1 we have  $\sigma((x-1)^2) := \sigma(|(x-1)^2|) \in F_0\text{Der}\mathbb{Z}\mathcal{C}(S, E)$  for any  $q \in S^*$  and  $x \in \pi_1(S^*, q)$ . Hence it induces a derivation of  $\mathcal{C}(S, E)^{\text{abel}} = \mathbb{Z}\mathcal{C}(S, E)/F_2\mathbb{Z}\mathcal{C}(S, E)$ , which we denote  $\bar{\sigma}((x-1)^2) \in \text{Der}\mathcal{C}(S, E)^{\text{abel}}$ .

**Lemma 4.2.1.** (1) *We have  $\bar{\sigma}((x-1)^2)(\gamma) = 2(x \cdot \gamma)\gamma(x-1) \in \mathcal{C}^{\text{abel}}(p_0, p_1)$  for any  $p_0, p_1 \in [q] \subset E$  and  $\gamma \in \mathcal{C}(p_0, p_1)$ . Here  $(x \cdot \gamma) \in K$  means the algebraic intersection number  $H_1(S^*; K) \times H_1(S, E; K) \rightarrow K$  and we regard  $x-1$  as an element of  $HC([q])$ . In particular,  $\sigma((x-1)^2)(\mathbb{Z}\mathcal{C}(p_0, p_1)) \subset F_1\mathbb{Z}\mathcal{C}(p_0, p_1)$ .*

(2) *The square  $\bar{\sigma}((x-1)^2)^2 \in \text{End}(\mathcal{C}^{\text{abel}}(p_0, p_1))$  vanishes for any  $p_0, p_1 \in [q] \subset E$ .*

*Proof.* (1) We remark  $0 = (x - 1)^2 = (x^2 - x) - (x - 1) \in HC([q])$ . Put  $x$  and  $\gamma$  in general position. Then we have

$$\begin{aligned} & \sigma((x - 1)^2)(\gamma) = \sigma(x^2)(\gamma) - 2\sigma(x)(\gamma) \\ &= 2 \sum_{p \in x \cap \gamma} \varepsilon(p; x, \gamma) \gamma_{p_0 p} x_p^2 \gamma_{pp_1} - 2 \sum_{p \in x \cap \gamma} \varepsilon(p; x, \gamma) \gamma_{p_0 p} x_p \gamma_{pp_1} \\ &= 2\gamma \sum_{p \in x \cap \gamma} \varepsilon(p; x, \gamma) \gamma_{pp_1}^{-1} (x_p^2 - x_p) \gamma_{pp_1}, \end{aligned}$$

and so

$$\bar{\sigma}((x - 1)^2)(\gamma) = 2(x \cdot \gamma) \gamma (x^2 - x) = 2(x \cdot \gamma) \gamma (x - 1).$$

(2) We write simply  $D = \bar{\sigma}((x - 1)^2)$ . Then we have

$$\begin{aligned} & D^2 \gamma = 2(x \cdot \gamma) D(\gamma x - \gamma) \\ &= 2(x \cdot \gamma) \cdot 2(x \cdot \gamma x) \gamma x (x - 1) - 2(x \cdot \gamma) \cdot 2(x \cdot \gamma) \gamma (x - 1) \\ &= 4(x \cdot \gamma)^2 \gamma (x - 1)^2 = 0. \end{aligned}$$

The last equality follows from  $(x - 1)^2 = 0 \in HC([q])$ . This completes the proof.  $\square$

### 4.3 Homological interpretation of the completed Goldman Lie algebra

Let  $S$  be a surface of finite type and not closed. In this subsection we give a natural identification of the completed Goldman Lie algebra of  $S$  with the first homology group of the surface with some twisted coefficients. As a corollary, we prove the completion map of the Goldman Lie algebra  $\rho: K\hat{\pi}(S)/K1 \rightarrow \widehat{K\hat{\pi}(S)}$  is injective. Here  $1 \in \hat{\pi}(S)$  is the constant loop. For the proof of this identification we use a group-like expansion of the fundamental group  $\pi_1(S)$  [18]. We adopt the same notation as in §3.3.

First of all, we introduce two local systems  $\mathcal{S}^c(S)$  and  $\widehat{\mathcal{S}}^c(S)$  on  $S$ . The stalks at  $p \in S$  are given by

$$\mathcal{S}^c(S)_p := K\pi_1(S, p), \quad \text{and} \quad \widehat{\mathcal{S}}^c(S)_p := \widehat{K\pi_1(S, p)},$$

respectively. Since  $\pi_1(S)$  is a free group, the completion map  $\mathcal{S}^c(S) \rightarrow \widehat{\mathcal{S}}^c(S)$  is injective [3], and  $H_2(S; \widehat{\mathcal{S}}^c(S)/\mathcal{S}^c(S)) = 0$ . Hence the induced homomorphism  $H_1(S; \mathcal{S}^c(S)) \rightarrow H_1(S; \widehat{\mathcal{S}}^c(S))$  is injective. In the sequel we regard the former as a submodule of the latter by this injection.

In [13] §3.4, we introduce a  $K$ -linear map

$$\lambda: K\hat{\pi}(S) \rightarrow H_1(S; \mathcal{S}^c(S)).$$

This maps  $\alpha \in \hat{\pi}(S)$  to the homology class induced by the section  $s_\lambda(\alpha) \in \Gamma(\alpha^* \mathcal{S}^c(S))$  given by  $s_\lambda(\alpha)(t) := \alpha_{\alpha(t)} \in K\pi_1(S, \alpha(t))$ ,  $t \in S^1$ . The kernel of the map  $\lambda$  is spanned by the constant loop 1,  $\text{Ker} \lambda = K1$  [13] Proposition 3.4.3 (1).

The  $K$ -bilinear map  $\mathcal{B}_p: K\pi_1(S, p) \otimes K\pi_1(S, p) \rightarrow K\hat{\pi}(S)$ ,  $u \otimes v \mapsto |uv|$ , with the intersection form on the surface  $S$  defines the pairing  $\mathcal{B}(\cdot): H_1(S; \mathcal{S}^c(S))^{\otimes 2} \rightarrow K\hat{\pi}(S)$ . As was shown in [13] Proposition 3.4.3 (2), we have

$$[u, v] = \mathcal{B}(\lambda(u) \cdot \lambda(v)) \tag{4.3.1}$$

for any  $u$  and  $v \in K\hat{\pi}(S)$ . Similarly we have the pairing  $\widehat{\mathcal{B}}(\cdot): H_1(S; \widehat{\mathcal{S}}^c(S))^{\otimes 2} \rightarrow \widehat{K\hat{\pi}}(S)$  induced by the  $K$ -bilinear map  $\widehat{\mathcal{B}}_p: K\widehat{\pi_1}(S, p) \otimes K\widehat{\pi_1}(S, p) \rightarrow \widehat{K\hat{\pi}}(S)$ ,  $u \otimes v \mapsto |uv|$ .

**Theorem 4.3.1.** *Let  $S$  be a surface of finite type and not closed, and  $K$  a commutative ring including  $\mathbb{Q}$ . Then the map  $\lambda$  extends to an isomorphism*

$$\widehat{\lambda}: \widehat{K\hat{\pi}}(S) \xrightarrow{\cong} H_1(S; \widehat{\mathcal{S}}^c(S)).$$

This satisfies

$$\widehat{\lambda}([u, v]) = \widehat{\mathcal{B}}(\widehat{\lambda}(u) \cdot \widehat{\lambda}(v))$$

for any  $u$  and  $v \in \widehat{K\hat{\pi}}(S)$ .

As a corollary we have

**Corollary 4.3.2.** *Let  $S$  be a surface of finite type and not closed, and  $K$  a commutative ring including  $\mathbb{Q}$ . Then the kernel of the completion map of the Goldman Lie algebra  $\rho: K\hat{\pi}(S) \rightarrow \widehat{K\hat{\pi}}(S)$  is spanned by the constant loop  $1 \in \hat{\pi}(S)$ .*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} K\hat{\pi}(S) & \xrightarrow{\lambda} & H_1(S; \mathcal{S}^c(S)) \\ \downarrow & & \downarrow \\ \widehat{K\hat{\pi}}(S) & \xrightarrow{\widehat{\lambda}} & H_1(S; \widehat{\mathcal{S}}^c(S)). \end{array}$$

From Theorem 4.3.1 the map  $\widehat{\lambda}$  is an isomorphism, while the right vertical arrow is injective. Hence the kernel of the completion map equals  $\text{Ker}\lambda = K1$ .  $\square$

To prove Theorem 4.3.1, we use a group-like expansion of the fundamental group  $\pi_1(S)$ . Choose a basepoint  $*$   $\in S$  and denote  $\pi := \pi_1(S, *)$ . Let  $\theta: \pi \rightarrow \widehat{T}$  be a group-like expansion. See Definition 3.3.8.

We define a linear map  $N: \widehat{T} \rightarrow \widehat{T}$  by  $N|_{H^{\otimes 0}} := 0$  and

$$N(X_1 \cdots X_n) := \sum_{i=1}^n X_i \cdots X_n X_1 \cdots X_{i-1}, \quad (4.3.2)$$

for  $X_j \in H$ ,  $n \geq 1$ . Then  $\theta$  induces an isomorphism  $\theta_*: H_1(S; \widehat{\mathcal{S}}^c(S)) \xrightarrow{\cong} N(\widehat{T}_1)$  [13] (5.3.1), Lemma 6.1.1. Moreover the composite  $\theta_* \circ \lambda: K\hat{\pi} \rightarrow N(\widehat{T}_1)$  equals the map  $\lambda_\theta: K\hat{\pi} \rightarrow N(\widehat{T}_1)$  defined by  $\lambda_\theta(|x|) := N\theta(x)$ ,  $x \in \pi$  [13] Lemma 6.3.2. Here we should remark that the proofs of the lemma and the proposition in [13] work well for group-like expansions over a commutative ring including  $\mathbb{Q}$  as well as for symplectic expansions over the rationals  $\mathbb{Q}$ . The key to proving the injectivity of the map  $\widehat{\lambda}$  is the following lemma.

**Lemma 4.3.3.** *For any  $n \geq 1$  we have*

$$\lambda_\theta^{-1}(N(\widehat{T}_n)) = |K1 + I\pi^n| (= K\hat{\pi}(S)(n)).$$

To prove this lemma, we need the following.



**Lemma 4.3.4.**  $0 \rightarrow K1 \oplus [\widehat{T}, \widehat{T}] \hookrightarrow \widehat{T} \xrightarrow{N} N(\widehat{T}_1) \rightarrow 0$  (exact).

*Proof.* Since  $N1 = 0$  and  $N([u, v]) = N(uv - vu) = 0$  for  $u, v \in \widehat{T}$ , we have  $K1 \oplus [\widehat{T}, \widehat{T}] \subset \text{Ker}N$ . Since  $N$  is homogeneous it suffices to show

$$(\text{Ker}N) \cap H^{\otimes n} \subset (K1 \oplus [\widehat{T}, \widehat{T}]) \cap H^{\otimes n},$$

for any  $n \geq 0$ . It is clear in the case  $n = 0$ . For any  $X_j \in H$  and  $n \geq 1$ , we have

$$\begin{aligned} X_1 \cdots X_n - \frac{1}{n}N(X_1 \cdots X_n) &= \frac{1}{n} \sum_{i=1}^n X_1 \cdots X_n - X_i \cdots X_n X_1 \cdots X_{i-1} \\ &= \frac{1}{n} \sum_{i=2}^n [X_1 \cdots X_{i-1}, X_i \cdots X_n] \in [\widehat{T}, \widehat{T}]. \end{aligned}$$

Hence  $u - \frac{1}{n}Nu \in [\widehat{T}, \widehat{T}]$  for any  $u \in H^{\otimes n}$ . This proves the lemma.  $\square$

*Proof of Lemma 4.3.3.* Since  $\theta(I\pi^n) \subset \widehat{T}_n$ , we have  $\lambda_\theta(|I\pi^n|) \subset N(\widehat{T}_n)$ . It is clear  $\lambda_\theta(1) = N(1) = 0$ . Hence  $|K1 + I\pi^n| \subset \lambda_\theta^{-1}(N(\widehat{T}_n))$ .

Suppose  $u \in K\pi$  satisfies  $N\theta(u) \in N(\widehat{T}_n)$ . From Lemma 4.3.4,  $\theta(u) \in \widehat{T}_n + K1 + [\widehat{T}, \widehat{T}]$ . This means  $\theta(u - \varepsilon(u)1) - \sum_{i=1}^m [v'_i, w'_i] \in \widehat{T}_n$  for some  $v'_i, w'_i \in \widehat{T}$ . There exist  $v_i$  and  $w_i \in K\pi$  such that  $v'_i - \theta(v_i)$  and  $w'_i - \theta(w_i) \in \widehat{T}_n$ , since  $\widehat{T}/\widehat{T}_n \cong_{\theta} \widehat{K\pi}/\widehat{I\pi^n} = K\pi/I\pi^n$  by (3.3.3). This implies  $\theta(u - \varepsilon(u)1 - \sum_{i=1}^m [v_i, w_i]) \in \widehat{T}_n$ . Since  $\widehat{T}/\widehat{T}_n \cong K\pi/I\pi^n$ ,  $z := u - \varepsilon(u)1 - \sum_{i=1}^m [v_i, w_i] \in I\pi^n$ . Hence we have  $|u| = |\varepsilon(u)1 + z| \in |K1 + I\pi^n|$ . This completes the proof.  $\square$

*Proof of Theorem 4.3.1.* From Lemma 4.3.3 the map  $\lambda_\theta$  induces an injective linear map  $\lambda_\theta: K\widehat{\pi}(S)/K\widehat{\pi}(S)(n) \hookrightarrow N(\widehat{T}_1)/N(\widehat{T}_n)$  for any  $n \geq 1$ . This map is surjective. In fact, for any  $u \in \widehat{T}_1$ , there exists some  $z \in K\pi$  such that  $u - \theta(z) \in \widehat{T}_n$ , and so  $Nu - \lambda_\theta(z) \in N(\widehat{T}_n)$ . Hence we have an isomorphism

$$\lambda_\theta: K\widehat{\pi}(S)/K\widehat{\pi}(S)(n) \xrightarrow{\cong} N(\widehat{T}_1)/N(\widehat{T}_n)$$

for any  $n \geq 1$ . Taking the projective limits, we obtain an isomorphism  $\widehat{\lambda}_\theta: \widehat{K\pi}(S) \xrightarrow{\cong} \varprojlim_{n \rightarrow \infty} N(\widehat{T}_1)/N(\widehat{T}_n) = N(\widehat{T}_1)$ , which preserves the filtrations. Hence we obtain an isomorphism  $\widehat{\lambda} := \theta_*^{-1} \circ \widehat{\lambda}_\theta: \widehat{K\pi}(S) \rightarrow H_1(S; \widehat{\mathcal{S}^c}(S))$ , which is independent of the choice of a group-like expansion  $\theta$ . The latter half of the theorem follows immediately from (4.3.1). This completes the proof.  $\square$

## 5 Dehn twists

In this section we suppose  $K$  is a commutative ring including the rationals  $\mathbb{Q}$ . We shall generalize results in [13] to any oriented surface  $S$ , and those in [17] to any surface of finite type with non-empty boundary. Let  $E$  be a non-empty closed subset of  $S$  with the property  $E \setminus \partial S$  is closed in  $S$ . We consider the completed Goldman Lie algebra  $\widehat{K\pi}(S^*)$  of  $S^* = S \setminus (E \setminus \partial S)$  and the homomorphism of Lie algebras  $\sigma: \widehat{K\pi}(S^*) \rightarrow \text{Der}(\widehat{K\mathcal{C}}(S, E))$ .

## 5.1 An invariant of unoriented free loops

We begin by defining an invariant of an unoriented free loop  $C$  in  $S \setminus (E \cup \partial S) = S^* \setminus \partial S$ .

**Lemma 5.1.1.** *Let  $f(t) \in K[[t-1]]$  be a formal power series in  $t-1$  with  $f(1) = f'(1) = 0$ .*

- (1) *For  $\alpha \in \widehat{\pi}(S^*)$ , choose a point  $q \in S^*$  and a based loop  $x \in \pi_1(S^*, q)$  representing the free loop  $\alpha$ . Then*

$$f(\alpha) := |f(x)| \in \widehat{K\pi}(S^*)(2)$$

*is well-defined. In other words,  $|f(x)|$  does not depend on the choice of  $q$  and  $x$ .*

- (2) *If  $f(t) = f(t^{-1})$ , then  $f(\alpha) = f(\alpha^{-1})$ . In particular, we may define  $f(C) := f(\alpha) \in \widehat{K\pi}(S^*)(2)$  if  $C = \alpha^{\pm 1}$ , namely, the unoriented free loop  $C$  is represented by an oriented free loop  $\alpha$ .*

- (3)  *$\sigma(f(\alpha)) \in \text{Der}\widehat{K\mathcal{C}}$  satisfies the three conditions (i)-(iii) in Lemma 1.3.2. In particular, we can define the exponential  $\exp(\sigma(f(\alpha))) \in \text{Aut}\widehat{K\mathcal{C}}$ . It satisfies the conditions (i)-(iii) in Definition 3.3.1.*

*Proof.* (1) Suppose  $q_1 \in S^*$  and  $x_1 \in \pi_1(S^*, q_1)$  satisfy  $|x| = |x_1|$ . Then we have  $x_1 = \gamma^{-1}x\gamma$  for some  $\gamma \in \Pi S^*(q, q_1)$ , and so  $f(x_1) = \gamma^{-1}f(x)\gamma \in \widehat{K\Pi S^*}(q_1, q_1)$ . This implies  $|f(x_1)| = |f(x)| \in \widehat{K\pi}(S^*)$ .

(2) is clear.

(3) Since  $f(\alpha) \in \widehat{K\pi}(S^*)(2)$ , we have  $\sigma(f(\alpha)) \in F_0\text{Der}\widehat{K\mathcal{C}}$  by Theorem 4.1.1. On the other hand,  $\sigma((x-1)^2)$  satisfies the conditions (ii) and (iii) from Lemma 4.2.1. Now we have

$$f(x) \equiv c(x-1)^2 \pmod{(x-1)^3},$$

for some constant  $c \in K$ , and any element in  $(x-1)^3K[[x-1]]$  induces an element of  $F_1\text{Der}\widehat{K\mathcal{C}}$  by Theorem 4.1.1. Hence  $\sigma(f(\alpha))$  satisfies all the conditions (i)-(iii) in Lemma 1.3.2. The condition (i) in Definition 3.3.1 for  $\exp(\sigma(f(\alpha)))$  follows from the fact  $\sigma(f(\alpha)) \in F_0\text{Der}\widehat{K\mathcal{C}}$ , (ii) from  $\alpha \cap E = \emptyset$ , and (iii) from Lemma 4.2.1.  $\square$

Now we define

$$L(t) := \frac{1}{2}(\log t)^2 \in \mathbb{Q}[[t-1]].$$

Here we remark  $tL'(t) = \log t$ . From Lemma 5.1.1 we obtain  $L(C) \in \widehat{K\pi}(S^*)(2)$  and  $\exp(\sigma(L(C))) \in \text{Aut}\widehat{K\mathcal{C}}$  for any unoriented free loop  $C$  in  $S^*$ . Furthermore we have

**Lemma 5.1.2.**

$$\exp(\sigma(L(C))) \in A(S, E).$$

*Proof.* It suffices to prove  $\sigma(L(C)) \in \text{Der}_\Delta\widehat{K\mathcal{C}}$ , see §2.1. Choose  $\alpha \in \widehat{\pi}(S^*)$  such that  $C = \alpha^{\pm 1}$ . For any  $\gamma \in \mathcal{C}(p_0, p_1)$ ,  $p_0, p_1 \in E$ , and  $n \geq 0$ , we have  $\sigma(\alpha^n)(\gamma) = \sum_{p \in \alpha \cap \gamma} n\varepsilon(p; \alpha, \gamma)\gamma_{*p}\alpha_p^n\gamma_{p*}$ , so that

$$\sigma(f(\alpha))(\gamma) = \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma)\gamma_{*p}\alpha_p f'(\alpha_p)\gamma_{p*}$$

for any  $f(t) \in K[[t-1]]$ . In particular,

$$\sigma(L(C))(\gamma) = \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma)\gamma_{*p}(\log \alpha_p)\gamma_{p*}. \quad (5.1.1)$$

On the other hand, we have  $\Delta(\log \alpha_p) = (\log \alpha_p) \widehat{\otimes} 1 + 1 \widehat{\otimes} (\log \alpha_p) \in K \widehat{\pi_1}(S, p)$ . Hence

$$\begin{aligned} \Delta\sigma(L(C))\gamma &= \sum_{p \in \alpha \cap \gamma} \varepsilon(p; \alpha, \gamma) (\gamma_{*p} \widehat{\otimes} \gamma_{*p}) (\log \alpha_p \widehat{\otimes} 1 + 1 \widehat{\otimes} \log \alpha_p) (\gamma_{p*} \widehat{\otimes} \gamma_{p*}) \\ &= \sigma(L(C))\gamma \widehat{\otimes} \gamma + \gamma \widehat{\otimes} \sigma(L(C))\gamma = (\sigma(L(C)) \widehat{\otimes} 1 + 1 \widehat{\otimes} \sigma(L(C))) \Delta\gamma. \end{aligned}$$

This means  $\sigma(L(C)) \in \text{Der}_{\Delta} \widehat{K\mathcal{C}}$ , and proves the lemma.  $\square$

Let us go back to the situation of Proposition 3.3.4. We take  $\eta_i \in \pi_1(N, *_i)$  in the positive direction. Then we define  $F^U \in \widehat{K\hat{\pi}}(N)$  by

$$F^U := \sum_{i=1}^n a_i^U L(\partial_i N).$$

By Lemmas 5.1.1 and 5.1.2,  $F^U \in \widehat{K\hat{\pi}}(N)(2)$  and  $\exp \sigma(F^U) \in A(N, \partial N)$ . From the construction, we have

$$U = \exp \sigma(F^U) \in A(N, \partial N). \quad (5.1.2)$$

## 5.2 The logarithm of Dehn twists

Recall the Dehn-Nielsen homomorphism  $\widehat{\text{DN}}: \mathcal{M}(S, E) \rightarrow \text{Aut}(K \widehat{\mathcal{C}}(S, E))$ . The following theorem is a generalization of a part of our previous result [13] Theorem 1.1.1, which does not involve a symplectic expansion and the total Johnson map, to any oriented surfaces.

**Theorem 5.2.1.** *Let  $S$  be an oriented surface and  $E$  a non-empty closed subset of  $S$  with the property  $E \setminus \partial S$  is closed in  $S$ . Then the Dehn-Nielsen homomorphism  $\widehat{\text{DN}}$  maps the right handed Dehn twist  $t_C$  along a simple closed curve  $C$  in  $S \setminus (E \cup \partial S)$  to*

$$\widehat{\text{DN}}(t_C) = \exp(\sigma(L(C))) \in \text{Aut} \widehat{K\mathcal{C}}(S, E).$$

*Proof.* We begin by computing  $\widehat{\text{DN}}(t_C)$  in the case  $S$  is an annulus  $\mathbf{a} = S^1 \times [0, 1]$ . We regard  $S^1 = [0, 1]/(0 \sim 1)$ , and define  $p_0 := (0 \bmod \sim, 0)$ ,  $p_1 := (0 \bmod \sim, 1)$ ,  $E := \{p_0, p_1\}$  and  $\mathcal{C} = \mathcal{C}(\mathbf{a}, E)$ . Consider a path  $\gamma_0: [0, 1] \rightarrow \mathbf{a}$  given by  $t \in [0, 1] \mapsto (0 \bmod \sim, t)$ , a based loop  $x \in \pi_1(\mathbf{a}, p_1)$  given by  $t \in [0, 1] \mapsto (t \bmod \sim, 1)$ , and a simple closed curve  $C = |x^{\pm 1}|$ . We have  $\sigma(|x^n|)(\gamma_0) = n\gamma_0 x^n$  for any  $n \geq 0$ , and  $\sigma(|x^n|)$  acts trivially on  $K\mathcal{C}(p_0, p_0)$  and  $K\mathcal{C}(p_1, p_1)$ . Hence, for any formal power series  $f(x) \in K \widehat{\pi_1}(\mathbf{a}, p_1)$  in  $x - 1$ , the derivation  $\sigma(f(x))$  acts trivially on  $\widehat{K\mathcal{C}}(p_0, p_0)$  and  $\widehat{K\mathcal{C}}(p_1, p_1)$ , and  $\sigma(f(x))(\gamma_0) = \gamma_0 x f'(x) \in \widehat{K\mathcal{C}}(p_0, p_1)$ . Since  $tL'(t) = \log(t)$ ,  $\sigma(L(C))(\gamma_0) = \sigma(L(x))(\gamma_0) = \gamma_0 \log x$ . Clearly  $\exp(\sigma(L(C)))(x) = x = \text{DN}(t_C)(x)$ . Hence we have

$$\exp(\sigma(L(C)))(\gamma_0) = \gamma_0 x = \text{DN}(t_C)(\gamma_0).$$

This proves

$$\exp(\sigma(L(C))) = \widehat{\text{DN}}(t_C) \in \text{Aut} \widehat{K\mathcal{C}}(\mathbf{a}, E), \quad (5.2.1)$$

namely, the theorem in the case  $S$  is an annulus.

Next we consider the general case. Choose a closed tubular neighborhood  $\mathbf{a}$  of the simple closed curve  $C$  in the surface  $S \setminus (E \cup \partial S)$ . The boundary  $\partial \mathbf{a}$  has two connected

components  $\partial_0 \mathbf{a}$  and  $\partial_1 \mathbf{a}$ . Choose a point  $q_i$  on each  $\partial_i \mathbf{a}$ ,  $i = 0, 1$ . We define  $S_1 := S \setminus \text{inta}$ ,  $S_2 := \mathbf{a}$ ,  $E_1 := E \cup \{q_0, q_1\}$  and  $E_2 := \{q_0, q_1\}$ . Then, in the setting of Proposition 3.2.1, we have  $S_3 = S_1 \cup S_2 = S$ ,  $E_3 = E_1$  and  $K\mathcal{C}_3 = K\mathcal{C}(S, E_1)$  is generated by  $K\mathcal{C}_1$  and  $K\mathcal{C}_2$ . We may regard  $L(C) \in \widehat{K\hat{\pi}(S_3^*)}(2)$  for  $S_3^* = S^* \setminus \{q_0, q_1\}$ . Both of  $\exp(\sigma(L(C)))$  and  $\widehat{DN}(t_C)$  act trivially on  $K\mathcal{C}_1$ , and coincide with each other on  $K\mathcal{C}_2$  by (5.2.1). Hence, by Proposition 3.2.1, they coincide with each other on  $K\mathcal{C}_3 = K\mathcal{C}(S, E_1)$ . Since both of them are continuous and  $K\mathcal{C}_3$  is dense in  $\widehat{K\mathcal{C}_3}$ , they coincide with each other on  $\widehat{K\mathcal{C}_3}$ . Since  $\widehat{K\mathcal{C}(S, E)}$  is a full subcategory of  $\widehat{K\mathcal{C}_3}$ , they coincide with each other on  $\widehat{K\mathcal{C}(S, E)}$ . This completes the proof of the theorem.  $\square$

### 5.3 Generalized Dehn twists and their localization

Theorem 5.2.1 motivates us to define a generalization of Dehn twists for not necessarily simple loops. Let  $C$  be an unoriented free loop in  $S^* \setminus \partial S$ .

**Definition 5.3.1.** *The generalized Dehn twist along  $C$  is defined as*

$$t_C := \exp(\sigma(L(C))) \in A(S, E) \subset \text{Aut}\widehat{K\mathcal{C}(S, E)}.$$

The case  $S = \Sigma_{g,1}$  and  $E = \{*\}$ , where  $* \in \partial S$ , is treated in [17].

It is natural to ask whether  $t_C$  is realizable as a diffeomorphism, i.e., is in the image of  $\widehat{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut}\widehat{K\mathcal{C}(S, E)}$ . In this subsection we shall give a criterion for the realizability of  $t_C$ . To restrict ourself to the case  $\widehat{DN}$  is injective, hereafter we assume  $S$  is of finite type with non-empty boundary and work under the assumption of Theorem 3.1.1.

Let  $\text{End}(K\hat{\pi}(S))$  be the space of filter-preserving endomorphisms of  $K\hat{\pi}(S)$ . Also let  $\text{Aut}(K\hat{\pi}(S))$  be the group of filter-preserving  $K$ -linear automorphisms of  $K\hat{\pi}(S)$ .

**Lemma 5.3.2.** (1) *Let  $D \in \text{Der}K\mathcal{C}(S, E)$ . For  $a \in K\mathcal{C}(p, p)$ , where  $p \in E$ , set  $|D|(|a|) := |D(a)|$ . Then this defines a well-defined  $K$ -linear map  $| \cdot | : \text{Der}K\mathcal{C}(S, E) \rightarrow \text{End}(K\hat{\pi}(S))$ .*

(2) *Let  $U \in \text{Aut}(K\mathcal{C}(S, E))$ . For  $a \in K\mathcal{C}(p, p)$ , where  $p \in E$ , set  $|U|(|a|) := |U(a)|$ . Then this defines a well-defined group homomorphism  $| \cdot | : \text{Aut}K\mathcal{C}(S, E) \rightarrow \text{Aut}(K\hat{\pi}(S))$ .*

*Proof.* Let  $p, q \in E$  and assume  $[p] = [q] = \lambda \in \pi_0 \mathcal{C}$ . Recall that  $| \cdot | : K\mathcal{C}(p, p) \rightarrow K\hat{\pi}(S_\lambda)$  is surjective for any  $p \in E$ . Take some  $\gamma \in \Pi S(q, p)$  and let  $a \in K\mathcal{C}(p, p)$ .

(1) It is sufficient to prove  $|D(a)| = |D(\gamma a \gamma^{-1})|$ . First of all, since  $0 = D(1) = D(\gamma \gamma^{-1}) = (D\gamma)\gamma^{-1} + \gamma D(\gamma^{-1})$ , we have  $D(\gamma^{-1}) = -\gamma^{-1}(D\gamma)\gamma^{-1}$ . We compute  $D(\gamma a \gamma^{-1}) = (D\gamma)a\gamma^{-1} + \gamma(Da)\gamma^{-1} + \gamma a D(\gamma^{-1}) = \gamma(Da)\gamma^{-1} + (D\gamma)a\gamma^{-1} - \gamma a \gamma^{-1}(D\gamma)\gamma^{-1}$ . Notice that  $|(D\gamma)a\gamma^{-1}| = |\gamma a \gamma^{-1}(D\gamma)\gamma^{-1}|$ . Hence  $|D(\gamma a \gamma^{-1})| = |\gamma(Da)\gamma^{-1}| = |D(a)|$ , as desired.

(2) This is clear from  $|U(\gamma a \gamma^{-1})| = |U(\gamma)UaU(\gamma)^{-1}| = |U(a)|$ .  $\square$

Any filter-preserving endomorphism (resp. automorphism) of  $K\hat{\pi}(S)$  naturally extends to an endomorphism (resp. automorphism) of  $\widehat{K\hat{\pi}(S)}$ . Consequently we have a  $K$ -linear map  $\text{End}(K\hat{\pi}(S)) \rightarrow \text{End}(\widehat{K\hat{\pi}(S)})$  and a group homomorphism  $\text{Aut}(K\hat{\pi}(S)) \rightarrow \text{Aut}(\widehat{K\hat{\pi}(S)})$ . The diagrams

$$\begin{array}{ccc} \text{Der}K\mathcal{C}(S, E) & \longrightarrow & \text{End}(K\hat{\pi}(S)) & & \text{Aut}K\mathcal{C}(S, E) & \longrightarrow & \text{Aut}(K\hat{\pi}(S)) \\ \downarrow & & \downarrow & \text{and} & \downarrow & & \downarrow \\ \text{Der}\widehat{K\mathcal{C}(S, E)} & \longrightarrow & \text{End}(\widehat{K\hat{\pi}(S)}) & & \text{Aut}\widehat{K\mathcal{C}(S, E)} & \longrightarrow & \text{Aut}(\widehat{K\hat{\pi}(S)}) \end{array}$$

commute.

The following theorem is a generalization of [17] §3.3.

**Theorem 5.3.3.** *Suppose  $S$  is of finite type with non-empty boundary,  $E \subset \partial S$ , and any connected component of  $\partial S$  has an element of  $E$ . Let  $C$  be an unoriented immersed free loop in  $S \setminus \partial S$  and assume the generalized Dehn twist  $t_C$  is in the image of  $\widehat{\text{DN}}$ . Then there is an orientation preserving diffeomorphism of  $S$  fixing  $\partial S$  pointwise, such that  $\widehat{\text{DN}}(\varphi) = t_C$  and the support of  $\varphi$  lies in a regular neighborhood of  $C$ .*

*Proof.* Take a diffeomorphism  $\varphi$  such that  $\widehat{\text{DN}}(\varphi) = t_C$ . We shall deform  $\varphi$  by isotopies until it has the desired property.

We claim that if  $\delta$  is a proper arc or an oriented loop in  $S$  that is disjoint from  $C$ , then  $\varphi(\delta)$  is isotopic to  $\delta$ . The case  $\delta$  is a proper arc is clear from  $\sigma(L(C))\delta = 0$ . To prove the case  $\delta$  is an oriented loop, consider the composite  $\mathcal{M}(S, E) \xrightarrow{\widehat{\text{DN}}} \text{Aut} \widehat{K\mathcal{C}(S, E)} \xrightarrow{\downarrow} \text{Aut}(\widehat{K\hat{\pi}(S)})$ . As we have noted in the proof of Theorem 4.1.2,  $|\sigma(u)(v)| = [u, |v|]$  for  $u \in K\hat{\pi}(S)$  and  $v \in K\mathcal{C}(p, p)$ , where  $p \in E$ . This implies that  $|\sigma(\alpha)| = \text{ad}(\alpha)$  for  $\alpha \in K\hat{\pi}(S)$ . Therefore  $|\widehat{\text{DN}}(\varphi)| = \exp(\text{ad}(L(C))) \in \text{Aut}(\widehat{K\hat{\pi}(S)})$ . Since  $\delta$  is disjoint from  $C$ ,  $\text{ad}(L(C))\delta = [L(C), \delta] = 0$ . Thus  $|\widehat{\text{DN}}(\varphi)|\delta = \delta \in \widehat{K\hat{\pi}(S)}$ . By Corollary 4.3.2, this implies  $|\text{DN}(\varphi)|\delta - \delta \in K1$ . Since the action of  $\mathcal{M}(S, E)$  on  $K\hat{\pi}(S)$  preserves the augmentation  $K\hat{\pi}(S) \rightarrow K, \hat{\pi} \ni x \mapsto 1$ , we conclude  $|\text{DN}(\varphi)|\delta = \delta$ . The claim is proved.

Let  $N = N(C)$  be a closed regular neighborhood of  $C$ . The Euler characteristic of  $N$  must be non-positive. If  $C$  is simple, then the assertion is clear from Theorem 5.2.1. Thus we may assume  $N$  is neither diffeomorphic to a disk nor an annulus. Let  $S \setminus \text{Int}(N) \cong \coprod_{\lambda} S_{\lambda}$  be the decomposition into connected components. Note that for any  $\lambda$  we have  $S_{\lambda} \cap \partial N \neq \emptyset$ . We shall take a system  $\mathcal{B}_{\lambda}$  of simple closed curves and proper arcs in  $S_{\lambda}$  by the following way. Let  $\chi(S_{\lambda})$  be the Euler characteristic of  $S_{\lambda}$ .

*Case 1.*  $\chi(S_{\lambda}) \geq 0$ . Then  $S_{\lambda}$  is one of the following: (a) a closed disk, (b) an annulus of which both the boundary components are in  $\partial N$ , (c) an annulus of which one of the boundary component is in  $\partial N$ , and the other component is in  $\partial S$ , (d) a once punctured disk. In these cases we let  $\mathcal{B}_{\lambda}$  to be empty.

*Case 2.*  $\chi(S_{\lambda}) = -1$ . Then  $S_{\lambda}$  is one of the following: (e) a torus with one boundary component, (f) a pair of pants of which the three boundary components are in  $\partial N$ , (g) a pair of pants of which two boundary components are in  $\partial N$ , and the other component is in  $\partial S$ , (h) a pair of pants of which one boundary component is in  $\partial N$ , and the other two components are in  $\partial S$ , (i) a once punctured annulus of which both the boundary components are in  $\partial N$ , (j) a once punctured annulus of which one of the boundary component is in  $\partial N$ , and the other component is in  $\partial S$ , (k) a twice punctured annulus. In cases (e)(g)(h)(j), let  $\mathcal{B}_{\lambda}$  be as in Figure 2. In cases (f)(i)(k), let  $\mathcal{B}_{\lambda}$  be empty.

*Case 3.*  $\chi(S_{\lambda}) \leq -2$ . Let  $r$  and  $r'$  be the cardinality of  $\pi_0(\partial N \cap S_{\lambda})$  and  $\pi_0(\partial S \cap S_{\lambda})$ , respectively, and let  $g$  be the genus of  $S_{\lambda}$  and  $n$  the number of punctures of  $S_{\lambda}$ . We have  $r \geq 1$  and  $2g + r + r' + n \geq 4$ . If  $r' > 0$ , let  $\mathcal{B}_{\lambda}$  be as in Figure 3. If  $r' = 0$  and  $g > 0$ , let  $\mathcal{B}_{\lambda}$  be as in Figure 4. If  $r' = g = 0$ , then  $r + n \geq 4$ . We let  $\mathcal{B}_{\lambda}$  be as in Figure 5.

Finally we set  $\mathcal{B} = \bigcup_{\lambda} \mathcal{B}_{\lambda}$ . Then  $\mathcal{B}$  has the following properties.

- (1) Any member of  $\mathcal{B}$  is disjoint from  $\partial N$ .
- (2) Any simple closed curve in  $\mathcal{B}$  is not parallel to a component of  $\partial N$  and the ends of any arcs in  $\mathcal{B}$  are in  $\partial S$ ,

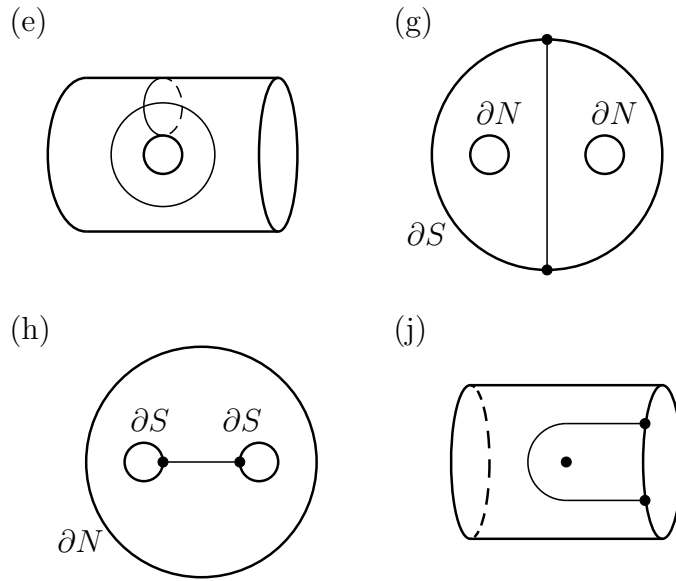


Figure 2:  $\mathcal{B}_\lambda$  for (e), (g), (h), and (j)

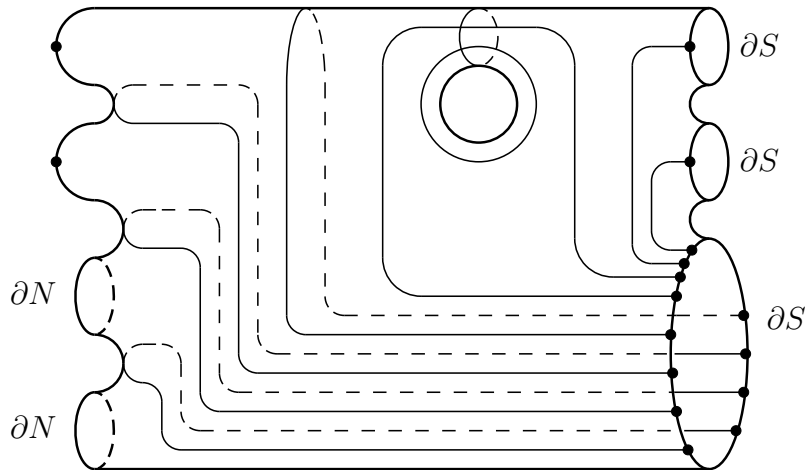


Figure 3:  $\mathcal{B}_\lambda$  for  $r' > 0$

- (3) Members of  $\mathcal{B}$  are pairwise non-isotopic and pairwise in minimal position in  $S$ .
- (4) The surface obtained from  $S \setminus \text{Int}(N)$  by cutting along  $\mathcal{B}$  is a disjoint union of surfaces of the types (a), (b), (c), (d), (f), (i), and (k).

It is clear that members of  $\mathcal{B}_\lambda$  are pairwise non-isotopic and pairwise in minimal position in  $S_\lambda$ . The property (3) for  $\mathcal{B}$  also follows since  $N$  is not a disk or an annulus, as we remarked before.

From the claim, each component of  $\partial N$  and  $\mathcal{B}$  is preserved by  $\varphi$  up to isotopy. By [8] Proposition 2.8, we may assume that  $\varphi$  is the identity on  $\partial N$  and  $\mathcal{B}$ . Consider the restriction of  $\varphi$  to each component of the result of cutting  $S$  along  $\partial N$  and  $\mathcal{B}$ . Each restriction is a self homeomorphism, and by the property (4) for  $\mathcal{B}$ , is isotpic to the identity or to a product of Dehn twists along the boundary components. This implies that  $\varphi|_{S \setminus \text{Int}(N)}$  is

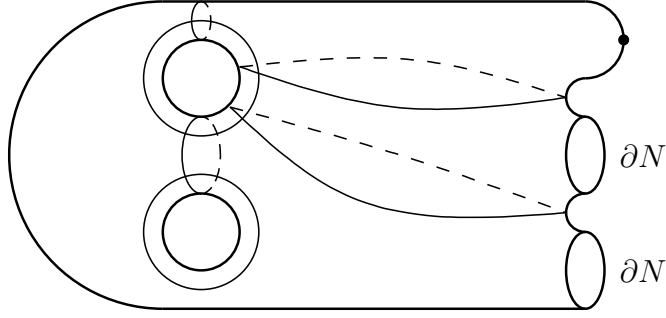


Figure 4:  $\mathcal{B}_\lambda$  for  $r' = 0, g > 0$

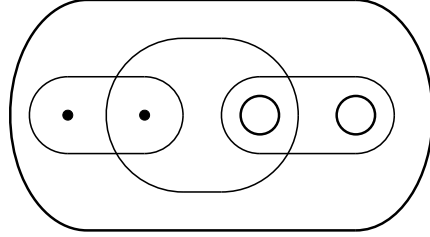


Figure 5:  $\mathcal{B}_\lambda$  for  $r' = g = 0$

isotopic to a product of Dehn twists along the boundary components of  $N$ . This completes the proof.  $\square$

## 5.4 The generalized Dehn twist along a figure eight

In this subsection we give a generalization of [17] Theorem 5.2.1. We suppose  $S$  is of finite type with non-empty boundary,  $E \subset \partial S$ , and any connected component of  $\partial S$  has an element of  $E$ .

Let  $C$  be an unoriented immersed free loop in  $S^* \setminus \partial S$ . We say  $C$  is a *figure eight* if the self-intersections of  $C$  consist of a single double point and the inclusion homomorphism  $\pi_1(C) \rightarrow \pi_1(S)$  is injective.

**Theorem 5.4.1.** *Let  $C$  be a figure eight on the surface  $S$ . Then  $\exp(\sigma(zL(C))) \in A(S, E)$  is not in the image of  $\widehat{DN}$  for any  $z \in K \setminus \{0\}$ . In particular the generalized Dehn twist  $t_C$  is not realizable as a diffeomorphism.*

*Proof.* Take a regular neighborhood  $N$  of  $C$  in  $S^* \setminus \partial S$ , which satisfies the assumptions of Proposition 3.3.4. Assume  $\exp(\sigma(zL(C))) \in A(S, E)$  is realized by a diffeomorphism  $\varphi$ . Then, by Theorem 5.3.3, we may take  $\varphi$  as a diffeomorphism whose support is included in  $N$ . In fact, Theorem 5.3.3 only treat the case  $z = 1$ , but the proof works as well as for general  $z$ . Then  $U := \exp(-\sigma(zL(C)))\varphi \in A(N, \partial N)$  satisfies the condition of Proposition 3.3.4. Hence, by (5.1.2), we have

$$\varphi = \exp(\sigma(zL(C) + F^U)) \in A(N, \partial N).$$

Here we remark  $[L(C), F^U] = 0$  since  $C \cap \partial N = \emptyset$ . The surface  $N$  is diffeomorphic to a pair of pants. We take  $\eta_i, \gamma_i, 1 \leq i \leq 3$ , as in Figure 6.

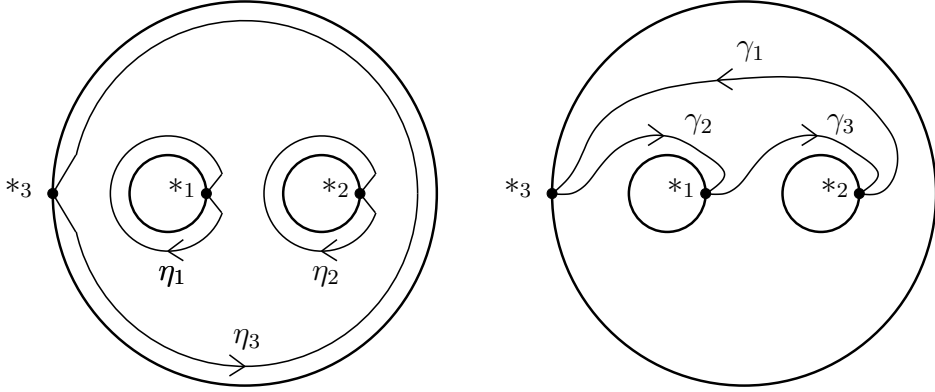


Figure 6:  $\eta_i$  and  $\gamma_i$

The mapping class group of the pair of pants is free abelian of rank 3, generated by boundary-parallel Dehn twists (see, e.g., [8] §3.6). By Theorem 5.2.1, we have  $\varphi = \exp(\sigma(\sum_{i=1}^3 a_i L(\eta_i))) \in A(N, \partial N)$  for some  $a_i \in \mathbb{Z}$ . By Proposition 1.3.3 (3) we obtain

$$\sigma((zL(C) + \sum_{i=1}^3 (a_i^U - a_i)L(\eta_i))) = 0 \in \text{Der}K\widehat{\mathcal{C}}(N, \partial N).$$

On the other hand, we have  $C = |(\gamma_3^{-1}\eta_1\gamma_3\eta_2^{-1})^{\pm 1}|$  and

$$\sigma((\gamma_3^{-1}\eta_1\gamma_3\eta_2^{-1})^m)(\gamma_1) = -m(\gamma_3^{-1}\eta_1\gamma_3\eta_2^{-1})^m\gamma_1$$

for any  $m \geq 0$ . Hence

$$\sigma(L(C))(\gamma_1) = \log(\eta_2\gamma_3^{-1}\eta_1^{-1}\gamma_3)\gamma_1.$$

Further there exist some  $b$  and  $c \in K$  such that

$$\begin{aligned} \sigma(\sum_{i=1}^3 (a_i^U - a_i)L(\eta_i))(\gamma_1) &= b(\log \eta_2)\gamma_1 + c\gamma_1(\log \eta_3^{-1}) \\ &= (b \log \eta_2 + c \log(\eta_2\gamma_3^{-1}\eta_1\gamma_3))\gamma_1. \end{aligned}$$

Here note that  $\gamma_1\eta_3^{-n} = (\eta_2\gamma_3^{-1}\eta_1\gamma_3)^n\gamma_1$  for  $n \geq 0$ . Hence we obtain

$$z \log(\eta_2\gamma_3^{-1}\eta_1^{-1}\gamma_3) + b \log \eta_2 + c \log(\eta_2\gamma_3^{-1}\eta_1\gamma_3) = 0 \in K\widehat{\pi_1(N, *_2)}. \quad (5.4.1)$$

The fundamental group  $\pi_1(N, *_2)$  is a free group of rank 2, so that there exists an isomorphism of  $K$ -algebras  $\theta: K\widehat{\pi_1(N, *_2)} \xrightarrow{\cong} K\langle\langle X, Y \rangle\rangle$  which satisfies  $\theta(\log \eta_2) = X$  and  $\theta(\log \gamma_3^{-1}\eta_1\gamma_3) = Y$ . Here  $K\langle\langle X, Y \rangle\rangle = \prod_{m=0}^{\infty} (KX \oplus KY)^{\otimes m}$  is the ring of non-commutative formal power series in indeterminates  $X$  and  $Y$ . In other words,  $\theta$  is a group-like expansion of the free group  $\pi_1(N, *_2)$ . Then the equation (5.4.1) is equivalent to

$$zX * (-Y) + bX + cX * Y = 0 \in K\langle\langle X, Y \rangle\rangle. \quad (5.4.2)$$

Here  $X * Y$  is the Hausdorff series in  $X$  and  $Y$ . The degree 1 part of (5.4.2) is  $z(X - Y) + bX + c(X + Y) = 0$ , so that we have  $c = z$  and  $b = -2z$ . The degree 2 part is  $-\frac{z}{2}[X, Y] + \frac{1}{2}c[X, Y] = 0$ . Thus the degree 3 part is

$$\frac{z}{6}[Y, [Y, X]] = 0.$$

This contradicts  $\frac{z}{6}[Y, [Y, X]] \neq 0$ , and proves the theorem.  $\square$



## 6 Kontsevich's Lie algebras and Johnson homomorphisms

In this section we suppose  $K$  is a field of characteristic 0. Let  $S$  be an oriented surface, and  $E$  a non-empty closed subset of  $S$  with the property  $E \setminus \partial S$  is closed in  $S$ . In §4.1 we introduced the Lie algebra homomorphism

$$\sigma: \widehat{K\hat{\pi}(S^*)} \rightarrow \text{Der}(\widehat{K\mathcal{C}(S, E)}),$$

where  $S^* = S \setminus (E \setminus \partial S)$ , while  $\text{Der}_\Delta \widehat{K\mathcal{G}}$  is the Lie subalgebra of  $\text{Der} \widehat{K\mathcal{G}}$  consisting of all the continuous derivations  $D$  stabilizing the coproduct  $\Delta$  for any groupoid  $\mathcal{G}$ . See §2.1. We define Lie subalgebras  $L(S, E)$  and  $L^+(S, E)$  of  $\widehat{K\hat{\pi}(S^*)}$  by

$$\begin{aligned} L(S, E) &:= \widehat{K\hat{\pi}(S^*)}(2) \cap \sigma^{-1}(\text{Der}_\Delta \widehat{K\mathcal{C}(S, E)}) \subset \widehat{K\hat{\pi}(S^*)}, \quad \text{and} \\ L^+(S, E) &:= \widehat{K\hat{\pi}(S^*)}(3) \cap L(S, E). \end{aligned}$$

$L^+(S, E)$  is an ideal of  $L(S, E)$ .

### 6.1 Geometric interpretation of Kontsevich's 'associative' and 'Lie'

In this subsection we study the case  $S = \Sigma_{g,1}$  and  $E \subset \partial \Sigma_{g,1}$ . Then we prove  $\widehat{K\hat{\pi}(\Sigma_{g,1})}(2)$  is isomorphic to a completion of Kontsevich's 'associative'  $a_g$ , and  $L(S, E)$  a completion of Kontsevich's 'Lie'  $l_g$  [15]. These results are essentially due to our previous work [13]. Anyway this means Kontsevich's 'associative' and 'Lie' are constructed in a geometric context, and  $\widehat{K\hat{\pi}(S^*)}(2)$  and  $L(S, E)$  for a general  $(S, E)$  can be regarded as generalizations of Kontsevich's 'associative' and 'Lie', respectively.

To state our previous results [13], we need some notations. Choose a basepoint  $* \in E$ . Adopting the notations in §3.3 we denote  $\pi := \pi_1(\Sigma_{g,1}, *)$ ,  $H := H_1(\Sigma_{g,1}; K)$  and  $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$ . Let  $\zeta \in \pi$  be a boundary loop in the opposite direction, and  $\{A_i, B_i\}_{i=1}^g \subset H$  a symplectic basis. The symplectic form  $\omega := \sum_{i=1}^g A_i B_i - B_i A_i \in H^{\otimes 2}$  is independent of the choice of a symplectic basis.

**Definition 6.1.1** (Massuyeau [18]). *A symplectic expansion  $\theta: \pi \rightarrow \widehat{T}$  is a group-like expansion which satisfies the equation  $\theta(\zeta) = \exp \omega$ .*

As was stated in §3.3, the algebra  $\widehat{T}$  has a filtration defined by the ideals  $\widehat{T}_p := \prod_{m=p}^{\infty} H^{\otimes m}$ ,  $p \geq 1$ . By the Poincaré duality, we identify  $H$  with  $H^* = \text{Hom}(H, K)$  via the isomorphism  $H \xrightarrow{\cong} H^*$ ,  $x \mapsto (y \mapsto (y \cdot x))$ . Then  $\widehat{T}_1$  is identified with  $H \otimes \widehat{T} = H^* \otimes \widehat{T} = \text{Der}(\widehat{T})$ , the (continuous) derivation Lie algebra of the (filtered)  $K$ -algebra  $\widehat{T}$ . Recall the linear map  $N: \widehat{T} \rightarrow \widehat{T}$  introduced in §4.3. Then the image  $N(\widehat{T}_1) = N(\widehat{T})$  equals the Lie subalgebra of  $\widehat{T}_1 = \text{Der}(\widehat{T})$  consisting of symplectic derivations, namely, derivations annihilating the symplectic form  $\omega$  (see [13] §2.7), which we denote  $\mathfrak{a}_g^- = \text{Der}_\omega(\widehat{T})$ . The Lie subalgebra  $\mathfrak{a}_g := N(\widehat{T}_2)$  is a completion of Kontsevich's 'associative'  $a_g$  [15]. Our previous results are

**Theorem 6.1.2** ([13] Theorem 1.2.1). *Let  $\theta: \pi \rightarrow \widehat{T}$  be a symplectic expansion. Then the map*

$$-\lambda_\theta: K\widehat{\pi}(\Sigma_{g,1}) \rightarrow \mathfrak{a}_g^-, \quad \lambda_\theta(|x|) := N\theta(x), \quad x \in \pi,$$

*is a Lie algebra homomorphism. The kernel is the subspace  $K1$  spanned by the constant loop 1, and the image is dense in  $N(\widehat{T}_1) = \mathfrak{a}_g^-$  with respect to the  $\widehat{T}_1$ -adic topology.*

**Theorem 6.1.3** ([13] Theorem 1.2.2). *Let  $\theta$  be a symplectic expansion. Then, for  $u \in K\widehat{\pi}(\Sigma_{g,1})$  and  $v \in K\pi$ , we have the equality*

$$\theta(\sigma(u)v) = -\lambda_\theta(u)\theta(v).$$

*Here the right hand side means minus the action of  $\lambda_\theta(u) \in \mathfrak{a}_g^-$  on the tensor  $\theta(v) \in \widehat{T}$  as a derivation. In other words, the diagram*

$$\begin{array}{ccc} K\widehat{\pi}(\Sigma_{g,1}) \times K\pi & \xrightarrow{\sigma} & K\pi \\ -\lambda_\theta \times \theta \downarrow & & \downarrow \theta \\ \mathfrak{a}_g^- \times \widehat{T} & \longrightarrow & \widehat{T}, \end{array}$$

*where the bottom horizontal arrow means the derivation, commutes.*

Note that for any  $p \geq 2$ ,  $N(\widehat{T}_p)$  is a Lie subalgebra of  $\mathfrak{a}_g^-$  and an ideal of  $\mathfrak{a}_g$ . Immediately from the proof of Theorem 4.3.1 together with Theorem 6.1.2 we obtain

**Theorem 6.1.4.** *Let  $\theta$  be a symplectic expansion. The map  $\lambda_\theta: K\widehat{\pi}(\Sigma_{g,1}) \rightarrow \mathfrak{a}_g^-$  induces isomorphisms of Lie algebras*

$$-\lambda_\theta: \widehat{K\widehat{\pi}(\Sigma_{g,1})} \xrightarrow{\cong} \mathfrak{a}_g^- \quad \text{and} \quad -\lambda_\theta: \widehat{K\widehat{\pi}(\Sigma_{g,1})(2)} \xrightarrow{\cong} \mathfrak{a}_g.$$

Under the identification  $\widehat{T}_1 = H \otimes \widehat{T} = \text{Der}(\widehat{T})$ , the Lie subalgebra consisting of all (continuous) derivations stabilizing the coproduct  $\Delta$  on  $\widehat{T}$  coincides with  $H \otimes \widehat{\mathcal{L}}$ . Here  $\widehat{\mathcal{L}}$  is the space of all Lie-like elements in the completed tensor algebra  $\widehat{T}$ . The Lie algebra  $H \otimes \widehat{\mathcal{L}}$  can be regarded as the (continuous) derivation Lie algebra of the Lie algebra  $\widehat{\mathcal{L}}$ . We define the Lie algebra  $\mathfrak{l}_g$  by the intersection

$$\mathfrak{l}_g := \mathfrak{a}_g^- \cap (H \otimes \widehat{\mathcal{L}}),$$

which is a completion of Kontsevich's 'Lie'  $l_g$  [15]. Here it should be remarked the Lie algebra  $l_g$  was introduced earlier by Morita [20] [21] as a target of the Johnson homomorphisms of the (higher) Torelli groups. Moreover we define  $\mathfrak{l}_g^+$  by

$$\mathfrak{l}_g^+ := \mathfrak{l}_g \cap \widehat{T}_3 \quad (\subset \mathfrak{l}_g \subset \widehat{T}_1 = \text{Der}\widehat{T}),$$

which is an ideal of  $\mathfrak{l}_g$ .

**Theorem 6.1.5.** *Let  $\theta$  be a symplectic expansion. The map  $\lambda_\theta: K\widehat{\pi}(\Sigma_{g,1}) \rightarrow \mathfrak{a}_g^-$  induces isomorphisms of Lie algebras*

$$-\lambda_\theta: L(\Sigma_{g,1}, *) \xrightarrow{\cong} \mathfrak{l}_g, \quad \text{and} \quad -\lambda_\theta: L^+(\Sigma_{g,1}, *) \xrightarrow{\cong} \mathfrak{l}_g^+.$$

*Here  $*$  is a basepoint on the boundary  $\partial\Sigma_{g,1}$ .*

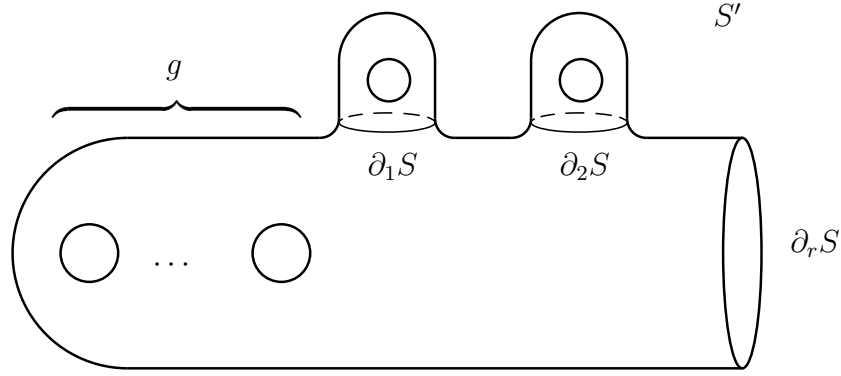


Figure 7: capping

*Proof.* Taking the projective limits of the diagram in Theorem 6.1.3, we obtain the commutative diagram

$$\begin{array}{ccc}
 \widehat{K\hat{\pi}}(\Sigma_{g,1}) \times \widehat{K\pi} & \xrightarrow{\sigma} & \widehat{K\pi} \\
 \parallel & & \parallel \\
 \mathfrak{a}_g^- \times \widehat{T} & \longrightarrow & \widehat{T}.
 \end{array}$$

The isomorphism  $\theta: \widehat{K\pi} \xrightarrow{\cong} \widehat{T}$  preserves the coproducts. Hence the theorem follows from the definition of  $\mathfrak{l}_g$  and  $\mathfrak{l}_g^+$ , and the fact  $-\lambda_\theta: \widehat{K\hat{\pi}}(\Sigma_{g,1}) \xrightarrow{\cong} \mathfrak{a}_g^-$  preserves the filtration.  $\square$

## 6.2 Infinitesimal Dehn-Nielsen theorem

In §3.1 we discussed the injectivity of the Dehn-Nielsen homomorphism  $\text{DN}: \mathcal{M}(S, E) \rightarrow \text{Aut}(K\mathcal{C}(S, E))$ . In view of the results in §5, we may regard the Lie algebra homomorphism  $\sigma: \widehat{K\hat{\pi}}(S) \rightarrow \text{Der}(\widehat{K\mathcal{C}}(S, E))$  as an infinitesimal analogue of the Dehn-Nielsen homomorphism. This subsection is devoted to the proof of the following, an infinitesimal Dehn-Nielsen theorem.

**Theorem 6.2.1.** *Let  $S$  be a compact connected oriented surface with non-empty boundary,  $E \subset \partial S$ , and suppose any connected component of  $\partial S$  has an element of  $E$ . Then the homomorphism  $\sigma: \widehat{K\hat{\pi}}(S) \rightarrow \text{Der}(\widehat{K\mathcal{C}}(S, E))$  is injective.*

The completion maps  $K\hat{\pi}(S)/K1 \rightarrow \widehat{K\hat{\pi}}(S)$  and  $K\mathcal{C}(S, E) \rightarrow \widehat{K\mathcal{C}}(S, E)$  are injective from Corollary 4.3.2 and [3]. Hence, as a corollary, we have

**Corollary 6.2.2.** *Under the assumption of Theorem 6.2.3 the homomorphism  $\sigma: K\hat{\pi}(S)/K1 \rightarrow \text{Der}(K\mathcal{C}(S, E))$  is injective.*

If  $S = \Sigma_{g,1}$ , Theorem 6.2.1 follows immediately from Theorems 6.1.3 and 6.1.4. Otherwise, we have  $S = \Sigma_{g,r}$  for  $r \geq 2$ . We number the boundary components  $\partial S = \coprod_{j=1}^r \partial_j S$ . Capping each  $\partial_j S$  by a compact surface diffeomorphic to  $\Sigma_{1,1}$ , we obtain  $S' = \Sigma_{g+r-1,1}$  as in Figure 7.

Then we have

**Lemma 6.2.3.** 1. The inclusion homomorphism  $K\widehat{\pi}_1(S, p) \rightarrow K\widehat{\pi}_1(S', p)$  is injective for any  $p \in S$ .

2. The inclusion homomorphism  $\widehat{K\hat{\pi}}(S) \rightarrow \widehat{K\hat{\pi}}(S')$  is injective.

Theorem 6.2.1 follows from this lemma. We denote  $S'' := \overline{S'} \setminus S$  and  $E'' := E \setminus \partial_r S \subset S''$ . By the assumption on  $E$ , each connected component of  $S \cap S''$  has some point in  $E''$ . Hence, from Proposition 3.2.1,  $K\mathcal{C}(S', E)$  is generated by  $K\mathcal{C}(S, E)$  and  $K\mathcal{C}(S'', E'')$ .

Let  $u \in \widehat{K\hat{\pi}}(S)$  satisfy  $\sigma(u) = 0 \in \text{Der}(K\mathcal{C}(S, E))$ . Clearly  $\sigma(u) = 0 \in \text{Der}(K\mathcal{C}(S'', E''))$ . Hence we have  $\sigma(u) = 0 \in \text{Der}(K\mathcal{C}(S', E))$ . By the injectivity of  $\sigma$  for  $S' = \Sigma_{g+r-1,1}$ , we have  $u = 0 \in \widehat{K\hat{\pi}}(S')$ . Using Lemma 6.2.3(2), we obtain  $u = 0 \in \widehat{K\hat{\pi}}(S)$ . This proves Theorem 6.2.1 modulo Lemma 6.2.3.  $\square$

*Proof of Lemma 6.2.3.* (1) We denote  $\pi := \pi_1(S, p)$  and  $\pi' := \pi_1(S', p)$ . The groups  $\pi$  and  $\pi'$  are finitely generated free groups. We can choose free generator systems  $\{\alpha_i, \beta_i\}_{i=1}^g \cup \{\gamma_j\}_{j=1}^{r-1} \subset \pi$  and  $\{\alpha'_i, \beta'_i\}_{i=1}^{g+r-1} \subset \pi'$ , such that the inclusion homomorphism maps

$$\begin{cases} \alpha_i \mapsto \alpha'_i, & (1 \leq i \leq g), \\ \beta_i \mapsto \beta'_i, & (1 \leq i \leq g), \\ \gamma_j \mapsto \alpha'_{g+j} \beta'_{g+j} \alpha'_{g+j}^{-1} \beta'_{g+j}^{-1}, & (1 \leq j \leq r-1). \end{cases}$$

We write simply  $H := H_1(S; K)$ ,  $H' := H_1(S'; K)$ ,  $A_i := [\alpha_i]$ ,  $B_i := [\beta_i]$ ,  $C_j := [\gamma_j] \in H$ , and  $A'_i := [\alpha'_i]$ ,  $B'_i := [\beta'_i] \in H'$ . Then the sets  $\{A_i, B_i\}_{i=1}^g \cup \{C_j\}_{j=1}^{r-1} \subset H$  and  $\{A'_i, B'_i\}_{i=1}^{g+r-1} \subset H'$  are  $K$ -free bases of  $H$  and  $H'$ , respectively. Let  $\theta: \pi \rightarrow \widehat{T}$  and  $\theta': \pi' \rightarrow \widehat{T}'$  be group-like expansions, where  $\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m}$  and  $\widehat{T}' = \prod_{m=0}^{\infty} (H')^{\otimes m}$ . By the isomorphisms (3.3.3)  $\theta: \widehat{K\pi} \cong \widehat{T}$  and  $\theta': \widehat{K\pi}' \cong \widehat{T}'$ , the inclusion homomorphism induces an algebra homomorphism  $\iota: \widehat{T} \rightarrow \widehat{T}'$  such that

$$\begin{cases} \iota(A_i) = A'_i + \text{higher terms}, & (1 \leq i \leq g), \\ \iota(B_i) = B'_i + \text{higher terms}, & (1 \leq i \leq g), \\ \iota(C_j) = [A'_{g+j}, B'_{g+j}] + \text{higher terms}, & (1 \leq j \leq r-1). \end{cases} \quad (6.2.1)$$

Hence it suffices to show any (continuous) algebra homomorphism  $\iota: \widehat{T} \rightarrow \widehat{T}'$  satisfying the condition (0.2.1) is injective.

To prove this, we introduce some filtrations on the algebras  $\widehat{T}$  and  $\widehat{T}'$ . We have  $H = H_{(1)} \oplus H_{(2)}$  where  $H_{(1)}$  and  $H_{(2)}$  are the linear spans of the sets  $\{A_i, B_i\}_{i=1}^g$  and  $\{C_j\}_{j=1}^{r-1}$ , respectively. We consider the decreasing filtration on  $H$  defined by  $F_1 H := H$  and  $F_2 H := H_{(2)}$ . It induces a decreasing filtration on the algebra  $\widehat{T}$  such that  $F_0 \widehat{T} / F_1 \widehat{T} = K$  and

$$F_n \widehat{T} / F_{n+1} \widehat{T} = \bigoplus_{q=1}^n \bigoplus_{\delta_1 + \dots + \delta_q = n} H_{(\delta_1)} \otimes \dots \otimes H_{(\delta_q)}$$

for any  $n \geq 1$ . On  $\widehat{T}'$  we introduce the filtration we consider usually,  $\widehat{T}'_n := \prod_{m=n}^{\infty} (H')^{\otimes m}$ ,  $n \geq 0$ . We have

$$\widehat{T}'_n / \widehat{T}'_{n+1} = \bigoplus_{\delta_1, \dots, \delta_n} H'_{(\delta_1)} \otimes \dots \otimes H'_{(\delta_n)},$$

where  $H'_{(1)}$  and  $H'_{(2)}$  are the linear spans of the sets  $\{A'_i, B'_i\}_{i=1}^g$  and  $\{A'_{g+j}, B'_{g+j}\}_{j=1}^{r-1}$ , respectively.

The condition (6.2.1) implies  $\iota(F_n\widehat{T}) \subset \widehat{T}'_n$  for any  $n \geq 0$ . The map  $\iota$  induces an isomorphism  $H_{(1)} \xrightarrow{\cong} H'_{(1)}$  and an injective map  $H_{(2)} \rightarrow H'_{(2)} \otimes H'_{(2)}$  whose image is a direct summand of the target. Hence the induced map

$$\iota_n: F_n\widehat{T}/F_{n+1}\widehat{T} \rightarrow \widehat{T}'_n/\widehat{T}'_{n+1}$$

is injective for any  $n \geq 0$ .

Now assume there exists a non-zero element  $u$  of the map  $\iota: \widehat{T} \rightarrow \widehat{T}'$ . Since  $\bigcap_{n=1}^{\infty} F_n\widehat{T} = 0$ , we have  $u \in F_n\widehat{T} \setminus F_{n+1}\widehat{T}$  for some  $n \geq 0$ . On the other hand, we have  $\iota_n(u \bmod F_{n+1}\widehat{T}) = 0 \in \widehat{T}'_n/\widehat{T}'_{n+1}$ . This implies  $u \in F_{n+1}\widehat{T}$ , which contradicts  $u \in F_n\widehat{T} \setminus F_{n+1}\widehat{T}$ . Hence the kernel of the map  $\iota: \widehat{T} \rightarrow \widehat{T}'$  is zero. This proves the part (1).  $\square$

(2) In view of Theorem 4.3.1, it suffices to show both of the inclusion homomorphisms

$$\begin{aligned} H_1(S; \widehat{\mathcal{S}}^c(S)) &\rightarrow H_1(S; \widehat{\mathcal{S}}^c(S')), \quad \text{and} \\ H_1(S; \widehat{\mathcal{S}}^c(S')) &\rightarrow H_1(S; \widehat{\mathcal{S}}^c(S)) \end{aligned}$$

are injective. The injectivity of the former follows from (1) and  $H_2(S; \widehat{\mathcal{S}}^c(S')/\widehat{\mathcal{S}}^c(S)) = 0$ .

To prove the injectivity of the latter, it suffices to show

$$H_2(S', S; \widehat{\mathcal{S}}^c(S')) = 0. \tag{6.2.2}$$

By the excision and the Poincaré duality theorem, the second homology group is isomorphic to  $H^0(S''; \widehat{\mathcal{S}}^c(S'))$ . Each connected component  $S''_j$ ,  $1 \leq j \leq r-1$ , of  $S''$  is of genus 1, and so has two distinct non-separating simple closed curves. Hence, by Proposition 3.3.3, we have  $H^0(S''_j; \widehat{\mathcal{S}}^c(S')) = 0$ ,  $1 \leq j \leq r-1$ . This means (6.2.2) and proves the part (2).  $\square$

This completes the proof of Lemma 6.2.3 and Theorem 6.2.1.

### 6.3 Higher Johnson homomorphism

Let  $S$  be an oriented surface, and  $E$  a non-empty closed subset of  $S$  with the property  $E \setminus \partial S$  is closed in  $S$ . The group  $\mathcal{M}(S, E)$  acts on the  $K$ -SAC  $K\mathcal{C}(S, E)/F_2K\mathcal{C}(S, E)$  in an obvious way. We define *the Torelli group*  $\mathcal{I}(S, E)$  of the pair  $(S, E)$  to be the kernel of this action

$$\mathcal{I}(S, E) := \text{Ker}(\mathcal{M}(S, E) \rightarrow \text{Aut}K\mathcal{C}/F_2K\mathcal{C}),$$

which is independent of the choice of  $K$ , a field of characteristic 0. When  $S = \Sigma_{g,1}$  and  $E \subset \partial\Sigma_{g,1}$ ,  $\mathcal{I}(S, E) = \mathcal{I}_{g,1}$ , the Torelli group of genus  $g$  with 1 boundary component. But, in general, as will be shown later,  $\mathcal{I}(S, E)$  is the “smallest” Torelli group in the sense of Putman [23]. It was studied also in [11] and [1]. In order to obtain Putman’s Torelli group [23] of other kinds, it seems to be needed to change the filtration on  $K\mathcal{C}(S, E)$  to that induced from a capping by surfaces of positive genus.

From Lemma 1.3.2, the exponential  $\exp(D)$  converges as an element in  $A(S, E)$  for any derivation  $D \in \sigma(L^+(S, E))$ . Clearly the image  $\sigma(L^+(S, E))$  is a Lie subalgebra of  $\text{Der}\widehat{K\mathcal{C}(S, E)}$ .

**Lemma 6.3.1.** *The exponential*

$$\exp: \sigma(L^+(S, E)) \rightarrow A(S, E)$$

*is injective, and its image is a subgroup of  $A(S, E)$ .*

*Proof.* The injectivity follows from Proposition 1.3.3 (3). As was shown in Proposition 1.3.3,  $(\exp D)^{-1} = \widehat{\exp}(-D) \in A(S, E)$  for  $D \in \sigma(L^+(S, E))$ . From the definition,  $\sigma(L^+(S, E)) \subset F_1 \widehat{\text{Der}} \widehat{K\mathcal{C}}$ . Moreover  $\widehat{K\hat{\pi}}(S^*)$  is complete with respect to the filtration  $\{\widehat{K\hat{\pi}}(S^*)(n)\}_{n \geq 1}$ . Hence the Hausdorff series  $u * u'$  of  $u$  and  $u' \in L^+(S, E)$  converges as an element of  $L^+(S, E)$ . This implies  $\exp \sigma(L^+(S, E))$  is a subgroup of  $A(S, E)$ .  $\square$

Thus the exponential  $\exp$  induces a group structure on the set  $\sigma(L^+(S, E))$ , which is a pro-nilpotent group.

For the rest of this subsection, we suppose  $S$  is compact with non-empty boundary,  $E \subset \partial S$ , and each component of  $\partial S$  has an element of  $E$ . From Theorems 3.1.1 and 6.2.1, both of the Dehn-Nielsen homomorphism  $\widehat{\text{DN}}: \mathcal{M}(S, E) \rightarrow A(S, E)$  and the homomorphism  $\sigma: L^+(S, E) \rightarrow \widehat{\text{Der}}(\widehat{K\mathcal{C}}(S, E))$  are injective. In particular, the Lie algebra  $L^+(S, E)$  has the structure of a pro-nilpotent group.

Assume the inclusion

$$\widehat{\text{DN}}(\mathcal{I}(S, E)) \subset \exp \sigma(L^+(S, E)) \tag{6.3.1}$$

holds. Then we have a unique injective map

$$\tau: \mathcal{I}(S, E) \rightarrow L^+(S, E)$$

such that  $\widehat{\text{DN}}|_{\mathcal{I}(S, E)} = \exp \circ \sigma \circ \tau: \mathcal{I}(S, E) \rightarrow A(S, E)$ , which is a group homomorphism with respect to the pro-nilpotent group structure on  $L^+(S, E)$ . The homomorphism  $\tau$  is a generalization of the higher Johnson homomorphism of  $\mathcal{I}_{g,1}$  introduced by Johnson [10] and improved by Morita [22]. At present we have no suitable presentation of the Lie algebra  $L^+(S, E)$ . In order to obtain it, we need to generalize Magnus expansions [12] [18] of free groups to those for groupoids.

The Zariski closure of the image of  $\tau$  does not equal the whole  $L^+(S, E)$ . This fact was discovered by Morita [22]. For recent progress on this problem, see [6] and its reference. From Theorem 5.4.1, the generalized Dehn twist along a null-homologous figure eight is in the complement of the closure. In our forthcoming paper [14] we will give a candidate for the defining equation of the Zariski closure.

Now suppose  $S = \Sigma_{g,r}$ ,  $g, r \geq 1$ , and each component of  $\partial S$  has a unique point of  $E$ . Then Putman [23] gave an explicit generator system of  $\mathcal{I}(S, E)$  for this  $(S, E)$ . Using this remarkable theorem together with our formula for Dehn twists (Theorem 5.2.1), we prove the inclusion (6.3.1) in this case.

To state Putman's theorem, we number the boundary components as  $\partial S = \coprod_{i=1}^r \partial_i S$ , and define a partition  $P$  of  $\pi_0(\partial S)$  by  $P = \{\{1, \dots, r\}\}$ . Our  $\mathcal{I}(S, E)$  is identified with Putman's  $\mathcal{I}(S, P)$  for this  $P$ , as follows. For this partition  $P$ , Putman's  $\partial H_1^P(S)$  vanishes, because  $\sum_{i=1}^r [\partial_i S] = 0 \in H_1(S; \mathbb{Z})$ . Hence we have  $H_1^P(S) = H_1(S, E; \mathbb{Z})$ . By [23] Theorem 3.3, the group  $\mathcal{I}(S, P)$  is exactly the subgroup of  $\mathcal{M}(S, E)$  which acts trivially on  $H_1(S, E; \mathbb{Z})$ . On the other hand, if  $*_0 \neq *_1 \in E$ , we have  $\mathcal{C}(S, E)^{\text{abel}}(*_0, *_1) = \mathbb{Z} \oplus H_1(S, \{*_0, *_1\}; \mathbb{Z})$  as  $\mathcal{M}(S, E)$ -modules. Thus a mapping class  $\varphi \in \mathcal{M}(S, E)$  acts trivially on  $K\mathcal{C}/F_2 K\mathcal{C}$  if and only if it acts trivially on  $H_1(S, E; \mathbb{Z})$ . In other words, our  $\mathcal{I}(S, E)$  equals Putman's  $\mathcal{I}(S, P)$  with  $P = \{\{1, \dots, r\}\}$ . Putman proved the following.

**Theorem 6.3.2** (Putman [23] Theorem 1.3). *For any partitioned surface  $(\Sigma_{g,r}, P)$  with  $g \geq 1$ , the group  $\mathcal{I}(\Sigma_{g,r}, P)$  is generated by twists about  $P$ -separating curves and twists about  $P$ -bounding pairs.*

For our  $P$ , a  $P$ -separating curve is a simple closed curve  $C$  with  $[C] = 0 \in H_1(S; \mathbb{Z})$ , and a  $P$ -bounding pair is a pair of disjoint, non-isotopic simple closed curves  $C_1$  and  $C_2$  with  $\pm[C_1] = \pm[C_2] \in H_1(S; \mathbb{Z})$ . The main theorem in this subsection is

**Theorem 6.3.3.** *The inclusion (6.3.1) holds for  $g, r \geq 1$ .*

*Proof.* In view of Putman's theorem, it suffices to show the  $\widehat{\text{DN}}$ -images of twists about  $P$ -separating curves and twists about  $P$ -bounding pairs are in  $\exp(\sigma(L^+(S, E)))$ .

Fix an element  $* \in E$ . Let  $C$  be a  $P$ -separating curve represented by some  $x \in \widehat{\pi_1(S, *)}$ . Since  $\pm[x] = \pm[C] = 0 \in H_1(S; \mathbb{Z})$ ,  $x - 1 \in I\pi_1(S, *)^2$ . Hence we have  $L(x) \in I\pi_1(S, *)^4$ , and so  $L(C) \in \widehat{K\hat{\pi}}(S)(4)$ . It follows from Theorem 5.2.1 that  $\widehat{\text{DN}}(t_C) = \exp(\sigma(L(C))) \in \exp(\sigma(L^+(S, E)))$ .

Let  $C_1$  and  $C_2$  form a  $P$ -bounding pair. Choose  $x_1$  and  $x_2 \in \pi_1(S, *)$  such that  $C_1 = |x_1^{\pm 1}|$ ,  $C_2 = |x_2^{\pm 1}|$  and  $[x_1] = [x_2] \in H_1(S; \mathbb{Z})$ . We have  $x_2 = x_1 z$  for some  $z \in [\pi_1(S, *), \pi_1(S, *)]$ . Since  $z - 1 \in I\pi_1(S, *)^2$ , we have  $(x_2 - 1)^2 - (x_1 - 1)^2 = x_1(z - 1)x_1(z - 1) + x_1(z - 1)(x_1 - 1) + (x_1 - 1)x_1(z - 1) \in I\pi_1(S, *)^3$ , while  $L(x_1) - \frac{1}{2}(x_1 - 1)^2$  and  $L(x_2) - \frac{1}{2}(x_2 - 1)^2$  are in  $I\pi_1(S, *)^3$ . Hence  $L(x_2) - L(x_1) \in I\pi_1(S, *)^3$ , and so  $L(C_2) - L(C_1) \in \widehat{K\hat{\pi}}(S)(3)$ . Since  $C_1 \cap C_2 = \emptyset$ , we have  $[L(C_1), L(C_2)] = 0$ . It follows from Theorem 5.2.1  $\widehat{\text{DN}}(t_{C_1}^{-1}t_{C_2}) = \exp(\sigma(L(C_2) - L(C_1))) \in \exp(\sigma(L^+(S, E)))$ .

This completes the proof of the theorem.  $\square$

Consequently we obtain the higher Johnson homomorphism

$$\tau: \mathcal{I}(\Sigma_{g,r}, E) \rightarrow L^+(\Sigma_{g,r}, E)$$

in the case  $g \geq 1$  and  $r \geq 2$ , which is an embedding of the Torelli group  $\mathcal{I}(\Sigma_{g,r}, E)$  into the pro-nilpotent group  $L^+(\Sigma_{g,r}, E)$ . This homomorphism is natural in the following sense.

Let  $S$  and  $S'$  be compact oriented surfaces with non-empty boundary, and  $i: S \hookrightarrow S'$  an embedding. Assume the genus of each surface is  $\geq 1$ . Choose a single point on each boundary component of these surfaces, and let  $E \subset \partial S$  and  $E' \subset \partial S'$  be the sets of these points. Then we have the natural homomorphism

$$\iota: \mathcal{M}(S, E) \rightarrow \mathcal{M}(S', E')$$

extending diffeomorphisms by the identity on the complement  $S \setminus S'$ . Then

**Proposition 6.3.4.** *There exists a Lie algebra homomorphism  $i: L^+(S, E) \rightarrow L^+(S', E')$ , which depends on the embedding  $i: S \hookrightarrow S'$ , such that the diagram*

$$\begin{array}{ccc} \mathcal{I}(S, E) & \xrightarrow{\iota} & \mathcal{I}(S', E') \\ \tau \downarrow & & \tau \downarrow \\ L^+(S, E) & \xrightarrow{i} & L^+(S', E') \end{array}$$

*commutes.*

*Proof.* First we check the top horizontal arrow is well-defined. Denote  $S_1 := S$ ,  $E_1 := E$ ,  $S_2 := \overline{S' \setminus S}$ , and  $E_2 := (E \cup E') \cap S_2$ . Then the inclusion homomorphism  $H_1(S, E; \mathbb{Z}) \oplus H_1(S_2, E_2; \mathbb{Z}) \rightarrow H_1(S', E \cup E'; \mathbb{Z})$  is surjective, while the inclusion homomorphism  $H_1(S', E'; \mathbb{Z}) \rightarrow H_1(S', E \cup E'; \mathbb{Z})$  is injective. Hence the homomorphism  $\iota$  maps  $\mathcal{I}(S, E)$  into  $\mathcal{I}(S', E')$ .

Moreover the pairs  $(S_1, E_1)$  and  $(S_2, E_2)$  satisfy the assumption of the van Kampen theorem (Proposition 3.2.1). Hence  $K\mathcal{C}_3 = K\mathcal{C}(S', E \cup E')$  is generated by  $K\mathcal{C}(S, E)$  and  $K\mathcal{C}(S_2, E_2)$ . Hence, if  $u \in \widehat{K\hat{\pi}}(S^*)$  satisfies  $\sigma(u) = 0 \in \text{Der}K\widehat{\mathcal{C}}(S, E)$ , then we have  $\sigma(i(u)) = 0 \in \text{Der}K\mathcal{C}(S', E \cup E')$ . Here  $i: \widehat{K\hat{\pi}}(S^*) \rightarrow \widehat{K\hat{\pi}}(S'^*)$  is the inclusion homomorphism. This implies the homomorphism

$$i: \sigma(\widehat{K\hat{\pi}}(S^*)) \rightarrow \sigma(\widehat{K\hat{\pi}}(S'^*)) \subset \text{Der}K\mathcal{C}(S', E \cup E')$$

extending derivations by 0 on  $S_2$  is well-defined. Post-composing the forgetful map  $\phi: \text{Der}K\mathcal{C}(S', E \cup E') \rightarrow \text{Der}K\mathcal{C}(S', E')$  and restrict  $\phi \circ i$  to  $L^+(S, E)$ , we obtain a Lie algebra homomorphism  $i: L^+(S, E) \rightarrow L^+(S', E')$ .

From Theorem 6.3.2, the group  $\mathcal{I}(S, E)$  is generated by twists about  $P$ -separating twists and twists about  $P$ -bounding pairs. It is clear that the diagram commutes on these twists. This proves the proposition.  $\square$

Recently Church [5] introduced the first Johnson homomorphism for all kinds of Putman's partitioned Torelli groups. It would be very interesting to describe an explicit relation between Church's homomorphisms and ours.

## References

- [1] B. van den Berg, On the abelianization of the Torelli group, PhD Thesis, University of Utrecht (2003)
- [2] J. Birman, Braids, links, and mapping class groups, Princeton University Press, New Jersey (1975)
- [3] N. Bourbaki, Groupes et algèbres de Lie, chapitre 2, Hermann, Paris (1972)
- [4] K.-T. Chen, Extension of  $C^\infty$  function algebra by integrals and Malcev completion of  $\pi_1$ , Adv. Math. **23**, 181-210 (1977)
- [5] T. Church, Orbits of curves under the Johnson kernel, preprint, arXiv:1108.4511 (2011)
- [6] N. Enomoto and T. Satoh, New series in the Johnson cokernels of the mapping class groups of surfaces, preprint, arXiv:1012.2175 (2010)
- [7] D. B. A. Epstein, Curves on 2-manifolds and isotopies, Acta Math. **115**, 83–107 (1966)
- [8] B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series, Princeton University Press (2011)
- [9] W. M. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface groups representations, Invent. Math. **85**, 263-302 (1986)



- [10] D. Johnson, A survey of the Torelli group, *Contemporary Math.* **20**, 165-179 (1983)
- [11] D. Johnson, The structure of the Torelli group - II: A characterization of the group generated by twists on bounding curves, *Topology* **24**, 113-126 (1985)
- [12] N. Kawazumi, Cohomological aspects of Magnus expansions, preprint, math.GT/0505497 (2005)
- [13] N. Kawazumi and Y. Kuno, The logarithms of Dehn twists, preprint, arXiv:1008.5017 (2010)
- [14] N. Kawazumi and Y. Kuno, Intersections of curves on surfaces and their applications to mapping class groups, preprint, arXiv:1112.3841 (2011)
- [15] M. Kontsevich, Formal (non)-commutative symplectic geometry, in: "The Gel'fand Mathematical Seminars, 1990-1992, Birkhäuser, Boston, 173-187 (1993)
- [16] Y. Kuno, A combinatorial construction of symplectic expansions, preprint, arXiv:1009.2219 (2011), to appear in *Proc. Amer. Math. Soc.*
- [17] Y. Kuno, The generalized Dehn twist along a figure eight, preprint, arXiv:1104.2107 (2011)
- [18] G. Massuyeau, Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant, preprint, arXiv:0809.4629 (2008)
- [19] G. Massuyeau and V. Turaev, Fox pairings and generalized Dehn twists, preprint, arXiv:1109.5248 (2011)
- [20] S. Morita, On the structure and the homology of the Torelli group, *Proc. Japan Acad. Ser. A* **65**, 147–150 (1989)
- [21] S. Morita, Mapping class groups of surfaces and three-dimensional manifolds, *Proceedings of the International Congress of Mathematicians (Kyoto 1990)*, Math. Soc. Japan, Tokyo, 665–674 (1991)
- [22] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, *Duke Math. J.* **70**, 699-726 (1993)
- [23] A. Putman, Cutting and pasting in the Torelli group, *Geometry and Topology* **11**, 829-865 (2007)
- [24] D. Quillen, Rational homotopy theory, *Ann. Math.* **90**, 205-295 (1969)
- [25] S. Smale, Diffeomorphisms of the 2-sphere. *Proc. Amer. Math. Soc.* **10** 621–626 (1959)

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The Graduate School of Mathematical Sciences was established in the University of Tokyo in April, 1992. Formerly there were two departments of mathematics in the University of Tokyo: one in the Faculty of Science and the other in the College of Arts and Sciences. All faculty members of these two departments have moved to the new graduate school, as well as several members of the Department of Pure and Applied Sciences in the College of Arts and Sciences. In January, 1993, the preprint series of the former two departments of mathematics were unified as the Preprint Series of the Graduate School of Mathematical Sciences, The University of Tokyo. For the information about the preprint series, please write to the preprint series office.

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