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Groups and Geometric Analysis

Integral Geometry,
Invariant Differential Operators,
and Spherical Functions

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