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GROUPS OF LINEAR OPERATORS DEFINED BY GROUP CHARACTERS

BY

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ABSTRACT. Some of the recent work on invariance questions can be regarded as follows: Characterize those linear operators on Hom(V, V) which preserve the character of a given representation of the full linear group. In this paper, for certain rational characters, necessary and sufficient conditions are described that ensure that the set of all such operators forms a group $\mathfrak{L}$. The structure of $\mathfrak{L}$ is also determined. The proofs depend on recent results concerning derivations on symmetry classes of tensors.

1. **Statements.** Let G be any subgroup of the full linear group $\mathrm{GL}(n, \mathbf{C})$ over the complex numbers, and let $\mathfrak{A}$ denote the linear closure of G in the total matrix algebra $M_n(\mathbf{C})$. Let $K: G \to \mathrm{GL}(N, \mathbf{C})$ be a representation which is extended to a representation of the multiplicative semigroup of $\mathfrak{A}$ in $M_n(\mathbf{C})$. Let $\mu_K(X) = \mathrm{tr}\, K(X)$ be the corresponding character. Next, let $\mathfrak{L}(G, K)$ denote the multiplicative semigroup of all linear transformations $\mathfrak{T}: \mathfrak{A} \to \mathfrak{A}$ having the property that $\mathfrak{T}$ preserves the character of the representation K; that is,

$$\mu_K(\mathfrak{T}(X)) = \mu_K(X), \qquad X \in \mathfrak{A}. \tag{1}$$

The two central questions which will concern us in this paper are:

(i) Under what circumstances is $\mathfrak{L}(G, K)$ a group, i.e., under what circumstances is it true that if (1) holds, then $\mathfrak{T}$ is nonsingular?

(ii) If $\mathfrak{L}(G, K)$ is a group, then what is its structure?

Probably the first instance of a question of this kind was discussed by Frobenius [3] who proved that if $G = \mathrm{GL}(n, \mathbf{C})$, so that $\mathfrak{A} = M_n(\mathbf{C})$, and if $K(X) = \det(X)$, then $\mathfrak{L}(G, K)$ is a group. He proved, in answer to question (ii), that for $\mathfrak{T} \in \mathfrak{L}(G, K)$ there exist fixed matrices U and V in $\mathrm{GL}(n, \mathbf{C})$ such that

$$\mathfrak{T}(X) = UXV, \quad X \in M_n(\mathbf{C}), \quad \text{or} \quad \mathfrak{T}(X) = UX^TV, \quad X \in M_n(\mathbf{C}),$$

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where $\det(UV) = 1$ (X^T is the transpose of X).

A related problem was discussed by I. Schur [12]. Let $3 \leq m \leq n$ and $\mathcal{T}$: $M_n(\mathbf{C}) \to M_n(\mathbf{C})$ be a linear transformation satisfying the following condition. For each $X \in M_n(\mathbf{C})$, the mth order subdeterminants of $\mathcal{T}(X)$ are fixed linearly independent linear homogeneous functions of the mth order subdeterminants of X. Schur proved that for such a $\mathcal{T}$ there exist fixed matrices $U, V \in \mathrm{GL}(n, \mathbf{C})$ such that

$$\mathcal{T}(X) = UXV, \quad X \in M_n(\mathbf{C}), \quad \text{or} \quad \mathcal{T}(X) = UX^TV, \quad X \in M_n(\mathbf{C}).$$

This problem can be reformulated in terms of the mth Grassmann compound $C_m(X)$ of X. Let S be a nonsingular linear transformation from $M_{\binom{n}{m}}(\mathbf{C})$ to itself. Characterize those linear transformations $\mathcal{T}$ on $M_n(\mathbf{C})$ which satisfy

$$C_m(\mathcal{T}(X)) = S(C_m(X)), \qquad X \in M_n(\mathbf{C}).$$

This reformulation and a proof depending on more recent results appear in [9].

Let $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and let G be the group consisting of all $X \in \mathrm{GL}(2, \mathbf{C})$ for which $X^* = PX^TP$. Let $K(X) = \det(X)$. As can be readily verified, $\mathfrak{L}(G, K)$ is isomorphic to the set of linear transformations mapping the real space $\mathbf{R}^4$ into itself and holding fixed the quadratic form

$$f(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

In [8] it is proved that $\mathfrak{A}$ consists of all $X \in M_2(\mathbf{C})$ of the form

$$\begin{bmatrix} z & w \\ \overline{w} & \overline{z} \end{bmatrix},$$

and that $\mathfrak{L}(G, K)$ consists of all $\mathcal{T}$ of the form

$$\mathcal{T}(X) = UXV, \quad X \in M_n(\mathbf{C}), \quad \text{or} \quad \mathcal{T}(X) = UX^TV, \quad X \in M_n(\mathbf{C}),$$

where $\det(UV) = 1$, and $U^* = PU^TP$, $V^* = PV^TP$.

In [10] it is proved that $\mathfrak{L}(G, K)$ is a group when $G = \mathrm{GL}(n, \mathbf{C})$ and $K(X) = C_m(X)$ for $3 < m \leq n$. In this instance, $\mu_K(X) = \operatorname{tr} C_m(X)$ is the mth elementary symmetric function of the eigenvalues of X, or equivalently, the sum of all $\binom{n}{m}$ m-square principal subdeterminants of X. In case $m < n$, the group $\mathfrak{L}(\mathrm{GL}(n, \mathbf{C}), C_m)$ consists of precisely those linear transformations $\mathcal{T}$ of the form

$$\mathcal{T}(X) = UXV, \quad X \in M_n(\mathbf{C}), \quad \text{or} \quad \mathcal{T}(X) = UX^TV, \quad X \in M_n(\mathbf{C}),$$

where $UV = e^{i\phi}I_n$ and $m\phi \equiv O(2\pi)$. This result was recently extended to include the case $m = 3$ in [2].

Another minor modification of our problem occurs in [5]. Let $U(n, \mathbf{C})$ denote the subgroup of $\mathrm{GL}(n, \mathbf{C})$ consisting of all unitary matrices. Then the semigroup

$\mathfrak{L}$ of all linear transformations $\mathcal{T}$ on $M_n(\mathbf{C})$ which satisfy $\mathcal{T}(U(n, \mathbf{C})) \subset U(n, \mathbf{C})$ is a group. It is shown that $\mathcal{T} \in \mathfrak{L}$ if and only if there exist fixed matrices $U, V \in U(n, \mathbf{C})$ such that

$$\mathcal{T}(X) = UXV, \quad X \in M_n(\mathbf{C}), \quad \text{or} \quad \mathcal{T}(X) = UX^TV, \quad X \in M_n(\mathbf{C}).$$

Observe that $\mathfrak{L}(\mathfrak{A}, K)$ is not always a group. Let $G = \mathrm{GL}(n, \mathbf{C})$ and $K(X)$ be the mth Kronecker power $\Pi^m(X)$ of X [14]. Then $\mu_K(X) = (\mathrm{tr}(X))^m$. The annihilator map $\mathcal{T}$ which sends each $X = (x_{ij})$ into $\mathcal{T}(X) = (y_{ij})$ where $y_{ij} = \delta_{ij}x_{ij}$ and which clearly belongs to $\mathfrak{L}(\mathfrak{A}, K)$, has no inverse.

In this paper we shall discuss problems (i) and (ii) for a certain class of rational representations of the multiplicative semigroup of $\mathfrak{A}$ which are in fact components of the mth Kronecker product representation $\Pi^m(X)$. It is somewhat easier to state our results in an invariant setting.

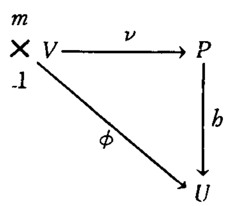
Let H be a subgroup of the symmetric group of degree m and let χ be a character of H of degree 1. Let V be an n-dimensional vector space over $\mathbf{C}$; let U be any vector space over $\mathbf{C}$ and $\phi(v_1, \cdots, v_m)$ an m-multilinear function on the Cartesian product $\times_1^m V$ to U. Then ϕ is said to be *symmetric with respect to H and χ* if

$$\phi(v_{\sigma(1)}, \cdots, v_{\sigma(m)}) = \chi(\sigma)\phi(v_1, \cdots, v_m)$$

holds for any $\sigma \in H$ and arbitrary vectors $v_i \in V$. A pair (P, ν) consisting of a vector space P over $\mathbf{C}$ and a fixed m-multilinear function $\nu: \times_1^m V \to P$, symmetric with respect to H and χ, is a *symmetry class of tensors* associated with H and χ if

(i) $\langle \mathrm{rng}\, \nu \rangle = P$, i.e., the linear closure of the range of ν is P;

(ii) (universal factorization property) for any vector space U over $\mathbf{C}$ and any m-multilinear function $\phi: \times_1^m V \to U$, symmetric with respect to H and χ, there exists a unique linear function $h: P \to U$ such that $\phi = h\nu$; i.e., the following diagram is commutative.



For any linear transformation $X: V \to V$ the preceding universal factorization property permits us to define a unique linear transformation $K(X): P \to P$, the *induced transformation* on P, which satisfies the following identity. For arbitrary vectors $v_1, \cdots, v_m$ in V

$$K(X)\nu(v_1, \cdots, v_m) = \nu(Xv_1, \cdots, Xv_m). \tag{2}$$

By the spanning property of the range of ν (i.e., (i) above), (2) immediately implies that $K(X)$ is multiplicative and in fact if $m \le n$ and $X \in \mathrm{GL}_n(V)$, the group of all linear bijections on V, then $K(X) \in \mathrm{GL}_N(P)$ where $N = \dim P$. If G is any subgroup of $\mathrm{GL}_n(V)$ and $\mathfrak{A}$ is the linear closure of G in $\mathrm{Hom}(V, V)$, we are thus in a position to discuss the structure of $\mathfrak{L}(G, K)$, which for the class of representations $K(X)$ just defined depends on the group H and the character χ. If we identify V with the space of n-tuples over $\mathbf{C}$, then of course $\mathrm{GL}_n(V)$ can be identified with $\mathrm{GL}(n, \mathbf{C})$ and we can ask for the structure of the semigroup $\mathfrak{L}(G, K)$ for the preceding class of representations $K(X)$ of $\mathfrak{A}$.

Our main results follow.

Theorem 1. *Let* $\dim V = n$, $H \subset S_m$, χ *a character of degree* 1 *on* H. *Let* (P, ν) *be the symmetry class associated with* H *and* χ *and* $X \to K(X)$ *be a representation of* $G = \mathrm{GL}_n(V)$ *by induced transformations on* P. *If* $m \le n$ *or* $\chi \equiv 1$, *then* $\mathfrak{L}(\mathrm{GL}_n(V), K)$ *is a group if and only if* $H \neq \{e\}$.

Theorem 2. *Let* $\dim V = n$, $H = S_m$, $m > 1$, $\chi \equiv 1$. *Let* G *be a subgroup of* $\mathrm{GL}_n(V)$. *If the algebra* $\mathfrak{A}$ *has the property that the conjugate transpose* X^* *of each* X *in* $\mathfrak{A}$ *is again in* $\mathfrak{A}$, *then* $\mathfrak{L}(G, K)$ *is a group.*

Theorem 3. *In Theorem* 1, *take* $H = S_m$, $m \ge 3$ *and* $\chi \equiv 1$. *Let* $\mathfrak{L}_1(\mathrm{GL}_n(V), K)$ *denote the subgroup of* $\mathfrak{L}(\mathrm{GL}_n(V), K)$ *of those* $\mathfrak{T}: \mathrm{Hom}(V, V) \to \mathrm{Hom}(V, V)$ *satisfying* $\mathfrak{T}(I_V) = \xi I_V$. *Then* $\mathfrak{L}_1(\mathrm{GL}_n(V), K)$ *consists precisely of those linear transformations* $\mathfrak{T}$ *which have the form*

$$\mathfrak{T}(X) = \xi U^{-1} X U, \qquad X \in \mathrm{Hom}(V, V), \tag{3}$$

or

$$\mathfrak{T}(X) = \xi U^{-1} X^T U, \qquad X \in \mathrm{Hom}(V, V). \tag{4}$$

Theorem 4. *In Theorem* 1, *take* $H = A_m \subset S_m$ *to be the alternating group,* $m \ge 3$, *and* $\chi \equiv 1$. *The group* $\mathfrak{L}_1(\mathrm{GL}_n(V), K)$ *consists precisely of those linear transformations* $\mathfrak{T}$ *of the form* (3) *or* (4).

Corollary 1. *Let* $\dim V = n$, $H = S_m$, $m > 1$, $\chi \equiv 1$. *If* G *is the group of all* $n \times n$ *permutation matrices (so that* $\mathfrak{A}$ *is the algebra of generalized doubly stochastic matrices), then* $\mathfrak{L}(G, K)$ *is a group.*

Let m and n be positive integers. Let $Q_{m,n}$ (resp. $G_{m,n}$) denote the set of all strictly increasing (resp. nondecreasing) sequences of length m chosen from the set $\{1, 2, \cdots, n\}$. If $f(\lambda_1, \cdots, \lambda_n)$ is a polynomial symmetric in the indeterminates $\lambda_1, \cdots, \lambda_n$ and $X \in \text{Hom}(V, V)$, we shall denote by $f(X)$ the value of f at the eigenvalues of X. For $m \geq 1$, let $h_m(\lambda_1, \cdots, \lambda_n)$ denote the mth completely symmetric polynomial

$$h_m(\lambda_1, \cdots, \lambda_n) = \sum_{\alpha \in G_{m,n}} \prod_{t=1}^{m} \lambda_{\alpha(t)};$$

and let $k_m(\lambda_1, \cdots, \lambda_n)$ denote the symmetric polynomial

$$k_m(\lambda_1, \cdots, \lambda_n) = h_m(\lambda_1, \cdots, \lambda_n) + \sum_{\alpha \in Q_{m,n}} \prod_{t=1}^{m} \lambda_{\alpha(t)}$$

$$= h_m(\lambda_1, \cdots, \lambda_n) + E_m(\lambda_1, \cdots, \lambda_n),$$

where $E_m(\lambda_1, \cdots, \lambda_n)$ is the mth elementary symmetric function of $\lambda_1, \cdots, \lambda_n$ when $m \leq n$ and 0 if $m > n$. •

Corollary 2. *Let* $m \geq 3$. *Any linear transformation* $\mathcal{T}$: Hom $(V, V) \rightarrow$ Hom (V, V) *satisfying* $\mathcal{T}(I_V) = \xi I_V$ *and* $h_m(\mathcal{T}(X)) = h_m(X)$, $X \in \text{Hom}(V, V)$, *has the form* (3) *or* (4).

Corollary 3. *Let* $m \geq 3$. *Any linear transformation* $\mathcal{T}$: Hom $(V, V) \rightarrow$ Hom (V, V) *satisfying* $\mathcal{T}(I_V) = \xi I_V$ *and* $k_m(\mathcal{T}(X)) = k_m(X)$, $X \in \text{Hom}(V, V)$, *has the form* (3) *or* (4).

We conjecture that in fact $\mathcal{L}_1(\text{GL}_n(V), K) = \mathcal{L}(\text{GL}_n(V), K)$ in Theorems 3 and 4. This amounts to showing that if $\mathcal{T}$: $\text{GL}_n(V) \rightarrow \text{GL}_n(V)$ and $\mu_K(\mathcal{T}(X)) = \mu_K(X)$ holds for all $X \in \text{Hom}(V, V)$, then $\mathcal{T}(I_V) = \xi I_V$ where $\xi^m = 1$.

2. **Partial derivations.** In [6] the standard notion of a derivation on a tensor algebra [4] is extended to higher order derivations on a general symmetry class (P, ν). We shall further extend the idea of a derivation induced by a single linear transformation to partial derivations induced by two linear transformations [11].

Let T and S be in Hom (V, V) and let $r + s = m$. For $\omega \in Q_{r,m}$ define

$$\Pi_\omega(T, S) = \bigotimes_{i=1}^{m} X_i \tag{5}$$

where $X_i = T$ for $i \in \text{rng}\,\omega$ and $X_i = S$ otherwise. In other words (5) is the tensor product of the linear transformations T and S in which T appears in positions numbered ω and S appears elsewhere. The linear transformation (5) acts on $\bigotimes_1^m V$, which of course is the symmetry class associated with $H = \{e\}$. Define

$$\delta_{r,s}(T, S) = \sum_{\omega \in Q_{r,m}} \Pi_\omega(T, S).$$

In order to simplify subsequent notation we make the following convention. Let $f: Q_{r,m} \times Q_{s,m} \to R$ be any function into a set R having an associative addition. We shall let $\Sigma' f(\omega)$ denote the summation of $f(\omega, \gamma)$ over all sequences $\omega \in Q_{r,m}$, $\gamma \in Q_{s,m}$ such that rng $\omega \cap$ rng $\gamma = \varnothing$. Next, define

$$M(X_1, \cdots, X_m) = \sum_{\phi \in S_m} X_{\phi(1)} \otimes \cdots \otimes X_{\phi(m)}.$$

Then it is easy to show that

$$M_{r,s}(T, S) = r!\,s!\,\delta_{r,s}(T, S), \tag{6}$$

where $M_{r,s}(T, S)$ denotes

$$M(\overbrace{T, \cdots, T}^{r}, \overbrace{S, \cdots, S}^{s}).$$

It is also a standard fact concerning symmetry classes that if the symmetry operator associated with H and χ (a linear transformation on $\bigotimes_1^m V$) is defined by

$$\tau_\chi = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma)\sigma$$

$(\sigma(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)})$, then the pair (P, ν), $P = \operatorname{rng} \tau_\chi \subset \bigotimes_1^m V$, $\nu(v_1, \cdots, v_m) = \tau_\chi(v_1 \otimes \cdots \otimes v_m)$ is the symmetry class associated with H and χ. It is also easy to show that the transformation $M_{r,s}(T, S)$ satisfies $M_{r,s}(T, S)\sigma = \sigma M_{r,s}(T, S)$ for all $\sigma \in S_m$; hence any symmetry class is an invariant subspace of $M_{r,s}(T, S)$. But then in view of (6), each symmetry class is an invariant subspace of $\delta_{r,s}(T, S)$.

We define the (r, s) *partial derivation* associated with T and S on (P, ν) to be the restriction of $\delta_{r,s}(T, S)$ to the invariant subspace P. We denote this by $\Omega_{r,s}(T, S)$. The reason for calling $\Omega_{r,s}(T, S)$ the (r, s) partial derivation on (P, ν) is the following formula:

$$K(x_1 T + x_2 S) = \sum_{r+s=m} x_1^r x_2^s \Omega_{r,s}(T, S). \tag{7}$$

In order to verify (7) we compute that

$$K(x_1T + x_2S)\nu(v_1, \cdots, v_m) = \nu((x_1T + x_2S)v_1, \cdots, (x_1T + x_2S)v_m)$$

$$= \sum_{r+s=m} x_1^r x_2^s \sum{}' \nu(\cdots, Tv_{\omega(1)}, \cdots, Sv_{\gamma(1)}, \cdots, Tv_{\omega(r)}, \cdots, Sv_{\gamma(s)}, \cdots), \tag{8}$$

where in the inside summand on the right side of (8) the T occurs in precisely the positions numbered ω and the S in positions numbered γ. On the other hand,

$$\Omega_{r,s}(T,S)\nu(v_1,\cdots,v_m)=\delta_{r,s}(T,S)r_\chi(v_1\otimes\cdots\otimes v_m)=r_\chi\delta_{r,s}(T,S)(v_1\otimes\cdots\otimes v_m)$$

$$(9)\qquad =r_\chi\sum{}'(\cdots\otimes Tv_{\omega(1)}\otimes\cdots\otimes Sv_{\gamma(1)}\otimes\cdots\otimes Tv_{\omega(r)}\otimes\cdots\otimes Sv_{\gamma(s)}\otimes\cdots)$$

$$=\sum{}'\nu(\cdots,Tv_{\omega(1)},\cdots,Sv_{\gamma(1)},\cdots,Tv_{\omega(r)},\cdots,Sv_{\gamma(s)},\cdots).$$

Replacing (9) in (8) we have (7).

We observe a number of elementary facts concerning the partial derivation $\Omega_{r,s}(T,S)$:

(i) If X and Y are in Hom (V,V), then

$$(10)\qquad K(X)\Omega_{r,s}(T,S)K(Y)=\Omega_{r,s}(XTY,XSY).$$

This follows immediately from (7).

(ii) If V is a unitary space, then $\bigotimes_1^m V$ is also a unitary space. Thus there is a natural inner product induced on the symmetry class (P,ν) associated with H and χ. Moreover, if T^* is the conjugate dual of $T\in$ Hom (V,V), then the conjugate dual of $\Omega_{r,s}(T,S)$ with respect to the induced inner product in (P,ν) is

$$(11)\qquad \Omega_{r,s}(T,S)^*=\Omega_{r,s}(T^*,S^*).$$

(iii) $\Omega_{1,m-1}(T,S)$ is linear in T and $\Omega_{m-1,1}(T,S)$ is linear in S.

A somewhat more combinatorially involved description is necessary to itemize the eigenvalues of $\Omega_{r,s}(T,S)$. In order to describe a basis for an arbitrary symmetry class associated with H and χ, we regard the elements of H as permutations acting on the functions (i.e., sequences) in $\Gamma_{m,n}=Z_n^{Z_m}$, where $Z_q=\{1,2,\cdots,q\}$ and for $\sigma\in H$, $\alpha\in\Gamma_{m,n}$,

$$\sigma((\alpha))(t)=\alpha(\sigma^{-1}(t)),\qquad t\in Z_m.$$

Let Δ denote a system of distinct representatives for the orbits in $\Gamma_{m,n}$ induced by H, and let $\overline{\Delta}$ denote the set of all of those elements $\alpha\in\Delta$ for which the character χ is identically 1 on the stabilizer subgroup $H_\alpha=\{\sigma\in H:\sigma(\alpha)=\alpha\}$. Let $n(\alpha)=|H_\alpha|$. It is routine to verify that if $\{e_1,\cdots,e_n\}$ is a basis of V, then the decomposable elements $\nu(e_{\alpha(1)},\cdots,e_{\alpha(m)})$, $\alpha\in\overline{\Delta}$ form a basis for P. In fact, if $\{e_1,\cdots,e_n\}$ is an orthonormal (hereafter abbreviated o.n.) basis of V, then the $|\overline{\Delta}|$ decomposable elements $(|H|/n(\alpha))^{1/2}\nu(e_{\alpha(1)},\cdots,e_{\alpha(m)})$ form an o.n. basis for P with respect to the induced inner product in $\bigotimes_1^m V$ defined by

$$(x_1\otimes\cdots\otimes x_m,\,y_1\otimes\cdots\otimes y_m)=\prod_{i=1}^m(x_i,y_i).$$

If we choose the system of distinct representatives Δ so that each sequence $\alpha \in \Delta$ is lowest in lexicographic order in the orbit in which it lies, then it is easy to see that $Q_{m,n} \subset \overline{\Delta}$ and $G_{m,n} \subset \Delta$, whatever the group H and character χ may be [7].

If we use the fact that for any pair (T, S) of commuting linear transformations there exists a common triangular o.n. basis, then it is not difficult to prove the following [11]:

(iv) If $ST = TS$ and the eigenvalues of T and S are $\lambda_1, \cdots, \lambda_n$ and $\kappa_1, \cdots, \kappa_n$ respectively, then after a suitable reordering of the κ_i's, the eigenvalues of $\Omega_{r,s}(T, S)$ are the numbers

$$\sum{}' \prod_{i=1}^{r} \lambda_{\alpha\omega(i)} \prod_{j=1}^{s} \kappa_{\alpha\gamma(j)}, \quad \alpha \in \overline{\Delta}. \tag{12}$$

In particular, if $r = 1$ and $s = m - 1$, then the eigenvalues of $\Omega_{1,m-1}(T, S)$ are

$$\sum_{t=1}^{m} \lambda_{\alpha(t)} \prod_{j \neq t}^{m} \kappa_{\alpha(j)}, \quad \alpha \in \overline{\Delta}. \tag{13}$$

Some additional combinatorial maneuvering will be required: if $\alpha \in \Gamma_{m,n}$ and $1 \leq t \leq n$, then let $m_t(\alpha)$ denote the number of integers i in $\{1, 2, \cdots, m\}$ for which $\alpha(i) = t$; i.e., $m_t(\alpha)$ is the multiplicity of occurrence of t in the range of α. More generally, if $p_1 + \cdots + p_r = n$ is a partition of n into positive parts, we define

$$\eta_t(\alpha) = \sum_{j=P_{t-1}+1}^{P_t} m_j(\alpha)$$

where $P_t = p_1 + \cdots + p_t$; i.e., $\eta_t(\alpha)$ is the number of times any integer k satisfying $P_{t-1} < k \leq P_t$ occurs in the range of α. We can write the eigenvalues (13) of $\Omega_{1,m-1}(T, S)$ in a form somewhat more suitable for our subsequent computations. Suppose that the eigenvalues of T are given by

$$\lambda_1 = \cdots = \lambda_{P_1} = l_1;\ \lambda_{P_1+1} = \cdots = \lambda_{P_2} = l_2;\ \cdots;\ \lambda_{P_{r-1}+1} = \cdots = \lambda_{P_r} = l_r \tag{14}$$

where the numbers l_t are distinct. Suppose, moreover, that $f(x)$ is an arbitrary scalar polynomial and $S = f(T)$. In this case the ordering (14) of the eigenvalues of T induces a corresponding ordering of the eigenvalues $\kappa_1, \cdots, \kappa_n$ of S, i.e.,

$$\kappa_1 = \cdots = \kappa_{P_1} = k_1;\ \kappa_{P_1+1} = \cdots = \kappa_{P_2} = k_2;\ \cdots;\ \kappa_{P_{r-1}+1} = \cdots = \kappa_n = k_r.$$

Since the ploynomial $f(x)$ can be chosen arbitrarily, it follows that the numbers k_t may be chosen arbitrarily. Regarding the k_t as momentarily all different from zero, we see that the eigenvalues (13) become

$$\sum_{t=1}^{m} \lambda_{\alpha(t)} \prod_{j\neq t}^{m} \kappa_{\alpha(j)} = \sum_{t=1}^{m} \frac{\lambda_{\alpha(t)}}{\kappa_{\alpha(t)}} \prod_{j=1}^{m} \kappa_{\alpha(j)} = \sum_{t=1}^{m} \frac{\lambda_{\alpha(t)}}{\kappa_{\alpha(t)}} \prod_{j=1}^{n} \kappa_j^{m_j(\alpha)}$$

$$(15) \qquad = \sum_{t=1}^{n} m_t(\alpha) \frac{\lambda_t}{\kappa_t} \prod_{j=1}^{n} \kappa_j^{m_j(\alpha)} = \sum_{t=1}^{r} \eta_t(\alpha) \frac{l_t}{k_t} \prod_{j=1}^{r} k_j^{\eta_j(\alpha)}$$

$$= \sum_{t=1}^{r} \eta_t(\alpha) l_t k_t^{\eta_t(\alpha)-1} \cdot \prod_{j\neq t}^{r} k_j^{\eta_j(\alpha)}.$$

If we interpret 0^0 as 1, then (15) holds even when some of the numbers k_t are zero. We now have the following formula for the trace of $\Omega_{1,m-1}(T, S)$:

$$\operatorname{tr} \Omega_{1,m-1}(T, S) = \sum_{\alpha\in\bar{\Delta}} \sum_{t=1}^{r} \eta_t(\alpha) l_t k_t^{\eta_t(\alpha)-1} \prod_{j\neq t}^{r} k_j^{\eta_j(\alpha)}$$

$$(16) \qquad = \sum_{t=1}^{r} l_t \left(\sum_{\alpha\in\bar{\Delta}} \eta_t(\alpha) k_t^{\eta_t(\alpha)-1} \prod_{j\neq t}^{r} k_j^{\eta_j(\alpha)} \right).$$

3. **Proofs.**

Lemma 1. *Let G be any subgroup of* $GL_n(V)$ *and* $\mathcal{T} \in \mathfrak{L}(G, K)$. *Then if* $A \in \ker \mathcal{T}$,

$$(17) \qquad \operatorname{tr} \Omega_{1,m-1}(A, X) = 0, \qquad X \in \mathfrak{A}.$$

Proof. Let x_1, x_2 be indeterminates over **C**. From (7) we have

$$K(x_1A + x_2X) = \sum_{r=0}^{m} x_1^r x_2^{m-r} \Omega_{r,m-r}(A, X).$$

Thus, if $\mathcal{T}(A) = 0$ we have

$$\operatorname{tr} \sum_{r=0}^{m} x_1^r x_2^{m-r} \Omega_{r,m-r}(A, X) = \mu_K(x_1A + x_2X) = \mu_K(\mathcal{T}(x_1A + x_2X))$$

$$(18) \qquad = \mu_K(\mathcal{T}(x_2X)) = \mu_K(x_2X) = x_2^m \mu_K(X).$$

If we equate coefficients in (18) we obtain

$$\operatorname{tr} \Omega_{r,m-r}(A, X) = 0, \qquad r = 1, 2, \cdots, m,$$

and hence (17) follows.

Proof of Theorem 1. Assume $H \neq \{e\}$ and let $A \in \ker \mathcal{T}$. We show that $A = 0$. By (10) we can assume that (17) holds for all X and that A is in Jordan normal form.

Lemma 2. *If every elementary divisor of A is linear and* $A \in \ker \mathcal{T}$, *then* $A = 0$.

Proof. By (iii) of § 2, (17) becomes

$$\sum_{i=1}^{n} a_{ii} \operatorname{tr} \Omega_{1,m-1}(E_{ii}, X) = 0, \qquad X \in M_n(\mathrm{C}). \tag{19}$$

We first assume that $\chi \equiv 1$. The eigenvalues of E_{ii} are $\lambda_i = 1$ and $\lambda_j = 0$, $j \neq i$; while those of E_{kk} are $\kappa_k = 1$ and $\kappa_j = 0$, $j \neq k$. Hence

$$\operatorname{tr} \Omega_{1,m-1}(E_{kk}, E_{kk}) = \sum_{\alpha \in \overline{\Delta}} \sum_{t=1}^{m} \kappa_{\alpha(t)} \prod_{j \neq t}^{m} \kappa_{\alpha(j)} = \sum_{t=1}^{m} \kappa_k^m = m, \tag{20}$$

because the only term which survives in the inner summation in (20) is the term corresponding to that α for which $\alpha(1) = \alpha(2) = \cdots = \alpha(m) = k$. We remark that this sequence is always in $\overline{\Delta}$ since, as we remarked, $G_{m,n} \subset \overline{\Delta}$ when $\chi \equiv 1$. If $i \neq k$, we again compute that

$$\operatorname{tr} \Omega_{1,m-1}(E_{ii}, E_{kk}) = \sum_{\alpha \in \overline{\Delta}} \sum_{t=1}^{m} \lambda_{\alpha(t)} \prod_{j \neq t} \kappa_{\alpha(j)} = \sum_{\beta \in \overline{\Delta}} 1 \cdot 1^{m-1} \tag{21}$$

where the inner summation in (21) is over precisely those $\beta \in \overline{\Delta}$ for which $m_k(\beta) = m - 1$ and $m_i(\beta) = 1$. Once again, since $G_{m,n} \subset \overline{\Delta}$, such sequences exist and we let p_{ik} denote their number. We assert that p_{ik} is independent of the pair (i, k); for if P is an arbitrary permutation matrix we have from (10)

$$p_{ik} = \operatorname{tr} \Omega_{1,m-1}(E_{ii}, E_{kk}) = \operatorname{tr} \Omega_{1,m-1}(P^T E_{ii} P, P^T E_{kk} P).$$

Obviously if $i \neq k$, we can choose P so that $P^T E_{ii} P = E_{i'i'}$ and $P^T E_{kk} P = E_{k'k'}$ for any preassigned distinct integers i', k'. We set p equal to the common value of the p_{ik}. We next assert that $p < m$, for since $H \neq \{e\}$, there must exist at least two sequences $\alpha \neq \beta$ in the same H-orbit for which $m_k(\alpha) = m_k(\beta) = m - 1$ and $m_i(\alpha) = m_i(\beta) = 1$. Thus there are at most $m - 1$ elements of $\overline{\Delta}$ with that property. If we set X successively equal to E_{kk}, $k = 1, 2, \cdots, n$, in (19) we obtain the following system of linear equations:

$$m a_{ii} + \sum_{k \neq i} p a_{kk} = 0, \qquad 1 \leq i \leq n.$$

Since $p < m$, the coefficient matrix in this system is nonsingular and we conclude that $a_{ii} = 0$ for $1 \leq i \leq n$. Thus since the elementary divisors of A are linear we conclude $A = 0$.

We now consider the case in which $\chi \neq 1$. If x is an indeterminate, then since E_{ii} and $I_n + xE_{11}$ commute we have from (13) that

$$\begin{aligned} \operatorname{tr} \Omega_{1,m-1}(E_{ii}, I_n + xE_{11}) &= \sum_{\alpha \in \overline{\Delta}} \sum_{t=1}^{m} \lambda_{\alpha(t)} \prod_{j \neq t} \kappa_{\alpha(j)} \\ &= \sum_{\alpha \in \overline{\Delta}} \prod_{j=1}^{n} \kappa_j^{m_j(\alpha)} \sum_{t=1}^{n} m_t(\alpha) \frac{\lambda_t}{\kappa_t} \end{aligned} \tag{22}$$

where $\lambda_1, \cdots, \lambda_n$ are again the eigenvalues of E_{ii} and $\kappa_1, \cdots, \kappa_n$ are the eigenvalues of $I_n + xE_{11}$. In case $i = 1$, the right side of (22) becomes

$$\sum_{\alpha\in\bar{\Delta}} m_1(\alpha)(1+x)^{m_1(\alpha)-1} \tag{23}$$

which we denote by $\phi_1(x)$. If $i > 1$, we see that the value of (22) (which is of course the same for $i = 2, 3, \cdots, n$) is

$$\phi_2(x) = \sum_{\alpha\in\bar{\Delta}} m_2(\alpha)(1+x)^{m_1(\alpha)}. \tag{24}$$

With $X = I_n + xE_{11}$ in (19) we have from (23) and (24) that

$$\begin{aligned} &a_{11}\phi_1(x) + \phi_2(x)\sum_{i=2}^{n} a_{ii} \\ &= a_{11}\sum_{\alpha\in\bar{\Delta}} m_1(\alpha)(1+x)^{m_1(\alpha)-1} + \left(\sum_{i=2}^{n} a_{ii}\right)\left(\sum_{\alpha\in\bar{\Delta}} m_2(\alpha)(1+x)^{m_1(\alpha)}\right) = 0. \end{aligned} \tag{25}$$

We observe that

$$q_j = \sum_{\alpha\in\bar{\Delta}} m_j(\alpha) \tag{26}$$

is independent of j and we denote this common value by q. A proof of this can be based on the fact that the eigenvalues of $K(X)$ are

$$\prod_{j=1}^{m} x_{\alpha(j)} = \prod_{j=1}^{n} x_j^{m_j(\alpha)}, \qquad \alpha \in \bar{\Delta},$$

where $x_1, \cdots, x_n$ are the eigenvalues of X. Thus

$$\det(K(X)) = \prod_{\alpha\in\bar{\Delta}} \prod_{j=1}^{n} x_j^{m_j(\alpha)} = \prod_{j=1}^{n} x_j^{q_j}.$$

Since $\det(K(P^TXP)) = \det(K(X))$ for any permutation matrix P, we conclude that

$$\prod_{j=1}^{n} x_j^{q_j} = \prod_{j=1}^{n} x_{\sigma(j)}^{q_j}, \qquad \sigma \in S_n.$$

And since $x_1, \cdots, x_n$ are arbitrary, $q_1 = q_2 = \cdots = q_n$. For example, it is easy to compute that

$$q = q_1 = \sum_{\alpha\in Q_{m,n}} m_1(\alpha) = \binom{n-1}{m-1};$$

and since $K(X)$ is the familiar mth compound martix $C_m(X)$ we see that

$$\det C_m(X) = \prod_{j=1}^{m} x_j^{q} = (\det(X))^{\binom{n-1}{m-1}},$$

and we have as a corollary to our computations the well-known Sylvester-Franke theorem [1]. For $x = 0$, (25) becomes q tr $A = 0$ and hence tr $A = 0$; we therefore rewrite (25) as

$$(\phi_1(x) - \phi_2(x))a_{11} = 0. \tag{27}$$

Clearly,

$$\deg \phi_1(x) = \max_{\alpha \in \overline{\Delta}} m_1(\alpha) - 1.$$

Now $\chi \not\equiv 1$, and thus there exists $\beta \in \overline{\Delta}$ and $1 < j \leq n$ such that

$$j \in \operatorname{rng} \beta, \qquad m_1(\beta) = \max_{\alpha \in \overline{\Delta}} m_1(\alpha),$$

for otherwise $\max_{\alpha \in \overline{\Delta}} m_1(\alpha) = m$ and $\alpha = (1, 1, \cdots, 1) \in \overline{\Delta}$. But the stabilizer of this α is obviously all of H and it would follow that $\sum_{\sigma \in H} \chi(\sigma) \neq 0$ so that $\sum_{\sigma \in H} \chi(\sigma) = |H|$. This can only happen if $\chi \equiv 1$, since χ is a character of degree 1. Hence

$$\deg \phi_1(x) = m_1(\beta) - 1 \quad \text{and} \quad \deg \phi_2(x) = m_1(\beta),$$

so that $\phi_1(x) - \phi_2(x) \neq 0$. From (27) it follows that a_{11} (and hence any a_{ii}) is 0. Thus $A = 0$, completing the proof of Lemma 2. We can now remove the condition that A has linear elementary divisors.

Lemma 3. *If* $A \in \ker \mathfrak{T}$, *then* $A = 0$.

Proof. From (11) we know that

$$\Omega_{1,m-1}(A, X)^* = \Omega_{1,m-1}(A^*, X^*).$$

Since (17) holds for all X, we have $\operatorname{tr} \Omega_{1,m-1}(A^*, X) = 0$ and hence $\operatorname{tr} \Omega_{1,m-1}(A \pm A^*, X) = 0$. Now $A \pm A^*$ is normal and hence has linear elementary divisors. Applying Lemma 2, we conclude that $A \pm A^* = 0$ and therefore $A = 0$.

We have proved that if $H \neq \{e\}$ and $m \leq n$ or $\chi \equiv 1$, then any $\mathfrak{T} \in \mathfrak{L}(\mathrm{GL}_n(V), K)$ is nonsingular; since we have observed that this set obviously forms a semigroup, it is in fact a group. Conversely, if $H = \{e\}$, then the symmetry class P is just the mth tensor space $\otimes_1^m V$ and $K(X) = \Pi^m(X)$. As we saw in § 1, $\mathfrak{L}(G, K)$ is not a group. This completes the proof of Theorem 1.

Proof of Theorem 2. We observe that if $A \in \mathfrak{A}$ and $f(x)$ is any scalar polynomial, then $f(A) \in \mathfrak{A}$. The eigenvalues of $\Omega_{1,m-1}(A, f(A))$ are given by (13) where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A and $\kappa_1, \cdots, \kappa_n$ are the eigenvalues of $f(A)$. Moreover, it is clear that the values of $f(x)$ may be arbitrarily assigned at the distinct eigenvalues of A.

Since $H = S_m$ and $\chi \equiv 1$, the symmetry class P is precisely the mth completely symmetric space, usually denoted by $V^{(m)}$ [4], and the sequence set $\overline{\Delta}$ is precisely $G_{m,n}$. Once again the problem is to show that if $\mathcal{T}(A) = 0$, then $A = 0$.

We have from Lemma 1 and formula (16) that

$$\sum_{t=1}^{r} l_t \left(\sum_{\alpha \in \overline{\Delta}} \eta_t(\alpha) k_t^{\eta_t(\alpha)-1} \prod_{j \neq t}^{r} k_j^{\eta_j(\alpha)} \right) = 0 \tag{28}$$

in which the distinct eigenvalues of A are described in (14) and the numbers $k_1, \cdots, k_r$ may be chosen arbitrarily. The first part of the proof is devoted to showing that all the eigenvalues of A are equal (i.e., that $r = 1$). Assume then that $r > 1$. For a fixed t, set $k_t = 1$ and $k_j = 0$ for $j \neq t$; observe that the coefficient of l_t in (28) is

$$m \sum_{\alpha \in \overline{\Delta},\ \eta_t(\alpha)=m} 1 = m \binom{p_t + m - 1}{m},$$

where the indicated binomial coefficient is just a count of the total number of sequences α in $G_{m,n}$ such that $\operatorname{rng} \alpha \subset \{P_{t-1} + 1, \cdots, P_t\}$. The coefficient of l_s, $s \neq t$, in (28) is

$$\sum_{\alpha \in \overline{\Delta},\ \eta_s(\alpha)=1,\ \eta_t(\alpha)=m-1} 1, \tag{29}$$

which is a count of the total number of nondecreasing sequences α of length m which have the property that $\operatorname{rng} \alpha$ contains precisely 1 integer in the interval $[P_{s-1} + 1, P_s]$ and $m - 1$ integers in $[P_{t-1} + 1, P_t]$. Since there are precisely

$$\binom{p_t + m - 2}{m - 1} \cdot p_s \tag{30}$$

such sequences, the value of (29) is (30). Thus (28) for the choice $k_t = 1$, $k_j = 0$ for $j \neq t$ is

$$l_t m \binom{p_t + m - 1}{m} + \sum_{s \neq t}^{r} l_s p_s \binom{p_t + m - 2}{m - 1} = 0,$$

or

$$l_t(p_t + m - 1) + \sum_{s \neq t}^{r} l_s p_s = 0. \tag{31}$$

Consider the system of homogeneous linear equations for $l_1, \cdots, l_r$ obtained by setting $t = 1, 2, \cdots, r$ in (31). By setting $k_1 = \cdots = k_r = 1$ in (28) we obtain the following additional condition on the l's:

$$\sum_{t=1}^{r} l_t \left(\sum_{\alpha \in \bar{\Delta}} \eta_t(\alpha) \right) = 0. \tag{32}$$

Now by (26)

$$\sum_{\alpha \in \bar{\Delta}} \eta_t(\alpha) = \sum_{\alpha \in \bar{\Delta}} \sum_{j=P_{t-1}+1}^{P_t} m_j(\alpha) = \sum_{j=P_{t-1}+1}^{P_t} q_j = p_t \cdot q.$$

Thus (32) becomes

$$\sum_{s=1}^{r} l_s p_s = 0. \tag{33}$$

Combining the system (31) with (33) we have $(m-1)l_t = 0$, $t = 1, \cdots, r$, and thus $l_t = 0$, $t = 1, \cdots, r$, contradicting the fact that the $l_1, \cdots, l_r$ are distinct. Hence $r = 1$. In other words, the condition

$$\operatorname{tr} \Omega_{1,m-1}(A, X) = 0, \qquad X \in \mathfrak{A}, \tag{34}$$

implies that all the eigenvalues of A are equal. If we set $X = I_n$ in (34) we then see that

$$\operatorname{tr} \Omega_{1,m-1}(A, I_n) = \sum_{\alpha \in \bar{\Delta}} \sum_{t=1}^{m} \lambda_{\alpha(t)} = \sum_{\alpha \in \bar{\Delta}} \sum_{t=1}^{n} m_t(\alpha) \lambda_t$$

$$= \sum_{t=1}^{n} q \cdot \lambda_t = q \operatorname{tr} A = 0.$$

Thus we conclude that if $\operatorname{tr} \Omega_{1,m-1}(A, X) = 0$, $X \in \mathfrak{A}$, then all the eigenvalues of A are zero. By repeating precisely the same argument as we gave in Lemma 3, we can conclude that the eigenvalues of both of the normal matrices $A \pm A^*$ are zero and hence $A = 0$. This completes the proof of Theorem 2.

The proof of Corollary 1 is now obvious, since the conjugate transpose of a generalized doubly stochastic matrix is a matrix of the same kind.

Proof of Theorem 3. We are assuming that $\mathcal{T}$: Hom $(V, V) \to$ Hom (V, V) satisfies

$$\mu_K(\mathcal{T}(X)) = \mu_K(X), \qquad X \in \operatorname{Hom}(V, V). \tag{35}$$

Since $\mu_K(\xi X) = \xi^m \mu_K(X)$ it is clear that we may assume $\mathcal{T}(I_n) = I_n$. From (35) we have

$$\operatorname{tr} K(\mathcal{T}(I_n + xX)) = \mu_K(\mathcal{T}(I_n + xX)) = \mu_K(I_n + xX) = \operatorname{tr} K(I_n + xX).$$

From (7) with $T = I_n$, $S = X$, $x_1 = 1$, $x_2 = x$ we have

$$\sum_{r=0}^{m} x^r \operatorname{tr} \Omega_{m-r,r}(I_n, \mathcal{T}(X)) = \sum_{r=0}^{m} x^r \operatorname{tr} \Omega_{m-r,r}(I_n, X)$$

and thus

$$\operatorname{tr} \Omega_{m-r,r}(I_n, \mathcal{T}(X)) = \operatorname{tr} \Omega_{m-r,r}(I_n, X). \tag{36}$$

Let $\kappa_1, \cdots, \kappa_n$ be the eigenvalues of X. Then as we know from (12), the eigenvalues of $\Omega_{m-r,r}(I_n, X)$ are

$$\sum{}' \prod_{j=1}^{r} \kappa_{\alpha\gamma(j)}, \qquad \alpha \in \bar{\Delta}, \tag{37}$$

where we recall that the prime indicates that the summation is over all $\gamma \in Q_{r,m}$. But the expression (37) is precisely the rth elementary symmetric function of $\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)}$, $E_r(\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)})$. Thus

$$\operatorname{tr} \Omega_{m-r,r}(I_n, X) = \sum_{\alpha \in G_{m,n}} E_r(\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)}), \tag{38}$$

and an elementary induction argument shows that the right-hand side of (38) is precisely

$$\binom{m+n-1}{m-r} h_r(\kappa_1, \cdots, \kappa_n), \tag{39}$$

where h_r denotes the rth completely symmetric polynomial in the κ's. From (36) and (39) it follows that

$$h_r(\kappa_1', \cdots, \kappa_n') = h_r(\kappa_1, \cdots, \kappa_n), \qquad r = 1, 2, \cdots, m,$$

where the $\kappa_1', \cdots, \kappa_n'$ are the eigenvalues of $\mathcal{T}(X)$. But by Wronski's relations [13] we know that the completely symmetric polynomials $h_1, \cdots, h_m$ form an integral polynomial basis for the space of all integral homogeneous symmetric polynomials of degree m. Thus it follows that

$$E_r(\kappa_1', \cdots, \kappa_n') = E_r(\kappa_1, \cdots, \kappa_n), \qquad r = 1, 2, \cdots, m.$$

In other words, we have proved that the rth elementary symmetric function of the eigenvalues of both X and $\mathcal{T}(X)$ are equal, $r = 1, 2, \cdots, m$; and we are in a position to apply a theorem of Marcus and Purves [10] (recently extended by Beasley [2]) which states that any such transformation must have one of the two forms indicated in (3) or (4). This completes the proof of Theorem 3.

The proof of Corollary 2 is an immediate consequence of Theorem 3. For, the eigenvalues of $K(X)$ when $H = S_m$, $\chi \equiv 1$, $P = V^{(m)}$ are the $\binom{n+m-1}{m}$

homogeneous products

$$\prod_{t=1}^{n} \lambda_t^{m_t(\alpha)}, \qquad \alpha \in G_{m,n},$$

and hence

$$\mu_K(X) = \operatorname{tr} K(X) = \sum_{\alpha\in G_{m,n}} \prod_{t=1}^{n} \lambda_t^{m_t(\alpha)} = h_m(\lambda_1, \cdots, \lambda_n).$$

Proof of Theorem 4. Once again we can assume that $\mathcal{T}(I_n) = I_n$. Our first task is to determine the structure of the $\overline{\Delta}$ set for $H = A_m$ and $\chi \equiv 1$. There are two cases to consider: $m \leq n$ and $m > n$. Let $\omega \in \Gamma_{m,n}$. If the integers in rng ω are distinct, then let α be the sequence such that rng $\alpha =$ rng ω and $\alpha(1) < \alpha(2) < \cdots < \alpha(m)$, and let β be the sequence such that rng $\beta =$ rng ω and $\beta(1) < \beta(2) < \cdots < \beta(m-2) < \beta(m) < \beta(m-1)$. It is clear that α and β lie in distinct A_m-orbits and that any other sequence $\gamma \in \Gamma_{m,n}$ for which rng $\gamma =$ rng ω is in the same A_m-orbit with either α or β. Thus each such sequence $\omega \in \Gamma_{m,n}$ gives rise to two elements in $\overline{\Delta}$, namely α and β. On the other hand, if $\omega \in \Gamma_{m,n}$ satisfies $m_t(\omega) \geq 2$ for some $1 \leq t \leq n$, then it is obvious that there exists a sequence α in $G_{m,n}$ in the same A_m-orbit with ω. Thus in the case $m \leq n$, the system of distinct representatives may be chosen to be $\overline{\Delta} = G_{m,n} \cup Q'_{m,n}$, where $Q'_{m,n}$ consists of precisely those sequences β for which $\beta(1) < \beta(2) < \cdots < \beta(m-2) < \beta(m) < \beta(m-1)$. In the case that $m > n$, then it is clear that any sequence $\omega \in \overline{\Delta}$ lies in the same A_m-orbit with a sequence $\alpha \in G_{m,n}$, so that we can choose $\overline{\Delta} = G_{m,n}$. In order to deal with both cases at once we will let $Q'_{m,n} = \varnothing$ if $m > n$.

Precisely as in (36) we see that if $\mathcal{T} \in \mathfrak{L}_1(\mathrm{GL}_n(V), K)$, then

$$\operatorname{tr} \Omega_{m-r,r}(I_n, \mathcal{T}(X)) = \operatorname{tr} \Omega_{m-r,r}(I_n, X), \qquad X \in M_n(\mathbf{C}). \tag{40}$$

The eigenvalues of $\Omega_{m-r,r}(I_n, X)$ are precisely the numbers

$$E_r(\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)}), \qquad \alpha \in \overline{\Delta},$$

and thus

$$\begin{aligned} \operatorname{tr} \Omega_{m-r,r}(I_n, X) &= \sum_{\alpha\in\overline{\Delta}} E_r(\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)}) \\ &= \sum_{\alpha\in G_{m,n}} E_r(\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)}) + \sum_{\alpha\in Q'_{m,n}} E_r(\kappa_{\alpha(1)}, \cdots, \kappa_{\alpha(m)}) \qquad (41) \\ &= \binom{n+m-1}{m-r} h_r(\kappa_1, \cdots, \kappa_n) + \binom{n-r}{m-r} E_r(\kappa_1, \cdots, \kappa_n). \end{aligned}$$

In case $m > n$, the elementary symmetric function does not appear in (41) and it is clear from (40) that

$$h_r(\mathcal{T}(X)) = \binom{n+m-1}{m-r}^{-1} \operatorname{tr} \Omega_{m-r,r}(I_n, \mathcal{T}(X))$$

$$= \binom{n+m-1}{m-r}^{-1} \operatorname{tr} \Omega_{m-r,r}(I_n, X)$$

$$= h_r(X), \qquad r = 1, 2, \cdots, m;\ X \in M_n(\mathrm{C}).$$

Hence by Corollary 2, $\mathcal{T}$ has the required form. On the other hand, if $3 \leq m \leq n$, then using Wronski's relations we have

$$\begin{aligned} \operatorname{tr} \Omega_{m-1,1}(I_n, X) &= aE_1(X), \\ \operatorname{tr} \Omega_{m-2,2}(I_n, X) &= bh_2(X) + cE_2(X) \\ &= b(E_1^2(X) - E_2(X)) + cE_2(X) \\ &= (c-b)E_2(X) + bE_1^2(X), \\ \operatorname{tr} \Omega_{m-3,3}(I_n, X) &= dh_3(X) + eE_3(X) \\ &= d(E_1^3(X) - 2E_1(X)E_2(X) + E_3(X)) + eE_3(X) \\ &= (d+e)E_3(X) - 2dE_1(X)E_2(X) + dE_1^3(X), \end{aligned} \tag{42}$$

where $a = \binom{m+n-1}{m-1} + \binom{n-1}{m-1}$, $b = \binom{n+m-1}{m-2}$, $c = \binom{n-2}{m-2}$, $d = \binom{n+m-1}{m-3}$ and $e = \binom{n-3}{m-3}$. Observe that $d + e > 0$, $c - b \neq 0$, $a \neq 0$; thus the relations (42) allow us to express $E_3(X)$ as a polynomial in $\operatorname{tr} \Omega_{m-r,r}(I_n, X)$, $r = 1, 2, 3$. It follows from (40), then, that $E_3(\mathcal{T}(X)) = E_3(X)$, $X \in M_n(\mathrm{C})$ and hence we can conclude as before that $\mathcal{T}$ has the required form. This completes the proof of Theorem 4.

By similar arguments Corollary 3 follows from Theorem 4. For, the eigenvalues of $K(X)$ when $H = A_m$, $\chi \equiv 1$ are the numbers

$$\prod_{t=1}^{n} \lambda^{m_t(\alpha)}, \qquad \alpha \in G_{m,n} \cup Q'_{m,n},$$

and hence

$$\mu_K(X) = \operatorname{tr} K(X) = \sum_{\alpha \in G_{m,n}} \prod_{t=1}^{n} \lambda_t^{m_t(\alpha)} + \sum_{\alpha \in Q_{m,n}} \prod_{t=1}^{n} \lambda_t^{m_t(\alpha)}$$

$$= k_m(\lambda_1, \cdots, \lambda_n).$$

REFERENCES

1. A. C. Aitken, *Determinants and matrices*, Oliver and Boyd, Edinburgh; Interscience, New York, 1962, pp. 90–110.

2. L. B. Beasley, *Linear transformations on matrices: The invariance of the third elementary symmetric function*, Canad. J. Math. **22** (1970), 746–752. MR **42** #3100.

3. G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen.* I, S.-B. Preuss. Akad. Wiss. Berlin 1897, 994–1015.

4. W. H. Greub, *Multilinear algebra*, Die Grundlehren der math. Wissenschaften, Band 136, Springer-Verlag, New York, 1967. MR **37** #222.

5. M. Marcus, *All linear operators leaving the unitary group invariant*, Duke Math. J. **26** (1959), 155–163. MR **21** #54.

6. ———, *Spectral properties of higher derivations on symmetry classes of tensors*, Bull. Amer. Math. Soc. **75** (1969), 1303–1307. MR **41** #245.

7. M. Marcus and W. R. Gordon, *The structure of bases in tensor spaces*, Amer. J. Math. **92** (1970), 623–640. MR **42** #7684.

8. M. Marcus and N. A. Kahn, *A note on a group defined by a quadratic form*, Canad. Math. Bull. **3** (1960), 143–148. MR **23** #A1653.

9. M. Marcus and F. May, *On a theorem of I. Schur concerning matrix transformations*, Arch. Math. **11** (1960), 401–404. MR **24** #A134.

10. M. Marcus and R. Purves, *Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions*, Canad. J. Math. **11** (1959), 383–396. MR **21** #4167.

11. R. Merris, *A generalization of the associated transformation*, Linear Algebra and Appl. **4** (1971), 393–406.

12. I. Schur, *Einige Bemerkungen zur Determinantentheorie*, S.-B. Preuss. Akad. Wiss. Berlin **25** (1925), Satz II, 454–463.

13. H. W. Turnbull, *Theory of equations*, Oliver and Boyd, Edinburgh; Interscience, New York, 1952, pp. 71–72.

14. J. H. M. Wedderburn, *Lectures on matrices*, Amer. Math. Soc. Colloq. Publ., vol. 17, Amer. Math. Soc., Providence, R. I., 1934, 79pp.

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