GROUPS WITH PARAMETRIC EXPONENTS

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1. **Introduction.** Elsewhere[2] we consider the problem of characterizing all the solutions x, in a free group on generators $a_1, a_2, \cdots, a_r$, of a given equation $w(x, a_1, a_2, \cdots, a_r)=1$. The totality of solutions can be described as the set of all values assumed by a certain finite set of group theoretic expressions upon substituting integers for certain parameters appearing in these expressions as exponents. As a trivial example, the solutions of $x^{-1}a_1^{-1}xa_1=1$ are all values assumed by a_1^{ν} as ν runs through the integers. The general result, and our method of obtaining it, has led us to the study of such expressions, or "words," that contain certain parameters $\nu_1, \nu_2, \cdots, \nu_d$ as exponents. These words may be taken in a natural way as representing elements of a group G that admits additional algebraic operations of raising an element g of G to an exponent α, where α is any element of the ring $Z[\nu_1, \nu_2, \cdots, \nu_d]$ of all polynomials in indeterminates in $\nu_1, \nu_2, \cdots, \nu_d$ with integer coefficients.

More generally, if X is any associative ring with 1, we call a group G an *X-group* if it is equipped with additional operations $g \rightarrow g^{\cdot\alpha}$ for each α in X, subject to the following axioms:

$$g^{\cdot 1} = g, \qquad g^{\cdot(\alpha+\beta)} = g^{\cdot\alpha}g^{\cdot\beta}, \qquad g^{\cdot(\alpha\beta)} = (g^{\cdot\alpha})^{\cdot\beta}, \qquad g(hg)^{\cdot\alpha} = (gh)^{\cdot\alpha}g.$$

We hasten to note that if n is an integer, the axioms imply $g^{\cdot n}=g^n$, so that we may omit all dots. Like the power maps, $g \rightarrow g^n$, these operations are not required to define endomorphisms: we do not require that $(gh)^\alpha = g^\alpha h^\alpha$. The last axiom requires that, like the power maps, these operations commute with all inner automorphisms: $(g^{-1}hg)^\alpha = g^{-1}h^\alpha g$.

The appropriateness of these axioms to our purpose is demonstrated by the fact, proved below, that for $X=Z[\nu_1, \nu_2, \cdots, \nu_d]$, a word ω in the letters $a_1, a_2, \cdots, a_r$ containing parameters $\nu_1, \nu_2, \cdots, \nu_d$ assumes the value 1 in the free group on $a_1, a_2, \cdots, a_r$ under all substitutions for the parameters if and only if it reduces to the "empty word" 1 by virtue of the given axioms.

The main result of this paper is the solution of an extended word problem for free X-groups, $X=Z[\nu_1, \nu_2, \cdots, \nu_d]$. An effective process is exhibited whereby, given a word ω, as above, an element α in X is determined such that, under any substitution of integers for the ν_i, ω assumes the value 1 if and only if α assumes the value 0.

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Although we study X-groups only for $X=Z[\nu_1, \nu_2, \cdots, \nu_d]$, $d\geqq 0$, and, indeed, only free X-groups, we believe the concept is of some more general interest. For $X=Z$, the ring of integers, the concept of X-group reduces to that of ordinary group. For $X=Z_m$, the integers modulo some m, the X-groups are just those ordinary groups in which all elements have order dividing m. For X the field of rationals, the X-groups are just those groups in which "extraction of roots" is always possible and unique; questions related to this have been studied by B. H. Neumann [7], A. I. Malcev [6], R. Baer [1], P. G. Kontorovich [4], and others, in particular, recently by G. Baumslag [2], who gives further references. Baumslag considers more generally the case that X consists of those rationals that can be written with denominator a product of primes belonging to some prescribed set. For X the field of real numbers, the formalism of X-groups is at least superficially suggestive of certain considerations in the local theory of Lie groups.

Taking a quite different approach, P. Hall [3] and also M. Lazard [5] have introduced groups admitting exponents from a ring more general than the ring of rational integers, and it is clear that there is some overlap between the groups considered by them and X-groups. It may also be noted that V. A. Tartakovski [8] has made use of a limited class of what we call parametric words.

Finally, I want to record that, although the connection is perhaps remote, my interest in the present problem derives from a question of A. Tarski, whether the "elementary theory" of free groups is decidable.

2. **Parametric words.** The existence of a free X-group F, for any X, on any number of generators $a_1, a_2, \cdots, a_r$ follows from the fact that the axioms for X-groups can be written as identical equations. However, confining ourselves henceforth to the case that $X=Z[\nu_1, \nu_2, \cdots, \nu_d]$, $d\geqq 0$, we shall need a more constructive description of the free X-group F in terms of "parametric words" representing its elements.

The elements of the ordinary free group F_0 on generators $a_1, a_2, \cdots, a_r$ are represented by "words" or "formal products" $\omega=a_{i_1}^{\epsilon_1}\cdots a_{i_n}^{\epsilon_n}$, $n\geqq 0$, $\epsilon_i=\pm 1$, of *letters* a_i^{ϵ}. (Rigorously, one may define a *word* to be a sequence of letters.) A word is *reduced* if it contains no letter followed by its inverse. Every element of F_0 is represented by a unique reduced word. The obvious procedure for replacing a given word by a reduced word representing the same group element provides an affirmative solution to the word problem for ordinary free groups.

We shall attempt to parallel all this for the free X-group F, defining first the concept of "parametric word," then a subset of "normal words," and finally solving for F the generalization stated above of the word problem.

We begin with the observation that the ordinary subgroup F_0 of F generated by $a_1, a_2, \cdots, a_r$ is in fact free on these generators. To see this, let G be the ordinary free group on a set of generators $b_1, b_2, \cdots, b_r$. If ρ is any retraction of X onto Z, for example that defined by setting all $\rho\nu_i = 0$, then we can make G into an X-group G' by defining $g^\alpha = g^{\rho\alpha}$ for all g in G and α in X. Since F is the free X-group on generators $a_1, a_2, \cdots, a_r$, and G' is an X-group generated by $b_1, b_2, \cdots, b_r$, there exists an X-group homomorphism ϕ of F onto G' for which $\phi a_i = b_i$, $i = 1, 2, \cdots, r$. But then the restriction of ϕ to F_0 is an ordinary group homomorphism of F_0 onto G, and it is in fact clear that it is an isomorphism, whence it follows that F_0 is free on the generators $a_1, a_2, \cdots, a_r$. We remark in passing that, taking F_0 in the role of G, every retraction ρ of X onto Z induces a retraction $\bar{\rho}$ of F onto F_0, under which $\bar{\rho}(g^\alpha) = (\bar{\rho}g)^{(\rho\alpha)}$.

Starting with F_0, we define an ascending chain of ordinary subgroups of F by letting F_{t+1} be the ordinary subgroup of F generated by all g^α where g is in F_t and α is in X. The union of this chain, as a subset of F containing the generators and closed under all the X-group operations, must constitute all of F. It follows that each element w of F is contained in some F_t; if $t > 0$, w may be represented by an expression $\omega = \omega^{\alpha_1}\omega_2^{\alpha_2} \cdots \omega_n^{\alpha_n}$ where the ω_i are expressions representing elements of F_{t-1} and the α_i are in X. If we take the ordinary words ω representing elements of F_0 as *words of height* 0, the way is open to define inductively what we mean by a word of height t, representing an element w of F_t. In view of this induction, supposing the word problem solved for F_{t-1}, we may more conveniently replace the ω_i by the elements w_i in F_{t-1} that they represent, and, tentatively, define a *word of height t*, for $t > 0$, to be a "formal product" $\omega = w_1^{\alpha_1}w_2^{\alpha_2} \cdots w_n^{\alpha_n}$, $n \geqq 0$, where all w_i are in F_{t-1}, and all α_i are in X. Rigorously, we may interpret the "formal product" as a sequence of ordered pairs, $\omega = ((w_1, \alpha_1), (w_2, \alpha_2), \cdots, (w_n, \alpha_n))$; but in practice no ambiguity will arise from the more suggestive product notation.

The desirability of amplifying this tentative definition of a "word" becomes apparent as soon as we attempt to single out a subclass that is in some reasonable sense "reduced." For these we should like to ensure that, under any retraction ρ, there is no cancellation between adjacent factors $\bar{\rho}(w_i^{\alpha_i})$. The example $\omega = a_1(a_2a_1)^\alpha$ shows that we must consider separately retractions ρ according to the sign of $\rho\alpha$. We are led thus to replace ω by three "words with side conditions": $(\omega, \rho\alpha > 0)$, $(\omega, \rho\alpha = 0)$, $(\omega, \rho\alpha < 0)$. In anticipation of iterating this sort of trichotomy, we take as our formal definition of a *word* a pair (ω, C), where ω is as before and C is any finite set of formal expressions of the type $\rho\alpha < \rho\beta$ or $\rho\alpha = \rho\beta$, for α and β elements of X.

It is clear what it means for a given retraction to *satisfy* a given set of condition C. The *values* of a word (ω, C), where ω represents the element w of F, are all elements $\bar{\rho}w$ in F_0 for ρ satisfying C. The values of a set of words are those of its members. Our aim is to define a class of "normal words," and to show that any given word effectively determines a finite set of normal words which is equivalent to it in the sense of having exactly the same set of values. But we shall use a more constructive definition of equivalence. Two sets of words will be called *equivalent* if it is possible to pass from one to the other by a succession of steps of the following three kinds.

(E1) *replace* $(\omega, C_1), \cdots, (\omega, C_h)$ *by* $(\omega, D_1), \cdots, (\omega, D_k)$ *where exactly the same retractions satisfy one of the* C_i *as satisfy one of the* D_j;

(E2) *replace* (ω, C) *by* (ω', C) *where* C *contains the condition* $\rho\alpha=\rho\beta$ *for some* α *and* β *in* X, *and* ω *and* ω' *differ only in that one contains* α *at certain places where the other contains* β;

(E3) *replace* (ω, C) *by* (ω', C) *where* $\omega=\omega'$ *follows from the axioms for X-groups.*

We note that (E1) with $h=1$ and $k=0$ permits us to delete a word (ω, C) in case C is inconsistent, that is, is satisfied by no retraction ρ. Likewise the case $h=1$ and $k=1$ permits us to replace C by any equivalent set of conditions, with the result that we need not pay close attention to the precise manner of formulation of the conditions C.

We define a *normal word of height* 0 to be any pair (ω, C) where ω is an ordinary reduced word representing an element of F_0, and C any consistent set of conditions. The definition of a normal word of height t, for $t>0$, requires a number of clauses, which we now formulate.

In a normal word (ω, C), where $\omega=w_1^{\alpha_1}\cdots w_n^{\alpha_n}$, we shall require that $n\geqq 1$, and that $\alpha_i=1$ for odd i. Changing notation, we can write

$$\omega = u_1v_1^{\alpha_1}u_2v_2^{\alpha_2}\cdots u_mv_m^{\alpha_m}u_{m+1}$$

where all u_i and v_i are in F_{t-1}, all α_i are in X, and $m\geqq 0$.

Our first condition is the following:

(N0) *each* u_i *is represented by some* ξ_i, *and each* v_i *by some* η_i, *such that all the* (ξ_i, C) *and* (η_i, C) *are normal words of height* $t-1$.

The next condition enforces a certain measure of uniqueness on the parts $v_i^{\alpha_i}$. Before stating it, we define a word (η, C) to be *primitive* if it is not equivalent to any word (ζ^β, C) where $\beta\neq\pm 1$. We now state the next condition:

(N1) C *contains all the conditions* $\rho\alpha_i>0$, *and* C *does not imply a condition* $\rho\alpha_i<k$ *for any* $i=1, 2, \cdots, m$, *and* k *in* Z; *each* (η_i, C) *is primitive, and, if* $t\geqq 2$, *is not equivalent to any word* (ζ, C) *of height* $t-2$.

In particular, the latter part of Condition N1 will ensure that no $v_i^{\alpha_1}$ can be absorbed into adjoining u_i or u_{i+1}.

Our next condition will turn out to be an immediate consequence of Condition N0; but, since we are deferring all proofs to later sections, we state it here separately.

(N2) *For each* v_i *there exist two letters* $L(v_i)$ *and* $R(v_i)$ *such that, for all* ρ *satisfying* C, *the element* $\bar{\rho}v_i$ *in* F_0 *is represented by a nontrivial reduced word beginning with* $L(v_i)$ *as left letter and ending with* $R(v_i)$ *as right letter*; *and each* $u_i\neq 1$ *must satisfy the analogous condition.*

A product $a_1a_2\cdots a_n$ will be called *reduced* (relative to C) if each $a_i\neq 1$

possesses letters $L(a_i)$ and $R(a_i)$ in the manner of Condition N2, and if, for $1 \leqq i \leqq j \leqq n$, $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 1$ implies $R(a_i)L(a_j) \neq 1$; in short, in the product there is no cancellation. We shall often insert a dot, writing $a \cdot b$, to express that the product ab is reduced relative to C. In particular, writing $a \cdot a$ expresses that a is *cyclically reduced*, that is, it does not begin and end in letters that are inverse to each other. With this we can state the condition that excludes the possibility of cancellation between factors in a normal word:

(N3) $u_1 v_1 v_1 u_2 v_2 v_2 \cdots u_m v_m v_m u_{m+1}$ *is reduced relative to C.*

There remains some ambiguity in the normal form as stipulated thus far in that a part of one factor may be transferred to an adjacent factor. The equation $a_1(a_2a_1)^\alpha = (a_1a_2)^\alpha a_1$, which is an instance of one of the axioms, shows that we can not hope to resolve this ambiguity in a symmetric manner. We choose to "shift" the parts $v_i^{\alpha_i}$ as far to the left as possible:

(N4a) *if* $u_i \neq 1$, $1 \leqq i \leqq m$, $R(u_i) \neq R(v_i)$;

(N4b) *if* $u_i = 1$, $1 \leqq i \leqq m$, $R(v_i) \neq R(v_{i+1})$.

Condition N4 is compatible with a weakened form of the symmetric counterpart of N4a, requiring that no u_{i+1} begin with the *whole* of preceding v_i:

(N5) *for* $1 \leqq i \leqq m$, (ξ_{i+1}, C) *is not equivalent to any* $(\eta_i \zeta, C)$ *where* ζ *represents an element* z *of* F_{t-1}, *and* $u_{i+1} = v_i \cdot z$, *reduced relative to C.*

We note that the corresponding weakened counterpart of N4b, requiring that, if $u_i = 1$, not $v_{i+1} = v_i \cdot z$, is incompatible with the remaining conditions; for this would lead to the following vicious circle of "reductions":

$$x^\alpha(xy)^\beta \to x^{\alpha+1}y(xy)^{\beta-1} \to x^{\alpha+1}(yx)^{\beta-1}y \to x^\alpha(xy)^{\beta-1}xy \to x^\alpha(xy)^\beta.$$

A word of height t, for $t>0$, will be called *normal* if it satisfied all the conditions N0, N1, N2, N3, N4, N5 set forth above.

3. Reduction to normal form.

THEOREM I. *There exists an effective procedure for associating with each word an equivalent finite set of normal words.*

The theorem will be proved by induction on the height of the given word. For words of height 0 the assertion is obvious. Let $t>0$ and assume

PROPOSITION 1. *If* (ω, C) *is a word of height* $t-1$, *then* (ω, C) *is equivalent to a set of normal words* $(\omega_1, C_1), \cdots, (\omega_n, C_n)$.

From this we must derive the corresponding assertion, Proposition 1′, for words of height t. In fact, we shall take as our induction hypothesis not only Proposition 1 but also certain auxiliary Propositions 2, 3, $\cdots$, 7, concerning words of height $t-1$, from which we must derive not only Proposition 1′, but also the corresponding Propositions 2′, 3′, $\cdots$, 7′. All of these propositions are obvious for $t=0$.

In this section we derive Proposition 1′ from 1, 2, · · · , 7, and in the next section derive 2′, 3′, · · · , 7′ from the same hypothesis. It will be evident from inspection that the reduction process is effective.

At various stages of the reduction process that will be described below, we shall have occasion to replace a word (ω, C) by three words (ω, C_1), (ω, C_2), (ω, C_3) obtained by adjoining to C each of the conditions $\rho\alpha<0$, $\rho\alpha=0$, $\rho\alpha>0$. The argument from this point on will deal separately with each of these three words. To simplify notation we shall drop subscripts and instead of saying that we are in case C_1, C_2 or C_3, we shall say rather that C contains $\rho\alpha<0$, $\rho\alpha=0$ or $\rho\alpha>0$. Clearly this is permissible provided we can effect the reduction of (ω, C) to some normal (ω', C) by making, at a finite number of junctures well determined by ω, for α determined by ω, the assumption that C contains one or another of $\rho\alpha<0$, $\rho\alpha=0$ and $\rho\alpha>0$.

This convention, whereby we regard C as unchanged throughout the argument, enables us to suppress mention of C. In this spirit we shall speak of the word ω instead of the word (ω, C). We shall write $\alpha<0$, $\alpha=0$, $\alpha>0$ to express that $\rho\alpha<0$, $\rho\alpha=0$, $\rho\alpha>0$ belongs to C. We shall speak of all ρ, rather than of all ρ that satisfy C. It is worth emphasizing that we must not suppose that C contains one out of an infinite set of alternatives. Thus, although all $\rho\alpha=k$ for some k in Z, we can not suppose that C necessarily contains one of the inequalities $\rho\alpha\leqq k$. If C does contain such an inequality, we write $\alpha\in Z$, and we write $\alpha\notin Z$ to express that C contains no such inequality.

We now state those propositions that make up the rest of the induction hypothesis.

PROPOSITION 2. *If (ω, C) is a word of height $t-1$, representing $w\neq 1$ in F, and is not equivalent to* $(1, C)$, *where* 1 *here denotes the "empty" word, then there exist letters $L(\omega, C)$ and $R(\omega, C)$ such that, for all ρ satisfying C, the reduced word representing $\bar{\rho}w$ in F_0 begins with $L(\omega, C)$ as its left letter and ends with $R(\omega, C)$ as its right letter.*

Apart from our convention regarding C, we should have to say rather that arbitrary (ω, C) is effectively equivalent to a finite set of words (ω_i, C_i), each with the property asserted by the proposition. In view of our convention, we can simplify the statement of the proposition by suppressing all reference to C. In view of the fact that we are assuming Proposition 1 we can go further: not only does every ω of height $t-1$ represent an element w of F_{t-1}, but every w in F_{t-1} is represented by normal (ω, C) for some ω. (Literally, C has been chosen so that this is true for every w that comes up for our consideration.) Thus we can as well speak of all w as of all ω, and restate the proposition more perspicuously.

PROPOSITION 2 (*restated*). *If w is in F_{t-1} and $w\neq 1$, then there exist letters $L(w)$ and $R(w)$ such that all $\bar{\rho}w$ begin with $L(w)$ and end with $R(w)$.*

(The fact that $w=1$ implies $\omega=1$, as well as conversely, follows from the original statement of the proposition, since 1 has no first or last letter.)

PROPOSITION 3. *If a is in F_{t-1} there exist b and c in F_{t-1} such that $a=b^{-1}\cdot c\cdot b$ reduced, and c is cyclically reduced (that is, $c^2=c\cdot c$).*

PROPOSITION 4. *If a is in F_{t-1}, then there exist b in F_{t-1} and α in X such that $a=b^\alpha$, and b is primitive (that is, $b=c^\beta$ implies $\beta=\pm 1$).*

PROPOSITION 5. *If a and b are in F_{t-1}, b cyclically reduced, primitive, and not in F_{t-2}, then there exist a_1, b_1, b_2 in F_{t-1} and k in Z such that $b=b_1\cdot b_2$, $a=a_1\cdot b_2b^k$ reduced, $k\geqq 0$, $b_1\neq 1$, and, if $a_1\neq 1$, $R(a_1)\neq R(b_1)$.*

PROPOSITION 6. *If a and b are in F_{t-1}, $a\neq b$, both are cyclically reduced, primitive, and not in F_{t-2}, then there exist a_1, a_2, b_1, b_2 in F_{t-1} and h, k in Z such that*

$$a = a_1\cdot a_2, \qquad b = b_1\cdot b_2, \qquad a_2a^h = b_2b^k, \qquad h \geqq 0,\ k \geqq 0,$$

$$a_1 \neq 1, \qquad b_1 \neq 1, \quad \text{and} \quad R(a_1) \neq R(b_1).$$

PROPOSITION 7. *If a and b are in F_{t-1} and $ab=ba$, then there exist c in F_{t-1} and α, β in X such that $a=c^\alpha$, $b=c^\beta$.*

To begin the proof of Proposition 1′, we suppose given a word (ω, C) of height t, and must replace it by an equivalent normal word (ω', C). That is, we are given a product representation

$$w = w_1^{\beta_1}w_2^{\beta_2}\cdots w_n^{\beta_n}, \qquad n \geqq 0,\ w_i \in F_{t-1},\ \beta_i \in X,$$

for an element w of F_t, and, by means of the axioms, must transform this into a representation satisfying the conditions N0, 1, 2, 3, 4, 5 for normality.

Condition N0 follows immediately from Proposition 1 and our convention on C. For this proposition ensures that each w_i is represented by a finite set of normal (ω_j, C_j) equivalent to (ω_i, C), and under our convention we suppose C is already one of the C_j, so that w_i is represented by a normal word (ω_i, C).

To establish *Condition* N1, we first delete any $w_i^{\beta_i}=1$. By Proposition 3 we may replace each part $w_i^{\beta_i}$ by $z_iv_i^{\beta_i}z_i^{-1}$, where v_i is cyclically reduced, and, by Proposition 4, after changing notation, we may suppose v_i primitive. We now have a product of factors u in F_{t-1} and factors v^α, which we may suppose are not in F_{t-1}, where the v are cyclically reduced and primitive. Replacing any succession of factors $u_1, u_2, \cdots, u_k$ all in F_{t-1} by a single factor $u=u_1u_2\cdots u_k$ in F_{t-1}, and inserting factors 1 if necessary, we have

$$w = u_1v_1^{\alpha_1}u_2v_2^{\alpha_2}\cdots u_mv_m^{\alpha_m}u_{m+1},$$

satisfying all the clauses of Condition N1 except that possibly some $\alpha_i<0$. This is simply remedied by replacing $v_i^{\alpha_i}$ by $v_i'^{\alpha_i'}$ where $v_i'=v^{-1}$ and $\alpha_i'=-\alpha_i$.

Condition N2 follows immediately from N0 together with Proposition 2.

To establish *Condition* N4, we again proceed by induction on i, assuming $R(v_{j-1}u_j) \neq R(v_j)$ for all $j < i$ (and, if $i > 0$, $u_1 \neq 1$, that $R(u_1) \neq R(v_1)$. For $i = 0$ this is trivial, and for $i = m$ we are done. If the hypothesis holds for i and fails for $i+1$ there are two cases.

CASE 1. $u_i \neq 1$. By Proposition 5 we can write $v_i = b_1 \cdot b_2$, $u_i = a_1 b_2 v_i^k$ reduced, $k \in Z$, $k \geqq 0$, $b_1 \neq 1$, and, if $a_1 \neq 1$, $R(a_1) \neq R(b_1)$. Then

$$\begin{aligned} u_i v_i^{\alpha_i} u_{i+1} &= a_1 b_2 (b_1 b_2)^{\alpha_i + k} u_{i+1} \\ &= a_1 (b_2 b_1)^{\alpha_i + k} b_1 u_{i+1} \\ &= u_i' v_i'^{\alpha_i'} u_{i+1}' \end{aligned}$$

where $u_i' = a_1$, $v_i' = b_2 b_1$, $u_{i+1}' = b_1 u_{i+1}$, $\alpha_i' = \alpha_i + k$. Clearly Conditions N0, 1, 2, 3 are preserved, as well as the hypothesis for $j < i$. Moreover, if $u_i' = a_i \neq 1$, the hypothesis holds for $j = i$, since then $R(v_{i-1}u_i') = R(a_1) \neq R(b_1) = R(v_i')$. If $u_i' = 1$, we are reduced to Case 2, below.

CASE 2. $u_i = 1$. By Proposition 6 we can write $v_{i-1} = a_1 \cdot a_2$, $v_i = b_1 \cdot b_2$, $a_2 v_{i-1}^h = b_2 v_i^k$, $h, k \in Z$, $h, k \geqq 0$, $a_1, a_2 \neq 1$, $R(a_1) \neq R(b_1)$. Then

$$\begin{aligned} v_{i-1}^{\alpha_{i-1}} v_i^{\alpha_i} u_{i+1} &= v_{i-1}^{\alpha_{i-1}-h-1} a_1 b_2 (b_1 b_2)^{\alpha_i + k} u_{i+1} \\ &= v_{i-1}^{\alpha_{i-1}-h-1} a_1 (b_2 b_1)^{\alpha_i + k} b_2 u_{i+1} \\ &= v_{i-1}^{\alpha_i' - 1} u_i' v_i'^{\alpha_i'} u_{i+1}' \end{aligned}$$

where $u_i' = a_1$, $v_i' = b_2 b_1$, $u_{i+1}' = b_2 u_{i+1}$, $\alpha_{i-1}' = \alpha_{i-1} - h - 1$, $\alpha_i' = \alpha_i + k$. Clearly N0, 1, 2, 3 are preserved, and the condition on $j < i$. For $j = i$, we verify that $R(v_{i-1}u_i') = R(a_1) \neq R(b_1) = R(v_i')$. This completes the induction on i.

To establish *Condition* N5, by the symmetric counterpart of Proposition 5 we can write $v_i = b_2 \cdot b_1$, $u_{i+1} = v_i^k b_2 a_1$ reduced, $k \in Z$, $k \geqq 0$, $b_1 \neq 1$, and, if $a_1 \neq 1$, $L(a_1) \neq L(b_1)$. Then $v_i^{\alpha_i} u_{i+1} = v_i^{\alpha_i + k} b_1 a_1 = v_i^{\alpha_i'} u_{i+1}'$ where $\alpha_i' = \alpha_i + k$, $u_{i+1}' = b_1 a_1$. Clearly none of Conditions N0, 1, 2, 3, 4 are lost. It remains to show that $u_{i+1}' = v_i \cdot w$, w in F_{t-1}, is impossible. But this gives $b_1 \cdot a_1 = b_2 \cdot b_1 \cdot w$ reduced, whence $a_1 = b_1 \cdot w$, which contradicts the conditions that $b_1 \neq 1$ and that, if $a_1 \neq 1$, $L(a_1) \neq L(b_1)$.

This completes the proof of Proposition 1′ from Propositions 1, 2, · · · , 6 (and 7).

4. **Proof of the auxiliary propositions.** We continue to argue on the basis of Propositions 1 through 7, together with 1′, which was established in the preceding section.

To prove Proposition 2′, let w in F_t, $w \neq 1$. By Proposition 1′ we can suppose $w = u_1 v_1^{\alpha_1} \cdots u_m v_m^{\alpha_m} u_{m+1}$, normal. Then, for any ρ, $\bar{p}w = \bar{p}u_1(\bar{p}v_1)^{\rho\alpha_1} \cdots (\bar{p}v_m)^{\rho\alpha_m} \bar{p}u_{m+1}$, where, by N2, 3, this product is reduced. It follows that if

$u_1 \neq 1$, $\bar{\rho}w$ begins with $L(u_1)$, and with $L(v_1)$ otherwise; similarly $\bar{\rho}w$ ends with $R(u_{m+1})$ if $u_{m+1} \neq 1$, and with $R(v_m)$ otherwise.

COROLLARY. *If $a = a_1 \cdot a_2 \cdot \cdot \cdot a_n$ reduced, all a_i are in F_t, and not all $a_i = 1$, then $a \neq 1$.*

To prove this, after deleting any $a_i = 1$, we reason as before that for all ρ, $\bar{\rho}a = \bar{\rho}a_i \cdot \cdot \cdot \bar{\rho}a_n$, whence $L(a) = L(a_1)$, $R(a) = R(a_1)$, and consequently $a \neq 1$.

We state now a lemma whose proof is contained in the algorithm given above for establishing Condition N3.

LEMMA A. *If a and b are in F_t, then there exist a_1, c, and b_2 in F_t such that $a = a_1 \cdot c$, $b = c^{-1} \cdot b_2$, and $ab = a_1 \cdot b_2$.*

An important consequence follows.

LEMMA B. *If a, b, c and d are in F_t and $a \cdot b = c \cdot d$, then there exists e in F_t such that either*

$$a = c \cdot e \quad \text{and} \quad d = e \cdot b,$$

or

$$c = a \cdot e \quad \text{and} \quad b = e \cdot d.$$

Proof of Lemma B. By Lemma A, we can write $a = e \cdot a_2$, $c = e \cdot c_2$, $a^{-1}c = a_2^{-1} \cdot c_2$. If $a_2 \neq 1$, $c_2 \neq 1$, then $b^{-1}a^{-1}cd = b^{-1}a_2^{-1}c_2d$ is reduced and, by Proposition 2′ (since it clearly has left and right letters), $b^{-1}a^{-1}cd \neq 1$, contrary to hypothesis. By symmetry, we may suppose that $a_2 = 1$. Then $a = e$, $c = a \cdot c_2$, and $ab = cd = ac_2d$ implies that $b = c_2 \cdot e$.

To prove Proposition 3′, let $a \in F_t$. By Lemma A, we may write $a = a_1 \cdot c$, $a = c^{-1} \cdot a_2$, $a^2 = a_1 \cdot a_2$. Lemma B applied to the relation $a_1 \cdot c = c^{-1} \cdot a_2$ presents two alternatives. First, suppose $a_1 = c^{-1} \cdot e$, $a_2 = e \cdot c$ then $a = c^{-1} \cdot e \cdot c$ where $a^2 = a^{-1} \cdot e \cdot e \cdot c$, whence $e \cdot e$ and e is cyclically reduced, as required. Otherwise, $c^{-1} = a_1 \cdot e$, $c = e \cdot a_2$. Then Lemma B applied to $c = e^{-1} \cdot a_1^{-1}$, $c = e \cdot a_2$ gives, after replacing e by e^{-1} if necessary, $e = e^{-1} \cdot f$, and substituting, $e = f^{-1} \cdot e \cdot f$. Applying ρ to this equation gives $\bar{\rho}f = 1$, and then to the equation $e = e^{-1} \cdot f$ gives $\bar{\rho}e = \bar{\rho}e^{-1}$ whence $\bar{\rho}e = 1$, $e = 1$ and $a = 1$, cyclically reduced.

LEMMA C. *If $a = uv^\alpha$, $b = wz^\beta$, normal, in F_t, then either $u = w$ and $v = z$ or else there exist c in F_{t-1} and h, $k \in Z$, h, $k \geqq 0$, such that $a^{-1}b = v^{-(\alpha-h)} \cdot c \cdot z^{\beta-k}$ reduced.*

Proof. By Lemma A we can write $u = e \cdot u_2$, $w = e \cdot w_2$, $u^{-1}w = u_2^{-1} \cdot w_2$. If $u_2 \neq 1$, $w_2 \neq 1$, then $a^{-1}b = v^{-\alpha} \cdot u^{-1} \cdot w_2 \cdot z^\beta$ reduced and we are done. By symmetry, it suffices to consider the case that $u_2 = 1$, where $a^{-1}b = v^{-\alpha} \cdot u_2^{-1} \cdot w_2 \cdot z^\beta$. By Proposition 5, we can write $v = v_1 \cdot v_2$, $w_2 = v^k v_1 w_3$ reduced, where $v_2 \neq 1$ and, if $w_3 \neq 1$, $L(w_3) \neq L(v_2)$. Then $a^{-1}b = v^{-\alpha}w_2z^\beta = v^{-(\alpha-k-1)}v_2^{-1}w_3z^\beta$, reduced provided $w_3 \neq 1$. Henceforth suppose $w_3 = 1$, whence $a^{-1}b = v_2^{-1}(v_2v_1)^{\alpha-k-1}z^\beta$. If $z \neq v_2v_1$, Proposition 6 yields $a^{-1}b = v_2^{-1}(v_2v_1)^{-\alpha}c = z^\beta = v^{-\alpha'}v_2^{-1}cz^{\beta'}$ reduced, for some c in F_{t-1} and $\alpha' = \alpha - h'$, $\beta' = \beta - k'$. Finally, suppose also that $z = v_2v_1$. Since $w_3 = 1$, $w = uv^kv_1$, while by Condition N4 for the normality of wz^β,

$R(w) \neq R(v_2v_1)$. We conclude first that $v_1=1$, whence $z=v_1=v$, and second that $k=0$, whence $w=u$.

LEMMA D (*Uniqueness of normal form*). *If* $a = u_1v_1^{\alpha_1} \cdots v_m^{\alpha_m}u_{m+1}$, $b=w_1z_1^{\beta_1} \cdots z_n^{\beta_n}w_{n+1}$, *both normal and in* F_t, *then* $a=b$ *implies that* $m=n$ *and that* $u_i=w_i$, $v_i=z_i$, *and* $\alpha_i=\beta_i$ *for all* $i=1, 2, \cdots, m$.

Proof. Induction on $m+n$. If $m=n=0$, then $a=u_1$, $b=w_1$, and the conclusion is trivial. We also consider separately the case that only one of m, n is 0, say $m=0$ and $n>0$. In this case $u_1^{-1}w_1z_1^{\beta_1}$ has, using Proposition 5, a reduced form $u'z_1^{\beta'}$ for some u' and β', hence $a^{-1}b$ has a reduced form different from 1, and, by the corollary, $a^{-1}b\neq 1$. For the induction argument, suppose m, $n>0$. Unless $u_1=w_1$ and $v_1=z_1$, Lemma C provides for $a^{-1}b$ a nontrivial reduced form, whence $a^{-1}b\neq 1$. Suppose then that $u_1=w_1$, $v_1=z_1$. If also $\alpha_1=\beta_1$ the conclusion follows by the induction hypothesis applied to $a'=u_2v_2^{\alpha_2} \cdots u_{m+1}$ and $b'=w_2z_2^{\beta_2} \cdots w_{n+1}$. In the remaining case we may suppose by symmetry that $\gamma=\alpha_1-\beta_1>0$, whence, after cancelling, we have $a'=v_1^{\gamma}u_2v_2^{\alpha_2} \cdots u_{m+1}$ equal to b'. If $\gamma\notin Z$, the product for a' is normal, and the induction hypothesis gives $1=w_2$, and $v_1=z_2$, which together with $v_1=z_1$, contradicts N4 for b. If $\gamma\in Z$, the product for a' becomes reduced when we replace $v_1^{\gamma}u_2z_1^{\beta_2}$ by a reduced product $u_2'z_1^{\beta'}$, and again the induction hypothesis gives $w_2=u_2'=v_1^{\gamma}u_2$, $\gamma>0$, which, together with $z_1=v_1$, contradicts N5 for b.

Define a' to be a *cyclic conjugate* of a if, for some b and c, $a=b\cdot c$ and $a'=cb$. This relation is transitive, although not symmetric. For suppose $a=b\cdot c$, $a'=cb=d\cdot e$, and $a''=ed$. By Lemma A, $c=c'\cdot f$, $b=f^{-1}\cdot b'$, and $a'=c'\cdot b'=d\cdot e$. By Lemma B, and symmetry, we may suppose that $c'=d\cdot g$, $e=g\cdot b'$. Then $a=f^{-1}\cdot b'\cdot d\cdot g\cdot f$ reduced, while $a''=gb'd=(gf)(f^{-1}b'd)$.

LEMMA E. *Let* a, *in* F_t, *not have any cyclic conjugate a power of an element of* F_{t-1}. *Then there exist integers* $t\geqq 0$ *and* $m>0$, *and, for each positive integer* i, *elements* u_i, v_i *in* F_{t-1} *and an element* α_i *in* X *such that, writing* $a_i=u_iv_i^{\alpha_i}$,

(1) *for all* $h\geqq k+2$, *the normal form of* a^h *begins*

$$a^h = a_1a_2 \cdots a_k \cdots ;$$

(2) *for all* $i>t$, $a_{i+m}=a_i$;

(3) *for* $s\geqq t$,

$$(a_1a_2 \cdots a_s)^{-1}a(a_1a_2 \cdots a_s) = a_{s+1}a_{s+2} \cdots a_{s+m}, \text{ normal.}$$

Proof. For a in F_t, define $l(a)=n$, the "length" of its normal form $a=w_1z_1^{\beta_1} \cdots w_{n+1}$. For $l(a)\geqq 1$, define $a'=(w_1z_1^{\beta_1})^{-1}a(w_1z_1^{\beta_1})$. If no cyclic conjugate of a is in F_{t-1}, evidently $l(a)\geqq l(a')\geqq 1$.

CASE 1. $l(a)=1$. It will suffice to prove the assertion of the lemma for any cyclic conjugate $u^{-1}au$ of a, where u is in F_{t-1}. Replacing $a=w_1z_1^{\beta_1}w_2$, normal, by the normal form of $z_1^{\beta_1}w_2w_1$ we can suppose $w_1=1$. If $w_2=wb_1^{-1}z_1^{-k}$

where $z_1 = b_1b_2$, in accordance with Proposition 5, conjugating by $b_1^{-1}z_1^{-k}$ we can suppose that a is cyclically reduced. If $w_2 = wb_2z_1^k$ in accordance with Proposition 5, conjugating by z_1^k we can suppose that $k=0$. Thus we may assume $a=(b_1b_2)^\beta wb_2$, normal and cyclically reduced, where $b_1 \neq 1$ and, if $w \neq 1$, $R(w) \neq R(b_1)$.

If $w \neq 1$, conditions (1) and (2) are realized with

$$a^h = (b_1b_2)^\beta w(b_2b_1)^\beta b_2 w(b_2b_1)^\beta b_2 w(b_2b_1)^\beta \cdots .$$

Suppose $w=1$. Since $a=(b_1b_2)^\beta b_2$ is not a power of the element b_1b_2 in F_{t-1}, we conclude that $b_2 \neq 1$ as well as $b_1 \neq 1$. We show next that $b_1b_2 \neq b_2b_1$. For $b_1b_2 = b_2b_1$ would imply that both b_1 and b_2 commuted with the primitive element $z=b_1b_2$, hence $b_1 = z_1^\gamma$, $b_2 = z_1^\delta$, where γ, $\delta \neq 0$ and both have the same sign and sum 1; but γ, $\delta > 0$ and $\gamma+\delta=1$ is impossible. It follows therefore by Proposition 6 that we can write

$$b_1b_2 = c_1 \cdot c_2, \qquad b_2b_1 = d_1 \cdot d_2, \qquad c_2(b_1b_2)^h = d_2(b_2b_1)^k,$$

with $h, k \in Z$, $h, k \geqq 0$, $c_1, d_1 \neq 1$, $R(c_1) \neq R(d_1)$.

Comparison of lengths in F_0 of images of $c_2(b_1b_2)^h$ and $c_2(b_2b_1)^k$ shows that $h=k$; moreover, if $h=k>0$, the terminal segments $\bar{\rho}(b_1b_2)$ and $\bar{\rho}(b_2b_1)$ of equal length must coincide for all ρ, giving $\bar{\rho}(b_1^{-1}b_2^{-1}b_1b_2)=1$, whence, by Proposition 2′, $b_1^{-1}b_2^{-1}b_1b_2=1$, which has been shown impossible. We conclude that $h=k=0$, whence $c_2=d_2$. If $c_2=d_2=1$, then $R(b_1b_2) \neq R(b_2b_1)$ and

$$a^h = (b_1b_2)^\beta(b_2b_1)^\beta b_2(b_2b_1)^\beta b_2(b_2b_1)^\beta \cdots, \text{ normal.}$$

Otherwise, the normalizing process gives

$$\begin{aligned} a^2 &= (b_1b_2)^\beta b_2(b_1b_2)^\beta b_2 = (b_1b_2)^\beta(b_2b_1)^\beta b_2^2 \\ &= (b_1b_2)^{\beta-1}c_1(d_2d_1)^\beta b_2^2 \\ &= (b_1b_2)^{\beta-1}c_1(d_2d_1)^{\beta'} b', \text{ normal,} \end{aligned}$$

for some β' and b'. Similarly,

$$a(b_1b_2)^{\beta-1}c_1 = (b_1b_2)^{\beta-1}c_1(d_2d_1)^{\beta''}b'', \text{ normal,}$$

for some β'' and b''. It follows that

$$a^h = (b_1b_2)^{\beta-1}c_1(d_2d_1)^{\beta''}[b''(d_2d_1)^{\beta''}]^{h-3}b''(d_2d_1)^{\beta'}b',$$

normal, realizing (1) and (2).

For $l(a)=1$ this establishes that, for some a_1, a_2, a_3 and some d in F_t, for all $h \geqq 4$,

$$a^h = a_1a_2a_3^{h-4}d \text{ normal.}$$

Hence

$$aa_1a_2a_3^{h-4}d = aa^h = a^{h+1} = a_1a_2a_3^{h-3}d,$$

whence $aa_1a_2 = a_1a_2a_3$, that is $(a_1a_2)^{-1}a(a_1a_2) = a_3$, which establishes (3).

CASE 2. $l(a) = l(a') > 1$. Write $a = b_1c_1^{\gamma_1} \cdots b_mc_m^{\gamma_m}b_{m+1}$ normal, $m > 1$. Now

$$a' = b_2c_2^{\gamma_2} \cdots b_mc_m^{\gamma_m}b_{m+1}b_1c_1^{\gamma_1}$$

is reduced to its normal form

$$a' = d_2e_2^{\epsilon_2} \cdots d_me_m^{\epsilon_m}d_{m+1}e_{m+1}^{\epsilon_{m+1}}d_{m+2}$$

by normalizing the part $c_m^{\gamma_m}b_{m+1}b_1c_1^{\gamma_1}$. The hypothesis that $l(a) = l(a')$ implies that this part has length 2, whence from the nature of the normalizing process it follows that $e_m = c_m$ and that $R(e_{m+1}d_{m+2}) = R(c_1)$.

We iterate this to find the normal form

$$a'' = u_3v_3^{\alpha_3} \cdots u_{m+1}v_{m+1}^{\alpha_{m+1}}u_{m+2}v_{m+2}^{\alpha_{m+2}}w$$

of $a'' = d_3e_3^{\epsilon_3} \cdots e_{m+1}^{\epsilon_{m+1}}d_{m+1}d_2e_2^{\epsilon_2}$ by replacing $e_{m+1}^{\epsilon_{m+1}}d_{m+1}d_2e_2^{\epsilon_2}$ by its normal form $v_{m+1}^{\alpha_{m+1}}u_{m+1}v_{m+2}^{\alpha_{m+2}}w$. From $e_m = c_m$ we infer that, even if $m = 2$, $d_2 = b_2$ and $e_2 = c_2$. From this and the normality of $c_1^{\gamma_1}b_2c_2^{\gamma_2}$, together with $R(e_{m+1}d_{m+2}) = R(c_1)$ we conclude that the part in question can fail of normality only by N5, and the process for establishing N5 shows that $v_{m+1} = e_{m+1}$, $v_{m+2} = e_{m+2}$, and $w = 1$.

If we define $a_1 = b_1c_1^{\gamma_1}$, $a_2 = d_2e_2^{\epsilon_2}$, and $a_{i+m} = a_i = u_iv_i^{\alpha_i}$ for $i \geqq 3$, the normality of the products $a_1a_2a_3 \cdots a_k$ requires only the observation that, since $e_2^{\epsilon_2}d_3e_3$ was normal, and $v_{m+2} = e_2$, $u_3 = d_3$, $v_{m+3} = e_3$, then $v_{m+2}^{\alpha_{m+2}}u_3v_{m+3}^{\alpha_{m+3}}$ is normal as it stands. It is now clear that (1) and (2) hold, and that $a'' = (a_1a_2)^{-1}a(a_1a_2) = a_3a_4 \cdots a_{m+2}$, reduced.

CASE 3. $l(a) > l(a')$. Since $l(a) \geqq l(a') \geqq l(a'') \geqq \cdots \geqq 1$, clearly for some $k < l(a)$ and $b = a^{(k)}$, $l(b) = l(b')$, whence b satisfies (1), (2), (3). Thus $b^h = b_1b_2 \cdots$ normal, $b_i = w_iz_i^{\beta_i}$, with $b_{i+m} = b_i$ for $i > t$ and

$$c = (b_1b_2 \cdots b_t)^{-1}b(b_1b_2 \cdots b_t) = b_{t+1}b_{t+2} \cdots b_{t+m}.$$

Replacing b by its cyclic conjugate c, we have the same conditions with $t = 0$, and b cyclically reduced. Thus, for some p and q, $a = p \cdot q$ and $c = qp$. By Lemma A, $q = q' \cdot e$, $p = e^{-1} \cdot p'$, $c = q' \cdot p'$. Further, $a = e^{-1} \cdot p' \cdot q' \cdot e$, reduced, whence $a^h = e^{-1} \cdot p' \cdot (q' \cdot p')^{h-1} \cdot q' \cdot e = e^{-1} \cdot p' \cdot b^{h-1} \cdot q' \cdot e = e^{-1} \cdot p' \cdot b_1b_2 \cdots$, reduced. Since $e^{-1} \cdot p' \cdot b_1 \cdot b_2$ is reduced, the normal form of $e^{-1} \cdot p' \cdot b_1 \cdot w_2$ ends with the same letters as b_1w_2, whence the normal form $a_1a_2 \cdots a_n$ of $e^{-1} \cdot p' \cdot b_1 \cdot b_2$ ends in $z_2^{\beta_2}$, and $a^h = a_1a_2 \cdots a_kb_3b_4 \cdots$, normal. Thus a satisfies (1) and (2), while for (3) it suffices to note that $d = a_1a_2 \cdots a_kb_3b_4 \cdots b_{3m} = e^{-1}p'b^3$, whence $dbd^{-1} = e^{-1}p'bp'^{-1}e = e^{-1}p'qpp'^{-1}e = e^{-1}p'q'e = a$, and $b = d^{-1}ad$.

Proof of Proposition 4′. Let a be in F_t. If some conjugate $p^{-1}ap = b^{\alpha}$, b in

F_{t-1}, then by Proposition 4, $b=c^\beta$ for some c primitive, whence $c'=pcp^{-1}$ also is primitive and $a=c'^{\beta\alpha}$. Otherwise we suppose that a has no cyclic conjugate a power of an element in F_{t-1}, whence, by Lemma E, $l(a)\geqq 1$. If $a=b^\alpha$ for some b in F_t, then b satisfies the same condition, and hence (1), (2), and (3) of Lemma E. In view of (3), we can replace b by a cyclically reduced conjugate whence, if $b=b_1b_2\cdots b_m$ normal, then $b^2=b_1b_2\cdots b_mb_1\cdots b_m$ normal, and $l(b^2)=2l(b)$. Now, if $\alpha>1$, $a=b^2\cdot b^{\alpha-2}$. In general, $l(u\cdot v)\geqq l(u)+l(v)-1$, $l(u\cdot v)\geqq l(u)$. Thus we have $l(a)\geqq l(b^2)=2l(b)>l(b)$. We conclude first that if $l(a)=1$, then $a=b^\alpha$ for $\alpha>1$ is impossible, and a is primitive. Second, if a is not primitive, then $a=b^\alpha$ for some b and some $\alpha>1$, and we argue by induction on length that $b=c^\beta$ for some primitive c, whence $a=c^{\alpha\beta}$, as required.

Proof of Proposition 5′. We show first that if b in F_t is cyclically reduced and primitive, and b is not in F_{t-1}, then no cyclic conjugate b' of b is a power of an element in F_{t-1}. For, if b' were such a power, since b' is primitive, b' itself would be in F_{t-1}. Then, for some p and q, $b=p\cdot q$, $b'=qp$, and, since b is cyclically reduced, $b'=q\cdot p$. From the fact that b' is in F_{t-1}, that is, $l(b')=0$, we conclude that p and q are in F_{t-1}, whence b is in F_{t-1}, contrary to hypothesis.

To prove Proposition 5′, we suppose a in F_t, and b in F_t, cyclically reduced, primitive, and not in F_{t-1}. It follows from the above and Lemma E that $l(b^h)\geqq h$ for all h in Z. Choose $h>l(a)$, whence $l(b^h)>l(a)$. By Lemma A we may write $a=c\cdot d$, $b^{-h}=d^{-1}\cdot e$, $ab^{-h}=c\cdot e$. Now $e=1$ would imply $d^{-1}=b^{-h}$, hence $a=c\cdot d=c\cdot b^h$ and $l(a)\geqq l(b^h)$, a contradiction. Hence $e\neq 1$. Now choose a new $h\geqq 0$, minimal such that, for some c, d, e, $a=c\cdot d$, $b^{-h}=d^{-1}\cdot e$, $ab^{-h}=c\cdot e$, and $e\neq 1$. Since $b^{-h}=d^{-1}\cdot e$ and $e\neq 1$, we conclude that $h>0$. Now $b\cdot b^{h-1}=e^{-1}\cdot d$, and one alternative under Lemma B is that $e^{-1}=b\cdot f$, $b^{h-1}=f\cdot d$. Then $b^{-(h-1)}=d^{-1}\cdot f^{-1}$ and, since $ab^{-h}=c\cdot e=c\cdot f^{-1}\cdot b^{-1}$, $ab^{-(h-1)}=c\cdot f^{-1}$, whence the minimality of h implies that $f=1$. This gives $a=c\cdot d=c\cdot b^h$, and, since $c\cdot e=c\cdot b^{-1}$, $R(c)\neq R(b)$, as required for Proposition 5′. The other alternative is that $b=e^{-1}\cdot f$ and $d=f\cdot b^{h-1}$. In this case we have $a=c\cdot d=c\cdot f\cdot b^{h-1}$, $b=e^{-1}\cdot f$, $e^{-1}\neq 1$, and, since $c\cdot e$, $R(c)\neq R(e^{-1})$, again satisfying Proposition 5′.

LEMMA F. *Let a and b be in F_t, and neither have as cyclic conjugate a power of an element of F_{t-1}. Then either $a=c^h$, $b=c^k$ for some $c\in F_t$, $h, k\in Z$, $h, k\geqq 0$, or $a^{-h}b^h=a^{-1}\cdot c\cdot b$ for some $c\in F_t$, $h\in Z$, $h\geqq 0$.*

Proof. By Lemma E, paralleling the notation of the lemma, $a^h=a_1a_2\cdots$, $b^h=b_1b_2\cdots$, normal, where, for some integers $t\geqq 0$ and $m, m'>0$ we have that $a_{i+m}=a_i$ and $b_{i+m'}=b_i$ for all $i>t$, and also that, writing $p=a_1a_2\cdots a_t$, $q=b_1b_2\cdots b_t$, $a'=p^{-1}ap=a_{t+1}a_{t+2}\cdots a_{t+m}$ and $b'=q^{-1}bq=b_{t+1}b_{t+2}\cdots b_{t+m'}$. If any first $a_i\neq b_i$, then Lemma C yields $a_{i+2}^{-1}a_{i+1}^{-1}a_i^{-1}b_ib_{i+1}b_{i+2}=a_{i+2}^{-1}\cdot c\cdot b_{i+2}$ reduced, hence, for large h, $a^{-h}b^h=a^{-1}\cdot c'\cdot b$ reduced, as required for Lemma F. If, on the other hand, $a_i=b_i$ for all i, we have $a_{i+n}=a_i$ where $n=(m, m')$, and also $p=q$. Writing $d=a_{t+1}a_{t+2}\cdots a_{t+n}$ it follows that $a'=d^{m/n}$ and $b'=d^{m'/n}$, whence $a=c^h$, $b=c^k$ where $c=pdp^{-1}$, $h=m/n$, and $k=m'/n$.

Proof of Proposition 6′. We are given a and b in F_t, cyclically reduced, primitive, and neither in F_{t-1}. The reasoning that began the proof of Proposition 5′ shows that both a and b satisfy the hypotheses of Lemma F. If, in accordance with that lemma, $a=c^h$, $b=c^k$, then, since a and b are primitive, $a=b^{\pm 1}$. Either $a=b$, or $a=b^{-1}$ and $a^{-1}b=a^{-1}\cdot 1\cdot b$ reduced, whence the conclusion of Proposition 6′ holds. Otherwise, by Lemma F, some $a^{-h}b^k=a^{-1}\cdot c\cdot b$ and again Proposition 6′ holds.

Proof of Proposition 7′. We are given a and b in F_t such that $ab=ba$.

Case 1. Suppose neither a nor b has as cyclic conjugate a power of an element in F_{t-1}. Then the same is true of a^{-1} and b^{-1}. Unless $a=c^h$ and $b=c^k$ for some c in F_t and h, k in Z, three applications of Lemma G yield an integer $h>0$ such that

$$a^hb^h = a\cdot c_1\cdot b, \qquad b^ha^{-h} = b\cdot c_2\cdot a^{-1}, \qquad a^{-h}b^{-h} = a^{-1}\cdot c_3\cdot b^{-1},$$

all reduced. We note that it follows from Proposition 3′ that, if $a\neq 1$, $h>0$, then a^h begins and ends with $L(a)$ and $R(a)$. It follows that

$$a^{2h}b^{2h}a^{-2h}b^{-2h} = a^{h+1}c_1b^2c_2a^{-2}c_3b^{-(h+1)}$$

without cancellation between the seven factors displayed, whence, by the Corollary, $a^{2h}b^{2h}a^{-2h}b^{-2h}\neq 1$, a^{2h} and b^{2h} do not commute, which contradicts $ab=ba$. This establishes the conclusion in Case 1.

Case 2. One of a and b, say a, is conjugate to a power of an element in F_{t-1}. Conjugating a and b simultaneously, we can suppose a itself if a power of an element in F_{t-1}, $a=v^\alpha$, v in F_{t-1}, and we can suppose further, apart from the trivial case $a=1$, that $\alpha>0$ and v is cyclically reduced and primitive.

Case 2A. *α in Z.* Then a is in F_{t-1}. If $l(b)=0$, then b is also in F_{t-1} and the conclusion follows by Proposition 7; in view of Proposition 4, in fact, $b=v^\beta$ for some β. Suppose $l(b)=n>0$, and $b=w_1z_1^{\beta_1}\cdots w_{n+1}$ normal. From the normalizing process we see that ba will have a normal form beginning with $w_1z_1^\beta$ for some β. The normal form of ab will begin with some $r(qp)^{\beta'}$ where $z_1=p\cdot q$, $p\neq 1$, and $aw_1=r\cdot q\cdot (p\cdot q)^k$ reduced, for some k in Z. By Lemma D, from $ab=ba$ we conclude that $w_1=r$ and $z_1=qp$. An argument in the proof of Lemma E shows that $z_1=p\cdot q=q\cdot p$ primitive, $p\neq 1$, implies $q=1$. Thus $aw_1=r(pq)^k=w_1z_1^k$, and $a=w_1z_1^kw_1^{-1}$. Since w_1z_1 is normal, if $w_1\neq 1$ then $R(w_1)\neq L(z_1)^{-1}$ and $L(w_1^{-1})=R(w_1)^{-1}\neq R(z_1)^{-1}$, and $a=w_1z^kw_1^{-1}$ is reduced, but not cyclically reduced, contrary to the hypothesis on a. Thus $w_1=1$, $a=z_1^k$, whence a commutes with $b'=w_2z_2^{\beta_2}\cdots w_{n+1}$. By induction on the length of b we conclude that $b'=v^\beta$ for some β, whence $b=v^{\beta+\beta_1}$, as required.

Case 2B. *α is not in Z.* Then $a=v^\alpha$ is normal. Again we reason by induction on $l(b)$. The case $l(b)=0$ falls under Case 2A with a and b interchanged. Let b have normal form as before, with $l(b)=n>0$. If $w_1=1$ and $z_1=v$ the conclusion follows by induction on $l(b)$ as before. Otherwise, by Lemma C, $v^\alpha w_1 z_1^{\beta_1} = v^{\alpha'} \cdot c \cdot z_1^\beta$ reduced, and the normal form of ab begins with $v^{\alpha''}$ for some α''. To examine the normal form of ba, we observe that the normal form of $w_n z_n^{\beta_n} w_{n+1} v^\alpha$ will begin with $w_n z_n^\beta$ for some β', unless $z_n^{\beta_n} w_{n+1} v^\alpha = z_n^{\beta_n} q(pq)^\gamma = z_n^{\beta_n}(qp)^\gamma q = (qp)^\delta q$ where $v = p \cdot q$, and $z^{\pm 1} = q \cdot p$. In this case, $z_n^{\beta_n} w_{n+1} = (qp)^\delta q(pq)^{-\alpha} = q(pq)^\epsilon = qv^\epsilon$, and $b = w_1 q v^\epsilon$, whence a commutes with $w_1 q$ in F_{t-1}, and the initial case of the induction gives $w_1 q = v^\beta$, whence $b = v^{\beta+\epsilon}$. Thus, even if $n=1$, the normal form of ba will begin with $w_1 z_1^{\beta''}$ for some β''. Since $ab = ab$, this contradicts Lemma D unless $w_1 = 1$ and $z_1 = v_1$.

This completes the proof of Proposition 7′, and with it the proof of Theorem I.

5. **Conclusions.** Our main result is the following.

THEOREM II. *There exists an effective procedure that associates with each expression* ϵ, *made up out of the symbols* $a_1, a_2, \cdots, a_r$ *by means of the* X-*group operations, where* $X = Z[\nu_1, \nu_2, \cdots, \nu_d]$, *an element* α *in* X, *in such a way that, if* w *is the element of the free* X-*group* F *represented by* ϵ, *and* ρ *is any retraction of* X *onto* Z, *then* $\bar{\rho}w = 1$ *if and only if* $\rho\alpha = 0$.

Taking into account the X-group axioms, we can replace ϵ by an expression ϵ' representing the same group element w, where $\epsilon' = \epsilon_1^{\alpha_1}\epsilon_2^{\alpha_2} \cdots \epsilon_n^{\alpha_n}$, $n \geqq 0$, all α_i in X, and where each ϵ_i represents an element w_i of F_t, for some fixed $t \geqq 0$. Let $\omega = w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_n^{\alpha_n}$, and let C be the empty set of conditions. Then (ω, C) is a word of height $t+1$, and, by Theorem I, (ω, C) is equivalent to a finite set of normal words $(\omega_1, C_1), (\omega_2, C_2), \cdots, (\omega_k, C_k)$. We may suppose these words indexed in such a way that, for some h, $0 \leqq h \leqq k$, $\omega_1 = 1$, $\omega_2 = 1, \cdots, \omega_h = 1$, while none of $\omega_{h+1}, \omega_{h+2}, \cdots, \omega_k$ is the empty word. Since every retraction ρ satisfies the vacuous set of conditions C, it follows from the definition of equivalence that every retraction satisfies one of $C_1, C_2, \cdots, C_k$. If ρ satisfies some C_i for $1 \leqq i \leqq h$, then $\bar{\rho}w = \bar{\rho}1 = 1$; while if ρ satisfies some C_i for $i > h$, then $\bar{\rho}w = \bar{\rho}w_i$, where w_i is the element represented by ω_i, and from conditions N2 and N3 for the normality of ω_i, it follows that $\bar{\rho}w_i$ has a left letter and a right letter, and hence $\bar{\rho}w_i \neq 1$.

For each i, after transposing, we can write the finite set of equations in C_i in the form $\rho\alpha_{i1} = 0, \rho\alpha_{i2} = 0, \cdots, \rho\alpha_{im_i} = 0$, $m_i \geqq 0$. Define $\alpha_i = \alpha_{i1}^2 + \alpha_{i2}^2 + \cdots + \alpha_{im_i}^2$, and set $\alpha = \alpha_1\alpha_2 \cdots \alpha_h$. If $\bar{\rho}w = 1$, then ρ satisfies some C_i, for $1 \leqq i \leqq h$, hence $\rho\alpha_{ij} = 0$ for $j = 1, 2, \cdots, m_i$, $\rho\alpha_i = 0$, and thus $\rho\alpha = 0$. Suppose conversely that $\rho\alpha = 0$. Then $h \geqq 1$, and $\rho\alpha_i = 0$ for some i, $1 \leqq i \leqq h$, whence $\rho\alpha_{ij} = 0$ for $j = 1, 2, \cdots, m_i$. Now (ω, C_i) is equivalent to $(1, C_i)$, by virtue of (E2) and (E3) alone, which make no reference to any inequalities contained in C_i; that is, ω can be transformed into 1 by means of the X-group axioms together with substitutions implied by the equations $\alpha_{i1} = 0$, $\alpha_{i2} = 0, \cdots, \alpha_{im_i} = 0$. Since $\rho\alpha_{ij} = 0$ for $j = 1, 2, \cdots, m_i$, it follows that $\bar{\rho}w$

$=\bar{\rho}1=1$. Thus we have shown that, for every retraction ρ, $\bar{\rho}w=1$ if and only if $\rho\alpha=0$, which completes the proof to the theorem.

We conclude by justifying an assertion made in the introduction that, for $X=Z[\nu_1, \nu_2, \cdots, \nu_d]$, $d\geqq 0$, the axioms given for X-groups are neither too strong nor too weak in the sense that an expression ϵ representing an element of the free X-group F reduces to 1 by virtue of these axioms if and only if, under every substitution of integers for $\nu_1, \nu_2, \cdots, \nu_d$, ϵ is transformed into an ordinary word representing the element 1 of the ordinary free group F_0.

It is clear that the axioms are not too strong: if they permit us to transform one expression ϵ into a second ϵ', surely ϵ and ϵ' represent the same element of F_0 under each substitution. To show that the axioms are strong enough we must show that if ϵ represents an element w of F such that $\bar{\rho}w=1$ under all retractions ρ, then in fact $w=1$, the trivial element of F. Returning to the notation used in the proof of Theorem II, we have that $\rho\alpha=0$ for all ρ. This implies that $\alpha=0$, identically in X, whence $h\geqq 1$ and $\alpha_i=0$ for some i, $1\leqq i\leqq h$. But $\alpha_i=\alpha_{i1}^2+\alpha_{i2}^2+\cdots+\alpha_{im_i}^2=0$ implies that $\alpha_{ij}=0$ for all $j=1, 2, \cdots, m_i$, so that C_i in fact contains no nontrivial conditions. But now, as before, ω reduces to 1 by means of the axioms together with the trivial condition $0=0$, that is, by means of the axioms alone.

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