GROVES' SCHEME ON RESTRICTED DOMAINS

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Abstract

It is proved that Groves' scheme is unique on restricted domains which are smoothly connected, in particular convex domains. This generalizes earlier uniqueness results by Green and Laffont and Walker. An example shows that uniqueness may be lost if the domain is not smoothly connected.

1 Introduction

In a classical paper Groves [7] introduced a scheme which induces individuals to reveal true information about preferences for public decisions under the assumption that preferences are additively separable and linear in money. This was done in the context of a game of incomplete information with partial communication. When full communication is feasible, Groves and Loeb [8] showed that the same scheme makes true revelation of preferences a dominant strategy.[1] Subsequently, Green and Laffont [5] proved that Groves' scheme is unique in this respect when a universal domain of preference profiles (or a domain containing all continuous profiles) is allowed.

Since Green and Laffont's proof rests crucially on the assumption of such a large domain of profiles (cf. Walker [12]), it is conceivable that uniqueness would be lost when the domain would be further restricted.[2] The purpose of the present paper is to investigate this issue. The main result (Theorem 1) states that Groves' scheme will be unique on any domain which is smoothly connected.

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[1] Clarke [4] had earlier proved this result for profile domains which are one-dimensional.

[2] Compare with Arrow's impossibility result in social choice theory, which is no longer valid on restricted domains (Arrow [1]).

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A domain is smoothly connected if for any two preference profiles in the domain there exists a differentiable deformation of one profile into the other within the domain (*i.e.,* the domain is a differentiably homotopic class). An example shows that, in general, smooth connectedness of the domain is indispensable for the uniqueness result to hold.

An important class of domains which are smoothly connected is the class of convex domains (Theorem 2). Thus, one may a priori take preferences to be convex, differentiable, polynomial, etc., without altering the uniqueness result. This generalizes substantially previous results by Green and Laffont [5, 6] and Walker.

The outline of the paper is as follows. Section 2 recalls the basic problem structure, Section 3 contains the uniqueness results, and the final section contains some concluding remarks.

2 Groves' Scheme Revisited[3]

Let D be a set of alternative public decisions containing at least two elements and $N = \{1, ..., n\}$ a set of n individuals or agents. Each agent $i \in N$ possesses a utility function $u_i(z, d)$ defined over money $z \in R$ and public decisions $d \in D$, which is linearly additively separable in its arguments, so that we can write:

$$u_i(z, d) = z + v_i(d), \qquad i \in N. \tag{1}$$

We call $v_i(d)$ agent i's *valuation function*, and it is assumed to belong to a prescribed set of valuation functions, V_i, called the *domain* or *admissible* set of valuation functions for agent i.

The objective is to choose an efficient public decision from D. If $v = (v_1, ..., v_n) \in V = \times_{i=1}^{n} V_i$ are the true valuation functions of the agents, it follows from (1) that an efficient decision $d^*(v) \in D$ must satisfy:

$$d^*(v) \in \underset{d \in D}{\arg\max} \sum_{i \in N} v_i(d).^4 \tag{2}$$

We will always assume V and D are such that (2) is well-defined for all $v \in V$.[5]

Only agent i knows his own valuation function. Hence, decisions must be made based on what he reports about this valuation function. In a *direct revelation mechanism* each agent is asked to report his true valuation function and subsequently a public decision is made according to a decision rule satisfying (2). In order to achieve efficient decisions in this way, agents must be induced to tell the truth. This may be done by employing a monetary *transfer scheme* $\{t_i(w)\}_{i \in N}$. If the joint reported valuations (or messages) are $w = (w_1, ..., w_n) \in V$, this scheme pays individual i $t_i(w)$ units of money.

[3] The terminology is taken from Green and Laffont [5].

[4] The notation "arg max" means the set of arguments that maximize the objective.

[5] A sufficient condition would be: all v_i's u.s.c. and D compact in a suitable topology.

From now on, let us assume we have fixed a particular efficient decision rule $d^*(\cdot)$ satisfying (2) for all $v \in V$. A transfer scheme which induces each individual to tell the truth as a dominant strategy (when the public decision is made according to $d^*(\cdot)$) is called *strongly individually incentive compatible* (s.i.i.c.). Truth will be a dominant strategy for all agents if and only if

$$v_i \in \arg\max_{w_i \in V_i} v_i(d^*(w_i, v^i)) + t_i(w_i, v^i), \qquad \forall v \in V, \qquad \forall i \in N.^6 \tag{3}$$

To see that condition (3) can be satisfied by a proper choice of t_i's, observe first that (2) implies:

$$v_i \in \arg\max_{w_i \in V_i} \sum_{j \in N} v_j(d^*(w_i, v^i)), \qquad \forall v \in V\,. \tag{4}$$

This simply says that truth-telling is socially optimal. Writing

$$t_i(w) = \sum_{j \in N \setminus \{i\}} w_j\big(d^*(w)\big) + h_i(w), \qquad \forall w \in V, \qquad \forall i \in N, \tag{5}$$

as we may by choosing h_i's appropriately, we can state (3) equivalently as:

$$v_i \in \arg\max_{w_i \in V_i} \sum_{j \in N} v_i(d^*(w_i, v^i)) + h_i(w_i, v^i), \qquad \forall v \in V, \qquad \forall i \in N. \tag{6}$$

Compare (4) and (6). It is immediate that (6) is true whenever h_i is independent of w_i, since (4) holds by the definition of $d^*(\cdot)$. By choosing the transfer scheme $\{t_i(w)\}_{i \in N}$ such that h_i is independent of w_i in (5), that is by using a *Groves' scheme*, the individual and social objective functions, expressed in terms of messages, coincide. As a result, a Groves' scheme is s.i.i.c., which was first proved in Groves and Loeb [8]. The reasoning given above, which uses the fact that (2) implies (4), is a variant of the argument in Groves and Loeb and illustrates quite clearly the rationale behind their result. It is also useful in discussing the uniqueness question, which we now turn to.[7]

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[6] We use the notation $x^i = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ and write $(\widehat{x}_i, x^i) = (x_1, ..., x_{i-1}, \widehat{x}_i, x_{i+1}, ..., x_n)$. Also, $V^i = \times_{j \in N \setminus \{i\}} V_j$.

[7] It could be mentioned that if one assumes that the domains V_i are one-dimensional, so that for any $v_i \in V_i$, $v_i(d) = v_i(d; x_i)$ for some parameter value $x_i \in R$, then Groves' scheme is easy to *derive* (under suitable differentiability assumptions) from (4) and (6). One may let agents report their parameter values instead of functions in this case. Let $m = (m_1, ..., m_n) \in R^n$ be their joint message. Then, if the respective derivatives exist, one gets from (4) and (6) as first-order conditions:

$$\frac{\partial}{\partial m_i} v_i(d^*(m_i, x^i); x_i) + \frac{\partial}{\partial m_i} t_i(m_i, x^i) = 0, \qquad \text{at} \quad m_i = x_i,$$

$$\frac{\partial}{\partial m_i} \sum_{j \in N} v_j(d^*(m_i, x^i); x_j) = 0, \qquad \text{at} \quad m_i = x_i.$$

3 Uniqueness on Restricted Domains

The question of when Groves' scheme is the only scheme which is s.i.i.c. can, by the previous discussion, be rephrased as follows: Under what conditions will consistency of (4) and (6) imply that each h_i has to be independent of w_i?

It is easy to see that generally consistency will not imply that h_i is constant in w_i. For instance, if V_i's are taken to be discrete domains, easy counterexamples can be given. Thus, a certain richness of the domains V_i is required. The following definition will provide the appropriate condition of richness.

Definition 1. *The domain $V = \times_{i \in N} V_i$ is said to be smoothly connected if for any two valuation functions v_i, $v_i' \in V_i$ and any $v^i \in V^i$, there exists a one dimensional parameterized family of valuation functions in V_i:*

$$V_i(v_i, v_i') = \{v_i(d; y_i) \in V_i \mid y_i \in [0,1]\},$$

such that for all $d \in D$

$$v_i(d;0) = v_i(d), \tag{i}$$

$$v_i(d;1) = v_i'(d), \tag{ii}$$

$$\frac{\partial v_i(d;y_i)}{\partial y_i} \quad \textit{exists for all} \quad y_i \in [0,1],^{8} \tag{iii}$$

and, moreover, for all $y_i \in [0,1]$ and all d in $D^*(v_i, v_i'; v^i) = \{d \in D \mid d = d^*(v_i(d;y_i), v^i) \text{ for some } y_i \in [0,1]\}$, we have

$$\left|\frac{\partial v_i(d;y_i)}{\partial y_i}\right| \leq K \quad \textit{for some} \quad 0 < K < \infty. \tag{iv}$$

The set $D^*(v_i, v_i'; v^i)$ is the set of public decisions, which according to our fixed decision function $d^*(\cdot)$ (satisfying (2)), maximize the social objective when the valuation functions are $(v_i(d;y_i), v^i)$ for some $y_i \in [0,1]$. It is rather innocuous to assume that condition (iv) is satisfied in view of the restriction to

Combining these equations gives:

$$\frac{\partial}{\partial m_i} t_i(m_i, x^i) = \frac{\partial}{\partial m_i} \sum_{j \in N \setminus \{i\}} v_j(d^*(m_i, x^i); x_j), \qquad \text{at} \quad m_i = x_i.$$

The unique solution is:

$$t_i(x) = \sum_{j \in N \setminus \{i\}} v_j(d^*(x); x_j) + h_i(x^i),$$

i.e., Groves' scheme.

This approach has been exhibited in more detail in Holmström [9]. Closely related approaches can be found in Laffont and Maskin [10] and Smets [11].

[8] At the endpoints $y_i = 0$ or 1 only existence of one-sided derivatives is assumed.

$D^*(v_i, v_i'; v^i)$, and it may well be that (iv) follows from (i)-(iii), but this question is still open. Of course, if $\partial v_i(d; y_i)/\partial y_i$ is continuous in both arguments that will be the case.

Our main result is the following.

Theorem 1. *If V is smoothly connected, any s.i.i.c. transfer scheme is a Groves' scheme.*

Proof: Fix $v^i \in V^i$ and let $v_i, v_i' \in V_i$ be arbitrary. If we can show that $h_i(v_i, v^i) = h_i(v_i', v^i)$ will follow from (4) and (6), the claim has been proved.

Consider the parameterized subdomain $V_i(v_i, v_i')$ in Definition 1. From (4) and (6) follows:

$$y_i \in \arg\max_{m_i \in [0,1]} \sum_{j \in N \setminus \{i\}} v_j(\widetilde{d}(m_i)) + v_i(\widetilde{d}(m_i); y_i), \tag{7}$$

$$y_i \in \arg\max_{m_i \in [0,1]} \sum_{j \in N \setminus \{i\}} v_j(\widetilde{d}(m_i)) + v_i(\widetilde{d}(m_i); y_i) + \widetilde{h}_i(m_i), \tag{8}$$

where we have written:

$$\begin{aligned} \widetilde{d}(m_i) &\equiv d^*(v_i(d; m_i), v^i), \\ \widetilde{h}(m_i) &\equiv h^*(v_i(d; m_i), v^i). \end{aligned}$$

By the lemma in Appendix A, (7) and (8) imply $\widetilde{h}_i(0) = \widetilde{h}_i(1)$. Hence $h_i(v_i, v^i) = h_i(v_i', v^i)$ using the definitions of $\widetilde{h}_i(m_i)$ and $v_i(d; m_i)$. ■

Remark 1. *It is self-evident from the proof that smooth connectedness could be replaced by piecewise smooth connectedness; i.e., for any $v_i, v_i' \in V_i$ there would exist a sequence of valuation functions $v_i^{(0)}, v_i^{(1)}, ..., v_i^{(k)} \in V_i$ such that $v_i^{(0)} = v_i, v_i^{(k)} = v_i'$ and each pair $(v_i^{(t)}, v_i^{(t+1)})$would would be smoothly connected.*

Remark 2. *In Appendix B an example is given which shows that uniqueness may be lost if the domain is not smoothly connected. In the example condition (iii) of Definition 1 is violated only at a single point for each d.*

The primary example of a smoothly connected domain is a convex domain as shown below. V is convex if $v, v' \in V$ implies $\lambda v + (1 - \lambda)v' \in V$ for all $\lambda \in [0, 1]$. (This should not be confused with the possible convexity of preferences themselves, though, of course, all convex preferences constitute a convex domain.)

Theorem 2. *If V is a convex domain, then V is smoothly connected and any s.i.i.c. transfer scheme is a Groves' scheme.*

Proof: Fix v_i, $v_i' \in V_i$ and $v^i \in V^i$. Let the functions

$$v_i(d; y_i) = (1 - y_i) \cdot v_i(d) + y_i \cdot v_i'(d), \qquad y_i \in [0, 1], \tag{9}$$

define the family $V_i(v_i, v_i')$ in Definition 1. By convexity this family is contained in V_i and obviously it satisfies i, ii, and iii. In fact, $\partial v_i(d; y_i)/\partial y_i = v_i'(d) - v_i(d)$.

It remains to be shown that for some $K > 0$

$$|v_i'(d) - v_i(d)| \leq K \qquad \text{on} \quad D^*(v_i, v_i'; v^i). \tag{10}$$

Make a contrapositive assumption that no K exists giving (10). Then there exists a sequence $\{d_k\}_{k=1}^{\infty} \subset D^*(v_i, v_i'; v^i)$ such that either $v_i'(d_k) - v_i(d_k) \to -\infty$ or $+\infty$. By symmetry it is enough to treat the former case.

Let y_i^k be such that $d_k = d^*(v_i(d; y_i^k), v^i)$, and take $\overline{d} \in D$ such that for $\forall k$

$$\sum_{j \in N} v_j(d_k) \leq \sum_{j \in N} v_j(\overline{d}).^9 \tag{11}$$

By definition of d_k and y_i^k,

$$\sum_{j \in N} v_j(d_k) + y_i^k \cdot (v_i'(d_k) - v_i(d_k)) \geq \sum_{j \in N} v_j(\overline{d}) + y_i^k \cdot (v_i'(\overline{d}) - v_i(\overline{d})), \qquad \forall k. \tag{12}$$

Combining (11) and (12), we get

$$y_i^k \cdot (v_i'(d_k) - v_i(d_k)) \geq y_i^k \cdot (v_i'(\overline{d}) - v_i(\overline{d})), \qquad \forall k. \tag{13}$$

Since we assumed $v_i'(d_k) - v_i(d_k) \to -\infty$ and $y_i^k \in [0, 1]$, (13) must imply the existence of a $k_0 > 0$ such that $y_i^k = 0$ for all $k \geq k_0$. Hence, $d_k = d_{k+1}$ for all $k \geq k_0$, contradicting the assumption that $v_i'(d_k) - v_i(d_k) \to -\infty$. ■

Examples of convex domains are the following:

1. V_i consists of all u.s.c. functions on D.
2. V_i consists of all continuous functions on D.
3. V_i consists of all strictly concave (or concave) functions on a convex subset of R^t.
4. V_i consists of all concave quadratic utility functions on a convex subset of R^t.

Thus Green and Laffont's [5] uniqueness results (case 1 and 2) and Walker's uniqueness result (case 3) follow from Theorem 2. Case 4 corresponds to a recent result by Green and Laffont [6], which is an improvement on Walker's result, since the domain in 4 is a subset of the domain in 3.

[9] We assumed earlier that D and V are such that a $\overline{d}$ satisfying (11) can be found.

4 Concluding Remarks

We have shown that only a Groves' scheme will be s.i.i.c. even when the domain of valuation functions is restricted, as long as this domain is smoothly connected, for instance convex. For all practical purposes this result shows that one has to be content with Groves' scheme if s.i.i.c. is desired. Since Groves' scheme possesses some undesirable properties, notably that the transfers need not sum up to zero, the result is a negative one. Restricting agents' admissible valuation functions to a smaller domain does not usually lead to any new classes of s.i.i.c. functions, which would possess some additional desirable features.

The methodology employed in this paper could also be used to study the uniqueness question when the problem is formulated as a game of incomplete information as was done originally in Groves [7]. Such a formulation would be called for if agents could not fully communicate their preferences, but it could also be used to weaken the notion of incentive compatibility and resolve some of the negative conclusions about Groves' scheme, as has been shown in d'Aspremont and Gérard-Varet [2].

In the context of a game of incomplete information the uniqueness result states that any incentive compatible transfer scheme has to equal a Groves' scheme in expectation (under appropriate independence assumptions). This was proved first by d'Aspremont and Gérard-Varet [2, 3] for a somewhat restricted model and has been generalized in Holmström [9].

Appendix A

Denote $I = [0,1], I_0 = (0,1)$.

Lemma 1. *Let $f\colon I^2 \to R$ and $g\colon I \to R$ satisfy:*
(i) $y \in \arg\max_{m\in I} f(m,y)$,
(ii) $y \in \arg\max_{m\in I} f(m,y) + g(m)$,
(iii) $\left|\frac{\partial f(m,y)}{\partial y}\right| \leq K < \infty$ *for all* $m, y \in I$.
Then $g(m) =$ constant on I.

Proof: Define the functions:

$$\eta(y) = f(y,y), \qquad \eta : I \to R,$$

$$\gamma(y) = f(y,y) + g(y), \qquad \gamma : I \to R,$$

$$\varphi(y,m) = f(y,m) - f(m,m), \qquad \varphi : I^2 \to R,$$

$$\pi(y,m) = f(y,m) + g(y) - f(m,m) - g(m), \qquad \pi : I^2 \to R.$$

Step 1: $\eta(y)$, $\gamma(y)$, $g(y)$ are Lipschitz on I. Take $y, y' \in I$ arbitrary. Suppose $\eta(y) \geq \eta(y')$. Then,

$$\begin{aligned} 0 &\leq \eta(y) - \eta(y') = f(y,y) - f(y,y') + f(y,y') - f(y',y') \\ &\leq f(y,y) - f(y,y') \leq K \cdot |y - y'|. \end{aligned}$$

The last two inequalities follow by (i) and (iii), respectively.
Arguing similarly if $\eta(y) \leq \eta(y')$, we get

$$|\eta(y) - \eta(y')| \leq K \cdot |y - y'|.$$

Analogously, we can show

$$|\gamma(y) - \gamma(y')| \leq K \cdot |y - y'|.$$

Since $g(y) = \gamma(y) - \eta(y)$, the inequalities above imply

$$|g(y) - g(y')| \leq 2K \cdot |y - y'|.$$

This establishes Step 1.

Step 2: Since η, γ, g are Lipschitz on I, they are a.e. differentiable on I. We claim $g' = 0$ a.e. on I_0. By (i) and (ii) we have for any m, $y \in I$,

$$\begin{aligned} \varphi(y,m) &\leq 0, \\ \pi(y,m) &\leq 0. \end{aligned} \tag{A.1}$$

From the definitions,

$$\frac{\varphi(y,m)}{y-m} = \frac{f(y,m)-f(y,y)}{y-m} + \frac{\eta(y)-\eta(m)}{y-m}, \tag{A.2}$$

and

$$\frac{\pi(y,m)}{y-m} = \frac{f(y,m)-f(y,y)}{y-m} + \frac{\gamma(y)-\gamma(m)}{y-m}. \tag{A.3}$$

Letting y be a point where $\eta'(y)$ and $\gamma'(y)$ exist, it follows from (iii) that the limits as $m \to y$ in (A.2) and (A.3) exist. Moreover, from (A.1) one gets (by taking $m \downarrow y$ and $m \uparrow y$):

$$\lim_{m\to y} \frac{\varphi(y,m)}{y-m} = \lim_{m\to y} \frac{\pi(y,m)}{y-m} = 0.$$

Thus, for a.e. $y \in I_0$,

$$\lim_{m\to y} \frac{g(y)-g(m)}{y-m} = \lim_{m\to y} \frac{\pi(y,m)-\varphi(y,m)}{y-m} = 0.$$

Step 3: Since $g' = 0$ a.e. on I_0 and g is Lipschitz on I, $g(y)$ is a constant on I. ■

Appendix B

We give an example for which schemes other than Groves' are s.i.i.c.

Let $n = 2$, $D = [0,1]$, and define V_i's via the following parametric representations:

$$\begin{aligned} v_1(d;x_1) &= \begin{cases} 0, & \text{if} \quad d \le x_1, \\ -(d-x_1), & \text{if} \quad d \ge x_1, \end{cases} \qquad x_1 \in [0,1]; \\ v_2(d;x_2) &= x_2 + \frac{1}{2}d, \qquad x_2 \in [0,1]. \end{aligned}$$

The maximizing d is equal to x_1, and hence is independent of x_2, though this is not essential. The critical fact is that $v_1(d;x_1)$ does not have a partial derivative with respect to x_1, at $d = x_1$, (though it has everywhere else), and this leads to a kink in the social objective at $d = x_1$. It is readily checked that the following transfer scheme is s.i.i.c. (we can take agents to report x_1, x_2 rather than the functions themselves):

$$\begin{aligned} t_1(x) &= g_1(x) + \frac{1}{4}x_1, \\ t_2(x) &= g_2(x). \end{aligned}$$

where $(g_1,\ g_2)$ is a Groves' transfer scheme. Clearly, $(t_1,\ t_2)$ is not a Groves' transfer scheme.

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