Growth and public debt: what are the relevant tradeoffs?*

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Abstract: *The interplay between growth and public debt is addressed considering a Barro-type [3] endogenous growth model where public spending is financed through taxes on income and public debt. The government has a target level of public debt relative to GDP, and the long run debt-to-GDP ratio is used as a policy parameter. We show that when debt is a large enough proportion of GDP, two distinct BGPs may co-exist, one being indeterminate. We exhibit two types of important tradeoff associated with self-fulfilling expectations. First, we show that the lowest BGP is always decreasing with respect to the debt-to-GDP ratio while the highest one is increasing. Second, we show that the highest BGP, which provides the highest welfare, is always locally indeterminate while the lowest is always locally determinate. Therefore, local and global indeterminacy may arise and self-fulfilling expectations appear as a crucial ingredient to understand the impact of debt on growth, welfare and macroeconomic fluctuations. Finally, a simple calibration exercise allows to provide an understanding of the recent experiences of many OECD countries.*

Keywords: *Endogenous growth, public spending, public debt, sunspot fluctuations.*

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1 Introduction

The last financial crisis has shed the light on the problem of large public debt in developed countries, in particular in Europe. As a response to the large negative shock that resulted from the drastic fall of asset prices, most governments have increased their public spending to support both the aggregate demand and the production process. As a consequence, in many advanced countries, debt levels have increased dramatically during the last decade. The control of the growth rate of public spending has then become a major concern for economists and policy-makers while public deficits have reached unsustainable levels. Indeed, a heavily indebted country may appear as fragile, for many reasons, among which solvency, or simply because it is unlikely to raise sufficient funds to deal with a large negative shock on its economy.

The Maastricht treaty introduced a rule on the maximal amount a country may contract, limiting the debt to 60% of GDP, but over the last decade this limit has been exceeded by most advanced countries, not only European ones, and even before 2008, as shown on the following Figure:

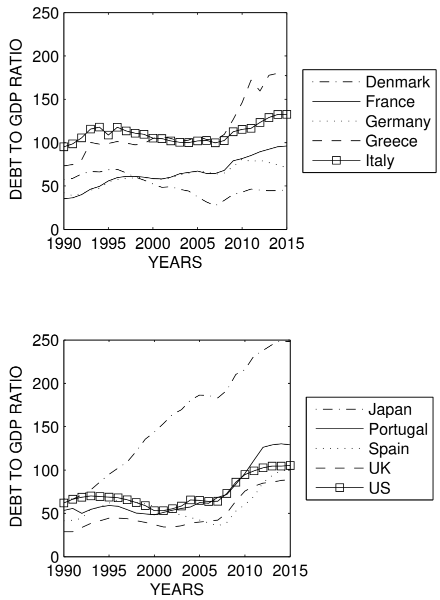

Figure 1: The debt/GDP ratio of OECD countries

At the same time, most of these countries are characterized over this period by low average growth rates and quite large fluctuations of GDP.[1] Given this fact, two types of questions become central:

1. What is the relationship between debt and growth?
2. What is the relationship between macroeconomic stability and debt?

While the literature has focused recently on the first question, mainly through empirical analyses, little attention has been paid to the second.

This paper proposes to fill this gap and to study the relationships between debt, growth and macroeconomic fluctuations in the simple framework of a Barro-type [3] endogenous growth model. Our aim is precisely to discuss the effect of public debt on the endogenous growth rate and on the fluctuations of this growth rate. We focus on government intervention as a source of macroeconomic fluctuations when government spending is financed through taxes on income and public debt. Public spending is useful because it improves, through the supply of a public good, households' utility of consumption and production. In order to focus on the recent public policies, we assume, as in Minea and Villieu [20],[2] that the government gradually adjusts the debt-to-GDP ratio in order to converge towards a target level in the long run in line with constraints like the Maastricht treaty. This long run target for the debt-to-GDP ratio is used as a policy parameter by the government.

The size of public debt as a proportion of GDP and its impact on growth has been widely discussed and debated from an empirical point of view over the recent years. The well-known paper of Reinhart and Rogoff [27] shows that a gross public debt exceeding 90% of nominal GDP on a sustained basis may have a significant negative impact on the growth rate.[3] While subject to a recent controversy,[4] this type of result has led the IMF after the starting of the global financial crisis to strongly advise European countries to decrease their debt. The main objective was to boost growth but also to stabilize the economies.

[1]See Section 6 for illustrative data on these facts.

[2]See also Futagami *et al.* [11] for a similar formulation but with a public debt target defined as a ratio to private capital.

[3]See also Reinhart *et al.* [26] and [28].

[4]Herndon *et al.* [16] have indeed identified a mistake in the methodology of Reinhart and Rogoff [27]. They show that when properly calculated, the average real GDP growth rate for countries carrying a public-debt over GDP ratio of over 90% is significantly larger than the level evaluated by Reinhart and Rogoff.

However, the recent empirical literature shows that the conclusions are not so clear-cut. While some contributions find evidences for a negative relationship between average debt over GDP ratios and long-run growth performance (Woo and Kumar [31]), others find several thresholds for the debt-to-GDP ratio and the size of the impact on growth may be small (Baum *et al.* [4]). Actually, the relationship between debt and growth appears to be based on complex non-linear effects and to be heterogeneous across countries.[5] In particular, Minea and Parent [18] show that for debt-to-GDP ratios above 115%, the correlation between growth and debt becomes positive.

The impact of debt on the stability of economies has also been discussed. For instance, Sutherland *et al.* [29] argue that the level of government debt has a significative impact on business cycle characteristics. They identify the characteristics of "low debt" and "high debt" business cycles aggregating the countries according to their level of debt. In countries with high debt, the cycle is more pronounced, with phases of expansions longer and larger and recessions also more pronounced. Such differences usually rely on the "vulnerability" of high public debt economies. Government then have less latitude to run the appropriate fiscal policy in case of negative shocks.

As explained in Panizza and Presbitero [25], beside all these empirical studies, a precise theoretical analysis of the existence of non-monotonicity or threshold effects in the relationship between public debt, economic growth and aggregate fluctuations is not yet available in the literature. Our paper provides a first step in that direction focusing on the role of global indeterminacy and self-fulfilling expectations.

While not directly concerned with the impact of debt on growth and macroeconomic instability, Collard *et al.* [7]-[8] provide an interesting focus with respect to our research agenda. They analyse the determinants of public debt under the assumptions that governments have limited horizons and default only when their income falls short of debt service requirements. They derive a government's maximum sustainable debt ratio, that strongly varies across countries. The difference between actual and maximum sustainable debt ratios then creates a "margin of safety" that allows governments to increase debt if necessary with little corresponding increase in default risk. In light of

[5] See Eberhart and Presbitero [9] and the survey of Panizza and Presbitero [25].

these results, when we focus on the recent experiences of Spain and Italy who have faced increasing interest rates generated by speculative attacks against their sovereign debt, we clearly see that an objective "margin of safety", based on countries' fundamentals, may not be enough to prevent a large increase of the perceived default risk by investors. This remark suggests a potential for expectation-driven effects. This is precisely the main focus of our paper.

Our framework allows to contribute to these different debates, i.e. the interplay between the debt-to-GDP ratio and both growth and expectation-driven macroeconomic fluctuations. Our central result is to show that when the fiscal pressure is strong enough and the debt-to-GDP ratio admits sufficiently large values, two distinct BGPs may co-exist. Global indeterminacy then arises and self-fulfilling expectations appear as a crucial ingredient to understand the impact of debt on growth. We also exhibit two types of important tradeoffs associated with self-fulfilling expectations. First, we show that the lowest BGP is always decreasing in the debt-to-GDP ratio, while the highest one is increasing. As suggested by the recent literature on non-linear relationships between debt and growth, depending on the BGP selected by agents' expectations, the relationship between debt and growth is not necessarily negative. This result contributes to the debate suggested by Panizza and Presbitero [25] and may in particular match the conclusion of Minea and Parent [18].

Second, we clearly exhibit a tradeoff between welfare and macroeconomic fluctuations: with a large enough debt-to-GDP ratio, the highest BGP, which provides the highest welfare, is locally indeterminate while the lowest is locally determinate. Our results then show non-trivial effects of debt on growth and macroeconomic fluctuations. Depending on agents' expectations, when debt is increasing, large fluctuations associated to self-fulfilling beliefs may occur and be associated at the same time with welfare losses if there is a coordination on the low steady-state. These results clearly show that, above a threshold, the size of debt has a dramatic impact on the dynamic properties of equilibria.

We also discuss the implications of our conclusions for the main OECD countries considering numerical illustrations based on realistic calibrations for the main fundamental parameters. We show that the OECD countries can be split into three sets: a first one with Denmark and Germany in which there exists a unique globally determinate long run growth rate, a second set with France and Spain in which there exist two long run growth rates respectively locally determinate and indeterminate, with one being associated to an endogenous recession, and a third set with Greece, Italy, Japan, Portugal, UK and US in which there exists a unique locally indeterminate long run growth rate. The last two sets contain countries that may be subject to strong macroeconomic fluctuations based on pessimistic expectations. Our analysis then provides a basis for understanding the recent experiences of many OECD countries relating macroeconomic instability to self-fulfilling expectations.

The rest of the paper is organized as follows. In the next Section, we discuss the related literature. In Section 3, we present the model and define the intertemporal equilibrium. In Section 4, we discuss the effect of debt in the long run analyzing the existence and multiplicity of steady-states, comparative statics and welfare. Section 5 provides an analysis of the effect of debt in the short run focusing on sunspot fluctuations and global indeterminacy. Section 6 presents data and numerical illustrations for the main OECD countries. Section 7 concludes the paper and a final Appendix contains all the proofs.

2 Related literature

There are few theoretical contributions that study the link between debt, growth and fluctuations in endogenous growth models. Futagami *et al.* [11] consider a government which focuses on a target for its policy based on the level of debt to the size of the economy, namely the private capital stock. Assuming a log-linear utility specification, they show that multiple BGPs may arise together with local indeterminacy and expectation-driven fluctuations. Beside the fact that the private capital stock is quite difficult to measure empirically, such a target does not appear as highly realistic as in most cases governments rely on GDP-based rules to characterize their fiscal policy. Famous examples are given by constraints imposed by the Maastricht treaty for the European Monetary Union or by the Code of Fiscal Stability in the UK.

Considering the same model as Futagami *et al.* [11] but with a debt over GDP target for the government, Minea and Villieu [20] prove that the multiplicity and indeterminacy conclusions do not survive.[6] They show indeed that

[6]Minea and Villieu [19] prove the possible existence of multiple BGPs and a form of

there exists a unique BGP which is locally determinate. However, as we will show later on, their results are not robust as they crucially depend on the log specification of consumers' preferences. Greiner and Semmler [13, 14] examine how government's financing method affects economies. They show that the link between public deficit and long run growth is not clear cut. However, they do not particularly focus on the impact of the debt-to-GDP ratio.

Our paper can also be seen as an extension of models with public spending but without public debt, like Cazzavillan [6]. He shows that a large public good elasticity in preferences, such that utility has increasing returns, is required to have indeterminacy and endogenous fluctuations. We assume instead that the utility function is characterized by a small public good elasticity and, thus, decreasing returns. This confirms that the main channel through which self-fulfilling expectations arise in our framework is the level of debt over GDP.

In two recent papers, we have studied the impact of the debt-to-GDP ratio on macroeconomic stability, but without endogenous growth. In Nishimura *et al.* [23], we consider a Ramsey economy with heterogeneous agents in which public spending does not affect production. We assume that the tax rate is constant while public spending is endogenously adjusted. We show that debt can be destabilizing or stabilizing, i.e. generating or ruling out damped or persistent macroeconomic fluctuations (period-two cycles), depending on whether the elasticity of utility with respect to public spending and the elasticity of capital labor substitution are low or large. However, indeterminacy can never occur. In Nishimura *et al.* [24], we consider a Ramsey economy with homogeneous agents. We assume a constant public spending that does not affect the fundamentals, and thus, through the government's budget constraint, an endogenous adjustment of the tax rate. We then show that multiple steady-states generically arise and that a large enough debt can be destabilizing through the occurrence of self-fulfilling expectations and sunspot-driven fluctuations.

3 The model

We consider a discrete time economy ($t = 0, 1, ..., \infty$), with three types of agents, a constant population of identical infinitely lived households, a large

global (but not local) indeterminacy considering instead a deficit over GDP target.

number of identical competitive firms and a government.

3.1 Households

Each consumer is endowed with one unit of labor and an initial stock of private physical capital which depreciates at a constant rate $\delta \in (0,1)$. We assume that the total population is normalized to one. The representative agent has separable preferences over time which depend on consumption c_t and public spending G_t. To be consistent with endogenous growth, we assume that the intertemporal utility function is given by

$$\sum_{t=0}^{+\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta} G_t^{\eta} \tag{1}$$

with $\beta \in (0,1)$ the discount factor, $\theta \in (0,1]$ the inverse of the elasticity of intertemporal substitution in consumption (EIS), and $\eta \in [0,\theta]$ the elasticity of utility with respect to public spending. The restriction on θ allows to consider some EIS larger than one, in accordance with the most recent estimates by Gruber [15] (see also Mulligan [21], Vissing-Jorgensen and Attanasio [30]) exhibiting values around 2. The restriction on η implies that the degree of homogeneity of the instantaneous utility function $1-\theta+\eta$ is lower than one, i.e. we do not allow increasing returns due to public spending.

Following Barro [2], the government is assumed here to provide a contemporaneous flow of public services to households. The assumption $\theta - \eta \in (0,1)$ implies also that private consumption and public spending are complement, as shown empirically for instance by Ni [22].[7] This formulation may provide a rationale for the fact that, following the financial crisis of 2008, most governments have increased public spending to boost the private demand.

Remark: As shown in Sections 4 and 5 below, all our main results will not depend on the consideration of a public spending externality into utility and still hold if $\eta = 0$. Since we assume $\theta - \eta \in (0,1)$, considering $\eta > 0$ allows to keep a realistic value for the EIS in the calibration exercise of Section 6.

[7] See also Amano and Wirjanto [1], Bouakez and Rebei [5], Evans and Karras [10] and Karras [17].

The representative household derives income from wage, capital and government bonds that allow to finance public debt. Denote r_t the real interest rate on physical capital, $\bar{r}_t$ the interest rate on government bonds and w_t the real wage. The representative household pays taxes on labor income, capital income and on the remuneration of bonds' holding, at a constant rate $\tau \in (0,1)$. He then maximizes (1) facing the budget constraint:

$$c_t + k_{t+1} + b_{t+1} = (1-\tau)[r_t k_t + w_t] + [1 + (1-\tau)\bar{r}_t]\, b_t + (1-\delta)k_t \quad (2)$$

Utility maximization gives:

$$\left(\tfrac{c_{t+1}}{c_t}\right)^{\theta} \left(\tfrac{G_{t+1}}{G_t}\right)^{-\eta} = \beta R_{t+1} = \beta\,[1 + (1-\tau)\bar{r}_{t+1}] \quad (3)$$

with the gross interest rate $R_{t+1} \equiv (1-\tau)r_{t+1} + 1 - \delta$ and the transversality conditions:

$$\lim_{t\to+\infty} \beta^t c_t^{-\theta} G_t^{\eta} k_{t+1} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \beta^t c_t^{-\theta} G_t^{\eta} b_{t+1} = 0 \quad (4)$$

Obviously, we get at the equilibrium the equality $R_{t+1} = [1 + (1-\tau)\bar{r}_{t+1}]$ since physical capital k_{t+1} and government bonds b_{t+1} are perfectly substitutable saving assets.

3.2 Firms

A representative firm produces the final good y_t using a Cobb-Douglas technology with constant returns at the private level but which is also affected by productive public services provided by the government, $y_t = Ak_t^s(L_t G_t)^{1-s}$, where $s \in (0, 1/2)$ is the share of capital income in GDP and $A > 0$ is a productivity parameter.

Following again Barro [2], we consider here that government also provides public services as input to private production processes. Standard examples are provided by the provision of a legal system, of national defense services, and in general of public infrastructures. This formulation also allows to take into account the fact that many governments have supported some industrial sectors after the starting of the financial crisis in 2008. For instance, in January 2009, the US Federal government created the Automotive Industry Finance Program that consisted in a $80 billion bailout of the U.S. auto industry. It started by providing operating cash for General Motors and Chrysler. These companies promised in return to fast-track development of energy-efficient vehicles and consolidate operations, and this had a direct impact on the productivity level.[8]

Remark: Under the assumption $\theta - \eta \in (0, 1)$, the public services externality in the production function ensures the existence of endogenous growth but does not allow per so to generate multiple equilibria. Indeed, we show in Sections 4 and 5 that uniqueness holds if there is no debt ($\alpha = 0$). Our main results on multiple equilibria and self-fulfilling business cycles fundamentally rely on the existence of a large enough debt-to-GDP ratio.

As population is normalized to one, $L_t = 1$, we get a standard Barro-type [3] formulation such that $y_t = Ak_t^s G_t^{1-s}$. Profit maximization then gives:

$$r_t = As\left(\frac{G_t}{k_t}\right)^{1-s} \quad \text{and} \quad w_t = A(1-s)k_t\left(\frac{G_t}{k_t}\right)^{1-s} \tag{5}$$

Note that we get the traditional eviction effect as the interest rate is an increasing function of government spending.

3.3 Government

Public spending G_t is financed by total income taxation and debt, through the following budget constraint:

$$G_t + (1 + \bar{r}_t)b_t = \tau(r_t k_t + w_t + \bar{r}_t b_t) + b_{t+1} \tag{6}$$

where $\tau \in (0, 1)$ is the constant proportional tax rate on households' total income. Total public expenditure is the sum of public spending G_t and the reimbursement of debt contracted the previous period $(1 + \bar{r}_t)b_t$, and b_{t+1} is the new issue of debt.

We observe that in our model, G_t corresponds to a flow of public spending. As we will see later, it is mainly determined by expectations on future debt emission. Following Glomm and Ravikumar [12], we could alternatively consider that the government aims to finance at time t a public investment G_{t+1}. In this case, the budget constraint (6) would write $G_{t+1} + (1 + \bar{r}_t)b_t = \tau(r_t k_t + w_t + \bar{r}_t b_t) + b_{t+1}$. In addition, G_t, which enters the utility and production functions, would become a predetermined variable instead of being a forward one. We conjecture that such a formulation would

[8] Barro [2] shows that defense purchases have a significant expansionary effect on GDP.

preserve the existence of multiple steady-states but should make the occurrence of local indeterminacy more difficult.[9]

Sustainability constraints like the one imposed by the Maastricht treaty for the European Monetary Union or the Code of Fiscal Stability in the UK mean that governments have to target their debt-to-GDP ratio $\alpha_t \equiv b_t/y_t$ at a certain level α in the long run. For instance, it corresponds to 0.6 in the European Union. According to Minea and Villieu [20], we consider a stabilizing rule which imposes that the debt-to-GDP ratio reduces if it exceeds the target α, but raises otherwise. We use a formulation in discrete time:

$$\alpha_{t+1} - \alpha_t = -\phi(\alpha_t - \alpha), \text{ with } \phi \in (0,1) \tag{7}$$

For instance, following the last global financial crisis, the level of debt in most European countries has well exceeded the bound of 60% of GDP as fixed by the Maastricht treaty. As a result, over the last years, governments have concentrated their efforts to control debt through reductions of the share α_t. The limit case without debt is of course obtained when $\alpha_t = 0$.

3.4 Intertemporal equilibrium

Let us consider equations (2), (3), (5), (6) and (7). Using (5) and (6), equation (2) becomes:

$$c_t + k_{t+1} + G_t = y_t + (1-\delta)k_t \tag{8}$$

Let us denote $x_t \equiv G_t/k_t$, $d_t \equiv c_t/k_t$, and $\gamma_t \equiv k_{t+1}/k_t$ the growth factor. We derive $y_t = Ak_t x_t^{1-s}$ and equations (3), (6), (7) and (8) can be written as:

$$\begin{aligned}
\left(\frac{d_{t+1}}{d_t}\right)^{\theta}\left(\frac{x_{t+1}}{x_t}\right)^{-\eta}\gamma_t^{\theta-\eta} &= \beta\left[(1-\tau)Asx_{t+1}^{1-s} + 1 - \delta\right] && (9)\\
x_t + \alpha_t A x_t^{1-s}\left[(1-\tau)Asx_t^{1-s} + 1 - \delta\right] &= \tau A x_t^{1-s} + \alpha_{t+1}Ax_{t+1}^{1-s}\gamma_t && (10)\\
d_t + \gamma_t + x_t &= Ax_t^{1-s} + 1 - \delta && (11)\\
\alpha_{t+1} &= (1-\phi)\alpha_t + \phi\alpha && (12)
\end{aligned}$$

From (11), we derive some expressions for d_t and d_{t+1}. Substituting these expressions into (9) allows to define an intertemporal equilibrium as a sequence $(x_t, \gamma_t, \alpha_t)_{t\geq 0}$ solution of the following system of three difference equations:

[9] Of course, having more precise results requires to conduct the whole dynamic analysis.

$$\begin{aligned}
&\beta x_{t+1}^{\eta}\frac{(1-\tau)Asx_{t+1}^{1-s}+1-\delta}{\left[Ax_{t+1}^{1-s}-x_{t+1}+1-\delta-\gamma_{t+1}\right]^{\theta}}=\frac{\gamma_t^{\theta-\eta}x_t^{\eta}}{\left[Ax_t^{1-s}-x_t+1-\delta-\gamma_t\right]^{\theta}}\\
&\alpha_{t+1}Ax_{t+1}^{1-s}\gamma_t=x_t-\tau Ax_t^{1-s}+\alpha_tAx_t^{1-s}\left[(1-\tau)Asx_t^{1-s}+1-\delta\right]\\
&\alpha_{t+1}=(1-\phi)\alpha_t+\phi\alpha
\end{aligned}\tag{13}$$

and the transversality conditions (4) with $k_0 \geqslant 0$, $b_0 \geqslant 0$ and $\alpha_0 \geqslant 0$ given. Note that there is only one predetermined variable in this dynamical system. Indeed, the growth factor γ_0 depends on k_1, and from (10), x_0 depends on x_1 and α_1. This means that x_t and γ_t are both forward variables.[10] In contrast, $\alpha_t = b_t/(Ak_tx_t^{1-s})$ is predetermined, since we already take into account that x_t is a forward variable.

Note that the dynamic path of α_t is driven independently by the third equation of the system (13). Since $1-\phi \in (0,1)$, the sequence $(\alpha_t)_{t\geqslant 0}$ monotonically converges to the steady-state α. The dynamic properties of the economy will therefore mainly depend on the first two equations of the system that drive the dynamic path of $(x_t,\gamma_t)_{t\geqslant 0}$, taking $(\alpha_t)_{t\geqslant 0}$ as given. When there is no debt, i.e. $\alpha_t = \alpha = 0$, the third equation becomes irrelevant and the second equation of the system (13) leads to a constant value for x. The dynamical system becomes one-dimensional with a difference equation characterizing the dynamics of the growth factor γ_t.

4 Effect of debt in the long run: Steady-states, comparative statics and welfare

A steady-state equilibrium corresponds to a stationary sequence $\{x_t,\gamma_t,\alpha_t\} = \{x^*,\gamma^*,\alpha^*\} \in \mathbb{R}^3_{++}$ for all t which satisfies system (13). Using the last equation of this system, we derive that $\alpha^* = \alpha$, and the stationary value of the growth factor γ^* allows to define a *balanced-growth path* (BGP) along which all the variables c_t, k_t and G_t grow at the common constant rate $g^* = \gamma^* - 1$.

Along a steady-state equilibrium, the first equation of system (13) gives:

$$x(\gamma) = \left(\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta As(1-\tau)}\right)^{\frac{1}{1-s}}\tag{14}$$

[10]Note also that the initial consumption c_0 has to be chosen in accordance with any initial choice of γ_0 and x_0 since, using (11), c_0 is given by $c_0 = k_0\left(Ax_0^{1-s} - x_0 + 1 - \delta - \gamma_0\right)$.

As, following Barro [3], endogenous growth is explained by productive public spending, this equation establishes a positive link between G/k and the growth factor γ. Indeed, a larger level of public spending per unit of capital increases productivity, which means a larger interest rate, that fosters growth. Substituting this expression into the second equation of system (13) implies that γ^* is a solution of the equation $\Delta(\gamma) = \Omega(\gamma)$ with

$$\Delta(\gamma) \equiv \tfrac{1}{A}\left(\tfrac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta A s(1-\tau)}\right)^{\frac{s}{1-s}} + \alpha\tfrac{\gamma^{\theta-\eta}}{\beta}, \text{ and } \Omega(\gamma) \equiv \tau + \alpha\gamma \tag{15}$$

Obviously, the expression of $x(\gamma)$ implies that any admissible solution must satisfy $\gamma > [\beta(1-\delta)]^{\frac{1}{\theta-\eta}} \equiv \gamma_{inf}$. This means that any BGP is characterized by strictly positive ratios of public spendings over capital and interest rate $r = sAx^{1-s}$. Moreover, as along a BGP all the variables c_t, k_t and G_t grow at a common constant rate, the transversality conditions (4) evaluated along such a BGP holds if $\gamma < \beta^{\frac{1}{\theta-\eta-1}} \equiv \gamma_{sup}$, with $\gamma_{sup} > \gamma_{inf}$. Note that the condition $\gamma < \gamma_{sup}$ is also equivalent to $\gamma^{\theta-\eta}/\beta = R > \gamma$ which means that the interest factor is larger than the growth factor.

In the following, we restrict our attention to tax rates that are not unrealistically large:

Assumption 1. *$\tau < 1 - s$.*

Since we assume that $s < 1/2$, this assumption is always fulfilled for tax rates lower or equal than 1/2. This corresponds to the empirical evidence found in OECD countries (see Table 2 in Section 6).

Any equilibrium $\gamma \in (\gamma_{inf}, \gamma_{sup})$ has also to satisfy $d(\gamma) = Ax(\gamma)^{1-s} - x(\gamma) + 1 - \delta - \gamma > 0$. This inequality is ensured by the following Lemma:

Lemma 1. *Under Assumption 1, there exists $\widehat{A} > 0$ such that if $A > \widehat{A}$, we have $d(\gamma) > 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$.*

Proof. See Appendix 8.1.

Taking into account this lemma, we first study the existence and multiplicity of steady-states, and we analyze the effects of a variation of the debt-to-GDP ratio on long-run growth and welfare.

4.1 Existence and multiplicity of BGP

The existence and the number of stationary solutions $\gamma \in (\gamma_{inf}, \gamma_{sup})$ are derived from the equation $\Delta(\gamma) = \Omega(\gamma)$. As a benchmark case, we start by considering the configuration without debt at a steady-state:

Proposition 1. *Under Assumption 1, let $A > \widehat{A}$ and $\alpha = 0$. Then, there exists $\hat{\tau} \in (0, 1-s)$ such that there is a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ if and only if $\tau \in (0, \hat{\tau})$.*

Proof. See Appendix 8.2.

This case corresponds to the BGP investigated in Barro [3]. To understand the intuition for this result, let us rewrite the budget constraint of the government (10) evaluated at the steady-state as $PS(\gamma) = DS(\gamma)$ with:

$$PS(\gamma) \equiv \tau - \tfrac{x(\gamma)^s}{A} \text{ and } DS(\gamma) \equiv \alpha(R(\gamma) - \gamma) \tag{16}$$

where $R(\gamma) \equiv \gamma^{\theta-\eta}/\beta$ and $x(\gamma)$ is given by (14).

The right-hand side $DS(\gamma)$ corresponds to the difference between the reimbursement of debt and the new debt issue deflated by the GDP. We will call it *debt service*, keeping in mind that the net interest factor $R - \gamma$ must be positive under the transversality condition. The left-hand side $PS(\gamma)$ corresponds to the difference between government revenue derived from income taxation and public spending, divided by GDP. We call this difference *primary surplus.* This is clearly a decreasing function of the ratio of government spending over capital x, which is positively correlated to the growth factor γ. Since government spending acts as a productive input, an increase in spending expands the output of the economy, which in turns improves tax revenue. However, the income tax rate being constant, the primary surplus expressed as a share of GDP always decreases in government spending over capital.

When $\alpha = 0$, there is no debt at a BGP and the primary budget is balanced ($PS(\gamma) = 0$). As seen in Proposition 1, the existence and uniqueness of the equilibrium $x = (A\tau)^{1/s}$ requires however a low enough tax rate. Indeed, if τ is too large, i.e. close to the labor share $1 - s$, the tax revenue is too high regarding possible public spendings. There is always a strictly positive primary surplus, which violates a balanced primary budget.

With positive debt at a steady-state, i.e. $\alpha > 0$, multiplicity of BGPs may easily arise (two steady-states) as it is illustrated in the following Figure:

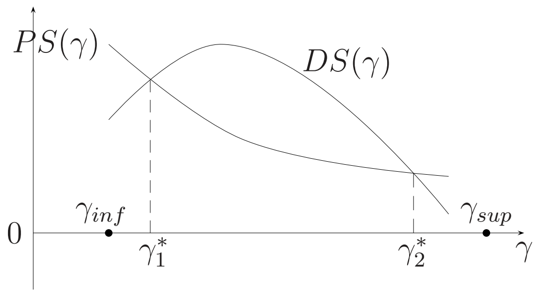

Figure 2: Multiplicity of steady-states.

It can be shown indeed that two steady-states exist when the debt-to-GDP ratio is sufficiently but not too large and the fiscal pressure is strong enough. The results are summarized in the following Proposition:

Proposition 2. *Under Assumption 1, let* $A > \widehat{A}$ *and* $\alpha > 0$*. Then, there exist* $\hat{\tau} \in (0, 1-s)$*,* $\widehat{\alpha} > \underline{\alpha} > 0$ *and* $\widetilde{\Theta} \in (0,1)$ *such that the following results hold:*

1. There are two steady-states $\gamma_1^*, \gamma_2^* \in (\gamma_{inf}, \gamma_{sup})$*, with* $\gamma_1^* < \gamma_2^*$*, if and only if* $\alpha \in (\underline{\alpha}, \widehat{\alpha})$*,* $\tau \in (\hat{\tau}, 1-s)$ *and* $\theta - \eta \in (0, \widetilde{\Theta})$*.*

2. There is a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ *if one of the following conditions is satisfied:*

(a) $\alpha < \widehat{\alpha}$ *and* $\tau \in (0, \hat{\tau})$*;*

(b) $\alpha > \widehat{\alpha}$ *and* $\tau \in (\hat{\tau}, 1-s)$*.*

Proof. See Appendix 8.3.

The following figure summarizes the conclusions of Proposition 2:

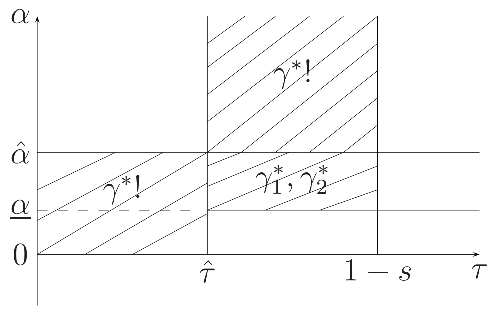

Figure 3: Uniqueness and multiplicity of steady-states.

Proposition 2 and Figure 3 show that when the stationary level of public debt over GDP is positive, a BGP exists in two main configurations: (i) when the debt-to-GDP ratio and the tax rate are both low enough ($\alpha < \widehat{\alpha}$, $\tau < \hat{\tau}$); (*ii*) when they are both large enough ($\alpha > \underline{\alpha}$, $\tau > \hat{\tau}$). Indeed, in case (*i*), the debt service to be financed is quite low, and thus a low tax revenue is sufficient to get a primary surplus able to finance the low debt reimbursement net of debt emission. In case (*ii*), we have exactly the opposite case. The debt service to be financed is quite high and requires a large primary surplus, which is obtained by a sufficiently high tax rate.

Interestingly, Proposition 2 also establishes that a positive stationary government debt may lead to a multiplicity of steady-states, as it is illustrated in Figure 3. This multiplicity refers of course to the existence of coordination failures. Optimistic expectations of a larger growth cause agents to believe, on the one hand, that debt emission will increase compared to debt reimbursement, implying thus a lower level of debt service, and on the other hand, that productive public spendings will increase, implying thus a lower primary surplus. As a result, optimistic expectations are compatible with the government budget constraint and are self-fulfilling. Similarly, pessimistic expectations of a lower growth cause agents to believe that debt emission will decrease compared to debt reimbursement, implying a higher debt service, and that public expenditures will decrease implying a lower primary surplus. These expectations are again compatible with the government budget constraint and are self-fulfilling. This mechanism explains the multiplicity of BGPs.

We also note that a sufficiently low value of $\theta - \eta$ is a crucial ingredient to get the multiplicity steady-states. The value of $\theta - \eta$ actually represents at a BGP the inverse of the EIS in consumption, and from the intertemporal arbitrage of consumers, affects the interest factor as given by $R = \gamma^{\theta-\eta}/\beta$. For $\theta - \eta$ larger or equal to one, optimistic expectations of a larger growth factor imply a too large increase of debt reimbursement with respect to debt emission. As a result, the debt service increases while the primary surplus significantly decreases due to the expectations of higher public spending. It follows that the government budget constraint cannot be satisfied and these expectations cannot be self-fulfilling. This case refers to the configuration studied in Minea and Villieu [20], who consider a model with a log-linear utility in consumption ($\theta = 1$) without public spending ($\eta = 0$), and where there is at most one BGP.

Remark: Minea and Villieu [20] actually consider a continuous time model with the debt adjustment rule $\dot{\alpha}(t) = -\phi[\alpha(t) - \alpha]$, but the same results as in discrete time can be obtained. Indeed, using our notations, and assuming a CES utility function with a non-unitary EIS, the government budget in their model becomes

$$\tau - \frac{x(t)^s}{A} = \frac{\alpha}{\theta}\left[s(1-\tau)(\theta-1)x(t)^{1-s} + \rho\right]$$

Although the right-hand-side becomes constant when $\theta = 1$ and uniqueness of the solution $x(t)$ is ensured, this conclusion is not robust when θ is sufficiently lower than 1 as the right-hand-side becomes a decreasing and convex function and multiple steady-states may then arise for appropriate parameters' values.

4.2 The effect of debt over GDP on growth and welfare

We examine how the different types of steady-states vary according to variations of the stationary debt-to-GDP ratio. It also allows us to focus on the welfare that can be easily computed along the BGP γ. We have indeed:

$$W(\gamma) = \frac{k_0^{1-(\theta-\eta)}}{1-\theta} \frac{d(\gamma)^{1-\theta} x(\gamma)^{\eta}}{1-\beta\gamma^{1-(\theta-\eta)}} \tag{17}$$

We get the following result:

Corollary 1. *Under Assumption 1, let $A > \widehat{A}$ and $\alpha > 0$. The following results hold:*

1. If $\alpha \in (\underline{\alpha}, \widehat{\alpha})$, $\tau \in (\hat{\tau}, 1-s)$ and $\theta - \eta \in (0, \widetilde{\Theta})$, then $d\gamma_1^*/d\alpha < 0$ and $d\gamma_2^*/d\alpha > 0$;

2. If $\alpha < \widehat{\alpha}$ and $\tau \in (0, \hat{\tau})$, then $d\gamma^*/d\alpha < 0$;

3. If $\alpha > \widehat{\alpha}$ and $\tau \in (\hat{\tau}, 1-s)$, then $d\gamma^*/d\alpha > 0$.

Moreover, there exist $\hat{\beta} \in (0,1)$ and $\hat{\delta} \in (0,1)$ such that if $\beta \in (\hat{\beta}, 1)$ and $\delta \in (0, \hat{\delta})$, then $W'(\gamma) > 0$.

Proof. See Appendix 8.4.

The following Figure allows to explain how we can understand the effect of α on the two steady-states:[11]

[11] The same explanation also allows to understand the cases with a unique steady-state.

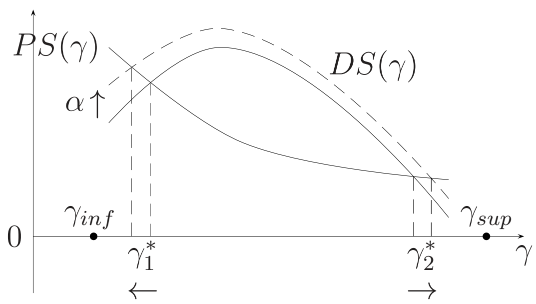

Figure 4: Comparative statics.

Following an increase of the stationary debt-to-GDP ratio, debt service becomes higher than the primary surplus. Two situations may then arise. In the first one, the primary surplus increases to restore the government budget constraint. Therefore, public spending decreases, and hence growth too. This happens at the BGP γ_1^*, with the lowest growth rate. In the second one, the intertemporal budget constraint is satisfied if there is a decrease of debt service, coming from a larger increase of debt emission compared to debt reimbursement. Since debt over GDP is constant at a BGP, it requires more growth. This happens at the BGP γ_2^* with the highest growth rate, because it reinforces the effect of growth on public emission compared to debt reimbursement.

The comparative statics of Corollary 1 provide quite different conclusions than the empirical results of Reinhart and Rogoff [27]. The relationship between debt and growth indeed strongly depends on both the level of the debt-to-GDP ratio and the level of the tax rate. First, the growth rate is always a decreasing function of the debt-to-GDP ratio when this share is small enough. Second, for intermediary values of α, multiple steady-states arise and the lower growth factor is negatively affected by α while the higher one is positively affected. Third, above a threshold level $\widehat{\alpha}$ and when the tax rate is large enough, the unique long-run growth rate becomes an increasing function of α. These results suggest a complex relationship between debt and growth in line with the recent literature surveyed by Panizza and Presbitero [25]. Moreover, they provide a theoretical justification for the positive correlation exhibited by Minea and Parent [18] with large enough debt-to-GDP ratios.

Our results then show that the negative impact of debt on growth as exhibited by Reinhart and Rogoff [27] and some of the recent literature has to be taken with caution. Above the threshold value $\underline{\alpha}$, when multiplicity holds, they only get part of the story as they miss the second possible equilibrium growth rate, which is an increasing function of α, and thus the possible strong impact of agents' expectations that select the long run equilibrium. They miss the fact that when the tax rate and the debt-to-GDP ratio are high enough, the growth factor increases. This can be linked to the concept of "debt tolerance" formulated by Reinhart *et al.* [28] that characterizes countries with strong fiscal structures, i.e. able to raise a large amount of taxes (see also Collard *et al.* [7]-[8]), or strong financial systems, i.e. on which a large amount of assets is traded.

Corollary 1 also shows that along a long-run balanced growth path, welfare is an increasing function of the stationary growth factor γ. We then exhibit a first important trade-off associated with self-fulfilling expectations. Depending on the long-run equilibrium selected by agents' expectations, increasing the debt-to-GDP ratio may have a positive or a negative impact on both the growth rate and welfare.

5 Effect of debt in the short run: Sunspot fluctuations and global indeterminacy

Focusing on the stability properties of the BGPs, we analyse now whether the debt-to-GDP ratio is destabilizing, through the occurrence of local indeterminacy and sunspot equilibria. As we have seen, two BGPs may coexist. Hence, in order to simplify the local stability analysis and to focus on a precise steady-state, we provide a normalization procedure of a BGP. Indeed, in order to get long run positive growth, we want to focus on the existence of a steady-state value of γ which is larger than 1. To simplify this analysis, we use the constant A to get the existence of a normalized BGP $\gamma^* > 1$.

This procedure is useful first to precisely calibrate the model on empirically relevant average values of the growth rate for the main OECD countries (see Section 6 below). Second, it is useful to ensure that γ^* remains invariant with respect to parameters' changes, in particular the targeted debt-to-GDP ratio

α. This invariance will allow us to study how the dynamical properties of the steady-state are affected when α is modified. As shown in the following Proposition, this property is obtained by choosing adequately the value of A that will adjust accordingly to keep γ^* constant. Third, this procedure allows to precisely locate the second BGP $\tilde{\gamma}$ with respect to the normalized one.

Proposition 3. *Under Assumption 1, let $\theta - \eta < \beta$. Then, there exists $\underline{\gamma} \in [1, \gamma_{sup})$ such that any given value $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$ is an admissible invariant BGP if and only if $\alpha \in (0, \alpha_{Max})$ and $A = A^*$ with*

$$\alpha_{Max} = \frac{\tau}{\gamma^* \left(\frac{\gamma^{*\theta - \eta} - 1}{\beta} - 1 \right)} (> \widehat{\alpha}) \quad \textit{and} \quad A^* = \frac{\left[\frac{\gamma^{*\theta - \eta} - \beta(1-\delta)}{\beta s (1-\tau)} \right]^s}{\left[\tau - \alpha \gamma^* \left(\frac{\gamma^{*\theta - \eta} - 1}{\beta} - 1 \right) \right]^{1-s}} (> \widehat{A}) \tag{18}$$

Moreover, there exist there exist $\hat{\alpha}$, $\underline{\alpha}$ and α_1 with $\underline{\alpha} < \overline{\alpha} < \alpha_1 < \widehat{\alpha} < \alpha_{Max}$ such that when $\alpha \in (\underline{\alpha}, \hat{\alpha})$, there is a second admissible BGP $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ such that:

i) $\tilde{\gamma} \in (\gamma^*, \gamma_{sup})$ when $\alpha \in (\underline{\alpha}, \bar{\alpha})$,
ii) $\tilde{\gamma} \in (1, \gamma^*)$ when $\alpha \in (\bar{\alpha}, \alpha_1)$,
iii) $\tilde{\gamma} \in (\gamma_{inf}, 1)$ when $\alpha \in (\alpha_1, \hat{\alpha})$.

Proof. See Appendix 8.5.

As it is illustrated in Figure 5, Proposition 3 shows, in accordance with Proposition 2, that there exist two admissible steady-states as long as the debt-to-GDP ratio α is large enough but not too big. Indeed, when α is too low, the second steady-state $\tilde{\gamma}$ is not admissible as it is larger than γ_{sup}. Proposition 3 also allows to locate precisely the second steady-state $\tilde{\gamma}$ with respect to γ^* and 1. The configuration with $\tilde{\gamma} < 1$ is clearly associated to endogenous recession.

Let us finally focus on the local stability properties of the steady-states. As explained in Section 3.4, our dynamical system (13) has only one predetermined variable α_t. Therefore, local determinacy is obtained if the steady-state is a saddle-point with only one stable dimension. On the contrary, when the steady-state is either a saddle-point with two stable dimensions or totally stable (a sink), there is local indeterminacy with the existence of a continuum of equilibrium paths. As a result, sunspot fluctuations also occur.

We show now that the stability properties of the NSS crucially depend on the value of the debt-to-GDP ratio.

Proposition 4. *Under Assumption 1, let $A = A^*$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Then, there exist $\bar{\delta} \in (0,1)$, $\underline{\beta} \in (0,1)$ and $\bar{\Theta} \in (0,\beta)$ such that if $\delta \in (0,\bar{\delta})$, $\beta \in (\underline{\beta},1)$ and $\theta - \eta \in (0,\bar{\Theta})$, the normalized steady-state γ^* is:*

i) locally determinate when $\alpha \in (0,\bar{\alpha})$,

ii) locally indeterminate when $\alpha \in (\bar{\alpha}, \alpha_{Max})$,

while the second admissible steady-state $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ is:

i) locally indeterminate if $\alpha \in (\underline{\alpha}, \bar{\alpha})$,

ii) locally determinate if $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

It follows that the unique normalized steady-state γ^* is globally determinate if and only if $\alpha \in [0,\underline{\alpha})$ while global indeterminacy arises if and only if $\alpha \in (\underline{\alpha}, \hat{\alpha})$.

Proof. See Appendix 8.6.

The following Figure summarizes Propositions 3 and 4:

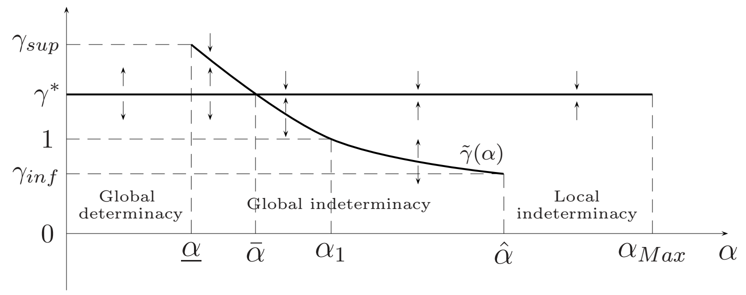

Figure 5: Local/global (in)determinacy of steady-states.

When the debt-GDP ratio is low enough with $\alpha \in [0,\underline{\alpha})$, the normalized BGP γ^* is unique, locally determinate and thus globally determinate as shown on Figure 5. This result is in line with Sutherland *et al.* [29] since low debt prevents the occurrence of business cycles. On the contrary, when the debt-to-GDP ratio is large enough ($\alpha > \underline{\alpha}$), the local indeterminacy of the BGP implies that there exist expectation-driven fluctuations. While Corollary 1 showed that a higher level of the debt-to-GDP ratio may foster the long run growth rate, this result exhibits a negative impact based on the fact that, over a precise threshold, public debt has a destabilizing role on the economy and leads to the existence of fluctuations based on self-fulfilling prophecies. As a result, debt may generate endogenous recessions associated to recurrent decreases of growth with the possible occurrence of negative growth rates.

Note also that our results are in line with Sutherland *et al.* [29] since large debt fosters macroeconomic instability.

Propositions 4 and Figure 5 also exhibit an important tradeoff when $\alpha > \underline{\alpha}$ as the largest BGP is always locally indeterminate, i.e. characterized by expectation-driven fluctuations, while the lowest BGP is always locally determinate. The larger the debt-to-GDP ratio is, the lower the locally determinate BGP. It can even be less than one, i.e. characterized by an economic recession. Because of multiplicity, there is also a global indeterminacy suggesting that depending on agents' expectations, the actual equilibrium may converge towards the lowest or the highest BGP, eventually with large fluctuations. As we know from Corollary 1 that the indeterminate BGP has the highest welfare, global indeterminacy can have strong implications in terms of welfare loss if agents coordinate their expectations on the lowest growth factor.

These results put a strong emphasis on self-fulfilling prophecies and reinforce the consistency of our analysis, already highlighted in the previous section, with the empirical facts exhibited by the recent literature. As shown by Sutherland *et al.* [29], increasing the debt-to-GDP ratio affects the business cycle of countries explained here by the existence of expectation-driven fluctuations. Depending on agents' beliefs, the impact on growth can be negative in the short or long run through the possible occurrence of recessions. Moreover, as suggested by Panizza and Presbitero [25], there exist some non-linear and threshold effects associated to the existence of multiple equilibria and global indeterminacy. Finally, the existence of self-fulfilling prophecies can explain the fact that while some countries have a potential maximal sustainable debt ratio higher than their actual one, the perceived default risk can increase dramatically without any fundamental reason (see Collard *et al.* [7]-[8]).

Note that Proposition 4 show that contrary to what is claimed by Minea and Villieu [20], considering a debt over GDP government rule does not allow to rule out the existence of local and global indeterminacy, and thus the existence of large macroeconomic fluctuations based on agents' expectations, as long as one departs from a log-linear utility function.

The intuition for the existence of local indeterminacy and sunspot equilibria is very similar to the one explaining the existence of multiple BGPs. Starting from the high BGP, assume that the households have optimistic expectations of a larger GDP growth factor y_{t+1}/y_t. According to the debt adjustment rule of the government, they believe that the level of future debt emission b_{t+1} will significantly increase implying a decrease of the debt service. As a result, a larger share of government revenue is expected to be devoted to productive public spending implying a larger ratio G/k and thus an increase of the interest rate r because growth comes from a Barro-type [3] externality. Then, the current growth factor raises and the initial expectations are self-fulfilling. The restriction on $\theta - \eta$ is explained by the fact that the existence of macroeconomic fluctuations based on initial self-fulfilling expectations requires that agents are able to substitute consumption over time in order to smooth their utility level. Such a behavior is obtained when the EIS in consumption $1/\theta$ is large enough.

It is important to note that this mechanism does not hold if we consider the lowest BGP. Indeed, optimistic expectations of a larger growth factor only cause in this case a slightly larger debt emission implying then an increase of the debt service. A larger of debt emission is used to debt reimbursement, rather than to improve the level of productive public spending. It follows that the ratio G/k and thus the interest rate r are basically constant or even decreasing. The current growth factor is therefore not affected and the initial expectation is no longer self-fulfilling.

6 A simple calibration on OECD countries

Let us now confront our theoretical conclusions to the experiences of OECD countries since 1990. We consider in Table 1 the debt to GDP ratios and report two types of statistics for GDP variations.[12] The first one is average growth rates, and it echoes to model average long run values γ^*. The second one is the standard deviation of the cyclical component of GDP (as presented in figure 8).[13]

[12] GDP are expressed at constant 2010 US dollar prices.

[13] To compute cyclical components we take logs of GDP and extract trends by using the Hodrick-Prescott filter with a conventional parameter value $\lambda = 100$ at annual frequency.

Countries	γ_{90-15} average 1990-2015	γ_{90-07} average 1990-2007	Debt-GDP ratio 1990-2015	Debt-GDP ratio 1990-2007	GDP s-d (cycle in %) 1990-2015	GDP s-d (cycle in %) 1990-2007
Denmark	1.0159	1.0222	50.5	53.9	1.69	1.60
France	1.015	1.0198	65.2	56.0	1.34	1.42
Germany	1.0147	1.0172	61.7	55.9	1.56	1.37
Greece	1.0086	1.0305	115.6	97.6	4.48	3.57
Italy	1.0066	1.0144	110.5	106.1	1.50	1.37
Japan	1.0098	1.0131	158.8	127.7	1.58	1.38
Portugal	1.0126	1.0216	72.3	55.6	2.24	2.33
Spain	1.0194	1.0306	58.8	51.6	2.60	2.40
UK	1.02	1.0251	50.3	38.2	1.94	1.78
US	1.0244	1.0298	73.4	63.2	1.66	1.58

Table 1: Debt-to-GDP ratios and GDP variations (IMF data)

In addition we distinguish two samples depending on whether we take into account the last financial crisis or not. We therefore compute statistics for the whole period $1990-2015$ or for the pre-crisis period $1990-2007$. As we will show, the qualitative results do not depend on the period considered.

There are strong distinct features between growth rates and cyclical components. For instance the growth rate of Greece is low but this country exhibits large fluctuations (standard deviation of GDP). Actually, the highest volatility of GDP is observed in Greece, but also Spain and Portugal (over the all period $1990-2015$). We also derive from these data the two sets of Figures 6 and 7 which depict either growth rates or standard deviations of the cyclical component of GDP for our OECD panel.

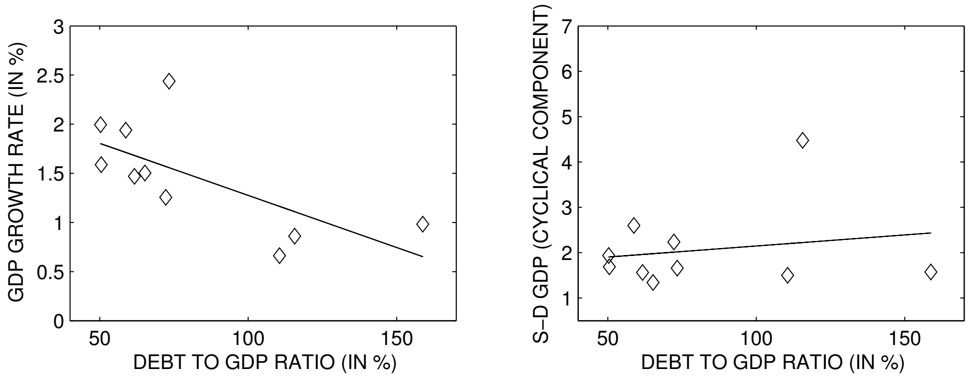

Figure 6: Debt-to-GDP, growth and volatility of GDP (1990-2015)

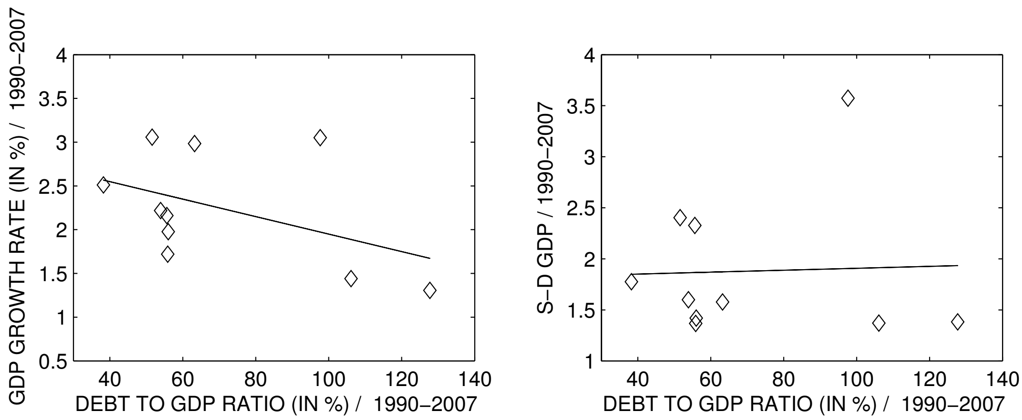

Figure 7: Debt-to-GDP, growth and volatility of GDP (pre-crisis period 1990-2007)

Although illustrative, Figure 6 emphasizes that over the whole period $1990 - 2015$ the GDP growth rate is typically negatively related to the debt to GDP ratio, while the volatility of the cyclical component of GDP appears as being slightly positively related to the debt to GDP ratio. The same results occur for the pre-crisis period $1990 - 2007$ as shown by Figure 7. These preliminary conclusions are in line with our theoretical results.

Let us now run a simple numerical exercise. Using a standard calibration consistent with quarterly data, we assume that $(\delta, \beta) = (0.025, 0.98)$. Following Gruber [15], Mulligan [21], Vissing-Jorgensen and Attanasio [30] who provide estimates for the elasticity of intertemporal substitution in consumption larger than unity and smaller than 2, we assume that $\theta = 0.52$. Concerning the elasticity η of utility with respect to public spending, Ni [22] provides some estimates that support the interval $\eta \in (0.32, 0.37)$. We will consider in the following $\eta = 0.34$ so that $\theta - \eta = 0.18$.

Countries	τ (% of GDP in 2015)	s (in 2015)	α (% of GDP in 2015)	γ_{inf} (over 1960-2015)
Denmark	46.6	0.36	39.5	0.951
France	45.3	0.31	96	0.9706
Germany	37.6	0.31	71.2	0.9438
Greece	33.8	0.4	176.9	0.9087
Italy	44.4	0.31	132.7	0.9452
Japan	28.6	0.4	229.2	0.9447
Portugal	32.5	0.36	129	0.9565
Spain	32.9	0.39	99.4	0.9603
UK	35.2	0.3	89.2	0.958
US	24.3	0.36	104.1	0.9722

Table 2: Main statistics (Source: IMF and OECD)

The Table 2 provides the fiscal pressure τ, the share of capital income into GDP s and the share of gross debt into GDP α in 2015.[14] As γ_{inf} is too low to be empirically plausible, this Table also provides an empirical lower bound for the growth factor which is given by the lowest value reached by each country over the period 1960-2015 during the worse recession. Except for Portugal that experienced its largest recession in 1975, for all other countries the lowest growth factor has been reached in 2009. As can be seen, there are huge variations of the fiscal pressure τ and the debt-to-GDP ratio α across countries. Also, while the share of capital income into GDP is most commonly fixed at 0.3, there are quite large differences across countries with a value often larger than 0.3. For each illustration we will use the parameters' values for τ, s, α, γ_{inf} and γ^* of the country we want to focus on.

Considering the numbers of Tables 1 and 2, the following Table 3 provides for each country all the critical values for the share α of debt over GDP identified in the theoretical analysis. Note however that the value $\hat{\alpha}$ has been computed considering the empirically relevant lower bound γ_{inf} instead of its theoretical value. Depending on the value of α for each country, the normal-

[14]The debt indicator is defined (in the Maastricht Treaty) as consolidated general government gross debt at nominal (face) value, outstanding at the end of the year in the following categories of government liabilities (as defined in ESA 2010): currency and deposits, debt securities and loans. The general government sector comprises the subsectors: central government, state government, local government and social security funds.

ized growth factor γ^* can be the low-growth or the high-growth BGP and thus can be characterized by local determinacy or indeterminacy.

Countries	Period	α	$\underline{\alpha}$	$\bar{\alpha}$	α_1	$\hat{\alpha}$
Denmark	$90-15$	39.5	116.6	117.9	120.4	120.2
	$90-07$		115.6	116	119.4	125.7
France	$90-15$	96	90.8	92.1	94.3	98.5
	$90-07$		90.1	90.8	93.6	98.1
Germany	$90-15$	71.2	75.4	76.5	78.3	84.6
	$90-07$		75.1	776	78	84.7
Greece	$90-15$	176.9	100.8	102.6	103.6	122.5
	$90-07$		98.4	97.8	101.3	112.4
Italy	$90-15$	132.7	90.1	92.6	93.6	108.8
	$90-07$		89.1	90.5	92.5	105.2
Japan	$90-15$	229.2	85.2	86.6	87.5	114.7
	$90-07$		84.9	86	87.2	109.7
Portugal	$90-15$	129	81.6	82.9	84.3	95.9
	$90-075$		80.7	81	83.4	90.5
Spain	$90-15$	99.4	93	93.6	95.8	100.3
	$90-07$		91.9	91.3	94.6	99.6
UK	$90-15$	89.2	66.8	67.3	69.5	75.6
	$90-07$		66.3	66.2	68.9	74.1
US	$90-15$	104.1	60.1	60.2	62.1	65.2
	$90-07$		59.7	59.3	61.7	61.3

Table 3: Critical values of α

Table 3 immediately shows that depending on whether $\alpha \in [0, \underline{\alpha})$, $\alpha \in (\alpha_1, \hat{\alpha})$ or $\alpha > \hat{\alpha}$, the OECD countries can be splited into three groups: group 1 with Denmark and Germany characterized by $\alpha \in [0, \underline{\alpha})$ in which the normalized growth factor is the unique globally determinate admissible steady-state, group 2 with France and Spain characterized by $\alpha \in (\alpha_1, \hat{\alpha})$ in which there exists a second negative long run growth rate, i.e. associated to an endogenous recession, and group 3 with Greece, Italy, Japan, Portugal, UK and the US characterized by $\alpha > \hat{\alpha}$ in which the normalized growth rate is the unique locally indeterminate admissible steady-state. These results appear to be true for the two sub-periods.

Consider the case of Denmark and Germany (group 1), where $\alpha \in [0, \underline{\alpha})$, for which the normalized growth factor is the unique globally determinate steady-state. In these two countries there is no fluctuations based on self-fulfilling prophecies which illustrates the large confidence that international investors have in particular with respect to Germany. The same kind of conclusions hold for Denmark. It is also worth noting that both countries have a debt over GDP ratio that is larger than the Maastricht upper bound but are still characterized by a strong macroeconomic stability. This result illustrates the concept of debt tolerance discussed by Reinhart *et al.* [28].

Let now turn to group 2 and first consider the case of France with $\alpha \in (\alpha_1, \hat{\alpha})$. Over the periods $1990 - 2015$ and $1990 - 2007$, the normalized steady-states $\gamma^*_{90-15} = 1.015$ and $\gamma^*_{90-15} = 1.0198$ respectively are both locally indeterminate. In both cases, global indeterminacy also arises as there exists a second steady-state $\tilde{\gamma}_{90-15} = 0.9888$ and $\tilde{\gamma}_{90-07} = 0.9845$ corresponding respectively to a recession of -1.12% and -1.54%, which is locally determinate These results then suggest that large fluctuations can emerge, provided that modification of beliefs are strong enough to imply a coordination on the low steady-state.

Very similar conclusions are obtained for Spain, where $\gamma^*_{90-15} = 1.0194$ and $\tilde{\gamma}_{90-15} = 0.9717$ while $\gamma^*_{90-07} = 1.0306$ and $\tilde{\gamma}_{90-07} = 0.963$, leading to a potential recession of respectively -2.83% and -3.7%. It is worth noting that while France and Spain have quite similar debt-to-GDP ratios, the potential recessions for Spain are significantly larger than those for France suggesting more potential volatility. This fact can be explained by the larger tax pressure in France which provides additional income to the government to face a negative shock. Such a property may be taken into account by agents' expectations. It is also interesting to remark that for France and Spain, a decrease of their debt ratio below 90%, i.e. far less restrictive than the Maastricht constraint, would be enough to ensure $\alpha < \underline{\alpha}$ and thus to guarantee the uniqueness of the long-run growth equilibrium without any fluctuations.

Lastly, consider the group-3 case (all the other countries), namely Greece, Italy, Japan, Portugal, UK and US, where $\alpha > \hat{\alpha}$, for which the normalized growth factor is the unique locally indeterminate steady-state leading to expectation-driven fluctuations. It is also worth noting that Greece, Italy, Japan and Portugal could completely stabilize their economies decreasing their debt over GDP ratio to a level much higher than the Maastricht constraint (between 80% to 100% depending on the country). But this would require a drastic reduction effort that could impact negatively their economies (between −32% to −63% depending on the country). Similarly, UK and US could also eliminate all possibility of expectation-driven macroeconomic instability by basically respecting a constraint of 60% like the one imposed by the Maastricht treaty.

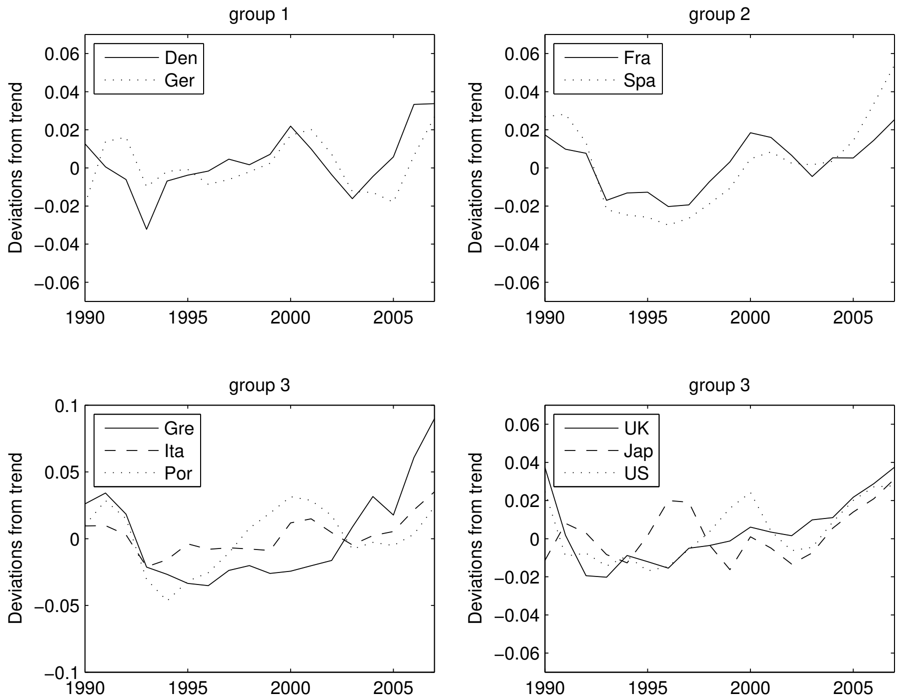

Figure 8: Cyclical components of GDP

Many of these conclusions are recovered into the data. The Figure 8 first splits in those 3 groups cyclical components of GDP over the period 1990-2015. Group 3 countries are clearly those associated to the largest volatility of GDP while group 1 countries are characterized by much less volatility. In group 2, Spain appears as much volatile than France which is quite close to Germany in group 1.

Accordingly, Table 4 reports the standard deviations of GDP of each country with respect to that of Germany (as a benchmark of group 1). The standard deviation of GDP of Greece, Portugal, UK and US in group 3-countries is typ-

	Period	
	1990-2015	1990-2007
	group 1	
Denmark	1.08	1.17
Germany	1.00	1.00
	group 2	
France	0.86	1.04
Spain	1.66	1.76
	group 3	
Greece	2.87	2.62
Italy	0.96	1.00
Japan	1.01	1.01
Portugal	1.43	1.70
UK	1.24	1.30
US	1.06	1.15

Table 4: Standard deviation of GDP wrt. that of Germany

ically higher than that of Germany, in line with our model predictions. On the other hand, Japan and Italy do not seem to match our theoretical results since related standard deviations of GDP are close to that of Germany. In particular, the case of Japan might be explained by the fact that most of the public debt is hold by Japanese institutions and residents, which might reduce the possibility of expectations volatility.

Regarding group 2-countries, global indeterminacy can explain the high volatility as observed for Spain (related to the possibility of a long run recession). But this result does not hold for France. Again, it might the case high fiscal pressure in France plays an important role by giving rise to some automatic stabilizing effects, hence reducing expectations volatility. Obviously, such an issue would require to devote a specific attention to stabilizing mechanisms.

7 Conclusion

In this paper, we consider a simple Barro-type [3] endogenous growth model where public spending is financed through taxes on income and public debt, and improves households' utility of consumption and production. The government is assumed to gradually adjust the debt-to-GDP ratio in order to converge towards a target level in the long run in line with constraints like the Maastricht treaty. This long run target for the debt-to-GDP ratio is used as a policy parameter by the government as in many European countries over the last decade.

We have proved that when the debt-to-GDP ratio is large enough, two distinct BGPs may co-exist, one being indeterminate. We have exhibited two types of important tradeoffs associated with self-fulfilling expectations. First, we have shown that the lowest BGP is always a decreasing function of the debt-to-GDP ratio while the highest one is an increasing function. As a result, depending on the BGP selected by agents' expectations, the relationship between debt and growth is not necessarily negative.

Second, local and global indeterminacy may arise and self-fulfilling expectations appear as a crucial ingredient to understand the impact of debt on growth and on macroeconomic fluctuations. There is clearly a tradeoff between welfare and business cycles: the highest BGP, which provides the highest welfare, is always locally indeterminate while the lowest is always locally determinate. Our results then show non-trivial effects of debt on growth, welfare and macroeconomic fluctuations. Depending on the expectations of agents, large fluctuations associated to self-fulfilling beliefs may occur and be associated at the same time with welfare losses if there is a coordination on the low steady-state.

Our paper then provides a theoretical analysis that improves our understanding of the complex non-linear and threshold effects between public debt, growth and macroeconomic fluctuations. We have also discussed the implications of our findings for the main OECD countries considering numerical illustrations based on realistic calibrations for the size of debt, the growth rate and the main fundamentals. We have shown that the existence of local indeterminacy and of multiple equilibria with global indeterminacy can provide a basis for understanding the recent experiences of many OECD countries relating the occurrence of endogenous recessions and macroeconomic instability to self-fulfilling expectations.

8 Appendix

8.1 Proof of Lemma 1

Substituting (14) in $d(\gamma)$, we get:

$$d(\gamma) = \frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta s(1-\tau)} - \left[\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta A s(1-\tau)}\right]^{\frac{1}{1-s}} + 1 - \delta - \gamma \tag{19}$$

We derive that $d(\gamma_{inf}) = 1 - \delta - [\beta(1-\delta)]^{\frac{1}{\theta-\eta}} > 0$. Moreover,

$$d(\gamma_{sup}) = \frac{\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)}{s(1-\tau)} \left[1 - \frac{1}{A^{\frac{1}{1-s}}} \left(\frac{\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)}{s(1-\tau)}\right)^{\frac{s}{1-s}} - s(1-\tau)\right] > 0 \tag{20}$$

if and only if $A > A_1(\tau)$, with

$$A_1(\tau) \equiv \frac{[\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)]^s}{[s(1-\tau)]^s [1 - s(1-\tau)]^{1-s}} \tag{21}$$

Note that $A_1'(\tau) > 0$. Since $\tau < 1 - s$, we get $A_1(1-s) > A_1(\tau)$ for any $\tau < 1 - s$ with

$$A_1(1-s) = \frac{[\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)]^s}{s^{2s}[(1-s)(1+s)]^{1-s}} \equiv \widehat{A}_1 \tag{22}$$

Computing now the first and second derivatives of $d(\gamma)$, we get:

$$\begin{aligned} d'(\gamma) &= (\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)}\left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}}\left(\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta s(1-\tau)}\right)^{\frac{s}{1-s}}\right] - 1 \\ d''(\gamma) &= (\theta-\eta)(\theta-\eta-1)\frac{\gamma^{\theta-\eta-2}}{\beta s(1-\tau)}\left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}}\left(\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta s(1-\tau)}\right)^{\frac{s}{1-s}}\right] \\ &- (\theta-\eta)^2\left(\frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)}\right)^2 \frac{s}{(1-s)^2 A^{\frac{1}{1-s}}}\left(\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\beta s(1-\tau)}\right)^{\frac{2s-1}{1-s}} \end{aligned} \tag{23}$$

Since $\theta - \eta < 1$, we easily derive that $d''(\gamma) < 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$ if the term between brackets on the first line of the expression of $d''(\gamma)$ is positive. As this term is a decreasing function of γ we conclude that it is positive for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$ if it is positive when evaluated at γ_{sup}. This is obtained if $A > A_2(\tau)$, with

$$A_2(\tau) \equiv \frac{[\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)]^s}{[s(1-\tau)]^s (1-s)^{1-s}} \tag{24}$$

Since $\tau < 1 - s$, we get $A_2(1-s) > A_2(\tau)$ for any $\tau < 1 - s$ with

$$A_2(1-s) = \frac{[\beta^{\frac{1}{\theta-\eta-1}} - (1-\delta)]^s}{s^{2s}(1-s)^{1-s}} \equiv \widehat{A}_2 \tag{25}$$

and $\widehat{A}_2 > \widehat{A}_1$. Let $\widehat{A}_2 = \widehat{A}$. We conclude that if $A > \widehat{A}$, then $d''(\gamma) < 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$. Therefore, the concavity of $d(\gamma)$ over $(\gamma_{inf}, \gamma_{sup})$ together with $d(\gamma_{inf}) > 0$ and $d(\gamma_{sup}) > 0$ ensure that when $A > \widehat{A}$, $d(\gamma) > 0$ for all $\gamma \in (\gamma_{inf}, \gamma_{sup})$ and all $\tau < 1 - s$. $\square$

8.2 Proof of Proposition 1

When $\alpha = 0$, we derive from $\Delta(\gamma) = \Omega(\gamma)$ that $\Omega(\gamma) = \tau$ is constant and

$$\Delta(\gamma) = \frac{1}{A}\left(\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta As(1-\tau)}\right)^{\frac{s}{1-s}}$$

is increasing from $\Delta(\gamma_{inf}) = 0$ to

$$\Delta(\gamma_{sup}) = \frac{1}{A}\left[\frac{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}{As(1-\tau)}\right]^{\frac{s}{1-s}}$$

There exists a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{sup})$ if and only if the inequality $\Delta(\gamma_{sup}) > \tau$ is satisfied, i.e.

$$\frac{1}{A}\left[\frac{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}{As}\right]^{\frac{s}{1-s}} > \tau(1-\tau)^{\frac{s}{1-s}} \equiv \varphi(\tau) \tag{26}$$

$\varphi(\tau)$ is increasing for all $\tau < 1-s$, and reaches its maximum value for $\tau = 1-s$, meaning that $\varphi(\tau) < \varphi(1-s) = (1-s)s^{\frac{s}{1-s}}$. Since we need to assume $A > \widehat{A}$ in order to ensure $d(\gamma) > 0$, where $\widehat{A}$ is given by (25), it follows that inequality cannot hold when $\tau = 1-s$. As it obviously holds when $\tau = 0$, we conclude that there exists $\hat{\tau} \in (0, 1-s)$ such that $\Delta(\gamma_{sup}) > \tau$ for $\tau \in (0, \hat{\tau})$, whereas $\Delta(\gamma_{sup}) < \tau$ for $\tau \in (\hat{\tau}, 1-s)$. $\square$

8.3 Proof of Proposition 2

By direct inspection of equation $\Delta(\gamma) = \Omega(\gamma)$, we see that $\Omega(\gamma)$ is linearly increasing in γ, with $\Omega'(\gamma) = \alpha > 0$, and

$$\Delta'(\gamma) = (\theta-\eta)\gamma^{\theta-\eta-1}\left[\frac{s}{A(1-s)}\frac{[\gamma^{\theta-\eta}-\beta(1-\delta)]^{\frac{2s-1}{1-s}}}{[\beta As(1-\tau)]^{\frac{s}{1-s}}} + \frac{\alpha}{\beta}\right] > 0 \tag{27}$$

Using $s \in (0, 1/2)$, we also easily derive that $\Delta(\gamma)$ is concave, i.e. $\Delta''(\gamma) < 0$. Since $\Delta(\gamma_{inf}) = \alpha(1-\delta)$ and $\Omega(\gamma_{inf}) = \tau + \alpha[\beta(1-\delta)]^{\frac{1}{\theta-\eta}}$, we have $\Delta(\gamma_{inf}) < (>)\Omega(\gamma_{inf})$ if and only if $\alpha < (>)\widehat{\alpha}$, with:

$$\widehat{\alpha} \equiv \frac{\tau}{1-\delta-[\beta(1-\delta)]^{\frac{1}{\theta-\eta}}} \tag{28}$$

Similarly, using

$$\Delta(\gamma_{sup}) = \frac{1}{A}\left[\frac{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}{As(1-\tau)}\right]^{\frac{s}{1-s}} + \alpha\beta^{\frac{1}{\theta-\eta-1}}$$

and $\Omega(\gamma_{sup}) = \tau + \alpha\beta^{\frac{1}{\theta-\eta-1}}$, we get $\Delta(\gamma_{sup}) > \Omega(\gamma_{sup})$ if and only if:

$$\frac{1}{A}\left[\frac{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}{As}\right]^{\frac{s}{1-s}} > \tau(1-\tau)^{\frac{s}{1-s}} \equiv \varphi(\tau) \tag{29}$$

Using the same argument as in the proof of Proposition 1, we conclude that when $A > \widehat{A}$, there exists $\hat{\tau} \in (0, 1-s)$ such that $\Delta(\gamma_{sup}) > \Omega(\gamma_{sup})$ for $\tau \in (0, \hat{\tau})$, whereas $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$ for $\tau \in (\hat{\tau}, 1-s)$.

We can then prove case 1 of the Proposition. Let us assume $\alpha < \widehat{\alpha}$, i.e. $\Delta(\gamma_{inf}) < \Omega(\gamma_{inf})$, and $A > \widehat{A}$ with $\tau \in (\hat{\tau}, 1-s)$, i.e. $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$. There are two steady-states if and only if there exists $\widehat{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ defined by $\Delta'(\widehat{\gamma}) = \Omega'(\widehat{\gamma})$ that satisfies $\Delta(\widehat{\gamma}) > \Omega(\widehat{\gamma})$. The equality $\Delta'(\gamma) = \Omega'(\gamma)$ is equivalent to

$$g(\gamma) \equiv (\theta-\eta)\frac{\gamma^{\theta-\eta-1}s}{\beta A(1-s)}\left(\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta As(1-\tau)}\right)^{\frac{2s-1}{1-s}}\frac{1}{As(1-\tau)} = \alpha\left[1-(\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta}\right] \equiv h(\gamma)$$

Note that $g'(\gamma) < 0$ and $h'(\gamma) > 0$. Moreover, we have

$$g(\gamma_{inf}) = +\infty > h(\gamma_{inf}) = \alpha\left[1-(\theta-\eta)\frac{[\beta(1-\delta)]^{\frac{\theta-\eta-1}{\theta-\eta}}}{\beta}\right]$$

Similarly, we have $h(\gamma_{sup}) = \alpha[1-(\theta-\eta)]$ and

$$g(\gamma_{sup}) = (\theta-\eta)\frac{s}{(1-s)A^{\frac{1}{1-s}}}\left(\frac{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}{s(1-\tau)}\right)^{\frac{s}{1-s}}\frac{1}{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}$$

Since $A > \widehat{A}$, we get

$$g(\gamma_{sup}) < \frac{(\theta-\eta)s}{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}$$

Therefore, we get $g(\gamma_{sup}) < h(\gamma_{sup})$ if and only if

$$\frac{(\theta-\eta)s}{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)} < \alpha[1-(\theta-\eta)]$$

or equivalently

$$\alpha > \frac{(\theta-\eta)s}{\left[\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)\right][1-(\theta-\eta)]} \equiv \tilde{\alpha}$$

It follows that there exists $\widetilde{\Theta} \in (0,1)$ such that $\tilde{\alpha} < \widehat{\alpha}$ if $\theta - \eta \in (0, \widetilde{\Theta})$.

Let us then assume that $\theta - \eta \in (0, \widetilde{\Theta})$ and $\alpha \in (\tilde{\alpha}, \widehat{\alpha})$. We know that there exists $\widehat{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ such that $\Delta'(\widehat{\gamma}) = \Omega'(\widehat{\gamma})$. We need finally to show that $\Delta(\widehat{\gamma}) > \Omega(\widehat{\gamma})$. When $\alpha = \tilde{\alpha}$ we get $\Delta'(\gamma_{sup}) = \Omega'(\gamma_{sup})$ and there is no steady-state. When $\alpha = \widehat{\alpha}$, we get $\Delta(\gamma_{inf}) = \Omega(\gamma_{inf})$ and $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$. In this case there exists two steady-states but the lower one is equal to γ_{inf}. Therefore, since $\partial\Delta(\gamma)/\partial\alpha - \partial\Omega(\gamma)/\partial\alpha = \gamma^{\theta-\eta}/\beta - \gamma > 0$, it follows that there exists a unique $\underline{\alpha} \in (\tilde{\alpha}, \widehat{\alpha})$ such that there are two steady-states γ_1^* and γ_2^* for $\alpha \in (\underline{\alpha}, \widehat{\alpha})$.

Let us finally prove case 2 of the Proposition. (a) Assume first that $\alpha < \widehat{\alpha}$, i.e. $\Delta(\gamma_{inf}) < \Omega(\gamma_{inf})$. The existence of a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{\mathrm{sup}})$

is ensured if $\Delta(\gamma_{sup}) > \Omega(\gamma_{sup})$ i.e. if $A > \widehat{A}$ and $\tau \in (0, \hat{\tau})$. (b) Assume now that $\alpha > \widehat{\alpha}$, i.e. $\Delta(\gamma_{inf}) > \Omega(\gamma_{inf})$. The existence of a unique steady-state $\gamma^* \in (\gamma_{inf}, \gamma_{\sup})$ is ensured if $\Delta(\gamma_{sup}) < \Omega(\gamma_{sup})$ i.e. if $A > \widehat{A}$ and $\tau \in (\hat{\tau}, 1-s)$.

□

8.4 Proof of Corollary 1

To determine the comparative statics of each type of steady-state with respect to α, we differentiate equation $\Delta(\gamma) = \Omega(\gamma)$ to get:

$$\frac{d\gamma^*}{d\alpha} = \frac{\partial\Delta(\gamma^*)/\partial\alpha - \partial\Omega(\gamma^*)/\partial\alpha}{\Omega'(\gamma^*) - \Delta'(\gamma^*)} \tag{30}$$

Since $\partial\Delta(\gamma)/\partial\alpha - \partial\Omega(\gamma)/\partial\alpha = \gamma^{\theta-\eta}/\beta - \gamma > 0$, the sign of $d\gamma^*/d\alpha$ is given by the sign of $\Omega'(\gamma^*) - \Delta'(\gamma^*)$, i.e. the difference between the slopes of $\Omega(\gamma)$ and $\Delta(\gamma)$ evaluated at each BGP. Using Proposition 2, we derive that $d\gamma_1^*/d\alpha < 0$ and $d\gamma_2^*/d\alpha > 0$ in case 1, $d\gamma^*/d\alpha < 0$ in case 2 and $d\gamma^*/d\alpha > 0$ in case 3.

Consider now the expression of $W(\gamma)$ as given by (17). We get:

$$W'(\gamma) = \frac{k_0^{1-(\theta-\eta)}d(\gamma)^{-\theta}x(\gamma)^\eta}{1-\theta}\frac{\left[(1-\theta)d'(\gamma)+\eta d(\gamma)\frac{x'(\gamma)}{x(\gamma)}\right]\left[1-\beta\gamma^{1-(\theta-\eta)}\right]+[1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)}d(\gamma)}{\left[1-\beta\gamma^{1-(\theta-\eta)}\right]^2}$$

with

$$\begin{aligned} d'(\gamma) &= (\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)}\left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}}\left(\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right)^{\frac{s}{1-s}}\right] - 1 \\ x'(\gamma) &= \frac{1}{(1-s)A^{\frac{1}{1-s}}}\left(\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right)^{\frac{s}{1-s}}(\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta s(1-\tau)} > 0 \end{aligned}$$

so that

$$W'(\gamma) > \frac{k_0^{1-(\theta-\eta)}d(\gamma)^{-\theta}x(\gamma)^\eta}{1-\theta}\frac{(1-\theta)d'(\gamma)\left[1-\beta\gamma^{1-(\theta-\eta)}\right]+[1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)}d(\gamma)}{\left[1-\beta\gamma^{1-(\theta-\eta)}\right]^2}$$

We have shown in the proof of Lemma 1 that when $A > \widehat{A}$, $d'(\gamma)$ is a monotone decreasing function with $d'(\gamma_{inf}) > d'(\gamma) > d'(\gamma_{sup})$. It follows that

$$W'(\gamma) > \frac{k_0^{1-(\theta-\eta)}d(\gamma)^{-\theta}x(\gamma)^\eta}{1-\theta}\frac{(1-\theta)d'(\gamma_{sup})\left[1-\beta\gamma^{1-(\theta-\eta)}\right]+[1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)}d(\gamma)}{\left[1-\beta\gamma^{1-(\theta-\eta)}\right]^2}$$

with

$$d'(\gamma_{sup}) = \frac{(\theta-\eta)}{s(1-\tau)}\left[1 - \frac{1}{(1-s)A^{\frac{1}{1-s}}}\left(\frac{\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)}{s(1-\tau)}\right)^{\frac{s}{1-s}}\right] - 1 \equiv \Psi(\beta,\delta) - 1$$

and $\Psi(\beta,\delta) > 0$. Let us denote

$$f(\gamma) \equiv (1-\theta)d'(\gamma_{sup})\left[1-\beta\gamma^{1-(\theta-\eta)}\right] + [1-(\theta-\eta)]\beta\gamma^{-(\theta-\eta)}d(\gamma)$$

Using $\gamma \in (\gamma_{inf}, \gamma_{sup})$, assume first that $d'(\gamma_{sup}) \geq 0$. We derive

$$\begin{aligned} f(\gamma) &> (1-\theta)\left[d'(\gamma_{sup})\left(1-\beta\gamma_{sup}^{1-(\theta-\eta)}\right)+\beta\gamma_{sup}^{-(\theta-\eta)}d(\gamma)\right] \\ &> (1-\theta)\beta^{\frac{1}{1-(\theta-\eta)}} > 0 \end{aligned}$$

for any $\gamma \in (\gamma_{inf}, \gamma_{sup})$. Assume now that $d'(\gamma_{sup}) < 0$. We derive

$$\begin{aligned} f(\gamma) &> (1-\theta)\left[d'(\gamma_{sup})\left(1-\beta\gamma_{inf}^{1-(\theta-\eta)}\right)+\beta\gamma_{sup}^{-(\theta-\eta)}d(\gamma)\right] \\ &> (1-\theta)\Big\{\Psi(\beta,\delta)\left(1-\beta^{\frac{1}{\theta-\eta}}(1-\delta)^{\frac{1-(\theta-\eta)}{\theta-\eta}}\right)-1+\beta^{\frac{1}{\theta-\eta}}(1-\delta)^{\frac{1-(\theta-\eta)}{\theta-\eta}} \\ &+ \beta^{\frac{1}{1-(\theta-\eta)}}d(\gamma)\Big\} \end{aligned}$$

When $\beta = 1$ and $\delta = 0$, we get $f(\gamma) > (1-\theta)d(\gamma) > 0$. Therefore, there exist $\hat{\beta} \in (0,1)$ and $\hat{\delta} \in (0,1)$ such that if $\beta \in (\hat{\beta}, 1)$ and $\delta \in (0, \hat{\delta})$, then $f(\gamma) > 0$ for any $\gamma \in (\gamma_{inf}, \gamma_{sup})$. The result follows. □

8.5 Proof of Proposition 3

A stationary solution $\gamma^* \in (1, \gamma_{sup})$ satisfies equation $\Delta(\gamma) = \Omega(\gamma)$ if

$$h(\gamma^*, \alpha) \equiv \left(\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right)^{\frac{s}{1-s}} A^{\frac{-1}{1-s}} - \left[\tau - \alpha\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right)\right] = 0 \tag{31}$$

This equation can hold only if $\alpha < \alpha_{Max}$, with:

$$\alpha_{Max} = \frac{\tau}{\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right)} \tag{32}$$

Note that if $\gamma^* = \gamma_{inf}$ then $\alpha_{Max} = \widehat{\alpha}$. Moreover, as $\gamma^* > 1$, straightforward computations show that if $\theta - \eta < \beta$, α_{Max} is an increasing function of γ^*. It follows that for any $\gamma^* \in (1, \gamma_{sup})$, $\alpha_{Max} > \widehat{\alpha}$.

Let us then assume that $\theta - \eta < \beta$ and $\alpha < \alpha_{Max}$. There is a unique $A = A^*$ solving equation (31), where A^* is given by (18). We immediately see that $\lim_{\gamma \to \gamma_{inf}} A^* = 0$, whereas

$$\lim_{\gamma \to \gamma_{sup}} A^* = \frac{\left[\beta^{\frac{1}{\theta-\eta-1}}-(1-\delta)\right]^s}{s^s(1-\tau)^s\tau^{1-s}} \equiv A^*_{\gamma_{sup}} \tag{33}$$

Since for any $\tau < 1 - s$ we have $A^*_{\gamma_{sup}} > \widehat{A}$, we conclude that there exists $\hat{\gamma} \in (\gamma_{inf}, \gamma_{sup})$ such that when $\gamma^* \in (\hat{\gamma}, \gamma_{sup})$, $A^* > \widehat{A}$. Let us then denote $\underline{\gamma} = \max\{1, \hat{\gamma}\}$ and let us choose $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$ for which $A^* > \widehat{A}$. Since $\widehat{A} > \widetilde{A}$, it follows from Lemma 1 that $d(\gamma^*) > 0$.

We need now to establish a first technical Lemma. Consider indeed equation $\Delta(\gamma) = \Omega(\gamma)$ with $A = A^*$ that can be simplified as follows

$$\begin{array}{rcl} h(\gamma,\alpha) & = & \left[\tau - \alpha\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right)\right]\left(\frac{\gamma^{\theta-\eta}-\beta(1-\delta)}{\gamma^{*\theta-\eta}-\beta(1-\delta)}\right)^{\frac{s}{1-s}} \\ & - & \left[\tau - \alpha\gamma\left(\frac{\gamma^{\theta-\eta-1}}{\beta} - 1\right)\right] = 0 \end{array} \tag{34}$$

We obviously get $h(\gamma^*, \alpha) = 0$. The followingLemma characterizes the slope of the function $h(\gamma, \alpha)$ when $\gamma = \gamma^*$. In the following we denote $h_1'(\gamma, \alpha) = \partial h(\gamma, \alpha)/\partial\gamma$ and $h_2'(\gamma, \alpha) = \partial h(\gamma, \alpha)/\partial\alpha$.

Lemma 8.1. *Assume that $\theta - \eta < \beta$, $A = A^*$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Then there exists $\bar{\alpha} \in (0, \alpha_{Max})$ such that*

$$h_1'(\gamma^*, \alpha) \gtreqless 0 \quad \Leftrightarrow \quad \alpha \lesseqgtr \bar{\alpha} \tag{35}$$

Proof. Straightforward computations using the fact that $h(\gamma, \alpha) = 0$ along a steady-state yields

$$\begin{array}{rcl} h_1'(\gamma,\alpha) & = & \frac{\tau(\theta-\eta)s}{1-s}\frac{\gamma^{\theta-\eta-1}}{\gamma^{\theta-\eta}-\beta(1-\delta)} \\ & - & \frac{\alpha\left\{\left[1-(\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta}\right]\left[\gamma^{\theta-\eta}-\beta(1-\delta)\right]+\gamma^{\theta-\eta}\left(\frac{\gamma^{\theta-\eta-1}}{\beta}-1\right)\frac{(\theta-\eta)s}{1-s}\right\}}{\gamma^{\theta-\eta}-\beta(1-\delta)} \end{array}$$

Consider the term between braces multiplying α

$$\varphi(\gamma) = \left[1 - (\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta}\right]\left[\gamma^{\theta-\eta} - \beta(1-\delta)\right] + \gamma^{\theta-\eta}\left(\frac{\gamma^{\theta-\eta-1}}{\beta} - 1\right)\frac{(\theta-\eta)s}{1-s}$$

We get $\varphi(\gamma) > 0$ if

$$(\theta-\eta)\frac{\gamma^{\theta-\eta-1}}{\beta} < 1 \quad \Leftrightarrow \quad \gamma > \left(\frac{\theta-\eta}{\beta}\right)^{\frac{1}{1-(\theta-\eta)}}$$

Under $\theta - \eta < \beta$ we get $[(\theta-\eta)/\beta]^{\frac{1}{1-(\theta-\eta)}} < 1$. Since $\gamma^* > 1$, it follows that $h_1'(\gamma^*, \alpha) \gtreqless 0$ if and only if $\alpha \lesseqgtr \bar{\alpha}$ with

$$\bar{\alpha} = \frac{\frac{\tau(\theta-\eta)s}{1-s}\gamma^{*\theta-\eta-1}}{\left[1-(\theta-\eta)\frac{\gamma^{*\theta-\eta-1}}{\beta}\right]\left[\gamma^{*\theta-\eta}-\beta(1-\delta)\right]+\gamma^{*\theta-\eta}\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right)\frac{(\theta-\eta)s}{1-s}} \tag{36}$$

Straightforward computations show that $\bar{\alpha} < \alpha_{Max}$.

□

We derive from equation (34) that when $\alpha = 0$, the unique admissible steady-state is γ^*. On the contrary, as soon as $\alpha > 0$, we get $\lim_{\gamma\to+\infty} h(\gamma, \alpha) = -\infty$. Since, when $\alpha \in (0, \bar{\alpha})$ we have $h_1'(\gamma^*, \alpha) > 0$, there necessarily exists a second solution $\tilde{\gamma} > \gamma^*$ of $h(\gamma, \alpha) = 0$. When $\alpha = \bar{\alpha}$ we get $\tilde{\gamma} = \gamma^*$. When $\alpha > \bar{\alpha}$, straightforward computations yield

$$h(\gamma_{inf}, \alpha) = -\left[\tau - \alpha\left(1 - \delta - [\beta(1-\delta)]^{\frac{1}{\theta-\eta}}\right)\right]$$

and thus $h(\gamma_{inf}, \alpha) < 0$ if and only if

$$\alpha < \frac{\tau}{1-\delta-[\beta(1-\delta)]^{\frac{1}{\theta-\eta}}} \equiv \hat{\alpha} \in (\bar{\alpha}, \alpha_{Max}) \tag{37}$$

Therefore, when $\alpha \in (\bar{\alpha}, \hat{\alpha})$, the second solution exists and satisfies $\tilde{\gamma} \in (\gamma_{inf}, \gamma^*)$. Note that (37) provides the same expression as the bound given by (28).

We need now to check whether this second solution is admissible, i.e. if $d(\tilde{\gamma}) > 0$. From Lemma 1, we know that, as long as $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$, $d(\tilde{\gamma}) > 0$. We then have to provide conditions on α to ensure that $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$. Let us consider equation (34). We have shown that $h_1'(\gamma^*, \alpha) \neq 0$ and thus $h_1'(\tilde{\gamma}, \alpha) \neq 0$ as long as $\alpha \neq \bar{\alpha}$. More precisely, we have

$$h_1'(\tilde{\gamma}, \alpha) \gtreqless 0 \quad \Leftrightarrow \quad \alpha \gtreqless \bar{\alpha} \tag{38}$$

Applying the Implicit Function Theorem, we conclude that $\tilde{\gamma} = \tilde{\gamma}(\alpha)$ with $\tilde{\gamma}'(\alpha) = -h_2'(\tilde{\gamma}, \alpha)/h_1'(\tilde{\gamma}, \alpha)$ and $\tilde{\gamma}(\bar{\alpha}) = \gamma^*$. Straightforward computations using the fact that $h(\gamma, \alpha) = 0$ along a steady-state yields

$$h_2'(\gamma, \alpha) \quad = \quad \frac{\tau}{\alpha}\left[1 - \left(\frac{\gamma^{\theta-\eta} - \beta(1-\delta)}{\gamma^{*\theta-\eta} - \beta(1-\delta)}\right)^{\frac{s}{1-s}}\right]$$

When $\alpha < \bar{\alpha}$ we have $h_1'(\gamma^*, \alpha) > 0$ and thus $h_1'(\tilde{\gamma}, \alpha) < 0$ with $\gamma^* < \tilde{\gamma}$. It follows that $h_2'(\tilde{\gamma}, \alpha) < 0$ and thus $\tilde{\gamma}'(\alpha) < 0$. When $\alpha \in (\bar{\alpha}, \alpha_{Max})$ we have $h_1'(\gamma^*, \alpha) < 0$ and thus $h_1'(\tilde{\gamma}, \alpha) > 0$ with $\gamma^* > \tilde{\gamma}$. It follows that $h_2'(\tilde{\gamma}, \alpha) > 0$ and thus $\tilde{\gamma}'(\alpha) < 0$. Therefore, for any $\alpha \in (0, \bar{\alpha}) \cup (\bar{\alpha}, \alpha_{Max})$, $\tilde{\gamma}(\alpha)$ is a monotone decreasing function.

From this property, assuming $\beta \in (\underline{\beta}, 1)$, $\delta \in (0, \bar{\delta})$ and $\theta - \eta \in (\underline{\Theta}, s)$, we finally conclude that there exist $\underline{\alpha} \in (0, \bar{\alpha})$ and $\alpha_1 \in (\bar{\alpha}, \hat{\alpha})$ such that the second steady-state exists and is admissible, i.e. such that $\tilde{\gamma} \in (\gamma_{inf}, \gamma_{sup})$, if $\alpha \in (\underline{\alpha}, \hat{\alpha})$, and it satisfies:

- $\tilde{\gamma} \in (\gamma^*, \gamma_{sup})$ if $\alpha \in (\underline{\alpha}, \bar{\alpha})$,
- $\tilde{\gamma} \in (1, \gamma^*)$ if $\alpha \in (\bar{\alpha}, \alpha_1)$,
- $\tilde{\gamma} \in (\gamma_{inf}, 1)$ if $\alpha \in (\alpha_1, \hat{\alpha})$.

Using equation (34), the bounds $\underline{\alpha} \in (0, \bar{\alpha})$ and $\alpha_1 \in (\bar{\alpha}, \hat{\alpha})$ are respectively solutions of $h(\gamma_{sup}, \alpha) = 0$ and $h(1, \alpha) = 0$, and are equal to

$$\underline{\alpha} = \frac{\tau \left[\left(\frac{\beta^{\frac{\theta-\eta}{\theta-\eta-1}} - \beta(1-\delta)}{\gamma^{*\theta-\eta} - \beta(1-\delta)} \right)^{\frac{s}{1-s}} - 1 \right]}{\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \left(\frac{\beta^{\frac{\theta-\eta}{\theta-\eta-1}} - \beta(1-\delta)}{\gamma^{*\theta-\eta} - \beta(1-\delta)} \right)^{\frac{s}{1-s}}}, \quad \alpha_1 = \frac{\tau \left[\left(\frac{1-\beta(1-\delta)}{\gamma^{*\theta-\eta} - \beta(1-\delta)} \right)^{\frac{s}{1-s}} - 1 \right]}{\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1 \right) \left(\frac{1-\beta(1-\delta)}{\gamma^{*\theta-\eta} - \beta(1-\delta)} \right)^{\frac{s}{1-s}} - \frac{1-\beta}{\beta}}$$

Note finally that $h_1'(\gamma^*, \alpha)$ and $h_1'(\tilde{\gamma}, \alpha)$ have opposite sign and that this sign changes as α crosses $\bar{\alpha}$. $\square$

8.6 Proof of Proposition 4

To study the local stability properties of the normalized steady-state γ^*, we need to linearize the dynamical system (13) around the steady-state (x^*, γ^*, α) with x^* as given by (14).

Lemma 8.2. *Under Assumption 1, let $\theta - \eta < \beta$, $A = A^*$, $\alpha \in (0, \alpha_{Max})$ and $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. The characteristic polynomial is given by $P(\lambda) \equiv [\lambda - (1-\phi)](\lambda^2 - T\lambda + D) = 0$, where:*

$$\begin{aligned} D &= \left(1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}\right) \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} + \frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*(1-s)} \equiv D(\alpha) \\ T &= 1 + \frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} \equiv T(\alpha) \end{aligned} \tag{39}$$

with

$$\begin{aligned} d^*(\alpha) &= \left(\frac{\gamma^{*\theta-\eta} - \beta(1-\delta)}{\beta s(1-\tau)}\right) \left[1 - \tau + \alpha\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right)\right] + 1 - \delta - \gamma^* \geq 0 \\ B_1(\alpha) &= \eta - \frac{\theta}{d^*(\alpha)} \left(\frac{\gamma^{*\theta-\eta} - \beta(1-\delta)}{\beta s(1-\tau)}\right) \left[1 - s - \tau + \alpha\gamma^* \left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right)\right] \\ B_2(\alpha) &= s\tau + \alpha\gamma^* \left(1 - s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right) + \alpha(1-s)\frac{\gamma^{*\theta-\eta} - \beta(1-\delta)}{\beta} > 0 \\ B_3 &= (1-s)\frac{\gamma^{*\theta-\eta} - \beta(1-\delta)}{\gamma^{*\theta-\eta}} > 0 \end{aligned}$$

Proof. Linearizing the dynamic system (13) around the normalized steady-state, we obtain:

$$\begin{aligned} \frac{\theta\gamma^*}{d^*(\alpha)} \frac{\Delta\gamma_{t+1}}{\gamma^*} + [B_1(\alpha) + B_3] \frac{\Delta x_{t+1}}{x^*} &= \left[\theta - \eta + \frac{\theta\gamma^*}{d^*(\alpha)}\right] \frac{\Delta\gamma_t}{\gamma^*} + B_1(\alpha)\frac{\Delta x_t}{x^*} \\ \alpha\gamma^*(1-s)\frac{\Delta x_{t+1}}{x^*} + \alpha\gamma^* \frac{\Delta\alpha_{t+1}}{\alpha} &= -\alpha\gamma^* \frac{\Delta\gamma_t}{\gamma^*} + B_2(\alpha)\frac{\Delta x_t}{x^*} + \alpha\frac{\gamma^{*\theta-\eta}}{\beta}\frac{\Delta\alpha_t}{\alpha} \\ \frac{\Delta\alpha_{t+1}}{\alpha} &= (1-\phi)\frac{\Delta\alpha_t}{\alpha} \end{aligned}$$

with

$$\begin{array}{rcl} d^*(\alpha) & = & \left(\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right)\left[1-\tau+\alpha\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right)\right]+1-\delta-\gamma^*\geq 0 \\ B_1(\alpha) & = & \eta-\frac{\theta}{d^*(\alpha)}\left(\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right)\left[1-s-\tau+\alpha\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right)\right] \\ B_2(\alpha) & = & s\tau+\alpha\gamma^*\left(1-s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right)+\alpha(1-s)\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta} \\ B_3 & = & (1-s)\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\gamma^{*\theta-\eta}}>0 \end{array} \tag{40}$$

Note that $B_2(\alpha)$ can also be written as

$$B_2(\alpha)=s\tau+\alpha\left[\gamma^*-(1-s)(1-\delta)+(1-2s)\frac{\gamma^{*\theta-\eta}}{\beta}\right]>0$$

We then derive the following linear system

$$\begin{pmatrix} \frac{\Delta\gamma_{t+1}}{\gamma^*} \\ \frac{\Delta x_{t+1}}{x^*} \\ \frac{\Delta\alpha_{t+1}}{\alpha} \end{pmatrix}=J\begin{pmatrix} \frac{\Delta\gamma_t}{\gamma^*} \\ \frac{\Delta x_t}{x^*} \\ \frac{\Delta\alpha_t}{\alpha} \end{pmatrix}$$

with

$$J=\begin{pmatrix} 1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}+\frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)} & \frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*}-\frac{B_2(\alpha)[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\alpha\gamma^{*2}(1-s)} & -\frac{[B_1(\alpha)+B_3]d^*(\alpha)\left[\frac{\gamma^{*\theta-\eta-1}}{\beta}-(1-\phi)\right]}{\theta\gamma^*(1-s)} \\ -\frac{1}{1-s} & \frac{B_2(\alpha)}{\alpha\gamma^*(1-s)} & \frac{\left[\frac{\gamma^{*\theta-\eta-1}}{\beta}-(1-\phi)\right]}{1-s} \\ 0 & 0 & 1-\phi \end{pmatrix}$$

The result follows after straightforward simplifications. □

We then get one obvious characteristic root equal to $1-\phi$ associated to the pre-determined variable α_t which is less than 1. The two other characteristic roots associated to the two forward variables γ_t and x_t are thus solutions of the polynomial $\tilde{P}(\lambda)=\lambda^2-T\lambda+D=0$.

We need now to establish a second technical Lemma providing a property of the discriminant of the characteristic polynomial $\tilde{P}(\lambda)=0$ that applies for any $\alpha\in[0,\alpha_{Max})$.

Lemma 8.3. *Assume that $A=A^*$ and $\gamma^*\in(\underline{\gamma},\gamma_{sup})$. Then there exist $\bar{\Theta}\in(0,\beta)$, $\delta_1\in(0,1)$ and $\beta_1\in(0,1)$ such that when $\theta-\eta\in(0,\bar{\Theta})$, $\delta\in(0,\delta_1)$ and $\beta\in(\beta_1,1)$, both roots of the characteristic polynomial $\tilde{P}(\lambda)=0$ are real and positive for any $\alpha\in[0,\alpha_{Max})$.*

Proof. Let us first compute the discriminant of the characteristic polynomial $\tilde{P}(\lambda)=0$. Straightforward computations give

$$\begin{aligned}\Delta &= \left[1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}+\frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)}+\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\right]^2\\ &- 4\left[\left(1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}\right)\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}+\frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*(1-s)}\right]\end{aligned}$$

Assume first that $B_1(\alpha)\geq 0$. We then get $B_1(\alpha)d^*(\alpha)\alpha+\theta B_2(\alpha)>0$ and

$$\begin{aligned}\Delta &> \left[1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}+\frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)}+\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\right]^2\\ &- 4\left(1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}\right)\left(\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}+\frac{B_1(\alpha)d^*(\alpha)}{\theta\gamma^*(1-s)}\right)\\ &> \left[1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}+\frac{[B_3-B_1(\alpha)]d^*(\alpha)}{\theta\gamma^*(1-s)}-\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\right]^2+4\frac{B_3 d^*(\alpha)}{\theta\gamma^*(1-s)}\left[\frac{B_1(\alpha)d^*(\alpha)\alpha+\theta B_2(\alpha)}{\theta\alpha\gamma^*(1-s)}\right]\\ &> 0\end{aligned}$$

Assume now that $B_1(\alpha)<0$. We then get

$$\begin{aligned}\Delta &> \left[1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}+\frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)}+\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\right]^2-4\left(1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}\right)\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\\ &> \left[1+\frac{(\theta-\eta)d^*(\alpha)}{\theta\gamma^*}+\frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)}-\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\right]^2+4\frac{B_2(\alpha)}{\alpha\gamma^*(1-s)}\frac{[B_1(\alpha)+B_3]d^*(\alpha)}{\theta\gamma^*(1-s)}\end{aligned}$$

with

$$B_1(\alpha)+B_3=\frac{\theta\left[\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta(1-\tau)}+1-\delta-\gamma^*\right]+d^*(\alpha)\left[(1-s)\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\gamma^{*\theta-\eta}}-(\theta-\eta)\right]}{d^*(\alpha)} \tag{41}$$

Let us consider the term between brackets that is multiplied by θ. Obviously this term is positive if and only if

$$\gamma^{*\theta-\eta}\left[1-\gamma^{*\theta-\eta-1}\beta(1-\tau)\right]-\beta(1-\delta)\tau>0$$

Since $\gamma<\gamma_{sup}$ we get

$$1-\gamma^{*\theta-\eta-1}\beta(1-\tau)>1-\tau>0$$

Moreover, since $\gamma>\gamma_{inf}$ we get

$$\gamma^{*\theta-\eta}\left[1-\gamma^{*\theta-\eta-1}\beta(1-\tau)\right]-\beta(1-\delta)\tau>\beta(1-\delta)(1-\tau)\left[1-\gamma^{*\theta-\eta-1}\beta\right]>0$$

It follows therefore that

$$\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta(1-\tau)}+1-\delta-\gamma^*>0 \tag{42}$$

for any $\gamma\in(\gamma_{inf},\gamma_{sup})$.

Consider now equation (41) when $\theta-\eta=0$. We get

$$B_1(\alpha)+B_3=\frac{\theta\left[\frac{1-\beta(1-\delta)}{\beta(1-\tau)}+1-\delta-\gamma^*\right]+d^*(\alpha)(1-s)[1-\beta(1-\delta)]}{d^*(\alpha)}$$

with obviously

$$\frac{1-\beta(1-\delta)}{\beta(1-\tau)} + 1 - \delta - \gamma^* > 0$$

Recalling that $\underline{\gamma} > \gamma_{inf}$, it follows that $B_1(\alpha) + B_3 > 0$ for any $\alpha \in [0, \alpha_{Max})$ as $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$. Therefore, for any given $\gamma^* \in (\underline{\gamma}, \gamma_{sup})$, there exists $\bar{\Theta} \in (0, \beta]$, such that when $\theta - \eta \in (0, \bar{\Theta})$, $B_1(\alpha) + B_3 > 0$ and thus $\Delta > 0$ for any $\alpha \in [0, \alpha_{Max})$.

Let us focus now on the sign of $D(\alpha)$ and $T(\alpha)$ as given by (39). Since $B_1(\alpha) + B_3 > 0$ and $B_2(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max})$, we immediately get $T(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max})$. To derive the sign of $D(\alpha)$, we need to study the sign of $B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha)$. Obvious computations give

$$\begin{aligned} B_1(\alpha)d^*(\alpha)\alpha + \theta B_2(\alpha) &= -\alpha^2(\theta-\eta)\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta} - 1\right)\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)} \\ &+ \alpha\Bigg\{\eta\left[\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta s} - 1\right) - \frac{(1-\delta)(1-s)}{s}\right] \\ &+ \frac{\theta}{1-\tau}\Bigg[\gamma^*\left(1 - s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right) - \frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s}(1-s)^2 \\ &- \tau\left[\gamma^*\left(1 - s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right) - \frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s}[1-s(1-s)]\right]\Bigg]\Bigg\} \\ &+ \theta s\tau \equiv \phi(\alpha) \end{aligned}$$

The polynomial $\phi(\alpha)$ is concave with $\phi(0) = \theta s\tau > 0$ and $\lim_{\alpha\to\infty}\phi(\alpha) = -\infty$, so that if $\phi(\alpha_{Max}) > 0$, then $\phi(\alpha) > 0$ for any $\alpha \in [0, \alpha_{Max})$. When $\alpha = \alpha_{Max}$, we derive from Lemma 8.2:

$$\begin{aligned} d^*(\alpha_{Max}) &= \left(\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right) + 1 - \delta - \gamma^* \geq 0 \\ B_1(\alpha_{Max}) &= \eta - \frac{\theta(1-s)}{d^*(\alpha_{Max})}\left(\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right) \\ B_2(\alpha_{Max}) &= (1-s)\alpha_{Max}\left[\gamma^* + \frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta}\right] \end{aligned} \tag{43}$$

so that

$$\begin{aligned} \phi(\alpha_{Max}) &= \alpha_{Max}\Bigg\{\eta\left[\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)} + 1 - \delta - \gamma^*\right] \\ &+ \frac{\theta(1-s)}{s(1-\tau)}\left[\gamma^* s(1-\tau) - \frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta} + (1-\delta)[1-s(1-\tau)]\right]\Bigg\} \\ &> \alpha_{Max}\Bigg\{\eta\left[\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)} + 1 - \delta - \gamma^*\right] \\ &+ \frac{\theta(1-s)}{s(1-\tau)}\left[s(1-\tau) - \frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta} + (1-\delta)[1-s(1-\tau)]\right]\Bigg\} \end{aligned}$$

Note first from (42) that

$$\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}+1-\delta-\gamma^*>\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta(1-\tau)}+1-\delta-\gamma^*>0$$

Second note that when $\delta=0$, the inequality characterizing $\phi(\alpha_{Max})$ becomes

$$\phi(\alpha_{Max})>\alpha_{Max}\left\{\eta\left[\frac{\gamma^{*\theta-\eta}-\beta}{\beta s(1-\tau)}+1-\gamma^*\right]+\frac{\theta(1-s)}{s(1-\tau)}\left[1-\frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta}\right]\right\}$$

with

$$1-\frac{\gamma^{*\theta-\eta}[1-s(1-\tau)]}{\beta}>0\quad\Leftrightarrow\quad\gamma^*<\left(\frac{\beta}{1-s(1-\tau)}\right)^{\frac{1}{\theta-\eta}}$$

Note then that

$$\left(\frac{\beta}{1-s(1-\tau)}\right)^{\frac{1}{\theta-\eta}}>\gamma_{sup}\quad\Leftrightarrow\quad 1-s(1-\tau)<\beta^{\frac{1}{1-(\theta-\eta)}}$$

which is satisfied when $\beta=1$. Therefore, there exists $\delta_1\in(0,1)$ and $\beta_1\in(0,1)$ such that if $\delta\in(0,\delta_1)$, $\beta\in(\beta_1,1)$ and $\theta-\eta\in(0,\bar{\Theta})$, then $\phi(\alpha_{Max})>0$ and thus $B_1(\alpha)d^*(\alpha)\alpha+\theta B_2(\alpha)>0$ for any $\alpha\in[0,\alpha_{Max})$. Under these conditions, and since $d^*(\alpha),B_2(\alpha)\geq 0$, it follows that $D(\alpha)$ is also positive for any $\alpha\in[0,\alpha_{Max})$.

□

We may now study the local stability properties of $\gamma^*\in(\underline{\gamma},\gamma_{sup})$ with $\alpha\in(0,\alpha_{Max})$. Consider the characteristic polynomial $\tilde{P}(\lambda)=0$. Straightforward computations from Lemma 8.2 give

$$\tilde{P}(1)=-\frac{d^*(\alpha)}{\theta\gamma^*(1-s)}\left[(\theta-\eta)\left(1-s-\frac{B_2(\alpha)}{\alpha\gamma^*}\right)+B_3\right]$$

Considering (40), we get

$$\begin{aligned}(\theta-\eta)\left(1-s-\frac{B_2(\alpha)}{\alpha\gamma^*}\right)+B_3 \;=\; & (\theta-\eta)\Bigg[1-s-\frac{s\tau}{\alpha\gamma^*}-\left(1-s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right)\\ & -\frac{1-s}{\gamma^*}\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta}\Bigg]+(1-s)\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\gamma^{*\theta-\eta}}\end{aligned}\tag{44}$$

which is clearly a monotone increasing function of α. So if there exists a positive value of α for which this expression is equal to zero, that value must be unique. Solving equation (44) equal to zero gives after simplifications

$$\begin{aligned}\alpha \;&=\; \frac{\frac{(\theta-\eta)s\tau}{(1-s)\gamma}}{(\theta-\eta)\left[1-\frac{1}{1-s}\left(1-s\frac{\gamma^{*\theta-\eta-1}}{\beta}\right)-\frac{1}{\gamma^*}\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta}\right]+\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\gamma^{*\theta-\eta}}}\\ &=\; \frac{\frac{\tau(\theta-\eta)s}{1-s}\gamma^{*\theta-\eta-1}}{\left[1-(\theta-\eta)\frac{\gamma^{*\theta-\eta-1}}{\beta}\right]\left[\gamma^{*\theta-\eta}-\beta(1-\delta)\right]+\gamma^{*\theta-\eta}\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right)\frac{(\theta-\eta)s}{1-s}}=\bar{\alpha}\end{aligned}$$

with $\bar{\alpha}$ as given by Lemma 8.1 (see (36)). Therefore, since $d^*(\alpha)>0$, we conclude that $P(1)\gtreqless 0$ if and only if $\alpha\lesseqgtr\bar{\alpha}$. We also get $\lim_{\lambda\to\pm\infty}P(\lambda)=+\infty$.

Consider finally the expression of T as given by (39) in Lemma 8.2. Using the expressions given in (40), we compute

$$T'(\alpha) = \frac{d^{*'}(\alpha)}{\theta\gamma^*(1-s)}\left[\frac{\gamma^{*\theta-\eta}[1-s-s(\theta-\eta)]-\beta(1-\delta)(1-s)}{\gamma^{*\theta-\eta}}\right] - \frac{s\tau}{\alpha^2\gamma^*(1-s)}$$

with

$$d^{*'}(\alpha) = \left(\frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\beta s(1-\tau)}\right)\gamma^*\left(\frac{\gamma^{*\theta-\eta-1}}{\beta}-1\right) > 0$$

Since $\theta-\eta < 1$, we get $1-s-s(\theta-\eta) > 1-2s$. Moreover, $s \in (0, 1/2)$ implies $1-s-s(\theta-\eta) > 0$. As $\gamma^* < \gamma_{sup}$, we derive

$$\gamma^{*\theta-\eta}\left[1-s-s(\theta-\eta)\right] - \beta(1-\delta)(1-s)$$
$$< \beta^{\frac{\theta-\eta}{\theta-\eta-1}}\left[1-s-s(\theta-\eta)-\beta^{\frac{1}{1-(\theta-\eta)}}(1-\delta)(1-s)\right]$$

We then conclude that when $\beta = 1$ and $\delta = 0$ the right-hand-side of this equation is negative. Therefore there exist $\beta_2 \in (0,1)$ and $\delta_2 \in (0,1)$ such that if $\beta \in (\beta_2, 1)$ and $\delta \in (0, \delta_2)$, then $T'(\alpha) < 0$. It follows that the minimal value of $T(\alpha)$ is obtained when $\alpha = \alpha_{Max}$. From equation (32) in the proof of Proposition 3 and using (43) we then derive:

$$\begin{aligned} T(\alpha_{Max}) - 2 &= \frac{(\theta-\eta)d^*(\alpha_{Max})}{\theta\gamma^*} + \frac{[B_1(\alpha)+B_3]d^*(\alpha_{Max})}{\theta\gamma^*(1-s)} + \frac{B_2(\alpha_{Max})}{\alpha_{Max}\gamma^*(1-s)} - 1 \\ &= \frac{(\theta-\eta)d^*(\alpha_{Max})}{\theta\gamma^*} + \frac{[B_1(\alpha_{Max})+B_3]d^*(\alpha_{Max})}{\theta\gamma^*(1-s)} + \frac{\gamma^{*\theta-\eta}-\beta(1-\delta)}{\gamma^*\beta} \end{aligned}$$

We have shown in the proof of Lemma 8.3 that if $\theta-\eta \in (0, \bar{\Theta})$, then $B_1(\alpha) + B_3 > 0$. Therefore, $T(\alpha) > 2$ for any $\alpha \in [0, \alpha_{Max})$. Let $\bar{\delta} = \min\{\delta_1, \delta_2\}$ and $\underline{\beta} = \max\{\beta_1, \beta_2\}$. When $\delta \in (0, \bar{\delta})$, $\beta \in (\underline{\beta}, 1)$ and $\theta - \eta \in (0, \bar{\Theta})$, we conclude finally that for any $\alpha \in [0, \bar{\alpha})$, the normalized steady-state γ^* is locally determinate as both characteristic roots are larger than one, and for any $\alpha \in (\bar{\alpha}, \alpha_{Max})$, γ^* is locally indeterminate as one root is less than one.

The local stability properties of the second steady-state $\tilde{\gamma}$ are obviously derived considering Lemma 8.1. $\square$

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