

Growth-fragmentation processes and bifurcators*

Quan Shi[†]

Abstract

Markovian growth-fragmentation processes introduced by Bertoin model a system of growing and splitting cells in which the size of a typical cell evolves as a Markov process X without positive jumps. We find that two growth-fragmentations associated respectively with two processes X and Y (with different laws) may have the same distribution, if (X, Y) is a *bifurcator*, roughly speaking, which means that they coincide up to a bifurcation time and then evolve independently. Using this criterion, we deduce that the law of a self-similar growth-fragmentation is determined by its index of self-similarity and a cumulant function κ .

Keywords: growth-fragmentation; Lévy process; self-similarity.

AMS MSC 2010: 60G51; 60J80.

Submitted to EJP on March 28, 2016, final version accepted on January 10, 2017.

Supersedes arXiv:1603.08495.

1 Introduction

We consider the family of *Markovian growth-fragmentation processes* introduced by Bertoin [4]; see also [9, 10, 11, 12] for related works. This stochastic model describes the evolution of a particle system, in which each particle may grow or decay gradually and split randomly into smaller pieces, independently of the other particles. It is convenient to describe it in terms of a cell population. The size of a typical cell evolves as a Markov process $X = (X(t), t \geq 0)$ with values in $[0, \infty)$, with càdlàg path and only negative jumps. The process X also encodes the relationship between cell size and cell replication: at each jump time $t \geq 0$ of X with $\Delta X(t) = X(t) - X(t-) < 0$, a “daughter” cell with initial size $-\Delta X(t)$ is born, and the “mother” is still alive after this cell replication. Each daughter follows the same dynamics as the mother and evolves independently of the other cells. Starting at time 0 from a single cell with size $x > 0$, we construct in this way a population of cells. For every time $t \geq 0$, denote by $\mathbf{X}(t)$ the multiset (that allows multiple instances of the elements) of the sizes of the cells alive at t . Then the process $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ is called a *(Markovian) growth-fragmentation process starting from x associated with the cell process X* . We stress that \mathbf{X} maintains only the sizes, but not

*Support: Swiss National Science Foundation 200021_144325/1.

[†]Universität Zürich, Switzerland. E-mail: quanshi.math@gmail.com

the genealogy of the population. More formal constructions will be given in Section 2.3 and Section 3.1.

By construction, the law of \mathbf{X} is determined by the law of X . However, growth-fragmentations driven by cell processes with different laws may have the same distribution. A first instance of such processes appears in Pitman and Winkel [17], with X being a so-called *fragmenter* (that is, the exponential of the negative of a pure-jump subordinator). The main purpose of this work is therefore to provide a sufficient condition for growth-fragmentations driven by different cell processes to have the same distribution. Our main result can be informally described as follows:

If there exists a coupling of (the distributions of) two cell processes X and Y which is a bifurcator, in the sense that they almost surely coincide for a strictly positive time and evolve independently afterwards, then under some mild technical conditions, the growth-fragmentations driven respectively by X and Y have the same finite-dimensional distributions.

This will be stated rigorously in Theorem 3.9. The idea of bifurcator also goes back to [17], which provides an explicit construction of bifurcators of fragmenters, as well as a characterization of the laws of all bifurcators of fragmenters.

Therefore, to give a sufficient condition for two growth-fragmentations to have the same distribution, it suffices to understand when two cell processes can be coupled to form a bifurcator (in other words, when there exists a bifurcator whose two marginal distributions are the respective laws of these two cell processes). We do not have a complete answer to this question in general, however, we investigate a study of bifurcators for positive self-similar Markov processes, which further allows us to characterize the laws of growth-fragmentations driven by self-similar processes, so-called *self-similar growth-fragmentation processes*.

Self-similar growth-fragmentations have been previously studied in [4] and have interesting applications: this model is connected with certain growth-fragmentation equations, see [8]; besides, a distinguished case of self-similar growth-fragmentation appears as the re-scaled limit of the lengths of the cycles obtained by slicing random Boltzmann triangulations with a simple boundary at heights, see [6].

In order to state our results, let us recall some basic facts about Lévy processes, which are closely related to self-similar Markov processes; see e.g. [1, 14]. Let ξ be a Lévy process with no positive jumps, which is often referred to as a *spectrally negative Lévy process (SNLP)*. The SNLP ξ is possibly killed at some independent exponential time. The distribution of ξ is characterized by its Laplace exponent $\Phi : [0, \infty) \rightarrow \mathbb{R}$:

$$\mathbb{E} \left[e^{q\xi(t)} \right] = e^{\Phi(q)t}, \quad \text{for all } q, t \geq 0.$$

It is well-known that the convex function Φ can be expressed by the Lévy-Khintchine formula

$$\Phi(q) = -k + \frac{1}{2}\sigma^2 q^2 + cq + \int_{(-\infty, 0)} (e^{qz} - 1 + q(1 - e^z)) \Lambda(dz), \quad q \geq 0, \quad (1.1)$$

where $k \geq 0$ is the killing rate, $\sigma \geq 0$, $c \in \mathbb{R}$ and the Lévy measure Λ on $(-\infty, 0)$ satisfies

$$\int_{(-\infty, 0)} (|z|^2 \wedge 1) \Lambda(dz) < \infty. \quad (1.2)$$

To avoid uninteresting cases, we will always assume that Λ is non-trivial. Then we say ξ is a SNLP with characteristics (σ, c, Λ, k) . We also introduce $\kappa : [0, \infty) \rightarrow (-\infty, \infty]$ which plays an important role in this work:

$$\kappa(q) := \Phi(q) + \int_{(-\infty, 0)} (1 - e^z)^q \Lambda(dz), \quad q \geq 0. \quad (1.3)$$

So $\kappa \geq \Phi$. Note that κ is convex and $\kappa(q) < \infty$ for all $q \geq 2$ because of (1.2). We stress that κ does not characterize the law of ξ , see Lemma 2.1.

Let $X^{(0)} := \exp(\xi)$, and we write by convention $X^{(0)}(t) = \partial$ if ξ is killed before t , where ∂ denotes a cemetery point. Then the process $X^{(0)}$ is called a *homogeneous cell process*, which is a special case of self-similar process. Let $\tilde{X}^{(0)} := \exp(\tilde{\xi})$, where $\tilde{\xi}$ is another SNLP with $\tilde{\kappa}$ defined as in (1.3), and write $\mathbf{X}^{(0)}$ and $\tilde{\mathbf{X}}^{(0)}$ for two growth-fragmentations associated with $X^{(0)}$ and $\tilde{X}^{(0)}$ respectively (with the same initial size of ancestor $x > 0$); see Section 2.3 for their formal construction. The following result partially encompasses Proposition 5 and Corollary 25 in [17].

Theorem 1.1 (Homogeneous). *The following statements are equivalent:*

1. $\kappa = \tilde{\kappa}$;
2. $X^{(0)}$ and $\tilde{X}^{(0)}$ can be coupled to form a bifurcator (see Definition 2.6);
3. $\mathbf{X}^{(0)}$ and $\tilde{\mathbf{X}}^{(0)}$ have the same finite-dimensional distributions.

We hence say that $\mathbf{X}^{(0)}$ is a *homogeneous growth-fragmentation process with characteristic κ* . The function κ serves as *cumulant* for $\mathbf{X}^{(0)}$, in the sense that

$$\mathbf{E} \left[\sum_{x \in \mathbf{X}^{(0)}(t)} x^q \right] = \exp(\kappa(q)t) \quad \text{for all } q \geq 2 \text{ and } t \geq 0,$$

which is proved in Proposition 2.15.

In general, a self-similar cell process with index $\alpha \in \mathbb{R}$ is associated with a Lévy process by Lamperti's representation [15] as follows. Let us define a time-change by

$$\tau_t^{(\alpha)} := \inf \left\{ r \geq 0 : \int_0^r \exp(-\alpha\xi(s)) ds \geq t \right\}, \quad t \geq 0,$$

with the convention that $\exp(-\alpha\xi(s)) = 0$ if ξ is killed before s . For every $x > 0$, let us denote by P_x the law of the process

$$X^{(\alpha)}(t) := x \exp(\xi(\tau_{tx^\alpha}^{(\alpha)})), \quad t \geq 0, \tag{1.4}$$

with the convention that $X^{(\alpha)}(t) = \partial$ for every $t \geq x^{-\alpha} \int_0^\infty \exp(-\alpha\xi(s)) ds$. We know from [15] that for every $c > 0$,

the law of $(cX^{(\alpha)}(c^\alpha t), t \geq 0)$ under P_x is P_{cx} ,

so we call $X^{(\alpha)}$ a *self-similar cell process with index α* ¹. If $\alpha = 0$, then we simply have $X^{(0)} = x \exp(\xi)$ under P_x , which is indeed a homogeneous cell process.

For $\alpha \neq 0$, we further need to assume that

$$\text{there exists } q > 0 \text{ with } \kappa(q) < 0. \tag{1.5}$$

Let $\mathbf{X}^{(\alpha)}$ be a growth-fragmentation associated with $X^{(\alpha)}$ starting from an ancestor cell with initial size $x > 0$. The assumption (1.5) ensures the *non-explosion* of the growth-fragmentation $\mathbf{X}^{(\alpha)}$, which means that for every $t \geq 0$ and $a > 0$, the multiset $\mathbf{X}^{(\alpha)}(t)$ has only finitely many elements (counted with their multiplicity) in (a, ∞) ; see [4]. It is also known from a recent work [7] that if $\kappa(q) > 0$ for all $q \geq 0$ and $\alpha \neq 0$, then the growth-fragmentation $\mathbf{X}^{(\alpha)}$ explodes in finite time. Under (1.5), it is known

¹The way we define the index of self-similarity α is consistent with the theory of self-similar fragmentations. However, we stress that in the theory of self-similar processes, it is rather $-\alpha$ which is called the index of self-similarity.

from Theorem 2 in [4] that $\mathbf{X}^{(\alpha)}$ keeps the self-similarity: recall that $\mathbf{X}^{(\alpha)}$ starts from an ancestor with initial size x , then for every $c > 0$, the law of $(c\mathbf{X}^{(\alpha)}(c^\alpha t), t \geq 0)$ is the same as a growth-fragmentation associated with $X^{(\alpha)}$ starting from cx . So we call $\mathbf{X}^{(\alpha)}$ a *self-similar growth-fragmentation with index α* .

Let us now present our main result for the self-similar case. Denote $\tilde{X}^{(\tilde{\alpha})}$ for the self-similar cell process of index $\tilde{\alpha} \in \mathbb{R}$ associated with $\tilde{\xi}$ by Lamperti's representation (1.4) and let $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ be a growth-fragmentation driven by $\tilde{X}^{(\tilde{\alpha})}$. Suppose that the respective ancestors of $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ and $\mathbf{X}^{(\alpha)}$ have the same initial size $x > 0$.

Theorem 1.2 (Self-similar). *Suppose that (1.5) holds for both κ and $\tilde{\kappa}$, then the following statements are equivalent:*

1. $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$;
2. $X^{(\alpha)}$ and $\tilde{X}^{(\tilde{\alpha})}$ can be coupled to form a bifurcator (see Definition 3.7);
3. $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ have the same finite-dimensional distributions.

Therefore, we say that $\mathbf{X}^{(\alpha)}$ is a self-similar growth-fragmentation with characteristics (κ, α) . Note that it follows immediately from the self-similarity that if $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ and $\mathbf{X}^{(\alpha)}$ have the same finite-dimensional distributions, then $\alpha = \tilde{\alpha}$.

Let us outline our proofs. For the homogeneous case, Theorem 1.1, we provide a direct proof of the equivalence (i) \Leftrightarrow (iii), by drawing a connection between homogeneous growth-fragmentations and *branching Lévy processes* introduced in [3]. However, this proof cannot be easily extended to the self-similar case. Nevertheless, we can deduce the implication (iii) \Rightarrow (i) in Theorem 1.2 from the self-similarity mentioned above and a study of martingales in self-similar growth-fragmentations in [5]. Further, we can construct a bifurcator of $X^{(\alpha)}$ and $\tilde{X}^{(\tilde{\alpha})}$ when $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$ by extending the approach of Pitman and Winkel [17] and using Lamperti's transformation, which gives (i) \Rightarrow (ii). This motivates us to establish the general sufficient condition, Theorem 3.9, which is informally stated above. We hence get the implication (ii) \Rightarrow (iii) and complete the proof.

Besides the class of self-similar processes associated with Lévy processes by Lamperti's transformations, the stationary processes driven by Lévy processes, exponential Ornstein-Uhlenbeck type processes (see e.g. [19]), are also natural examples for cell processes. The techniques developed in this paper also open the way to study the growth-fragmentations associated with exponential Ornstein-Uhlenbeck type processes, which is discussed in [20].

The rest of this article is organized as follows. We start with working on homogeneous growth-fragmentations in Section 2. We first study the bifurcators of homogeneous cell processes, and then characterize the laws of homogeneous growth-fragmentations by using their connections with branching Lévy processes. In Section 3, we first provide a non-explosion condition of general Markovian growth-fragmentations, then we introduce the notion of bifurcators for general cell processes and establish our main result, Theorem 3.9, a general sufficient condition for two growth-fragmentations to have the same law. Applying Theorem 3.9, we complete the proofs of Theorem 1.1 and Theorem 1.2.

2 The homogeneous case

Throughout the rest of this work, we denote by ξ and γ two SNLPs with respective characteristics (σ, c, Λ, k) and $(\sigma_\gamma, c_\gamma, \Lambda_\gamma, k_\gamma)$, and define κ and κ_γ respectively for ξ and γ as in (1.3). We also define

$$\bar{z} := \log(1 - e^z), \quad z \in (-\infty, 0),$$

so that $e^z + e^{\bar{z}} = 1$. Note that $z \mapsto \bar{z}$ is an involution, i.e. $\bar{\bar{z}} = z$. For every Lévy measure Λ , we write $\bar{\Lambda}$ for the push-forward measure of Λ via the map $z \mapsto \bar{z}$. We remark that it follows from (1.2) that

$$\Lambda((-\infty, -\log 2]) < \infty \quad \text{and} \quad \bar{\Lambda}([-\log 2, 0)) < \infty.$$

This section is concerned with growth-fragmentations driven by homogeneous cell processes, and our investigation consists of two parts. We first depict the structure of the family of SNLPs that have the same κ in Section 2.1, specifically, we show that they can be derived from each other by the *switching transformations*, which are introduced by Pitman and Winkel [17] to study the bifurcators of fragmenters. We next show that the law of a homogeneous growth-fragmentation associated with $\exp(\xi)$ is characterized by κ . In this direction, we recall the construction of branching Lévy processes introduced by Bertoin [3] in Section 2.2 and then build a connection between homogeneous growth-fragmentations and branching Lévy processes in Section 2.3. These two results motivate us to extend the conception of bifurcator to general Markov processes and to study the relations between bifurcators and Markovian growth-fragmentations, which will become the object of investigation in Section 3.

We will often appeal to the following relation between the SNLPs that have the same κ in terms of their characteristics.

Lemma 2.1. *There is $\kappa = \kappa_\gamma$, if and only if*

$$\Lambda + \bar{\Lambda} = \Lambda_\gamma + \bar{\Lambda}_\gamma, \quad \sigma = \sigma_\gamma, \quad k = k_\gamma,$$

and
$$c + \int_{(-\infty, -\log 2)} (1 - 2e^z)\Lambda(dz) = c_\gamma + \int_{(-\infty, -\log 2)} (1 - 2e^z)\Lambda_\gamma(dz).$$

Proof. It is easy to check the *if* part by straightforward calculation. We now prove the *only if* part. If $\kappa = \kappa_\gamma$, then the third order derivatives of $\kappa(q)$ and $\kappa_\gamma(q)$ are equal for every $q > 2$, i.e.

$$\int_{(-\infty, 0)} (e^{q\bar{z}}z^3 + e^{qz}\bar{z}^3) \Lambda(dz) = \int_{(-\infty, 0)} (e^{q\bar{z}}z^3 + e^{qz}\bar{z}^3) \Lambda_\gamma(dz).$$

Therefore, for every $q > 2$ there is

$$\int_{(-\infty, 0)} e^{qz}z^3 (\Lambda(dz) + \bar{\Lambda}(dz)) = \int_{(-\infty, 0)} e^{qz}z^3 (\Lambda_\gamma(dz) + \bar{\Lambda}_\gamma(dz)),$$

which implies that $\Lambda + \bar{\Lambda} = \Lambda_\gamma + \bar{\Lambda}_\gamma$. Iterating this argument over the lower order derivatives of κ and κ_γ , we obtain the other identities in turn. □

2.1 Switching transformations and bifurcators

In order to give a construction of bifurcators of homogeneous cell processes, we now generalize the switching transformations between fragmenters in [17] to SNLPs. Let ξ be a SNLP with characteristics (σ, c, Λ, k) and $p : (-\infty, 0) \rightarrow [0, 1]$ be a measurable function, which will serve as switching probability, such that

$$\int_{(-\infty, 0)} p(z) \Lambda(dz) < \infty. \tag{2.1}$$

We shall derive another SNLP $\xi^{[p]}$ from ξ by switching according to p in the following way. At each jump time $t > 0$ of ξ with $z := \Delta\xi(t) = \xi(t) - \xi(t-) < 0$, we *mark* this jump

time with success probability $p(z)$ (so with failure probability $1 - p(z)$ we do not mark it), independently of the other jumps. We thus define a point process by the marked jumps:

$$\Delta_1(t) := \begin{cases} \Delta\xi(t) & \text{if } t \text{ is a marked time,} \\ 0 & \text{otherwise.} \end{cases}$$

Implicitly, the killing time ζ is never marked. We stress that the number of marked jump times is locally finite if and only if (2.1) holds. Indeed, observing from the property of Lévy processes (see e.g. [1]) that $(\Delta\xi(t), t \geq 0)$ is a Poisson point process with characteristic measure Λ , we have that Δ_1 is a Poisson point process with characteristic measure $\Lambda_1(dz) := p(z)\Lambda(dz)$. Next, we define a point process $\bar{\Delta}_1$ associated with Δ_1 by

$$\bar{\Delta}_1(t) := \begin{cases} \log(1 - e^{\Delta_1(t)}) & \text{if } \Delta_1(t) \neq 0, \\ 0 & \text{if } \Delta_1(t) = 0. \end{cases}$$

Then $\bar{\Delta}_1$ is a Poisson point process with characteristic measure $\bar{\Lambda}_1(dz) := p(\bar{z})\bar{\Lambda}(dz)$, where $\bar{\Lambda}$ is the image of Λ by the map $z \mapsto \bar{z}$. Therefore, as (2.1) holds, the processes

$$\xi_1(t) := \sum_{s \leq t} \Delta_1(s) \quad \text{and} \quad \bar{\xi}_1(t) := \sum_{s \leq t} \bar{\Delta}_1(s)$$

are compound Poisson processes with respective (finite) Lévy measures Λ_1 and $\bar{\Lambda}_1$. We finally define the switching transform of ξ according to p by the process

$$\xi^{[p]} := \xi - \xi_1 + \bar{\xi}_1.$$

Lemma 2.2. *Let ξ be a SNLP with characteristics (σ, c, Λ, k) and $p : (-\infty, 0) \rightarrow [0, 1]$ be a measurable function that satisfies (2.1). Then the switching transform $\xi^{[p]}$, derived from ξ according to p , is a SNLP with characteristics*

$$\begin{cases} \sigma^{[p]} & := \sigma, \\ \Lambda^{[p]}(dz) & := (1 - p(z))\Lambda(dz) + p(\bar{z})\bar{\Lambda}(dz), \\ c^{[p]} & := c + \int_{(-\infty, 0)} (1 - 2e^z)p(z)\Lambda(dz), \\ k^{[p]} & := k, \end{cases}$$

and Laplace exponent

$$\Phi^{[p]}(q) := \Phi(q) + \int_{(-\infty, 0)} ((1 - e^z)^q - e^{qz})p(z)\Lambda(dz), \quad q \geq 0.$$

Define $\kappa^{[p]}$ as in (1.3) for $\xi^{[p]}$, then $\kappa^{[p]} = \kappa$. Further,

$$\tau := \inf \left\{ t \geq 0 : \xi(t) \neq \xi^{[p]}(t) \right\}$$

has an exponential distribution with parameter $\int_{(-\infty, 0) \setminus \{-\log 2\}} p(z)\Lambda(dz) < \infty$. Moreover, if $\tau < \infty$ then τ is a jump time of both ξ and $\xi^{[p]}$ with

$$\exp(\xi(\tau)) + \exp(\xi^{[p]}(\tau)) = \exp(\xi(\tau-)).$$

Proof. We observe that ξ , ξ_1 and $\bar{\xi}_1$ are all Lévy processes with respect to the filtration $\sigma(\xi(s), \Delta_1(s), s \leq t)$. The Lévy processes $(\xi - \xi_1)$ and ξ_1 are independent since they never jump at the same time. For the same reason, the Lévy processes $(\xi - \xi_1)$ and $\bar{\xi}_1$ are also independent. Therefore, the Laplace exponent of $\xi^{[p]}$ is $\Phi - \Phi_1 + \bar{\Phi}_1$, where Φ_1

and $\bar{\Phi}_1$ are respective Laplace exponents of ξ_1 and $\bar{\xi}_1$. So we get the characteristics of $\xi^{[p]}$ and thus check that $\kappa^{[p]} = \kappa$ by straightforward calculation.

We next observe from the construction of $\xi^{[p]}$ that

$$\inf \left\{ t \geq 0 : \xi(t) \neq \xi^{[p]}(t) \right\} = \inf \{ t \geq 0 : \Delta_1(t) \neq 0 \text{ and } \Delta_1(t) \neq -\log 2 \},$$

which implies the second part of the statement. □

Remark 2.3. It follows from (1.2) that for every $a \geq 2$ the function $z \mapsto (1 - e^z)^a$ satisfies (2.1). However, the function $z \mapsto (1 - e^z)$, which would correspond to the size-biased pick between $\exp(\Delta\xi(t))$ and $(1 - \exp(\Delta\xi(t)))$ (see Section 2.2 in [17]) cannot satisfy (2.1) unless $\int_{(-\infty,0)} (|z| \wedge 1) \Lambda(dz) < \infty$.

Lemma 2.4. *If $\kappa_\gamma = \kappa$, then for every measurable function $p: (-\infty, 0) \rightarrow [0, 1]$ such that*

$$\int_{(-\infty,0)} p(z) \Lambda_\gamma(dz) < \infty \quad \text{and} \quad p(z) + p(\bar{z}) = 1 \text{ for } (\Lambda_\gamma + \bar{\Lambda}_\gamma)\text{-a.e. } z \in (-\infty, 0), \quad (2.2)$$

there is $\int_{(-\infty,0)} p(z) \Lambda(dz) < \infty$ and $\xi^{[p]} \stackrel{d}{=} \gamma^{[p]}$.

The function $z \mapsto \mathbb{1}_{\{z < -\log 2\}} + \frac{1}{2} \mathbb{1}_{\{z = -\log 2\}}$ gives an example that satisfies (2.2).

Proof of Lemma 2.4. As $\kappa_\gamma = \kappa$, it follows from Lemma 2.1 and (1.2) that $\Lambda - \Lambda_\gamma$ is a finite signed measure and hence we have

$$\int_{(-\infty,0)} p(z) \Lambda(dz) \leq \int_{(-\infty,0)} p(z) \Lambda_\gamma(dz) + \int_{(-\infty,0)} |\Lambda - \Lambda_\gamma|(dz) < \infty.$$

So the switching transforms $\xi^{[p]}$ and $\gamma^{[p]}$ are well-defined. As (2.2) holds, by combining Lemma 2.1 and Lemma 2.2, we get that the characteristics of $\xi^{[p]}$ are the same as those of $\gamma^{[p]}$. □

We next see that the SNLPs that have the same κ are related to each other via the switching transformations.

Proposition 2.5. *If $\kappa_\gamma = \kappa$, then $\gamma \stackrel{d}{=} \xi^{[p]}$, where p is the measurable function defined by the Radon-Nikodym derivative*

$$p := d\bar{\Lambda}_\gamma / d(\Lambda_\gamma + \bar{\Lambda}_\gamma).$$

Proof. Observe that

$$\int_{(-\infty,0)} p(z) \Lambda_\gamma(dz) \leq \int_{(-\infty, -\log 2)} \Lambda_\gamma(dz) + \int_{(-\log 2, 0)} \bar{\Lambda}_\gamma(dz) < \infty.$$

Since $p(\bar{z})$ is equal to the value of the Radon-Nikodym derivative $d\Lambda_\gamma / d(\Lambda_\gamma + \bar{\Lambda}_\gamma)(z)$ for $(\Lambda_\gamma + \bar{\Lambda}_\gamma)$ -a.e. $z \in (-\infty, 0)$, there is $p(z) + p(\bar{z}) = 1$ for $(\Lambda_\gamma + \bar{\Lambda}_\gamma)$ -a.e. $z \in (-\infty, 0)$. Then we have by Lemma 2.2 that the switching transform $\gamma^{[p]}$ is well-defined, and has the same characteristics as γ , i.e. $\gamma^{[p]} \stackrel{d}{=} \gamma$. As (2.2) holds, it follows from Lemma 2.4 that $\xi^{[p]}$ is also well-defined and $\xi^{[p]} \stackrel{d}{=} \gamma^{[p]}$. So we conclude that $\gamma \stackrel{d}{=} \xi^{[p]}$. □

We finally present a construction of a bifurcator of homogeneous cell processes, which has the following precise definition.

Definition 2.6. *A pair of homogeneous cell processes (X, Y) is a **bifurcator** if it satisfies the following properties:*

1. Let $\tau := \inf\{t \geq 0 : X(t) \neq Y(t)\}$. There is almost surely either $\tau = \infty$ or the identity

$$X(\tau) + Y(\tau) = X(\tau-) = Y(\tau-).$$

2. (Asymmetric Markov branching property) Conditionally given $\tau > t$, the pair $(X(r+t)/X(t), Y(r+t)/Y(t))_{r \geq 0}$ is a copy of (X, Y) ; conditionally given $\tau \leq t$, the two processes $(X(r+t)/X(t))_{r \geq 0}$ and $(Y(r+t)/Y(t))_{r \geq 0}$ are independent copies of X and Y respectively.

This definition generalizes bifurcators of fragmenters in [17]. We shall later extend this notion to general cell processes, see Definition 3.7.

Lemma 2.7. *If $\kappa = \kappa_\gamma$, then there exists a bifurcator of homogeneous cell processes (X, Y) , such that the marginal laws of X and Y are the laws of $\exp(\xi)$ and $\exp(\gamma)$ respectively.*

Proof. Since $\kappa = \kappa_\gamma$, we can build as in Proposition 2.5 the switching transform $\xi^{[p]}$ derived from ξ such that $\xi^{[p]} \stackrel{d}{=} \gamma$. We stress that ξ and $\xi^{[p]}$ are still coupled after the switching time $\tau := \inf\{t \geq 0 : \xi(t) \neq \xi^{[p]}(t)\}$. However, let us define a process Y by

$$Y(t) := \mathbb{1}_{\{t < \tau\}} \exp(\xi(t)) + \mathbb{1}_{\{t \geq \tau\}} \exp(\xi^{[p]}(\tau) + \gamma'(t - \tau)), \quad t \geq 0,$$

where γ' is a copy of γ , independent of $\xi^{[p]}$ and ξ . Then we easily check that $Y \stackrel{d}{=} \exp(\gamma)$ and the pair of homogeneous cell processes $(X := \exp(\xi), Y)$ satisfies Definition 2.6. \square

2.2 Binary branching Lévy processes

Let ξ_b be a SNLP with characteristics $(\sigma_b, c_b, \Lambda_b, k_b)$ and Π_b be a Lévy measure on $[-\log 2, 0)$ that satisfies

$$\int_{[-\log 2, 0)} (1 \wedge z^2) \Pi_b(dz) < \infty. \tag{2.3}$$

Informally speaking, a *binary branching Lévy process (BBLP)* introduced in [3] models the evolution of a particle system, in which each particle moves in \mathbb{R} according to the SNLP ξ_b , independently of the other particles, and at rate $\Pi_b(dz)$ each particle gives birth to two children scattered on \mathbb{R} , whose initial positions relative to the position of the parent at death are given by z and $\bar{z} = \log(1 - e^z)$. We further add a properly chosen positive drift for the entire system, which is an analogue of the compensation term in the Lévy-Khintchine formula (1.1), so that the particles in this system do not all shift to $-\infty$ instantaneously. Proposition 3 in [4] establishes a close connection between BBLPs and homogeneous growth-fragmentations. We will extend this connection in the next subsection. Before that, we recall some basic facts of BBLPs in this subsection.

Let us represent the formal construction of BBLPs in [3], starting with the case when the branching occurs with a finite intensity, i.e. $\Pi_b([-\log 2, 0)) < \infty$. Write $\mathcal{U}_2 := \bigcup_{n=0}^\infty \{\ell, r\}^n$ for the binary Ulam-Harris tree with $\{\ell, r\}^0 := \emptyset$ by convention, so for every $i \in \mathbb{N}$, an element in $\{\ell, r\}^i$ is a word $v = (n_1, n_2, \dots, n_i)$ composed of i letters of the alphabet $\{\ell, r\}$. We write $|v| := i$ for the generation of v and $(v\ell, vr)$ for its children, where $v\ell$ would be referred to as the *left* child and vr as the *right* child. For every $j \leq |v|$, we denote by $[v]_j := (n_1, n_2, \dots, n_j)$ the ancestor of v at the j -th generation.

Definition 2.8. *Let ξ_b be a SNLP with characteristics $(\sigma_b, c_b, \Lambda_b, k_b)$ and Π_b be a finite measure on $[-\log 2, 0)$. We consider three independent processes $(\lambda_v)_{v \in \mathcal{U}_2}$, $(L_v)_{v \in \mathcal{U}_2}$ and $(D_v)_{v \in \mathcal{U}_2}$ such that:*

- $(\lambda_v)_{v \in \mathcal{U}_2}$ is a family of i.i.d. exponential variables with parameter $\Pi_b([-\log 2, 0))$.

- $(L_v)_{v \in \mathcal{U}_2}$ is a family of independent SNLP distributed as

$$\xi_b(t) + \left(\int_{[-\log 2, 0)} (1 - e^z) \Pi_b(dz) \right) t, \quad t \geq 0.$$

- $(D_{v\ell}, D_{vr})_{v \in \mathcal{U}_2}$ is a family of i.i.d. random variables, such that $D_{v\ell}$ is distributed according to the conditional probability $\Pi_b(\cdot \mid [-\log 2, 0))$ and

$$D_{vr} = \overline{D_{v\ell}} = \log(1 - \exp(D_{v\ell})) \leq D_{v\ell}.$$

Define for every $v \in \mathcal{U}_2$ the birth time by $\beta_v := \sum_{j=0}^{|v|-1} \lambda_{[v]_j}$, and iteratively the positions of its children at birth by $(a_{vi} = a_v + L_v(\lambda_v) + D_{vi}, i \in \{\ell, r\})$, with $a_\emptyset = 0$. We agree that $L_v(s) = -\infty$ if L_v is killed before s . Then the positions of the particles alive at time $t \geq 0$ form a multiset of elements in \mathbb{R} :

$$\mathbf{Z}(t) := \{ \{ a_v + L_v(t - b_v) : v \in \mathcal{U}_2, \beta_v \leq t < \beta_v + \lambda_v \} \}.$$

The process $(\mathbf{Z}(t), t \geq 0)$ is a **binary branching Lévy process (BBLP) with characteristics** $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$.

Remark 2.9. A multiset \mathcal{I} could be equivalently viewed as the point measure $\sum_{i \in \mathcal{I}} \delta_i$, where δ stands for the Dirac mass. So we can identify \mathbf{Z} with a point process.

We next extend the construction to infinite branching intensity and suppose that

$$\Pi_b([-\log 2, 0)) = \infty.$$

For every $d \leq -\log 2$, let us set

$$\Pi_b^{\{d\}} := \mathbf{1}_{\{[-\log 2, d)\}} \Pi_b, \quad \Lambda_b^{\{d\}} := \Lambda_b + \mathbf{1}_{\{[d, 0)\}} \Pi_b. \tag{2.4}$$

We know from Lemma 3 in [3] that we can construct in the same probability space a family of processes

$$(\mathbf{Z}^d, \quad -\infty < d \leq -\log 2),$$

with each \mathbf{Z}^d being a BBLP with characteristics $(\sigma_b, c_b, \Lambda_b^{\{d\}}, k_b, \Pi_b^{\{d\}})$ in the sense of Definition 2.8 (we stress that (2.3) assures that $\Pi_b^{\{d\}}$ is a finite measure), such that for every $d \leq d' \leq -\log 2$ there is $(\mathbf{Z}^d)^{\{d'\}} = \mathbf{Z}^{d'}$, where $(\mathbf{Z}^d)^{\{d'\}}$ is the system derived from \mathbf{Z}^d by keeping at each branching event the child particle that is closer to the mother, and suppressing the other child particle (together with its offspring) whenever it is born at distance from its mother $\geq |d'|$.

Definition 2.10. In the notation above, suppose that Π_b is a Lévy measure on $[-\log 2, 0)$ that verifies (2.3). Then the limit process (by monotonicity in the sense of multiset inclusion)

$$\mathbf{Z}(t) := \lim_{d \rightarrow -\infty} \uparrow \mathbf{Z}^d(t), \quad t \geq 0$$

is a **BBLP with characteristics** $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$.

Remark 2.11. Our notation is slightly different from that of [3]. In this work we only consider the *binary* case, that is, in each branching event a particle splits into at most two fragments. However, *branching Lévy processes* constructed in [3] are more general: an offspring of more than two fragments in one dislocation is allowed. In the sense of Definition 2 in [3], a BBLP with characteristics $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$ is characterized by $(\sigma_b, c_b - k_b, \mu_b)$, where μ_b is a measure on the infinite vector space

$$\left\{ (r_1, r_2, r_3, \dots) : r_i \in [-\infty, 0] \text{ for } i \in \mathbb{N}, e^{r_1} + e^{r_2} + \dots \leq 1, \text{ and } r_1 \geq r_2 \geq \dots \right\},$$

and is given by the sum of the following three measures: the image of Λ_b by the map $z \mapsto (z, -\infty, -\infty, \dots)$, the image of Π_b by the map $z \mapsto (z, \bar{z}, -\infty, -\infty, \dots)$ and $k_b \delta_{(-\infty, -\infty, \dots)}$.

Let Φ_b be the Laplace exponent of the SNLP ξ_b with characteristics $(\sigma_b, c_b, \Lambda_b, k_b)$. Introduce $\kappa_b : [0, \infty) \rightarrow (-\infty, \infty]$ by

$$\kappa_b(q) := \Phi_b(q) + \int_{[-\log 2, 0)} (e^{qz} + (1 - e^z)^q - 1 + q(1 - e^z)) \Pi_b(dz), \quad q \geq 0,$$

then κ_b serves as cumulant for the BBLP \mathbf{Z} . Specifically, we know from Theorem 1 in [3] that for every $q \geq 2$, there is $\kappa_b(q) < \infty$ and

$$\mathbb{E} \left[\sum_{z \in \mathbf{Z}(t)} e^{qz} \right] = e^{\kappa_b(q)t} \quad \text{for all } t \geq 0. \tag{2.5}$$

We now check that if $\Lambda_b = 0$, then the cumulant determines the distribution of the BBLP in the following sense.

Lemma 2.12. *Let \mathbf{Z} and \mathbf{Z}' be two BBLPs with respective characteristics $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$ and $(\sigma'_b, c'_b, \Lambda'_b, k'_b, \Pi'_b)$. If $\Lambda_b = \Lambda'_b = 0$ and their cumulants $\kappa_b = \kappa'_b$, then \mathbf{Z} and \mathbf{Z}' have the same law.*

Proof. Since the third order derivatives of κ'_b and κ_b are equal for all $q > 2$, by a similar argument as in the proof of Lemma 2.1, we find that $\Pi'_b + \bar{\Pi}'_b = \Pi_b + \bar{\Pi}_b$. As Π'_b and Π_b are supported on $[-\log 2, 0]$, we hence find that $\Pi'_b = \Pi_b$. By iterating this argument over the lower order of derivatives, we conclude that \mathbf{Z} and \mathbf{Z}' have the same characteristics, thus the same law. \square

2.3 Homogeneous growth-fragmentations

For every $x > 0$, write P_x for the law of the homogeneous cell process $X := x \exp(\xi)$, where ξ is a SNLP with κ defined as in (1.3). If ξ is killed at a time ζ , then by convention we denote $X(t) = \partial$ for all $t \geq \zeta$, where ∂ is the cemetery state. Let \mathbf{X} be a homogeneous growth-fragmentation associated with X , which was informally described in the Introduction. By connecting to branching Lévy processes, we shall prove in this section that the law of \mathbf{X} is characterized by the cumulant function κ .

In that direction, let us present the rigorous construction of \mathbf{X} , which is only a slight modification of that in [4]. We start with listing the jumps of X in the following way. Fix $q > 2$ and $K > \Phi(q)$. Recalling that the jump process $\Delta\xi$ is a Poisson point process with characteristic measure Λ and using the compensation formula (see e.g. [1]), we get for every $x > 0$

$$\begin{aligned} E_x \left[\sum_{0 \leq s} |\Delta X(s)|^q e^{-Ks} \right] &= E_x \left[\sum_{0 \leq s} X(s-)^q (1 - e^{\Delta\xi(s)})^q e^{-Ks} \right] \\ &= E_x \left[\int_0^\infty e^{-Ks} X(s-)^q ds \int_{(-\infty, 0)} (1 - e^z)^q \Lambda(dz) \right], \end{aligned}$$

where E_x stands for mathematical expectation under P_x . Using the definition of Φ and κ , we deduce that

$$E_x \left[\sum_{0 \leq s} |\Delta X(s)|^q e^{-Ks} \right] = \frac{\kappa(q) - \Phi(q)}{K - \Phi(q)} x^q, \tag{2.6}$$

which implies that P_x -almost surely

$$\sum_{s \geq 0} |\Delta X(s)|^q e^{-Ks} < \infty.$$

We may therefore list the jump times of X in a sequence $(t_i)_{i \in \mathbb{N}}$, such that the value of $|\Delta X(t_i)|^q e^{-Kt_i}$ is decreasing with i . By convention, if X has a finite number of jumps, then the tail of this sequence is filled with ∞ with $\Delta X(\infty) = \partial$. In the sequel, **the i -th jump time of X** shall always refer to the i -th element t_i in this sequence.

Let us give some basic notations. Let $\mathcal{U} := \bigcup_{i=0}^{\infty} \mathbb{N}^i$ be the Ulam-Harris tree, by convention $\mathbb{N}^0 = \{\emptyset\}$. An element $u \in \mathcal{U}$ is a finite sequence of natural numbers $u = (n_1, \dots, n_{|u|})$ where $|u| \in \mathbb{N}$ stands for the generation of u . We write $u_- = (n_1, \dots, n_{|u|-1})$ for its mother and $uk = (n_1, \dots, n_{|u|}, k)$ for its k -th daughter with $k \in \mathbb{N}$. We also denote $[u]_i = (n_1, \dots, n_i)$ for every $i \leq |u|$ with $[u]_0 = \emptyset$ by convention.

We next construct the **cell system driven by X** , which is a family of homogeneous cell processes indexed by \mathcal{U}

$$\mathcal{X} := (\mathcal{X}_u, u \in \mathcal{U}),$$

where each \mathcal{X}_u depicts the evolution of the size of the cell indexed by u as time passes. Specifically, we fix an arbitrary $x > 0$, which is the initial size of the ancestor cell. Then we set the birth time of \emptyset at $b_\emptyset := 0$ and let the life career $\mathcal{X}_\emptyset = (\mathcal{X}_\emptyset(t), t \geq 0)$ be a process of law P_x . Given the life path of \mathcal{X}_\emptyset , then we generate the first generation. For $i \in \mathbb{N}$, say the i -th jump time of \mathcal{X}_\emptyset is t_i and $x_i := -\Delta \mathcal{X}_\emptyset(t_i)$, we then set $b_i = t_i$ and build a sequence of conditionally independent processes $(\mathcal{X}_i)_{i \in \mathbb{N}}$ with respective conditional distribution P_{x_i} . By convention, if $t_i = \infty$ (which means that \mathcal{X}_\emptyset has less than i jumps), then we agree that the cell i as well as all its progeny have degenerate life careers, i.e. for every $v \in \mathcal{U}$ we set $\mathcal{X}_{iv} \equiv \partial$ and $b_{iv} = \infty$. We continue in this way to construct higher generations recursively. More precisely, for every individual $u \in \mathcal{U}$, the laws of her daughters are determined by the trajectory of \mathcal{X}_u : given \mathcal{X}_u , say the i -th jump of \mathcal{X}_u is at time t with $y := -\Delta \mathcal{X}_u(t)$, then its i -th daughter ui is born at time $b_{ui} := t$ and ui 's size process $\mathcal{X}_{ui} = (\mathcal{X}_{ui}(r), r \geq 0)$ has conditional distribution P_y , independent of the size processes of the other individuals in the same generation.

Write \mathcal{P}_x for the law of this cell system \mathcal{X} (recall that $x > 0$ indicates the initial size of the Eve \emptyset , i.e. $\mathcal{X}_\emptyset(0) = x$). According to [13], the probability distribution \mathcal{P}_x indeed exists and is uniquely determined by the above description.

Finally, for every $t \geq 0$ let $\mathbf{X}(t)$ be the multiset whose elements are sizes of the cells alive at time t , i.e.

$$\mathbf{X}(t) := \{\{\mathcal{X}_u(t - b_u) : u \in \mathcal{U}, b_u \leq t < b_u + \zeta_u\}\},$$

where $\zeta_u := \inf\{t \geq 0 : \mathcal{X}_u(t) = \partial\}$ denotes the life length of u . Then we refer to $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$ as a **growth-fragmentation process driven by X** and write \mathbf{P}_x for the law of \mathbf{X} under \mathcal{P}_x .

Remark 2.13. The construction of the cell system \mathcal{X} is only a slight modification of that of a cell system in [4], and that of a general branching process (also called Crump-Mode-Jagers process) in [13]. The only difference lies in the fact that, in [4] daughters are listed in decreasing order of the sizes at birth, and in [13] daughters are enumerated by their birth times. However, in full generality, it is not always possible to enumerate the jumps of a homogeneous process X in decreasing order of jump sizes or increasing order of jump times.

Remark 2.14. If we use a different way to enumerate the jumps of X , it is intuitively clear that the new cell system is the same as the original one, up to a permutation of \mathcal{U} . Thus the growth-fragmentation \mathbf{X} obviously does not depend on the method of enumeration and the law of \mathbf{X} is determined by X .

We now present a connection between homogeneous growth-fragmentation processes and BBLPs.

Proposition 2.15. *Let ξ be a SNLP with characteristics (σ, c, Λ, k) and κ defined as in (1.3) and \mathbf{X} be a homogeneous growth-fragmentation process (starting from 1) driven by $X := \exp(\xi)$. Then the process $\log \mathbf{X}$ is the unique (in law) BBLP with cumulant κ and $\Lambda_b = 0$. Specifically, $\log \mathbf{X}$ has characteristics $(\sigma, c_b, 0, k, \Pi_b)$, where*

$$c_b = c + \int_{(-\infty, -\log 2)} (1 - 2e^z)\Lambda(dz), \text{ and } \Pi_b = \mathbb{1}_{\{(-\log 2, 0)\}}(\Lambda + \bar{\Lambda}) + \frac{1}{2}\mathbb{1}_{\{-\log 2\}}(\Lambda + \bar{\Lambda}).$$

In particular, we have

$$\mathbb{E}_1 \left[\sum_{y \in \mathbf{X}(t)} y^q \right] = \exp(\kappa(q)t) \text{ for all } q \geq 2 \text{ and } t \geq 0.$$

Remark 2.16. Taking account of Remark 2.11, we find that the homogeneous growth-fragmentation \mathbf{X} is a *compensated fragmentation process* in the sense of Definition 3 in [3]. Further, when $\sigma = 0$, $\int_{(-\infty, 0)} (1 - e^z)\Lambda(dz) < \infty$ and $c + \int_{(-\infty, 0)} (1 - e^z)\Lambda(dz) \leq 0$, \mathbf{X} is a homogeneous fragmentation process in the sense of [2]. Loosely speaking, a compensated fragmentation is the limit of properly dilated homogeneous fragmentations, see Corollary 4 in [3].

Proposition 2.15 extends Corollary 25 in [17] (which treats the case when ξ is the negative of a pure-jump subordinator) and Proposition 3 in [4] (for the case when the Lévy measure Λ of ξ has no mass on $(-\infty, -\log 2)$). Before tackling the proof of Proposition 2.15, let us provide a variation of Theorem 1.1, which summarizes the discussion in this section.

Corollary 2.17. *Let ξ and $\tilde{\xi}$ be two SNLPs with respective cumulants κ and $\tilde{\kappa}$ defined as in (1.3). Let \mathbf{X} and $\tilde{\mathbf{X}}$ be the homogeneous growth-fragmentations associated with ξ and $\tilde{\xi}$ respectively (with the same initial size of ancestor $x > 0$). The following statements are equivalent:*

1. $\kappa = \tilde{\kappa}$;
2. $\tilde{\xi}$ has the same law as a switching transform of ξ ;
3. \mathbf{X} and $\tilde{\mathbf{X}}$ have the same finite-dimensional distributions.

Proof of Corollary 2.17. (i) \Leftrightarrow (ii): The two directions follow respectively from Proposition 2.5 and Lemma 2.2.

(i) \Leftrightarrow (iii): We know from Proposition 2.15 that $\log \mathbf{X}$ and $\log \tilde{\mathbf{X}}$ are BBLPs with respective cumulants κ and $\tilde{\kappa}$. If \mathbf{X} and $\tilde{\mathbf{X}}$ have the same finite-dimensional distributions, then so do the BBLPs $\log \mathbf{X}$ and $\log \tilde{\mathbf{X}}$, and in particular their cumulants are the same. Conversely, if $\kappa = \tilde{\kappa}$, then we deduce from Lemma 2.1 or Lemma 2.12 that the BBLPs $\log \mathbf{X}$ and $\log \tilde{\mathbf{X}}$ have the same characteristics, thus the same finite-dimensional distributions. \square

The rest of this section is devoted to the proof of Proposition 2.15.

Proof of Proposition 2.15. The idea of the proof is similar to that of Proposition 3 in [4]. Let \mathbf{Z} be a BBLP with characteristics $(\sigma, c_b, 0, k, \Pi_b)$ and write $(\mathbf{Z}^d)_{d \in (-\infty, -\log 2]}$ for the family of BBLPs as in Definition 2.8, each \mathbf{Z}^d a BBLP with characteristics $(\sigma, c_b, \mathbb{1}_{\{(\bar{d}, 0)\}}\Pi_b, k, \mathbb{1}_{\{-\log 2, \bar{d}\}}\Pi_b)$, such that

$$\mathbf{Z}(t) = \lim_{d \rightarrow -\infty} \uparrow \mathbf{Z}^d(t), \quad t \geq 0.$$

We shall check for every $d \in (-\infty, -\log 2)$ that $\exp(\mathbf{Z}^d)$ has the same dynamics as a truncated cell system associated with the cell process $X = \exp(\xi)$, in which each cell $u \in \mathcal{U}$ is killed at the first instant s with $\mathcal{X}_u(s) \leq e^d \mathcal{X}_u(s-)$, together with her future descendants (born at time $> s$); furthermore, for each $j \in \mathbb{N}$ the daughter cell uj is killed at birth (together with its descendants) whenever her size is less than or equal to e^d times the size of her mother immediately before the birth event, i.e. $\mathcal{X}_{uj}(0) \leq e^d \mathcal{X}_u(b_{uj}-)$. Letting $d \rightarrow -\infty$, we conclude from Definition 2.8 and the monotonicity that $\log \mathbf{X}$ has the same distribution as \mathbf{Z} . Then it is straightforward to check that $\log \mathbf{X}$ indeed has cumulant κ and the identity in the proposition thus follows from (2.5). The uniqueness of $\log \mathbf{X}$ follows from Lemma 2.12.

So it remains to prove that $\exp(\mathbf{Z}^d)$ indeed has the same law as the truncated cell system. In this direction, let us construct an auxiliary particle system as follows, which is a minor modification of Definition 2.8. Fix an arbitrary $d < -\log 2$. Let us consider three independent sequences of processes $(\lambda_v)_{v \in \mathcal{U}_2}$, $(L_v)_{v \in \mathcal{U}_2}$ and $(D_{v\ell}, D_{vr})_{v \in \mathcal{U}_2}$ such that:

- $(\lambda_v)_{v \in \mathcal{U}_2}$ is a family of i.i.d. exponential variables with parameter $\Lambda((\infty, \bar{d}])$;
- $(L_v)_{v \in \mathcal{U}_2}$ is a family of i.i.d. SNLPs with characteristics $(\sigma, \tilde{c}, \mathbb{1}_{\{[\bar{d}, 0)\}} \Lambda, k)$, where $\tilde{c} := c + \int_{(-\infty, \bar{a})} (1 - e^z) \Lambda(dz)$.
- $(D_{v\ell}, D_{vr})_{v \in \mathcal{U}_2}$ is a family of i.i.d. pairs of random variables such that each $D_{v\ell}$ is distributed according to the conditional probability $\Lambda(\cdot \mid (-\infty, \bar{d}))$ and

$$D_{vr} = \overline{D_{v\ell}} = \log(1 - \exp(D_{vr})).$$

Write $\beta_v := \sum_{j=0}^{|v|-1} \lambda_{[v]_j}$ for the birth time, and define by induction $a_{vi} = a_v + L_v(\lambda_v) + D_{vi}$ for $i \in \{\ell, r\}$ with $a_\emptyset = 0$. So we define \mathbf{L} by

$$\mathbf{L}(t) := \{ \{ a_v + L_v(t - \beta_v) : v \in \mathcal{U}_2, \beta_v \leq t < \beta_v + \lambda_v \} \}, \quad t \geq 0.$$

We stress that unlike in Definition 2.8, $\mathbb{1}_{\{(-\infty, \bar{a})\}} \Lambda$ is not supported on $[-\log 2, 0)$, so $D_{v\ell}$ may be possibly smaller than D_{vr} . However, we may obtain a BBLP by changing the indices of the particles. Specifically, let us define a bijection $h : \mathcal{U}_2 \rightarrow \mathcal{U}_2$ in the following way. Let $h(\emptyset) := \emptyset$. Given $h(v)$ with $v \in \mathcal{U}_2$ by induction, then we assign the index of $\max(D_{h(v)\ell}, D_{h(v)r})$ to $h(v\ell)$ and let $h(vr)$ be the sister of $h(v\ell)$. We therefore define $(D'_{v\ell}, D'_{vr}) := (D_{h(v)\ell}, D_{h(v)r})$, $\beta'_v := \beta_{h(v)}$ and $L'_v := L_{h(v)}$ for each $v \in \mathcal{U}_2$, and further define recursively $a'_{vi} := a'_v + L'_v(\lambda'_v) + D'_{vi}$. As h is a bijection, it is plain that

$$\mathbf{L}(t) = \{ \{ a'_v + L'_v(t - \beta'_v) : v \in \mathcal{U}_2, \beta'_v \leq t < \beta'_v + \lambda'_v \} \}, \quad t \geq 0.$$

Let

$$\Pi_{\mathbf{L}} = \frac{1}{2} \mathbb{1}_{\{-\log 2\}}(\Lambda + \bar{\Lambda}) + \mathbb{1}_{\{(-\log 2, \bar{a})\}} \Lambda + \mathbb{1}_{\{(-\log 2, 0)\}} \bar{\Lambda},$$

then $\Pi_{\mathbf{L}}$ is supported on $[-\log 2, 0)$ and we observe that $((D'_{h(v)\ell}, D'_{h(v)r}), v \in \mathcal{U}_2)$ is a family of i.i.d. random variables such that $D'_{h(v)\ell}$ has conditional law $\Pi_{\mathbf{L}}(\cdot \mid [-\log 2, 0))$ and $D'_{h(v)r} = \overline{D'_{h(v)\ell}}$, that $(\beta'_v, v \in \mathcal{U}_2)$ is a family of i.i.d. exponential variables with parameter $\Lambda((-\infty, \bar{d})) = \Pi_{\mathbf{L}}([-\log 2, 0))$ and that $(L'_v, v \in \mathcal{U}_2)$ is a family of independent copies of $\tilde{\xi}$. Using this point of view, we hence deduce that \mathbf{L} is a BBLP as in Definition 2.8, with characteristics $(\sigma, c_b, \Lambda_{\mathbf{L}} := \mathbb{1}_{\{[\bar{a}, 0)\}} \Lambda, k, \Pi_{\mathbf{L}})$, where we have used the fact that

$$\tilde{c} - \int_{[-\log 2, 0)} (1 - e^z) \Pi_{\mathbf{L}}(dz) = c + \int_{(-\infty, -\log 2)} (1 - 2e^z) \Lambda(dz) = c_b.$$

Let us next give some remarks on the *left-most* branch of the particle system \mathbf{L} , that is the process obtained by concatenating the segments of size processes of particles $\{\emptyset, \ell, \ell\ell, \ell\ell\ell, \dots\} := \ell^{\mathbb{N} \cup \{0\}} := \ell^{\mathbb{N}_0} \subset \mathcal{U}_2$:

$$A_{\ell^{\mathbb{N}_0}}(t) := \sum_{v \in \ell^{\mathbb{N}_0}} \mathbf{1}_{\{\beta_v \leq t < \beta_v + \lambda_v\}} (a_v + L_v(t - \beta_v)), \quad t \geq 0. \tag{2.7}$$

Using elementary properties of Lévy processes, we find that $A_{\ell^{\mathbb{N}_0}}$ has the same distribution as ξ . We also notice that for every time $t \geq 0$ when $\Delta A_{\ell^{\mathbb{N}_0}}(t) < \bar{d}$, that is equivalently $\exp(A_{\ell^{\mathbb{N}_0}})$ has a jump of size $-\Delta \exp(A_{\ell^{\mathbb{N}_0}}(t)) > e^{\bar{d}} \exp(A_{\ell^{\mathbb{N}_0}}(t-))$, there is $t = \beta_v + \lambda_v$ for a certain $v \in \ell^{\mathbb{N}_0}$. A fortiori, for every $s \geq 0$ such that $\Delta A_{\ell^{\mathbb{N}_0}}(s) < \bar{d} < \bar{d}$, that is equivalently $\exp(A_{\ell^{\mathbb{N}_0}}(s)) \leq e^{\bar{d}} \exp(A_{\ell^{\mathbb{N}_0}}(s-))$, there is $s = \beta_w + \lambda_w$ for a certain $w \in \ell^{\mathbb{N}_0}$.

We finally consider the process $\hat{\mathbf{L}}$, which is associated with a system derived from \mathbf{L} , by suppressing for each $v \in \mathcal{U}_2$ the child that corresponds to $D_{v\ell}$ whenever $D_{v\ell} \leq d$. So we can explain the dynamics of $\exp(\hat{\mathbf{L}})$ as follows. This system starts with an Eve cell whose size evolves according to $\mathcal{X}_\emptyset := \exp(A_{\ell^{\mathbb{N}_0}})$, and the Eve cell is killed (together with her future descendants) at the first instant $s \geq 0$ when there is $\mathcal{X}_\emptyset(s) \leq e^{\bar{d}} \mathcal{X}_\emptyset(s-)$. Further, for each time $t \leq s$ when \mathcal{X}_\emptyset has a jump of size $y := -\Delta \mathcal{X}_\emptyset(t) > e^{\bar{d}} \mathcal{X}_\emptyset(t-)$, there is $t = \beta_v + \lambda_v$ for a certain $v \in \ell^{\mathbb{N}_0}$, then a daughter cell with initial size y is born and the size of this daughter cell evolves according to the process $\exp(A_{vr\ell^{\mathbb{N}_0}})$, where $A_{vr\ell^{\mathbb{N}_0}}$ is the process associated with $vr\ell^{\mathbb{N}_0} := \{vrw : w \in \ell^{\mathbb{N}_0}\}$ as in (2.7). Note that the process $\exp(A_{vr\ell^{\mathbb{N}_0}})$ has the same distribution as $y \exp(\xi)$. This daughter cell evolves independently of the other daughter cells, is killed at the first instant when her size drops suddenly by factor smaller than $e^{\bar{d}}$, and gives birth to grand-daughter cells each time her size drops suddenly by factor smaller than $e^{\bar{d}}$ (note that $e^{\bar{d}} > e^{\bar{d}}$) before being killed (with killing time included). We continue so on and so forth to obtain the higher generations. So we conclude that $\exp(\hat{\mathbf{L}})$ indeed has the same law as a truncated cell system associated with $X = \exp(\xi)$.

On the other hand, using the point of view that \mathbf{L} is a BBLP with characteristics $(\sigma, c_b, \Lambda_{\mathbf{L}}, k, \Pi_{\mathbf{L}})$, since $D_{vr} > d$ always holds by the construction, we may equivalently view $\hat{\mathbf{L}}$ as the system obtained from \mathbf{L} by suppressing for each $v \in \mathcal{U}_2$ the smaller child D'_{vr} whenever $D'_{vr} \leq d$. We hence deduce from Lemma 3 in [3] that $\hat{\mathbf{L}}$ is a BBLP with characteristics $(\sigma, c_b, \Lambda_{\mathbf{L}}^{\{\bar{d}\}}, k, \Pi_{\mathbf{L}}^{\{\bar{d}\}})$, where $\Lambda_{\mathbf{L}}^{\{\bar{d}\}}$ and $\Pi_{\mathbf{L}}^{\{\bar{d}\}}$ are derived from $\Lambda_{\mathbf{L}}$ and $\Pi_{\mathbf{L}}$ as in (2.4). We check that $(\Lambda_{\mathbf{L}}^{\{\bar{d}\}}, \Pi_{\mathbf{L}}^{\{\bar{d}\}}) = (\mathbf{1}_{\{[\bar{d}, 0)\}} \Pi_b, \mathbf{1}_{\{[-\log 2, \bar{d})\}} \Pi_b)$, so the two BBLPs $\hat{\mathbf{L}}$ and $\mathbf{Z}^{\bar{d}}$ have the same characteristics, which ends the proof. \square

3 Markovian growth-fragmentation processes and bifurcators

In this section, we shall extend the notion of bifurcator to general cell processes and further establish a sufficient condition for different Markovian growth-fragmentations to have the same distribution, which finally orients us toward the proofs of Theorem 1.1 and Theorem 1.2. Let us first present a sufficient condition for non-explosion of growth-fragmentations, which slightly generalizes the approach in [4].

3.1 A sufficient condition for non-explosion

A Feller process $X = (X(t), t \geq 0)$ is called a **cell process**, if it has càdlàg path on $(0, \infty) \cup \{\partial\}$ with no positive jumps. We refer to ∂ as a cemetery point and denote the lifetime of X by $\zeta := \inf \{t \geq 0 : X(t) = \partial\} \in [0, \infty]$. For every $x \geq 0$ we write P_x for the law of X with initial value $X(0) = x$ and E_x for mathematical expectation under P_x .

As we have discussed in Section 2.3, to study the growth-fragmentation associated with X , we first want an ordering of the jumps of X , which is necessary to rigorously build a cell system driven by X . Furthermore, we need a sufficient condition for the

non-explosion of the cell system, that is for every $t \geq 0$ the multiset of the sizes of all cells alive at time t is locally finite. For these purposes, we henceforth suppose the following hypothesis for X , which is reminiscent of that in Theorem 1 in [4].

(H) There exists a measurable function $f: [0, \infty) \times ([0, \infty) \cup \{\partial\}) \rightarrow [0, \infty)$ which fulfills

$$f(r, \partial) = f(r, 0) = 0, \quad \text{for every } r \geq 0,$$

and

$$\inf_{r < l, x > a} f(r, x) > 0, \quad \text{for every } a, l > 0, \tag{3.1}$$

such that for every $x > 0$ and every $s, t \geq 0$, there is

$$E_x \left[f(s+t, X(t)) + \sum_{0 \leq r \leq t} f(s+r, -\Delta X(r)) \right] \leq f(s, x).$$

Example 3.1. For $x > 0$, let P_x be the law of the homogeneous cell process

$$X^{(0)}(t) := x \exp(\xi(t)), \quad t \geq 0.$$

Fix $q \geq 2$ and $K \geq \kappa(q)$, we have by an analogue of (2.6) that for every $x > 0$ and every $s, t \geq 0$

$$\begin{aligned} & E_x \left[X^{(0)}(t)^q e^{-K(t+s)} + \sum_{0 \leq r \leq t} |\Delta X^{(0)}(r)|^q e^{-K(r+s)} \right] \\ &= \left(e^{(\Phi(q)-K)t} + \frac{\kappa(q) - \Phi(q)}{K - \Phi(q)} (1 - e^{(\Phi(q)-K)t}) \right) x^q e^{-Ks} \leq x^q e^{-Ks}. \end{aligned}$$

So $X^{(0)}$ satisfies (H) with the function $f(t, x) = x^q e^{-Kt}$.

From now on we fix a function f such that (H) holds for X . In particular, (H) entails that for every $x > 0$

$$\sum_{r \geq 0} f(r, -\Delta X(r)) < \infty \quad P_x\text{-almost surely.}$$

Hence we may naturally enumerate the jump times of X by listing them in a sequence $(t_i)_{i \in \mathbb{N}}$ such that $(f(t_i, -\Delta X(t_i)))_{i \in \mathbb{N}}$ is decreasing, and thus reproduce the construction in Section 2.3 to build a cell system $\mathcal{X} := (\mathcal{X}_u, u \in \mathcal{U})$ driven by X , starting from an ancestor of initial size $x > 0$, with birth time b_u and life duration ζ_u for each $u \in \mathcal{U}$. Denote the sizes of the cells alive at time $t \geq 0$ by the multiset

$$\mathbf{X}(t) := \{\!\! \{ \mathcal{X}_u(t - b_u) : u \in \mathcal{U}, b_u \leq t < b_u + \zeta_u \}\!\!\},$$

then $(\mathbf{X}(t), t \geq 0)$ is a **growth-fragmentation process driven by X** . We write \mathcal{P}_x for the law of \mathcal{X} and \mathbf{P}_x for the law of \mathbf{X} under \mathcal{P}_x . It is intuitively clear that the law of \mathbf{X} is independent of the enumeration method.

For every non-negative measurable function $h: (0, \infty) \rightarrow [0, \infty)$ and every multiset \mathcal{I} with elements in $(0, \infty)$, introduce the notation

$$\langle \mathcal{I}, h \rangle := \sum_{y \in \mathcal{I}} h(y) \in [0, \infty].$$

Let us define for every $s \geq 0$ a space \mathcal{M}_f^s : a multiset \mathcal{I} is in \mathcal{M}_f^s , if \mathcal{I} has elements in $(0, \infty)$ and $\langle \mathcal{I}, f(s, \cdot) \rangle < \infty$.

Lemma 3.2. *Suppose that X satisfies (H) with a function f . Then we have for every $x > 0$ that*

$$\mathbf{E}_x [\langle \mathbf{X}(t), f(s+t, \cdot) \rangle] \leq f(s, x), \quad \text{for all } t, s \geq 0,$$

where \mathbf{E}_x denotes the mathematical expectation under \mathbf{P}_x . So we have \mathbf{P}_x -almost surely $\mathbf{X}(t) \in \mathcal{M}_f^t$.

Lemma 3.2 encompasses Theorem 1 in [4] for the case when f only depends on the x variable, i.e. $f(t, x) \equiv f(x)$ for every $x, t \geq 0$. In that case f is a so-called *excessive function* for \mathbf{X} . In the same spirit, we may refer to f as a *time-dependent excessive function* for \mathbf{X} .

Proof. The proof is an adaptation of arguments of Theorem 1 in [4]. We may assume that \mathbf{X} is associated with a cell system \mathcal{X} of law \mathcal{P}_x and write \mathcal{E}_x for mathematical expectation under \mathcal{P}_x . We will prove that the sequence

$$\Sigma(i) := \sum_{|u| \leq i, b_u \leq t} f(s+t, \mathcal{X}_u(t-b_u)) + \sum_{|v|=i, b_v \leq t} \sum_{b_v \leq r \leq t} f(s+r, -\Delta \mathcal{X}_v(r-b_v)), \quad i \in \mathbb{N}$$

is a non-negative super-martingale, then $\Sigma(\infty) = \lim_{i \rightarrow \infty} \Sigma(i)$ exists almost surely and $\Sigma(\infty) \geq \langle \mathbf{X}(t), f(s+t, \cdot) \rangle$. We thus deduce from Fatou's lemma that

$$\begin{aligned} \mathbf{E}_x [\langle \mathbf{X}(t), f(s+t, \cdot) \rangle] &\leq \mathcal{E}_x [\Sigma(0)] \\ &= E_x \left[f(s+t, X(t)) + \sum_{0 \leq r \leq t} f(s+r, -\Delta X(r)) \right] \leq f(s, x), \end{aligned}$$

where the last inequality derives from (H).

So it remains to prove that $\Sigma(i)$ is a super-martingale. For every v with $|v| = i$, given $\mathcal{F}_{i-1} := \sigma(\mathcal{X}_u, |u| \leq i-1)$ we have by (H) that

$$\mathcal{E}_x \left[f(s+t, \mathcal{X}_v(t-b_v)) + \sum_{b_v \leq r \leq t} f(s+r, -\Delta \mathcal{X}_v(r-b_v)) \mid \mathcal{F}_{i-1} \right] \leq f(s+b_v, \mathcal{X}_v(0)).$$

Summing over v of i -th generation on the event $\{t \geq b_v\}$, we get that

$$\begin{aligned} &\mathcal{E}_x \left[\sum_{|v|=i, b_v \leq t} f(s+t, \mathcal{X}_v(t-b_v)) + \sum_{|v|=i, b_v \leq t} \sum_{b_v \leq r \leq t} f(s+r, -\Delta \mathcal{X}_v(r-b_v)) \mid \mathcal{F}_{i-1} \right] \\ &\leq \sum_{|v|=i, b_v \leq t} f(s+b_v, \mathcal{X}_v(0)) = \sum_{|u|=i-1, b_u \leq t} \sum_{b_u \leq r \leq t} f(s+r, -\Delta \mathcal{X}_u(r-b_u)). \end{aligned}$$

Adding $\sum_{|u| \leq i-1, b_u \leq t} f(s+t, \mathcal{X}_u(t-b_u))$ to both sides of inequality, we conclude that

$$\mathcal{E}_x [\Sigma(i) \mid \mathcal{F}_{i-1}] \leq \Sigma(i-1),$$

which means that $\Sigma(i)$ is a super-martingale. □

Let \mathcal{M}_+ be the class of all multisets \mathcal{I} on $(0, \infty)$, which has only finitely many elements in $[a, \infty)$ for every $a > 0$. Note that each $\mathcal{I} \in \mathcal{M}_+$ corresponds to a Radon measure on $(0, \infty)$ (in the sense of Remark 2.9) and (3.1) ensures that $\mathcal{M}_f^s \subset \mathcal{M}_+$ for every $s \geq 0$. On account of Lemma 3.2, we can hence view the growth-fragmentation \mathbf{X} as a stochastic process with values in \mathcal{M}_+ , which means that \mathbf{X} does not explode. The space \mathcal{M}_+ is endowed with the following topology:

Definition 3.3. We denote the cardinality of a multiset \mathcal{J} by $|\mathcal{J}|$. A sequence $(\mathcal{I}_n)_{n \in \mathbb{N}}$ in \mathcal{M}_+ converges to $\mathcal{I} \in \mathcal{M}_+$ if and only if for all $r \in (0, \infty)$ such that $\mathcal{I} \cap \{r\} = \emptyset$ there is $|\mathcal{I}_n \cap [r, \infty)| \rightarrow |\mathcal{I} \cap [r, \infty)|$.

The advantage of endowing \mathcal{M}_+ with this topology is that it is a Polish space (homeomorphic to a complete and separable metric space), see Theorem 2.1 and Theorem 2.2 in [16]. It is known from Lemma 2.1 in [16] that convergence in \mathcal{M}_+ implies vague convergence. See [16] for more properties of \mathcal{M}_+ .

We next introduce a *truncate* operation on \mathcal{X} tailored for our future purpose, which is different from the one in the proof of Proposition 2.15. For every $\epsilon > 0$, we obtain a truncated system $\mathcal{X}^{[\epsilon]} = (\mathcal{X}_u^{[\epsilon]}, u \in \mathcal{U})$, by killing each cell process at the first time $s \geq 0$ when its size is less than or equal to ϵ , together with its future (born at time $> s$) descendants. Specifically, let us denote for every $u \in \mathcal{U}$ its ancestral lineage by $A_u := (A_u(t), t \geq 0)$, i.e.

$$A_u(t) := \sum_{n \leq |u|-1} \mathcal{X}_{[u]_n}(t - b_{[u]_n}) \mathbb{1}_{\{b_{[u]_n} \leq t < b_{[u]_{n+1}}\}} + \mathcal{X}_u(t - b_u) \mathbb{1}_{\{b_u \leq t\}}, \quad t \geq 0,$$

where $[u]_n$ denotes u 's ancestor at the n -th generation for all $n \leq |u|$, then we have that

$$\mathcal{X}_u^{[\epsilon]}(t) := \begin{cases} \mathcal{X}_u(t), & \text{if } \inf_{0 \leq r \leq t+b_u} A_u(r) > \epsilon. \\ \partial, & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $\mathbf{X}^{[\epsilon]}$ be the point process on $(0, \infty)$ associated with $\mathcal{X}^{[\epsilon]}$:

$$\mathbf{X}^{[\epsilon]}(t) = \{ \mathcal{X}_u^{[\epsilon]}(t - b_u) : u \in \mathcal{U}, b_u \leq t, \mathcal{X}_u^{[\epsilon]}(t - b_u) \neq \partial \}, \quad t \geq 0.$$

Lemma 3.4. Suppose that X satisfies (H). Then for every $x \geq 0$ and every $t \geq 0$, under the topology of \mathcal{M}_+ the multiset $\mathbf{X}^{[\epsilon]}(t)$ converges \mathbf{P}_x -almost surely to $\mathbf{X}(t)$ as $\epsilon \downarrow 0+$.

Proof. We first note that if a cell $u \in \mathcal{U}$ is alive at time $t \geq 0$ with $\mathcal{X}_u(t - b_u) > 0$, then \mathbf{P}_x -almost surely its ancestral lineage has a size bounded away from 0 before time t , i.e. $\inf_{0 \leq r \leq t} A_u(r) > 0$. So there exists $\epsilon > 0$ small enough such that $\mathcal{X}_u^{[\epsilon]}(r - b_u) = \mathcal{X}_u(r - b_u)$ for all $b_u \leq r \leq t$, and we have \mathbf{P}_x -almost surely

$$\lim_{\epsilon \rightarrow 0+} \mathcal{X}_u^{[\epsilon]}(t - b_u) \mathbb{1}_{\{t \geq b_u\}} = \mathcal{X}_u(t - b_u) \mathbb{1}_{\{t \geq b_u\}}.$$

We hence obtain by the monotone convergence that for every $a > 0$, \mathbf{P}_x -almost surely

$$\lim_{\epsilon \rightarrow 0} \sum_{u \in \mathcal{U}} \mathbb{1}_{\{\mathcal{X}_u^{[\epsilon]}(t - b_u) \geq a\}} \mathcal{X}_u^{[\epsilon]}(t - b_u) \mathbb{1}_{\{t \geq b_u\}} = \sum_{u \in \mathcal{U}} \lim_{\epsilon \rightarrow 0} \mathbb{1}_{\{\mathcal{X}_u^{[\epsilon]}(t - b_u) \geq a\}} \mathcal{X}_u^{[\epsilon]}(t - b_u) \mathbb{1}_{\{t \geq b_u\}},$$

which means that \mathbf{P}_x -almost surely

$$\lim_{\epsilon \rightarrow 0} |\mathbf{X}^{[\epsilon]}(t) \cap [a, \infty)| = |\mathbf{X}(t) \cap [a, \infty)|.$$

So we conclude that $\mathbf{X}^{[\epsilon]}(t)$ converges \mathbf{P}_x -almost surely to $\mathbf{X}(t)$ in \mathcal{M}_+ . □

We observe that the truncated system $\mathcal{X}^{[\epsilon]}$ has a discrete temporal branching structure, since for each càdlàg process the set of jump times with sizes of jumps $< -\epsilon$ is discrete. By the same arguments as the proof of Proposition 2 in [4], we deduce from this observation and Lemma 3.4 that \mathbf{X} has the temporal branching property. To describe this property, let us define a family $(\rho_{s,t}, t \geq s \geq 0)$, where each $\rho_{s,t}$ is a probability kernel from \mathcal{M}_f^s to \mathcal{M}_f^t , in the following way. Given a multiset $\mathcal{J} \in \mathcal{M}_f^s$, we may construct a family of independent random multisets $(\mathcal{I}_y, y \in \mathcal{J})$, such that each \mathcal{I}_y has the law of

$\mathbf{X}(t - s)$ under \mathbf{P}_y . Define by the sum of the multisets $\mathcal{J}^t := \uplus_{y \in \mathcal{J}} \mathcal{I}_y$, then $\mathcal{J}^t \in \mathcal{M}_f^t$, since it follows from Lemma 3.2 that

$$\langle \mathcal{J}^t, f(t, \cdot) \rangle = \sum_{y \in \mathcal{J}} \mathbf{E}_y [\langle \mathbf{X}(t - s), f(t, \cdot) \rangle] \leq \sum_{y \in \mathcal{J}} f(s, y) = \langle \mathcal{J}, f(s, \cdot) \rangle < \infty.$$

We hence define $\rho_{s,t}(\mathcal{J}, \cdot)$ by the law of \mathcal{J}^t .

Proposition 3.5 (Temporal branching property). *Suppose that X satisfies (H) with a function f . Then for every $t \geq s \geq 0$ and every $x > 0$, the conditional distribution of $\mathbf{X}(t)$ under \mathbf{P}_x given $(\mathbf{X}(r), 0 \leq r \leq s)$ is $\rho_{s,t}(\mathbf{X}(s), \cdot)$.*

Remark 3.6. One may easily extend the analysis in this section to time-inhomogeneous Markov processes. Let X is a time-inhomogeneous cell process and write $P_{s,x}$ for the law of X starting at time $s \geq 0$ with initial size $x \geq 0$. Then the counterpart of condition (H) is that there exists a function f that satisfies (3.1) and for every $x > 0$ and every $s \geq 0$,

$$E_{s,x} \left[f(t, X(t)) + \sum_{s \leq r \leq t} f(r, -\Delta X(r)) \right] \leq f(s, x), \quad \text{for all } t \geq s,$$

where $E_{s,x}$ means mathematical expectation under $P_{s,x}$. Under this condition, one may easily build a cell system driven by X (with the life path of each \mathcal{X}_u scaled by the universal time) and check that the system does not explode by an analogue of Lemma 3.2. Details shall be left to interested readers.

3.2 Bifurcators

For every $x > 0$, let P_x and Q_x be respectively the laws of two cell processes X and Y , both starting from x . We now give a formal definition of bifurcators of cell processes, which extends both Definition 2 by Pitman and Winkel [17] and the present Definition 2.6 for homogeneous cell processes.

Definition 3.7. *A bivariate process (X', Y') is called a **bifurcator** of branches X and Y , if it satisfies the following properties:*

1. *For every $x > 0$, write \mathbb{P}_x for the joint distribution of (X', Y') with initial value $X'(0) = Y'(0) = x$. Under \mathbb{P}_x , each component X' and Y' has the law P_x and Q_x respectively, that is, the two marginal distributions of \mathbb{P}_x are P_x and Q_x .*
2. *Let $\tau := \inf\{t \geq 0 : X'(t) \neq Y'(t)\}$. For every $x > 0$, conditionally on $\{\tau < \infty\}$, there is*

$$X'(\tau) + Y'(\tau) = X'(\tau-) = Y'(\tau-), \quad \mathbb{P}_x - a.s. \tag{3.3}$$

3. *(Asymmetric Markov branching property) For every $x > 0$, the process*

$$(X'(t), Y'(t), \mathbf{1}_{\{\tau > t\}})_{t \geq 0}$$

is Markovian under \mathbb{P}_x . Specifically, conditionally given $\tau > t$, the bivariate process $(X'(r + t), Y'(r + t))_{r \geq 0}$ has distribution $\mathbb{P}_{X'(t)}$; conditionally given $\tau \leq t$, $(X'(r + t), Y'(r + t))_{r \geq 0}$ is a pair of independent processes of respective laws $P_{X'(t)}$ and $Q_{Y'(t)}$.

If such a bifurcator (X', Y') exists, then we say X and Y can be coupled to form a bifurcator.

Remark 3.8. We know from (3.3) that if $\tau < \infty$, then (3.3) implies that τ is a jump time of both X' and Y' , which is almost surely strictly positive and strictly smaller than the lifetimes of X' and Y' . Define a filtration $(\mathcal{G}_t)_{t \geq 0}$ by the usual augmentation (see e.g. Section 1.4 in [18]) of $\sigma(X'(r), Y'(r), 0 \leq r \leq t)$, note that τ is a (\mathcal{G}_t) -stopping time and each component X' or Y' satisfies the strong Markov property.

We next state a sufficient condition for growth-fragmentations based on different cell processes to have the same distribution, which is the main purpose of this work. Suppose that (H) holds for both X and Y , then we know from the preceding subsection that we can construct two non-explosive growth-fragmentations \mathbf{X} and \mathbf{Y} associated with X and Y respectively. Note that (H) entails that for every $x > 0$ and every $s \geq 0$,

$$E_x \left[\sum_{r \geq 0} f(s+r, -\Delta X(r)) \right] \leq f(s, x).$$

However, we shall need a stronger inequality and make the following assumption:

(H η) For a certain $\eta \in (0, 1)$, there exists a function g that satisfies

$$\inf_{r < l, x > a} g(r, x) > 0, \quad \text{for every } a, l > 0, \tag{3.4}$$

such that for every $x > 0$ and every $s \geq 0$,

$$E_x \left[\sum_{r \geq 0} g(s+r, -\Delta X(r)) \right] \leq \eta g(s, x).$$

Theorem 3.9. *Let X and Y be two cell processes that both satisfy (H) and (H η). Suppose that X and Y can be coupled to form a bifurcator, then for every $x > 0$, two Markovian growth-fragmentations \mathbf{X} and \mathbf{Y} driven respectively by X and Y , both starting from x , have the same finite-dimensional distributions.*

3.3 Proof of Theorem 3.9

Let us briefly explain the idea of the proof. Fix $x > 0$, let \mathcal{X} and \mathcal{Y} be cell systems associated with X and Y respectively, with respective laws \mathcal{P}_x and \mathcal{Q}_x . For every $\epsilon > 0$, let $\mathbf{X}^{[\epsilon]}$ be the process associated with the truncated cell system $\mathcal{X}^{[\epsilon]}$ derived from \mathcal{X} as in (3.2), by killing each cell together with its future descendants when its size becomes less than or equal to ϵ . Similarly we define $\mathbf{Y}^{[\epsilon]}$. We shall prove for every $\epsilon > 0$ that $\mathbf{X}^{[\epsilon]}$ under \mathcal{P}_x has the same law as $\mathbf{Y}^{[\epsilon]}$ under \mathcal{Q}_x . Then letting $\epsilon \rightarrow 0+$, we conclude from Lemma 3.4 that \mathbf{X} and \mathbf{Y} have the same finite-dimensional distributions.

Let us fix an arbitrary $\epsilon > 0$. To prepare for the proof that $\mathbf{X}^{[\epsilon]}$ and $\mathbf{Y}^{[\epsilon]}$ have the same law, we construct a family of bivariate processes $((X_v, Y_v), v \in \mathcal{U}_2)$ (recall that $\mathcal{U}_2 = \bigcup_{n \in \mathbb{N}} \{\ell, r\}^n$ is the binary tree) in the following way. Since X and Y can be coupled to form a bifurcator, there exists a bifurcator with distribution $(\mathbb{P}_y, y > 0)$, whose marginal distributions are P_y and Q_y under \mathbb{P}_y . Then we let $(X_\emptyset, Y_\emptyset)$ be a bifurcator with law \mathbb{P}_x and write $\beta_\emptyset := 0$ for the birth time of \emptyset . Suppose by induction that we have built for a certain $v \in \mathcal{U}_2$ a bifurcator (X_v, Y_v) with birth time β_v . Write

$$\tau_v := \inf\{t \geq 0 : X_v(t) \neq Y_v(t)\}$$

for the switching time of this bifurcator,

$$T_v^X := \inf\{t \geq 0 : X_v(t) \leq \epsilon\}$$

for the first time when X_v is smaller than ϵ , and

$$\tilde{T}_v^X := \inf\{t \geq 0 : -\Delta X_v(t) > \epsilon\}$$

for the first time when X_v has a jump of size greater than ϵ . Then we define the lifetime of v by

$$\lambda_v := \tau_v \wedge T_v^X \wedge \tilde{T}_v^X,$$

then λ_v is a (\mathcal{G}_t^v) -stopping time, where \mathcal{G}_t^v is the usual augmentation of

$$\sigma\left((X_v(r), Y_v(r)), 0 \leq r \leq t\right).$$

At the lifetime λ_v , we distinguish the following two situations.

- If $\lambda_v = T_v^X < \tilde{T}_v^X \wedge \tau_v$ or $\lambda_v = \infty$, then v is **killed** at its lifetime λ_v . Further, we agree that for every $w \in \mathcal{U}_2 \setminus \{\emptyset\}$, vw is also killed, with $\beta_{vw} = \infty$, $\lambda_{vw} = 0$ and $X_{vw} \equiv Y_{vw} \equiv \partial$. As $\tau_v \wedge \tilde{T}_v^X$ is almost surely strictly positive, this situation also covers the case when $\lambda_v = 0$ (if and only if $T_v^X = 0$, i.e. $X_v(0) \leq \epsilon$).

- Otherwise, v **branches** at its lifetime λ_v , giving birth to two independent bifurcators $(X_{v\ell}, Y_{v\ell})$ and (X_{vr}, Y_{vr}) with respective distributions $\mathbb{P}_{a_{v\ell}}$ and $\mathbb{P}_{a_{vr}}$, where

$$(a_{v\ell}, a_{vr}) := (X_v(\lambda_v), -\Delta X_v(\lambda_v)). \tag{3.5}$$

Set their birth time by $\beta_{v\ell} = \beta_{vr} := \beta_v + \lambda_v$. We further **mark** v if $\lambda_v = \tau_v \leq T_v^X \wedge \tilde{T}_v^X$ (we also say that we mark the branching event at the death of v), so v is **non-marked** if $\lambda_v = \tilde{T}_v^X < \tau_v$. Using the junction relation (3.3) of the bifurcator, we also have

$$(a_{v\ell}, a_{vr}) = \begin{cases} (-\Delta Y_v(\lambda_v), Y_v(\lambda_v)), & \text{if } v \text{ is marked,} \\ (Y_v(\lambda_v), -\Delta Y_v(\lambda_v)), & \text{if } v \text{ is non-marked.} \end{cases} \tag{3.6}$$

Note that if v is non-marked, then $a_{vr} > \epsilon$ always holds; but if v is marked, then it is possible that $a_{vr} \leq \epsilon$, which means that vr is immediately killed and $\lambda_{vr} = 0$. In both marked and non-marked cases, it is possible that $a_{v\ell} \leq \epsilon$ and $v\ell$ is immediately killed with $\lambda_{v\ell} = 0$.

We continue so on and so forth to construct all generations of the family $(X_v, Y_v)_{v \in \mathcal{U}_2}$ and denote its law by \mathcal{P}_x (recall that $x = X_\emptyset(0) = Y_\emptyset(0)$). Finally, we define a process

$$\mathbf{W}_{(X,Y)}(t) := \{\!\{ X_v(t - \beta_v) : v \in \mathcal{U}_2, \beta_v \leq t < \beta_v + \lambda_v \}\!\}, \quad t \geq 0.$$

Note by construction that every element of $\mathbf{W}_{(X,Y)}(t)$ is larger than ϵ . A notable feature of this system is that, roughly speaking, $\mathbf{W}_{(X,Y)}$ is symmetric, i.e. its law is invariant under the permutation of labels X and Y .

Lemma 3.10. $\mathbf{W}_{(X,Y)}$ has the same law as $\mathbf{W}_{(Y,X)}$.

Proof. Given the family $((X_v, Y_v), v \in \mathcal{U}_2)$ constructed as above, let us define recursively a bijection $h : \mathcal{U}_2 \rightarrow \mathcal{U}_2$ with $h(\emptyset) := \emptyset$, such that for every $v \in \mathcal{U}_2$ we have

$$(h(v\ell), h(vr)) := \begin{cases} (h(v)r, h(v)\ell), & \text{if } h(v) \text{ is marked,} \\ (h(v)\ell, h(v)r), & \text{if } h(v) \text{ is non-marked or killed.} \end{cases}$$

We next describe the dynamics of $((Y'_v, X'_v) := (Y_{h(v)}, X_{h(v)}), v \in \mathcal{U}_2)$ as a bivariate system generated by the bifurcator (Y, X) . Specifically, define T_v^Y and \tilde{T}_v^Y for Y_v in the same way as T_v^X and \tilde{T}_v^X , then the lifetime of each (Y'_v, X'_v) is $\lambda'_v := \tau_{h(v)} \wedge T_{h(v)}^Y \wedge \tilde{T}_{h(v)}^Y$, which is equal to $\lambda_{h(v)}$. Indeed, since $X_{h(v)}(t) = Y_{h(v)}(t)$ for all $t < \tau_{h(v)}$, we find that

- If $\lambda_{h(v)} = T_{h(v)}^X < \tilde{T}_{h(v)}^X \wedge \tau_{h(v)}$ or $\lambda_{h(v)} = \infty$, then

$$T_{h(v)}^Y = T_{h(v)}^X \quad \text{and} \quad T_{h(v)}^Y < \tilde{T}_{h(v)}^Y \wedge \tau_{h(v)};$$

- If $\lambda_{h(v)} = \tau_{h(v)} \leq T_{h(v)}^X \wedge \tilde{T}_{h(v)}^X$, then $\tau_{h(v)} \leq T_{h(v)}^Y \wedge \tilde{T}_{h(v)}^Y$;

- if $\lambda_{h(v)} = \tilde{T}_{h(v)}^X < \tau_{h(v)}$, then $\tilde{T}_{h(v)}^Y = \tilde{T}_{h(v)}^X < \tau_{h(v)}$ and $\tilde{T}_{h(v)}^Y \leq T_{h(v)}^Y$.

At the lifetime $\lambda_{h(v)}$, v is killed in the first case; in the other two cases, v generates two independent bifurcators $(Y'_{v\ell}, X'_{v\ell})$ and (Y'_{vr}, X'_{vr}) of respective laws $\mathbb{P}_{a_{h(v)\ell}}$ and $\mathbb{P}_{a_{h(v)r}}$. It follows from (3.6) and the construction of h that for every $v \in \mathcal{U}_2$:

$$(a_{h(v)\ell}, a_{h(v)r}) = (Y_{h(v)}(\lambda_{h(v)}), -\Delta Y_{h(v)}(\lambda_{h(v)})).$$

Define for every $t \geq 0$ that

$$\begin{aligned} \mathbf{W}'(t) &:= \{Y'_v(t - \beta'_v) : v \in \mathcal{U}_2, \beta'_v \leq t < \beta'_v + \lambda'_v\} \\ &= \{Y_{h(v)}(t - \beta_{h(v)}) : v \in \mathcal{U}_2, \beta_{h(v)} \leq t < \beta_{h(v)} + \lambda_{h(v)}\}, \end{aligned}$$

then the process $(\mathbf{W}'(t), t \geq 0)$ is a copy of $\mathbf{W}_{(Y,X)}$. On the other hand, since h is a bijection and recall that for every $v \in \mathcal{U}_2$, $X_v(t) = Y_v(t)$ for all $t < \lambda_v$, then clearly $\mathbf{W}' = \mathbf{W}_{(X,Y)}$. \square

We next consider the process associated with the left-most branch $\ell^{\mathbb{N}_0} = \{\emptyset, \ell, \ell\ell, \dots\}$, that is

$$\begin{aligned} A_{\ell^{\mathbb{N}_0}}(t) &:= \sum_{n \geq 0} \mathbb{1}_{\{\beta_{\ell^n} \leq t < \beta_{\ell^n} + \lambda_{\ell^n}\}} X_{\ell^n}(t - \beta_{\ell^n}) \\ &= \sum_{n \geq 0} \mathbb{1}_{\{\beta_{\ell^n} \leq t < \beta_{\ell^n} + \lambda_{\ell^n}\}} Y_{\ell^n}(t - \beta_{\ell^n}), \quad t \geq 0. \end{aligned} \tag{3.7}$$

where $\ell^n := (\ell \dots \ell) \in \{\ell, r\}^n$ and $\ell^0 := \emptyset$. Let $N := \inf \{n \in \mathbb{N} : \ell^n \text{ is killed}\}$, with convention $\inf \{\emptyset\} = \infty$. If $N < \infty$, then $A_{\ell^{\mathbb{N}_0}}(t) = \partial$ for all $t \geq \beta_{\ell^N} + \lambda_{\ell^N}$. By concatenating $A_{\ell^{\mathbb{N}_0}}$ with the segment of X_{ℓ^N} after its lifetime λ_{ℓ^N} , we define

$$A_{\ell^{\mathbb{N}_0}}^X(t) := \begin{cases} A_{\ell^{\mathbb{N}_0}}(t), & t < \beta_{\ell^N} + \lambda_{\ell^N}. \\ X_{\ell^N}(t - \beta_{\ell^N}), & t \geq \beta_{\ell^N} + \lambda_{\ell^N}. \end{cases} \tag{3.8}$$

We agree that $A_{\ell^{\mathbb{N}_0}}^X = A_{\ell^{\mathbb{N}_0}}$ if $N = \infty$.

Lemma 3.11. *Suppose that (H η) holds for both X and Y . Then the process $A_{\ell^{\mathbb{N}_0}}^X$ has the law of P_x (the law of X starting from x), and the process derived from $A_{\ell^{\mathbb{N}_0}}^X$ by killing at $\zeta_{\ell^{\mathbb{N}_0}}^X := \inf \{t \geq 0 : A_{\ell^{\mathbb{N}_0}}^X(t) \leq \epsilon\}$ is $A_{\ell^{\mathbb{N}_0}}$.*

Proof. It should be intuitive that $A_{\ell^{\mathbb{N}_0}}^X$ has the law of P_x because of the construction (3.5) and the strong Markov property of X ; however, it is a priori possible that none of $\ell^{\mathbb{N}_0}$ is killed and their birth times accumulate to a finite limit, i.e. $N = \infty$ and $\lim_{n \rightarrow \infty} \beta_{\ell^n} < \infty$, then $A_{\ell^{\mathbb{N}_0}}^X$ is killed at this limit time, thus does not have the law of P_x . We shall prove that this case does not happen, thanks to the assumption (H η). Therefore, almost surely there are only two possible situations: either $N < \infty$, or $N = \infty$ & $\lim_{n \rightarrow \infty} \beta_{\ell^n} = \infty$, so we deduce from the strong Markov property of X that $A_{\ell^{\mathbb{N}_0}}^X$ indeed has the law of P_x . Further, we easily check that $\zeta_{\ell^{\mathbb{N}_0}}^X = \beta_{\ell^N} + \lambda_{\ell^N}$ when $N < \infty$ and $\zeta_{\ell^{\mathbb{N}_0}}^X = \infty$ when $N = \infty$, then the second part of the claim follows.

So it remains to prove that if $N = \infty$, which means that none of $(\ell^n)_{n \in \mathbb{N}}$ is killed, then $\lim_{n \rightarrow \infty} \beta_{\ell^n} = \infty$. We consider separately the following two situations.

In the first situation there are infinitely many marked elements in $\ell^{\mathbb{N}_0}$, and we list all of them in a sequence $(\ell^{n_i})_{i \in \mathbb{N}} \subset \ell^{\mathbb{N}_0}$ with $n_i \uparrow \infty$. Let $\mathcal{G}_n := \sigma(X_{\ell^j}, Y_{\ell^j}, j \leq n)$ and g_Y be a function such that (H η) holds for Y with $\eta_Y < 1$, then

$$M_i := \eta_Y^{-i} \sum_{r \geq 0} g_Y(\beta_{\ell^{n_i}} + r, -\Delta Y_{\ell^{n_i}}(r)), \quad i \in \mathbb{N}$$

is a non-negative \mathcal{G}_{n_i} -super-martingale. Indeed, consider the ancestral lineage of $\ell^{n_{i+1}}$ for the Y -side, shifted to the left by $\beta_{\ell^{n_i \ell}}$ ($\ell^{n_i \ell}$ means ℓ^{n_i+1}), that is

$$A_{i+1}^Y(t) := \sum_{n_{i+1} \leq k < n_{i+1}} \mathbb{1}_{\{\beta_{\ell^k} \leq t + \beta_{\ell^{n_i \ell}} < \beta_{\ell^k} + \lambda_{\ell^k}\}} Y_{\ell^k}(t + \beta_{\ell^{n_i \ell}} - \beta_{\ell^k}) + \mathbb{1}_{\{t + \beta_{\ell^{n_i \ell}} \geq \beta_{\ell^{n_{i+1}}}\}} Y_{\ell^{n_{i+1}}}(t + \beta_{\ell^{n_i \ell}} - \beta_{\ell^{n_{i+1}}}), \quad t \geq 0,$$

with $A_{i+1}^Y(0) = Y_{\ell^{n_i \ell}}(0)$. Then

$$M_{i+1} \leq \eta_Y^{-(i+1)} \sum_{r \geq 0} g_Y(\beta_{\ell^{n_i \ell}} + r, -\Delta A_{i+1}^Y(r)).$$

Observing that these segments are connected by only non-marked branching events and using (3.6), we hence deduce by the strong Markov property of Y that conditionally on \mathcal{G}_{n_i} , A_i^Y has distribution Q_y with $y := Y_{\ell^{n_i \ell}}(0)$. As Y satisfies (H η), we have

$$\begin{aligned} \mathcal{E}_x[M_{i+1} | \mathcal{G}_{n_i}] &\leq \mathcal{E}_x \left[\eta_Y^{-(i+1)} \sum_{r \geq 0} g_Y(\beta_{\ell^{n_i \ell}} + r, -\Delta A_{i+1}^Y(r)) \middle| \mathcal{G}_{n_i} \right] \\ &\leq \eta_Y^{-i} g_Y(\beta_{\ell^{n_i \ell}}, Y_{\ell^{n_i \ell}}(0)) = \eta_Y^{-i} g_Y(\beta_{\ell^{n_i}} + \lambda_{\ell^{n_i}}, -\Delta Y_{\ell^{n_i}}(\lambda_{\ell^{n_i}})) \leq M_i, \end{aligned}$$

where \mathcal{E}_x denotes the mathematical expectation under the law \mathcal{P}_x of the system $(X_v, Y_v, v \in \mathcal{U}_2)$, and the equality follows from (3.6) as ℓ^{n_i} is marked. We conclude that M_i is a non-negative super-martingale and hence M_i converges almost surely to a limit as $i \rightarrow \infty$. Multiplying the last display by $\eta_Y^{n_i}$, we have

$$g_Y(\beta_{\ell^{n_i \ell}}, X_{\ell^{n_i \ell}}(0)) = g_Y(\beta_{\ell^{n_i}} + \lambda_{\ell^{n_i}}, -\Delta Y_{\ell^{n_i}}(\lambda_{\ell^{n_i}})) \leq \eta_Y^i M_i \rightarrow 0 \quad \text{almost surely.}$$

As g_Y satisfies (3.4), it follows that in the event $\lim_{n \rightarrow \infty} \beta_{\ell^n} < \infty$, there is

$$\lim_{i \rightarrow \infty} X_{\ell^{n_i \ell}}(0) = 0.$$

This is absurd as we have assumed that no element in $\ell^{\mathbb{N}_0}$ is killed.

In the second situation, there are infinitely many non-marked branching elements in $\ell^{\mathbb{N}_0}$. Consider for each $k \in \mathbb{N}$ the ancestral lineage of ℓ^k for the side of X , i.e.

$$A_{\ell^k}^X(t) := \sum_{n=0}^{k-1} \mathbb{1}_{\{\beta_{\ell^n} \leq t < \beta_{\ell^{n+1}}\}} X_{\ell^n}(t - \beta_{\ell^n}) + \mathbb{1}_{\{\beta_{\ell^k} \leq t\}} X_{\ell^k}(t - \beta_{\ell^k}), \quad t \geq 0.$$

Then for each $k \in \mathbb{N}$, we deduce from the strong Markov property of X that $A_{\ell^k}^X$ has law P_x under \mathcal{P}_x . Let g_X be a function such that (H η) holds for X with constant $\eta_X < 1$, then

$$E_x \left[\sum_{r \geq 0} g_X(r, -\Delta A_{\ell^k}^X(r)) \right] \leq g_X(0, x). \tag{3.9}$$

Suppose, by contradiction, that there exists a certain $M > 0$ such that with probability $p_M > 0$ there is $\lim_{n \rightarrow \infty} \beta_{\ell^n} < M$ and write $\inf_{t < M, y \geq \epsilon} g_X(t, y) =: c_{M, \epsilon} > 0$ as (3.4) holds for g_X . For every $k \in \mathbb{N}$, write m_k for the number of non-marked particles in the set $\{\ell^i, i < k\}$, then we get that

$$\begin{aligned} E_x \left[\sum_{r \geq 0} g_X(r, -\Delta A_{\ell^k}^X(r)) \right] &\geq \mathcal{E}_x \left[\sum_{1 \leq i \leq k-1} \mathbb{1}_{\{\ell^i \text{ is non-marked}\}} g_X(\beta_{\ell^i}, -\Delta X_{\ell^i}(\lambda_{\ell^i})) \right] \\ &\geq p_M m_k c_{M, \epsilon}, \end{aligned}$$

where the last inequality is obtained by restricting to the event $\lim_{n \rightarrow \infty} \beta_{\ell^n} < M$ and observing that $-\Delta X_{\ell^i}(\lambda_{\ell^i}) \geq \epsilon$ whenever ℓ^i is non-marked. Letting $k \rightarrow \infty$, we find a contradiction against (3.9) since $m_k \rightarrow \infty$. We have therefore proved the claim. \square

Remark 3.12. Given the system $((X_v, Y_v), v \in \mathcal{U}_2)$, let us define a branch

$$\bar{v} := (v_n \in \mathcal{U}_2)_{n \geq 0}$$

with $v_0 := \emptyset$ by setting recursively $v_{n+1} := v_n \ell$ if v_n is non-marked and $v_{n+1} := v_n r$ if v_n is marked. Then the branch $A_{\bar{v}}^Y$ associated with the system $((X_v, Y_v), v \in \mathcal{U}_2)$ has the same law as Y . Indeed, recall that the system $((Y_{h(v)}, X_{h(v)}), v \in \mathcal{U}_2)$ defined as in the proof of Lemma 3.10 can be viewed as a system generated by the bifurcator (Y, X) , then applying Lemma 3.11 to this system, we have that $A_{h(\ell^{\mathbb{N}_0})}^Y$ has the same law as Y . We observe that $\bar{v} = h(\ell^{\mathbb{N}_0})$ by the construction of h , which entails our claim.

Lemma 3.13. $\mathbf{W}_{(X,Y)}$ has the same law as $\mathbf{X}^{[\epsilon]}$.

Proof. We first give some remarks on the process $A_{\ell^{\mathbb{N}_0}}^X$ defined as in (3.8). We know from Lemma 3.11 that $A_{\ell^{\mathbb{N}_0}}^X$ has law P_x , and $A_{\ell^{\mathbb{N}_0}}^X$ killed at $\zeta_{\ell^{\mathbb{N}_0}} := \inf \{t \geq 0 : A_{\ell^{\mathbb{N}_0}}^X(t) \leq \epsilon\}$ is $A_{\ell^{\mathbb{N}_0}}$. For every $\ell^m \in \ell^{\mathbb{N}_0}$ such that $t := \beta_{\ell^m} + \lambda_{\ell^m} \leq \zeta_{\ell^{\mathbb{N}_0}}$, we have by (3.6) that the size of the jump at t is

$$y := -\Delta A_{\ell^{\mathbb{N}_0}}^X(t) = -\Delta X_{\ell^m}(\lambda_{\ell^m}) = X_{\ell^m r}(0).$$

Note that it is possible that $y > \epsilon$ or $y \leq \epsilon$: if $y \leq \epsilon$, then we know that the particle $\ell^m r$ (together with its progeny) is killed immediately, that is $\lambda_{\ell^m r} = 0$. On the other hand, for all $m \in \mathbb{N}$ and $t' \in (\beta_{\ell^m}, \beta_{\ell^m} + \lambda_{\ell^m})$, we have $-\Delta A_{\ell^{\mathbb{N}_0}}^X(t') \leq \epsilon$.

Let us now describe the dynamics of $\mathbf{W}_{(X,Y)}$ as the following truncated cell system. The cell system starts with a cell whose size evolves according to $\mathcal{X}_\emptyset := A_{\ell^{\mathbb{N}_0}}^X$ with law P_x . By killing \mathcal{X}_\emptyset at the time when entering $(0, \epsilon]$, we get $\mathcal{X}_\emptyset^{[\epsilon]} = A_{\ell^{\mathbb{N}_0}}$. We next build the first generation. The daughter cells in the first generation born (strictly) after $\zeta_{\ell^{\mathbb{N}_0}}$ are all killed. For each time $t \leq \zeta_{\ell^{\mathbb{N}_0}}$ with $y := -\Delta \mathcal{X}_\emptyset(t) > \epsilon$, we observe from the remarks above that there exists a certain $w \in \ell^{\mathbb{N}_0}$ such that $t = \beta_w + \lambda_w$ and $X_{wr}(0) = y$. So a daughter cell is born at t and its size evolves according to $A_{wr\ell^{\mathbb{N}_0}}^X$, which is the process associated with $wr\ell^{\mathbb{N}_0} := \{wrv, v \in \ell^{\mathbb{N}_0}\}$ as in (3.8). As $wr\ell^{\mathbb{N}_0}$ is the left-most branch in the sub-tree $(wrv, v \in \mathcal{U}_2)$, we deduce from Lemma 3.11 that $A_{wr\ell^{\mathbb{N}_0}}^X$ has distribution P_y and $A_{wr\ell^{\mathbb{N}_0}}$ defined as in (3.7) is $A_{wr\ell^{\mathbb{N}_0}}^X$ killed when entering $(0, \epsilon]$. On the other hand, for every time $t' \leq \zeta_{\ell^{\mathbb{N}_0}}$ with $y' := -\Delta \mathcal{X}_\emptyset(t') \in (0, \epsilon]$, we agree that the daughter cell born at t' is killed immediately. We hence conclude that those non-degenerate size processes $\mathcal{X}_i^{[\epsilon]}$ with $i \in \mathbb{N}$ are exactly those non-degenerate processes $A_{wr\ell^{\mathbb{N}_0}}$ with $w \in \ell^{\mathbb{N}_0}$. The proof is completed by iteration of this argument. \square

Proof of Theorem 3.9. For every $\epsilon > 0$, applying Lemma 3.13 to $\mathbf{W}_{(Y,X)}$, we deduce that $\mathbf{W}_{(Y,X)}$ and $\mathbf{Y}^{[\epsilon]}$ have the same law. Together with Lemma 3.10, this implies that $\mathbf{X}^{[\epsilon]}$ and $\mathbf{Y}^{[\epsilon]}$ have the same law. Letting $\epsilon \rightarrow 0+$, we conclude by Lemma 3.4 that \mathbf{X} and \mathbf{Y} have the same finite-dimensional distributions. \square

3.4 Proof of Theorem 1.1

Using Theorem 3.9, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The implication $(i) \Rightarrow (ii)$ follows from Lemma 2.7 and the equivalence $(iii) \Leftrightarrow (i)$ follows from Corollary 2.17. So it remains to prove that $(ii) \Rightarrow (iii)$. Suppose that $X^{(0)}$ and $\tilde{X}^{(0)}$ can be coupled to form a bifurcator. We can check that $X^{(0)}$ and $\tilde{X}^{(0)}$ satisfy (H) and $(H\eta)$, then we are led to the conclusion that the growth-fragmentations have the same finite dimensional distribution by Theorem 3.9. Indeed, fix $q \geq 2$ and $K > \kappa(q)$, then we know from Example 3.1 that $X^{(0)}$ satisfies (H) with the function $(t, x) \mapsto x^q e^{-Kt}$; further, it follows from (2.6) that $X^{(0)}$ also satisfies $(H\eta)$ with this function and any $\eta \in (\frac{\kappa(q) - \Phi(q)}{K - \Phi(q)}, 1)$. Similarly, we have that $\tilde{X}^{(0)}$ also satisfies both (H) and $(H\eta)$. This completes the proof. \square

3.5 Proof of Theorem 1.2

We now turn to self-similar growth-fragmentations. In order to prove Theorem 1.2, we first prove the following lemmas.

Lemma 3.14. *Let $X^{(\alpha)}$ be a self-similar cell process with index $\alpha \in \mathbb{R}$ related to a SNLP ξ as in (1.4). Suppose $\kappa(q) < 0$ for a certain $q > 0$, then $X^{(\alpha)}$ satisfies both (H) and (H η) (for any $\eta \in (1 - \frac{\kappa(q)}{\Phi(q)}, 1)$) with the function $(t, x) \mapsto x^q$.*

Proof. This follows directly from Lemma 2 and Lemma 3 in [4]. □

Lemma 3.15. *Let $X^{(\alpha)}$ and $Y^{(\alpha)}$ be two self-similar cell processes with index $\alpha \in \mathbb{R}$ related to SNLPs ξ and γ respectively as in (1.4). Suppose that $\kappa = \kappa_\gamma$, then $X^{(\alpha)}$ and $Y^{(\alpha)}$ can be coupled to form a bifurcator.*

Proof. By Proposition 2.5 we may assume that ξ is the switching transform of γ with switching time $\tau = \inf\{t \geq 0 : \xi(t) \neq \gamma(t)\}$. Say $X^{(\alpha)}(0) = Y^{(\alpha)}(0) = x > 0$, we set

$$\tau^{(\alpha)} := x^{-\alpha} \int_0^\tau \exp(-\alpha\xi(r))dr,$$

then we have by Lamperti's time-substitution (1.4) the identities

$$\tau^{(\alpha)} = \inf\{t \geq 0 : X^{(\alpha)}(t) \neq Y^{(\alpha)}(t)\}$$

and

$$X^{(\alpha)}(\tau^{(\alpha)}) + Y^{(\alpha)}(\tau^{(\alpha)}) = X^{(\alpha)}(\tau^{(\alpha)}-).$$

Let $\tilde{Y}^{(\alpha)}$ be an independent copy of $Y^{(\alpha)}$ and we build a process

$$\hat{Y}^{(\alpha)}(t) := Y^{(\alpha)}(t)\mathbb{1}_{\{t < \tau^{(\alpha)}\}} + y\tilde{Y}^{(\alpha)}(y^\alpha(t - \tau^{(\alpha)}))\mathbb{1}_{\{t \geq \tau^{(\alpha)}\}}, \quad t \geq 0,$$

where $y := -\Delta X^{(\alpha)}(\tau^{(\alpha)})$. Then $(X^{(\alpha)}, \hat{Y}^{(\alpha)})$ is a bifurcator. □

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) \Rightarrow (ii): This follows from Lemma 3.15.

(ii) \Rightarrow (iii): Since Lemma 3.14 ensures that (H) and (H η) hold under assumption (1.5), we have from Theorem 3.9 that the self-similar growth-fragmentations $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ have the same finite-dimensional distributions.

(iii) \Rightarrow (i): Suppose that the growth-fragmentations $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ have the same finite-dimensional distributions. We first know from the self-similarity (Theorem 2 in [4]) that $\alpha = \tilde{\alpha}$.

To prove $\kappa = \tilde{\kappa}$, we first deduce from Proposition 3.3 and its proof in [5] that for every $q > 0$, there is

$$\mathbf{E}_1 \left[\int_0^\infty \left(\sum_{y \in \mathbf{X}^{(\alpha)}(t)} y^{q+\alpha} \right) dt \right] = \begin{cases} -\frac{1}{\kappa(q)}, & \text{if } \kappa(q) < 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.10}$$

Note from Corollary 4 in [4] that the integrand possesses càdlàg paths under assumption (1.5). As $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\alpha)}$ have the same same finite-dimensional distributions, we thus deduce that for every $q > 0$ with $\kappa(q) < 0$, there is $\tilde{\kappa}(q) = \kappa(q) < 0$. Therefore, if there exists $q_0 > 0$ such that $\kappa(q) < 0$ for all $q > q_0$, then $\kappa(q) = \tilde{\kappa}(q)$ for all $q > q_0$. Otherwise, by the convexity of κ there exists $\omega > 0$, which is the largest root of κ , such that $\kappa(q) > 0$ for all $q > \omega$. It follows from (3.10) that ω is also the largest root for $\tilde{\kappa}$ and $\tilde{\kappa}(q) > 0$ for all $q > \omega$. We hence deduce from Theorem 3.5 in [5] that $\tilde{\kappa}(q) = \kappa(q)$ for all $q > \omega$. Summarizing the two cases, we conclude that there exists a certain constant $a > 0$ such that $\kappa(q) = \tilde{\kappa}(q)$ for all $q > a$, which entails that $\kappa = \tilde{\kappa}$. □

References

- [1] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR-1406564
- [2] Jean Bertoin, *Homogeneous fragmentation processes*, Probab. Theory Related Fields **121** (2001), no. 3, 301–318. MR-1867425
- [3] Jean Bertoin, *Compensated fragmentation processes and limits of dilated fragmentations*, Ann. Probab. **44** (2016), no. 2, 1254–1284. MR-3474471
- [4] Jean Bertoin, *Markovian growth-fragmentation processes*, Bernoulli (forthcoming).
- [5] Jean Bertoin, Timothy Budd, Nicolas Curien, and Igor Kortchemski, *Martingales in self-similar growth-fragmentations and their connections with random planar maps*, arXiv:1605.00581
- [6] Jean Bertoin, Nicolas Curien, and Igor Kortchemski, *Random planar maps & growth-fragmentations*, arXiv:1507.02265
- [7] Jean Bertoin and Robin Stephenson, *Local explosion in self-similar growth-fragmentation processes*, Electron. Commun. Probab. **21** (2016), 12 pp.
- [8] Jean Bertoin and Alexander R. Watson, *Probabilistic aspects of critical growth-fragmentation equations*, Adv. in Appl. Probab. **48** (2016), no. A, 37–61. MR-3539296
- [9] Fabien Campillo, Nicolas Champagnat, and Coralie Fritsch, *Links between deterministic and stochastic approaches for invasion in growth-fragmentation-death models*, Journal of Mathematical Biology (2016), 1–41.
- [10] Fabien Campillo and Coralie Fritsch, *Weak convergence of a mass-structured individual-based model*, Appl. Math. Optim. **72** (2015), no. 1, 37–73. MR-3369396
- [11] Marie Doumic, Marc Hoffmann, Nathalie Krell, and Lydia Robert, *Statistical estimation of a growth-fragmentation model observed on a genealogical tree*, Bernoulli **21** (2015), no. 3, 1760–1799. MR-3352060
- [12] Coralie Fritsch, Jérôme Harmand, and Fabien Campillo, *A modeling approach of the chemo-stat*, Ecological Modelling **299** (2015), 1 – 13.
- [13] Peter Jagers, *General branching processes as Markov fields*, Stochastic Process. Appl. **32** (1989), no. 2, 183–212. MR-1014449
- [14] Andreas E. Kyprianou, *Fluctuations of Lévy processes with applications*, second ed., Universitext, Springer, Heidelberg, 2014, Introductory lectures. MR-3155252
- [15] John Lamperti, *Semi-stable Markov processes. I*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **22** (1972), 205–225. MR-0307358
- [16] Filip Lindskog, Sidney I. Resnick, and Joyjit Roy, *Regularly varying measures on metric spaces: hidden regular variation and hidden jumps*, Probab. Surv. **11** (2014), 270–314. MR-3271332
- [17] Jim Pitman and Matthias Winkel, *Regenerative tree growth: Markovian embedding of fragmenters, bifurcators, and bead splitting processes*, Ann. Probab. **43** (2015), no. 5, 2611–2646. MR-3395470
- [18] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR-1725357
- [19] Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 2013, Translated from the 1990 Japanese original, Revised edition of the 1999 English translation. MR-3185174
- [20] Quan Shi, *A growth-fragmentation model related to Ornstein-Uhlenbeck type processes*, arXiv:1702.01091

Acknowledgments. The author is grateful to Jean Bertoin for suggesting this research and for his guidance throughout the work. The author also thanks Igor Kortchemski, Matthias Winkel and two anonymous referees for their helpful comments.

Electronic Journal of Probability

Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS¹)
- Easy interface (EJMS²)

Economical model of EJP-ECP

- Non profit, sponsored by IMS³, BS⁴, ProjectEuclid⁵
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund⁶ (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

¹LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

²EJMS: Electronic Journal Management System <http://www.vtex.lt/en/ejms.html>

³IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

⁴BS: Bernoulli Society <http://www.bernoulli-society.org/>

⁵Project Euclid: <https://projecteuclid.org/>

⁶IMS Open Access Fund: <http://www.imstat.org/publications/open.htm>