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GROWTH OF A COMPOSITE FUNCTION OF ENTIRE FUNCTIONS

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§ 1. Introduction.

Let $f(z)$ and $g(z)$ be entire functions. Then we have the well-known inequality

$$\log M(r, f(g)) \leqq \log M(M(r, g), f). \tag{1}$$

And it follows from Clunie [2] that if $g(0)=0$, then for $r \geqq 0$

$$\log M(r, f(g)) \geqq \log M(c(\rho)M(\rho r, g), f), \tag{2}$$

where $0<\rho<1$ and $c(\rho)=(1-\rho)^2/4\rho$. Furthermore, these inequalities (1) and (2) are best possible. We next wish to have similar estimations of $T(r, f(g))$. As an immediate consequence of (1) and well-known inequalities $T(r, f) \leqq \log^+ M(r, f) \leqq 3T(2r, f)$, we have

$$T(r, f(g)) \leqq 3T(2M(r, g), f). \tag{3}$$

The inequality (3), however, is not sharp.

The main purpose of this paper is to give an upper estimation of $T(r, f(g))$ and prove the following:

THEOREM 1. *Let $f(z)$ and $g(z)$ be entire functions. If $M(r, g)>((2+\varepsilon)/\varepsilon)|g(0)|$ for any $\varepsilon>0$, then we have*

$$T(r, f(g)) \leqq (1+\varepsilon)T(M(r, g), f). \tag{4}$$

In particular, if $g(0)=0$, then

$$T(r, f(g)) \leqq T(M(r, g), f) \tag{5}$$

for all $r>0$.

Since $T(r, f(z^n))=T(r^n, f(z))$ for any meromorphic function $f(z)$, Theorem 1 is best possible. In the above example $g(z)$ is a polynomial. However, we shall

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prove that

THEOREM 2. *Let $f(z)$ be a transcendental entire function of order zero and $g(z)$ a transcendental entire function of lower order zero. Suppose that for any $0<\sigma<1$ there are two numbers $\alpha>1$ and $r_0>1$ such that*

$$\frac{T(r^{\sigma}, f)}{T(r, f)}>\sigma^{\alpha} \tag{6}$$

holds for all $r>r_0$. Then we have

$$\limsup_{r\to\infty}\frac{T(r, f(g))}{T(M(r, g), f)}=1\,. \tag{7}$$

It is clear that there exist entire functions satisfying (6). For instance, it follows from a result of Clunie [1] that there is an entire function $f(z)$ satisfying $T(r, f)\sim(\log r)^{\beta}$ $(r\to\infty)$ with a constant $\beta>1$ and so $f(z)$ satisfies (6) with a suitable number $\alpha>1$.

We shall now give some lower estimations of $T(r, f(g))$. Firstly, for certain classes of entire functions, we shall show the following theorem, which we can deduce from $\cos\pi\lambda$-theorem (cf. Kjellberg [4], [5]) and the argument of the proof of Theorem 2:

THEOREM 3. *Let $f(z)$ be a transcendental entire function of order zero satisfying* (6) *and $g(z)$ a transcendental entire function of lower order λ $(\lambda<1/2)$. Then we have*

$$\limsup_{r\to\infty}\frac{T(r, f(g))}{T(M(r, g), f)}\geqq(\cos\pi\lambda)^{\alpha}\,.$$

In general we shall prove

THEOREM 4. *Let $f(z)$ and $g(z)$ be transcendental entire functions, $K\,(>0)$ an arbitrary number and $\beta(r)$ unbounded, strictly increasing, continuous function of $r(>0)$ satisfying*

$$\beta(r)\geqq r \quad \textit{and} \quad \log\beta(r)=o(T(\xi' r, g)) \qquad (r\to\infty)\,, \tag{8}$$

where ξ' is a constant satisfying $0<\xi'<1$. Then there is an unbounded increasing sequence $\{r_{\nu}\}$ such that

$$T(r_{\nu}, f(g))+O(1)\geqq N(r_{\nu}, 0, f(g))$$

$$\geqq K\Big(\frac{N(\beta(r_{\nu}), 0, f)}{\log\beta(r_{\nu})-O(1)}-O(1)\Big) \qquad (\nu\to\infty)$$

When $g(z)$ is of finite order, from a result of Valiron [7] and Edrei-Fuchs [3] and the argument of the proof of Theorem 4 we can deduce

THEOREM 5. *Let $f(z)$ be a transcendental entire function, $g(z)$ a transcendental entire function of finite order, c a constant satisfying $0<c<1$ and α a positive number. Then we have for all $r\geqq R_0$,*

$$T(r, f(g))+O(1)\geqq N(r, 0, f(g))$$

$$\geqq(\log(1/c))\left(\frac{N(M((cr)^{1/(1+\alpha)}, g), 0, f)}{\log M((cr)^{1/(1+\alpha)}, g)-O(1)}-O(1)\right) \qquad (r\to\infty).$$

§2. Proof of Theorem 1.

Let $u(z)$ be the harmonic function in the disk $\{|z|<r\}$ which has the boundary values $\log^+|f(g(re^{i\theta}))|$ on the circumference $\{|z|=r\}$. We define $u^*(z)$ by

$$u^*(z)=u(z) \qquad \text{in } \{|z|<r\}$$
$$=\log^+|f(g(z))| \qquad \text{in } \{r\leqq|z|<\infty\}.$$

Then it is clear that $u^*(z)$ is a subharmonic function in $\{|z|<\infty\}$. Let $v(w)$ be the harmonic function in the disk $\{|w|<M(r, g)\}$ with the boundary values $\log^+|f(M(r, g)e^{i\phi})|$ on $\{|w|=M(r, g)\}$. We denote by D_z the component of the set $\{z; g(z)=w, |w|<M(r, g)\}$, which contains the origin. Then we have $\{|z|<r\}\subset D_z$. Further $v(g(z))$ is harmonic in D_z and $v(g(z))=\log^+|f(g(z))|=u^*(z)$ on the boundary of D_z. Hence it follows from the maximum principle that $u^*(z)\leqq v(g(z))$ in D_z. In particular we have

$$u^*(0)\leqq v(g(0)). \tag{2.1}$$

By Gauss' mean value theorem we have

$$u^*(0)=u(0)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(g(re^{i\theta}))|d\theta=T(r, f(g)), \tag{2.2}$$

$$v(0)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|(f(M(r, g)e^{i\phi})|d\phi=T(M(r, g), f). \tag{2.3}$$

Hence, if $g(0)=0$, (5) follows from (2.1), (2.2) and (2.3). If $g(0)\neq 0$ and $M(r, g)>((2+\varepsilon)/\varepsilon)|g(0)|$, then it follows from Harnack's inequality that

$$v(g(0))\leqq\frac{M(r, g)+|g(0)|}{M(r, g)-|g(0)|}v(0)<(1+\varepsilon)v(0),$$

which, together with (2.1), proves (4).

Thus the proof of Theorem 1 is complete.

§ 3. Proof of Theorem 2.

In the first place we shall prove the following:

LEMMA 1. *Let $g(z)$ and $f(z)$ be two entire functions. Suppose that $|g(z)| > R > |g(0)|$ on the circumference $\{|z|=r\}$ for some $r>0$. Then we have*

$$T(r, f(g)) \geqq \frac{R-|g(0)|}{R+|g(0)|} T(R, f)$$

Proof. Let $u(z)$ be the harmonic function in the disk $\{|z|<r\}$ which has boundary values $\log^+|f(g(re^{i\theta}))|$ on the circumference $\{|z|=r\}$. Let $v(w)$ be the harmonic function in the disk $\{|w|<R\}$ which has the boundary values $\log^+|f(Re^{i\phi})|$ on $\{|w|=R\}$. We define $v^*(w)$ by

$$\begin{aligned} v^*(w) &= v(w) && \text{in } \{|w|<R\}, \\ &= \log^+|f(w)| && \text{in } \{|w| \geqq R\}. \end{aligned}$$

Then we deduce that $v^*(w)$ is subharmonic in $\{|w|<\infty\}$ and so $v^*(g(z))$ is subharmonic in $\{|z|<\infty\}$. Since $|g(z)|>R$ for $|z|=r$, it follows from the definitions of $u(z)$ and $v^*(w)$ that $v^*(g(z))=\log^+|f(g(z))|=u(z)$ on the circumference $\{|z|=r\}$. Hence by virtue of the maximum principle we have $u(z) \geqq v^*(g(z))$ in $\{|z| \leqq r\}$ and in paticular

$$u(0) \geqq v^*(g(0)). \tag{3.1}$$

Since $R>|g(0)|$, by Harnack's inequality we obtain

$$v^*(g(0)) = v(g(0)) \geqq \frac{R-|g(0)|}{R+|g(0)|} v(0). \tag{3.2}$$

On the other hand by Gauss' mean value theorem we have

$$u(0) = T(r, f(g)) \quad \text{and} \quad v(0) = T(R, f),$$

which, together with (3.1) and (3.2), proves our Lemma.

We are now ready to prove our Theorem 2. We deduce from Theorem 1 that

$$\limsup_{r\to\infty} \frac{T(r, f(g))}{T(M(r, g), f)} \leqq 1. \tag{3.3}$$

Since $g(z)$ is of lower order zero, it follows from a result of Kjellberg [5] that there is an increasing, unbounded, positive sequence $\{r_n\}$ such that

$$\min_{|z|=r_n} \log|g(z)| \sim \log M(r_n, g) \qquad (n\to\infty).$$

Hence for any $\varepsilon>0$ we have

$$|g(z)|>M(r_n, g)^{1-\varepsilon} \qquad \text{for } |z|=r_n,\ r_n>r_0 .$$

We may assume that $M(r_n, g)^{1-\varepsilon}>|g(0)|$ and (6) is valid for $r=M(r_n, g)$. Hence our Lemma 1 and (6) yield

$$T(r_n, f(g))\geqq \frac{M(r_n, g)^{1-\varepsilon}-|g(0)|}{M(r_n, g)^{1-\varepsilon}+|g(0)|}T(M(r_n, g)^{1-\varepsilon}, f)$$

$$\geqq (1-\varepsilon)^{\alpha}\frac{M(r_n, g)^{1-\varepsilon}-|g(0)|}{M(r_n, g)^{1-\varepsilon}+|g(0)|}T(M(r_n, g), f)$$

and consequently

$$\limsup_{r\to\infty}\frac{T(r, f(g))}{T(M(r, g), f)}\geqq \liminf_{n\to\infty}\frac{T(r_n, f(g))}{T(M(r_n, g), f)}\geqq (1-\varepsilon)^{\alpha} .$$

Since ε is arbitrary, (7) follows from this and (3.3).

Thus the proof of Theorem 2 is complete.

§4. Proof of Theorem 4.

We first need the following lemma, which we can deduce from the proof of Lemma 1 in Clunie [2] (cf. [4, Lemma 2]):

LEMMA 2. *Let $g(z)$ be a transcendental entire function, K a positive number and $\alpha(r)$ and $\beta(r)$ two unbounded, strictly increasing, continuous functions satisfying*

$$\alpha(r)\geqq r , \qquad \beta(r)\geqq r \qquad \text{and}$$

(4.1)

$$\log\beta(\eta\alpha(r))=o(T(\xi r, g)) \qquad (r\to\infty) ,$$

where η and ξ are constants satisfying $\eta>1$ and $0<\xi<1$. Let c satisfy $\xi<c\leqq 1$. Then there are a positive number R_0 and an unbounded increasing sequence $\{r_\nu\}_{\nu=1}^{\infty}$ with $r_1>R_0$ and $r_\nu\to\infty$ $(\nu\to\infty)$ such that for $\nu\geqq 1$ and for all r in $r_\nu\leqq r\leqq\alpha(r_\nu)$ and all w satisfying $\beta(R_0)\equiv R_1\leqq|w|\leqq\beta(r)$ we have

$$n(cr, w, g)>K .$$

We also need the following well-known inequalities:

LEMMA 3. *Let $f(z)$ be a meromorphic function and c a constant satisfying $0<c<1$. Then there are two positive constants r_0 and R_0 such that for all $r\geqq R_0$*

$$n(cr, 0, f)\log(1/c)\leqq N(r, 0, f)\leqq n(r, 0, f)(\log r-\log r_0)$$

Now we shall prove Theorem 4. Choose two constants η and ξ such that

$\eta>1$, $0<\xi<1$ and $\xi'=\xi/\eta$. Then (8) yields

$$\log \beta(\eta r)=\log \beta(\xi r/\xi')=o(T(\xi r, g)) \qquad (r\to\infty),$$

which shows that (4.1) is true with $\alpha(r)=r$. Hence Lemma 2 implies that there is an unbounded increasing sequence $\{r_\nu\}$ such that for all w satisfying $R_1<|w|\leqq\beta(r_\nu)$ we have

$$n(cr_\nu, w, g)>K/\log(1/c). \tag{4.2}$$

Let $\{w_\mu\}$ be the zeros of $f(z)$. Then taking Lemma 3 and (4.2) into account we have

$$\begin{aligned} N(r_\nu, 0, f(g)) &\geqq n(cr_\nu, 0, f(g))\log(1/c) \\ &= \sum_\mu n(cr_\nu, w_\mu, g)\log(1/c) \\ &\geqq K(n(\beta(r_\nu), 0, f)-n(R_1, 0, f)) \\ &\geqq K\Big(\frac{N(\beta(r_\nu), 0, f)}{\log\beta(r_\nu)-\log r_0}-n(R_1, 0, f)\Big). \end{aligned}$$

Using this and Nevanlinna's first main theorem, we obtain Theorem 4.

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