# *Growth of Degree for Iterates of Rational Maps in Several Variables*

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ABSTRACT. We investigate maps on $\mathbb{P}^2$ and $\mathbb{C}^3$ for which the algebraic degrees of the iterates, $d_m$, are not constant nor grow like $d^m$. These maps are called $\mathcal{B}$-maps. Answering a question posed by Ghys, we will show here that there are only countably many sequences $\{d_m\}$ of natural numbers for which there exists a rational map $f$ on $\mathbb{P}^2$ with deg $f^m = d_m$, for all $m \in \mathbb{N}$. We show here also that there are indeed infinitely many possible sequences $\{d_m\}$ with $d_1 > 2$.

## 1. INTRODUCTION

Friedland and Milnor ([6]) divided the polynomial automorphisms of $\mathbb{C}^2$ into two families: the elementary maps, consisting of maps for which the degree of the iterates remain constant; the other family, the Hénon maps, formed by maps for which the degree of the $m$-th iterate is $d^m$, where $d$ is the degree of the map. Bonifant ([1]) found to the contrary several rational maps on $\mathbb{P}^2$ for which the degrees of the iterates grow differently from the elementary maps and the Hénon maps. The problem remained whether the set of possible sequences $\{d_m\}$ of degrees of iterates was uncountable. Ghys pointed out to one of the authors that he thought this set should be countable.

The objective of this paper is to show that indeed the set of sequences $\{d_m\}$ of natural numbers for which there exists a rational self map $f$ of $\mathbb{P}^2$ such that deg $f^m = d_m$ is countable. We also show that there are infinitely many possibilities for the sequence of degrees of iterates of rational self maps of $\mathbb{P}^2$. Based in a



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result of Fornæss and Wu ([5]), we show that the number of sequences of degrees of iterates of degree 2 polynomial automorphisms of $\mathbb{C}^3$ is finite.

To simplify notation we call $\mathcal{A}$-maps, those rational self maps of $\mathbb{P}^n$ or those polynomial automorphisms of $\mathbb{C}^n$, that behave under iteration either like elementary maps or Hénon maps of $\mathbb{C}^2$. The complement of the $\mathcal{A}$-maps are the $\mathcal{B}$-maps. We give here the complete list of $\mathcal{A}$ and $\mathcal{B}$-maps of the polynomial automorphisms of $\mathbb{C}^3$ of degree 2. The $\mathcal{B}$-maps are classified as follows,

$\mathcal{B}$-linear, if the growth of the degree of its iterates is non constant and affine.
$\mathcal{B}$-Fibonacci, if the growth of the degree of its iterates follows a $d$-Fibonacci sequence, $\{dC_m\}$, i.e., a multiple of the Fibonacci sequence.
$\mathcal{B}$-flashing, if the degrees of its iterates grow every second time.
Basic-$\mathcal{B}$-maps, if after high enough iterates its degree remains constant.

We finish this paper showing that in $\mathbb{P}^2$ the sets of linear and Fibonacci $\mathcal{B}$-maps are also not empty. We study the dynamical features of two examples of such maps. At least in the Fibonacci case, one of the maps whose dynamics is studied here, is not birational. In both, the $\mathcal{B}$-linear and $\mathcal{B}$-Fibonacci cases, we give examples of maps with a finite and an infinite number of degree lowering curves.

## 2. BASIC FACTS

In this section we will give some definitions and prove some propositions that will form the basis of our work. We start with the set up for rational maps on $\mathbb{P}^n$.

Since $\mathbb{C}^n \subset \mathbb{P}^n$, every point $(x_1, \dots x_n)$ of $\mathbb{C}^n$ will be represented by $[x_1 : \cdots : x_n : 1]$ as an element of $\mathbb{P}^n$. Conversely, a point $[\zeta_1 : \cdots : \zeta_{n+1}] \in \mathbb{P}^n$ with $\zeta_{n+1} \neq 0$ corresponds to the point $(\zeta_1/\zeta_{n+1}, \dots, \zeta_n/\zeta_{n+1}) \in \mathbb{C}^n$. Points of the complementary set, i.e., those points for which $\zeta_{n+1} = 0$, are the points at infinity. This illustrates the fact that $\mathbb{P}^n$ contains $n+1$ copies $\mathbb{C}^n_j$, of $\mathbb{C}^n$. Each one of the $\mathbb{C}^n_j$ is given by $\zeta_j \neq 0$, $j = 1, \dots, n+1$.

Let us consider now rational functions from $\mathbb{C}^n_j$ to $\mathbb{C}$. From above, taking $x_k = \zeta_k/\zeta_{n+1}$ for all $k$, $1 \leq k \leq n$ and clearing denominators, we can rewrite a rational function $\rho = p(x_1, \dots, x_n)/q(x_1, \dots, x_n)$ on $\mathbb{C}^n_j$ in the form $P(\zeta_1, \dots, \zeta_{n+1})/Q(\zeta_1, \dots, \zeta_{n+1})$, where $P$ and $Q$ are homogeneous polynomials of the same degree. Hence, its value at a point $(\zeta_1, \dots, \zeta_{n+1})$ does not change if we multiply the homogeneous coordinates by a common multiple, and $\rho$ can be viewed then as a function on $\mathbb{P}^{n+1}$.

Let $R : \mathbb{C}^n_j \to \mathbb{C}^n_j$ for some $j$, $j = 1, \dots, n+1$ be a rational map, defined by $(x_1, \dots, x_n) \mapsto (u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n))$. We first rewrite each coordinate function $u_j(x_1, \dots, x_n)$, for all $j$, $j = 1, \dots, n$ in the form

$U_j(\zeta_1,\ldots,\zeta_{n+1})/V_j(\zeta_1,\ldots,\zeta_{n+1})$, where $U_j$, $V_j$ are homogeneous polynomials with $\deg U_j = \deg V_j$ for all $j$, $j = 1, \ldots, n$. Next we put all the components of $R$ over a common denominator, i.e., in the form $(A_1/C,\ldots,A_n/C)$, with $\deg A_1 = \cdots = \deg A_n = \deg C$. Finally, introducing homogeneous coordinates $\zeta'_j/\zeta'_{n+1} = A_j/C$ for all $j$, $j = 1, \ldots, n$, we rewrite $R$ as follows,

$$(\zeta_1,\ldots,\zeta_{n+1}) \mapsto [A_1(\zeta_1,\ldots,\zeta_{n+1}) : \cdots : A_{n+1}(\zeta_1,\ldots,\zeta_{n+1})],$$

where $A_j$ for all $j$, $j = 1, \ldots, n+1$ are homogeneous polynomials of the same degree. Now $R$ is a rational self-map of $\mathbb{P}^n$.

Since we are interested in studying the growth under iteration of the algebraic degree of an homogeneous self map of $\mathbb{P}^n$, we need to cancel all common factors.

***Definition 2.1.*** The *algebraic degree* $d$ of any homogeneous map

$$R : \mathbb{P}^n \to \mathbb{P}^n \quad \text{via} \quad [z_0 : \cdots : z_n] \mapsto [Q_0(z_0,\ldots,z_n) : \cdots : Q_n(z_0,\ldots,z_n)]$$

is defined to be the common degree of $Q_j$ for all $j = 0, \ldots, n$ after cancellation of all common factors.

Since the self maps of $\mathbb{P}^n$ with algebraic degree 1 do not have interesting dynamical properties, we will assume from now on that $d \geq 2$.

***Definition 2.2.*** The subset of rational self-maps of $\mathbb{P}^n$ for which the sequences $\{d_m\}$, of algebraic degrees of iterates remain constant or grow like $d^m$, will be called $\mathcal{A}$-*maps*. The complement of the $\mathcal{A}$-*maps*, i.e., the set of rational self-maps of $\mathbb{P}^n$ with the property that the algebraic degree of their iterates $d_m$, does not grow like $d^m$ or remains equal to the constant $d$ under iteration, will be called $\mathcal{B}$-*maps*.

In $\mathbb{P}^2$, elementary and generic meromorphic maps are examples of $\mathcal{A}$-maps. In $\mathbb{C}^2$, shears and Hénon maps are examples of these kind of maps.

***Definition 2.3.*** Let $\mathrm{Fix}_R$ be the set of fixed points of $R$,

$$\mathrm{Fix}_R = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid Q_j(z_0,\ldots,z_n) = \lambda z_j \text{ for all } j,\ 0 \leq j \leq n,\ \lambda \in \mathbb{C}\setminus\{0\}\}.$$

***Definition 2.4.*** Let $\mathcal{I}_R$ be the set of indeterminacy of $R$,

$$\mathcal{I}_R = \{[z_0 : \cdots : z_n] \in \mathbb{P}^n \mid Q_0(z_0,\ldots,z_n) = \cdots = Q_n(z_0,\ldots z_n) = 0\}.$$

***Definition 2.5.*** An irreducible algebraic variety $\mathcal{V}_j \not\subset \mathcal{I}_R$ of dimension $\dim \mathcal{V}_j \geq 1$ is *R-constant* if $R$ takes a constant value on $\mathcal{V}_j \setminus \mathcal{I}_R$, say $R(\mathcal{V}_j \setminus \mathcal{I}_R) = p_j$ for some $p_j \in \mathbb{P}^n$. We denote by $\mathcal{V}_R = \bigcup_j \mathcal{V}_j$ the union of all such maximal irreducible compact complex varieties $\mathcal{V}_j$ on each of which $R$ has a constant value.

We observe that the union of all such maximal irreducible compact $R$-constant complex varieties $\mathcal{V}_j$ is not necessarily finite, nor even countable.

***Proposition 2.6.*** *If $\mathcal{V}_R \neq \emptyset$, then also $\mathcal{I}_R \neq \emptyset$. In fact, each irreducible branch $\mathcal{V}_j$ of $\mathcal{V}_R$ must intersect $\mathcal{I}_R$. It can happen that $\mathcal{V}_R = \emptyset$, while $\mathcal{I}_R \neq \emptyset$* (see [3], Proposition 1.2).

***Definition 2.7.*** For a given point $p \in \mathbb{P}^n$, a point $q$ is said to be a *preimage* of $p$ if $R$ is defined at $q$, i.e., $q \notin \mathcal{I}_R$, and $R(q) = p$. We will denote the set of preimages of $p$ as $R^{-1}(p)$.

***Proposition 2.8.*** *Let $R : \mathbb{P}^n \to \mathbb{P}^n$ be a rational map of degree $d$. Assume that $\mathcal{I}_R \neq \emptyset$ but is finite, and assume also that $R$ is of maximal rank $n$. Then for any $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$, which is not the image of an $R$-constant variety, $R^{-1}(a) \subset \mathbb{P}^n \setminus \mathcal{I}_R$,* $\operatorname{card} R^{-1}(a) = d' < d^n$. *Here we count the number of points with multiplicity.*

*Proof.* Let us consider the map $\mathbb{C}^{n+2} \to \mathbb{C}^{n+1}$ such that $(z_0, \ldots, z_n, t) = (z, t) \mapsto R(z) - at^d$, where

$$R(z) - at^d = [P_0(z_0, \ldots, z_n) : \cdots : P_n(z_0, \ldots, z_n)] - [a_0 t^d : \cdots : a_n t^d].$$

So, we have $n+1$ polynomials in $\mathbb{P}^n$.

Let us take $t = 1$, by hypothesis we are assuming that there is no positive dimensional variety $\mathcal{V}$, $\mathcal{V} \subset \mathbb{P}^n \setminus \mathcal{I}_R$, such that $R(\mathcal{V}) = a$. Then the number of zeroes of $R(z) - at^d$, when $t = 1$, is finite in $\mathbb{P}^n$. So, by Bezout's theorem, the number of zeroes of $R(z) - at^d$ will be $d^{n+1}$, counting multiplicity. By hypothesis $\mathcal{I}_R \neq \emptyset$ and finite, so for $p \in \mathcal{I}_R$, $[p : 0]$ is a zero of multiplicity at least $d$. Hence the number of zeroes of $R(z) - at^d$ when $t = 1$ is less than $d^n$. Since rotation of $t$ by a $d$-th root of unity produces an equivalent solution in $\mathbb{P}^n$, we get that $d' < d^n$. ∎

***Definition 2.9.*** An irreducible variety $\mathcal{V}_j$ of codimension 1, $\mathcal{V}_j \subset \mathbb{P}^n$, is said to be a *degree lowering variety* if, for some (smallest) $n = n_j \geq 1$, $R^n(\mathcal{V}_j) \subset \mathcal{I}_R$.

In this paper we will discuss the growth of the degrees of the iterates of self-maps $R$ of $\mathbb{P}^n$ with degree lowering varieties. When there is a degree lowering variety $\mathcal{V}_| = \{h_j = 0\}$, all the components of the iterates of $R^{m_j+1}$ vanish on it.

Hence one needs to factor out a power of $h_j$ in order to describe the map properly. Hence the degree of the iterate will drop below $d^{m_j+1}$.

## 3. THE GHYS QUESTION

We show first in this section that there are at most countably many sequences $\{d_m\}$ of natural numbers, for which there exists a rational self map $R$ of $\mathbb{P}^n$ whose degree under iteration satisfies that deg $R^m = d_m$, for all $m \in \mathbb{N}$. Conversely, we show after that there are infinitely many possibilities for the sequence of degrees of iterates.

Let $R(z_0, \dots, z_n)$ be a homogeneous polynomial of degree $d$. Then $R$ is a finite sum of terms $c_{i_0,i_1,\dots,i_n} z_0^{i_0} z_1^{i_1} \dots, z_n^{i_n}$, where $i_0$ ranges from 0 to $d$ and, for a given $i_0$, $i_1$ ranges from 0 to $d - i_0$. And in general for given $i_0$, $i_1$, $\dots$, $i_{k-1}$, $i_k$ ranges from 0 to $d - (i_0 + i_1 + \cdots + i_{k-1})$, $0 \le k \le n$. Then $i_n$ is finally determined and the number of coefficients $C$ of $R$ is given by $\binom{d+n}{n} = (d+1)\cdots(d+n)/n!$.

Let

$$\mathcal{H}_d = \{R = [P_0 : \cdots : P_n] \mid P_i \text{ is a homogeneous polynomial in } \mathbb{C}^{n+1}, \\ \deg P_i = d,\ 0 \le i \le n\} \setminus \{[0 : \cdots : 0]\}.$$

We can identify $\mathcal{H}_d$ with the list of coefficients $C$. Then $\mathcal{H}_d$ is given by a set of $(n+1)\binom{d+n}{n}$ coefficients $C$. Moreover we only consider $C \neq 0$, and if we multiply all the coefficients with the same complex non zero number, we get the same map on $\mathcal{H}_d$. In other words, $\mathcal{H}_d = \mathbb{P}((V_d^{n+1})^{n+1})$ where $V_d^{n+1}$ is the set of homogeneous polynomials in $\mathbb{C}^{n+1}$ of degree $d$, $\dim V_d^{n+1} = \binom{d+n}{n}$ and hence $\dim \mathcal{H}_d = r(d) = (n+1)\binom{d+n}{n} - 1$.

Let $X = \{R \in \mathcal{H}_d \mid \operatorname{rank} R < n\}$.

***Proposition 3.1.*** *$X \subset \mathcal{H}_d$ is a closed analytic subvariety.*

*Proof.* Let $R$ be a map of degree $d$ with coefficients in $X$. Let $f(z_0, \dots, z_n)$ be the Jacobian determinant of its lifting in $\mathbb{C}^{n+1}$, $f(z_0, \dots, z_n)$ is a polynomial map of degree $(n+1)(d-1)$. Since by hypothesis $\operatorname{rank} R < n$, $f(z_0, \dots, z_n)$ must be equal to zero. Now, each of the coefficients of $f(z_0, \dots, z_n)$ is a homogeneous polynomial in $C$, therefore its common zero set $X$ is a closed analytic subvariety of $\mathcal{H}_d$. □

Now, let $\mathcal{P}_d$ denote the projective space of all homogeneous, not identically zero, polynomials of $\mathbb{C}^{n+1}$ of degree $d$, $\dim \mathcal{P}_d = s(d) = \binom{d+n}{n} - 1$.

Let $p, q$ be strictly positive integers. We define the canonical maps

$$\Phi_{p,q} : \mathcal{P}_d \times \mathcal{H}_q \to \mathcal{H}_{q+p} \quad \text{via} \quad (A, [P_0 : P_1 : P_2]) \mapsto [AP_0 : AP_1 : AP_2].$$

Let $Z_{p,q}$ denote the irreducible image of $\Phi_{p,q}$.

For any $k$, $d - 1 \geq k \geq 1$, let $Z^{d,k} = \bigcup_{\ell \leq k} Z_{d-\ell,\ell}$ be those maps in $\mathcal{H}_d$ of degree at most $k$; we set $Z^{d,d} = \mathcal{H}_d$. Then all the $Z^{d,k}$ are closed analytic subsets of $\mathcal{H}_d$ and $Z^{d,1} \subset Z^{d,2} \cdots \subset Z^{d,d}$. We say that the maps in $Z^d_k := Z^{d,k} \setminus Z^{d,k-1}$ have (reduced) degree $k$. Then the $Z^d_k$ stratify the space $\mathcal{H}_d$.

For every strictly positive integer $m$, we consider the holomorphic map

$$\Psi_{d,m} : \mathcal{H}_d \setminus X \to \mathcal{H}_{d^m} \quad \text{via} \quad F \mapsto F^m.$$

This induces a stratification of $\mathcal{H}_d \setminus X$ given by $W^{d,m}_k := \Psi^{-1}_{d,m}(Z^{d^m}_k)$. The $W^{d,m}_k$ are closed analytic subsets of $\mathcal{H}_d \setminus (X \cup_{\ell<k} W^{d,m}_\ell)$ and consists of those maps for which the degree of the $m^{th}$ iterate is exactly $k$.

Next we combine these stratifications of $\mathcal{H}_d \setminus X$ inductively. Let ${}_jW^{d,1}$ denote the irreducible components of the $\{W^{d,1}_\ell\}$. Let us assume next that we have a countable sequence of irreducible varieties ${}_jW^{d,m}$ stratifying $\mathcal{H}_d \setminus X$. We assume moreover that on each ${}_jW^{d,m}$, $g, h \in {}_jW^{d,m}$ implies that deg $g^k =$ deg $h^k$, for all $k \leq m$.

We stratify each ${}_jW^{d,m}$ into pieces ${}_jW^{d,m} \cap W^{d,m+1}_k$. Listing all the irreducible components of all these we obtain the sequence $\{{}_jW^{d,m+1}\}$.

We have shown:

***Theorem 3.2.*** *There exists for each $m$ a countable sequence of disjoint irreducible varieties ${}_jW^{d,m}$, such that $\mathcal{H}_d \setminus X = \bigcup {}_jW^{d,m}$.*

*If $h, g \in {}_jW^{d,m}$, then* deg $h^k =$ deg $g^k$ *for all $k \leq m$. Moreover, each ${}_\ell W^{d,m+1} \subset {}_{j(\ell)}W^{d,m}$ for some $j(\ell)$. Furthermore, either ${}_\ell W^{d,m+1}$ is a dense given set in ${}_{j(\ell)}W^{d,m}$, or* $\dim {}_\ell W^{d,m+1} < \dim {}_{j(\ell)}W^{d,m}$.

We will show now that there are only countable many sequences $\{d_m\} \subset \mathbb{N}$ for which there exists a rational map $R$ of degree $d$ on $\mathcal{H}_d$ for which deg $R^m = d_m$.

Let $\mathbb{S}$ denote the collection of all $s = \{W_m\} = \{{}_{j_m}W^{d,m}\}^\infty_{m=1}$, where ${}_{j_{m+1}}W^{d,m+1} \subset {}_{j_m}W^{d,m}$ as in the theorem. If $h$ is any map of degree $d$, then $h$ belongs to $\bigcap_m {}_{j_m}W^{d,m}$ for some sequence $s$. We say that $h \in s$ for short. Also note that if $h, g \in s$, then deg $h^k =$ deg $g^k$ for all $k$. Hence we only need to show that the set $\mathbb{S}$ is countable. For each $s = \{W_m\}$ let $\lambda = \{\lambda_m\}$ be the sequence of dimensions, $\lambda_m = \dim W_m$. Then $(m+1)(d+1)\cdots(d+m)/m! \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq \cdots \geq 0$. Observe that, for each $s = \{W_m\}$, there is a smallest integer $m = m(s)$ for which $\lambda_m = \lambda_{m+1} = \cdots$. Let $\tilde{s} = \{W_j\}_{j \le m(s)}$ to count the number of possible growth of degrees of iterates we only need to consider $\{\lambda_1, \dots, \lambda_{m(s)}\}$. But these are contained in

$$\bigcup_{m \ge 1} \{0, 1, \dots, (m+1)(d+1) \cdots (d+m)/m!\}^m,$$

which is a countable set. □

We will prove next that there are infinitely many possibilities for the growth of the degree of the iterates of a rational map $F$ of degree $d$, $d \ge 2$, in $\mathbb{P}^2$.

***Proposition 3.3.*** *Let us consider the map* $F : \mathbb{P}^2 \to \mathbb{P}^2$,

$$[z : w : t] \overset{F}{\mapsto} \left[ zt^{d-1} : (wt^{d-1} + z^d) \cos\left(\frac{\pi}{m}\right) - t^d \sin\left(\frac{\pi}{m}\right) \right.$$
$$\left. : (wt^{d-1} + z^d) \sin\left(\frac{\pi}{m}\right) + t^d \cos\left(\frac{\pi}{m}\right) \right].$$

*Then*

$$\begin{aligned} \deg F^k[z : w : t] &= d^k \quad \text{for all } k \le m,\ m \in \mathbb{N}, \\ \deg F^k[z : w : t] &< d^k \quad \text{for all } k \ge m+1,\ m \in \mathbb{N}. \end{aligned}$$

*So, there are infinitely many possibilities for the growth of the degree of a rational self map of* $\mathbb{P}^2$.

*Proof.* Let us observe that $F[z : w : t]$ is a composition of a rotation $h_{w,t}$, and a shear $R$, i.e., $F = (h_{w,t} \circ R)[z : w : t]$, where

$$h_{w,t}[z : w : t] = \left[ z : w \cos\left(\frac{\pi}{m}\right) - t \sin\left(\frac{\pi}{m}\right) : w \sin\left(\frac{\pi}{m}\right) + t \cos\left(\frac{\pi}{m}\right) \right],$$

and $R[z : w : t] = [zt^{d-1} : wt^{d-1} + z^d : t^d]$, $d \ge 2$. We will use the following dynamical features of $R$.

1. $\mathcal{I}_R = [0 : 1 : 0] = \mathcal{I}_F$.
2. $(t = 0)$ is the only degree lowering curve of $R$.
3. $(z = 0)$ is the set of fixed points of $R$, $\mathrm{Fix}_R$.

Since $F(t=0) = [0 : \cos(\pi/m) : \sin(\pi/m)] \in (z=0)$, and $(z=0)$ is fixed by $R$ and invariant under $h_{w,t}$, we only need to look at $h^k_{w,t}(z=0)$, for all $k \in \mathbb{N}$.

In $(z=0)$,

$$h_{w,t}[w:t] = \left[ \begin{pmatrix} \cos(\pi/m) & -\sin(\pi/m) \\ \sin(\pi/m) & \cos(\pi/m) \end{pmatrix} \begin{pmatrix} w \\ t \end{pmatrix} \right]^T = \left[ H \begin{pmatrix} w \\ t \end{pmatrix} \right]^T.$$

Since in $(z=0)$, $\binom{1}{0}$ is an eigenvector of $H^m$, i.e., $H^m \binom{1}{0} = \lambda \binom{1}{0}$, with $\lambda \neq 0$, $F^k(t=0)$, will hit the point of indeterminacy $[0:1:0]$ for the first time when $k = m$, i.e., in the $m$-th iterate. In other words, we have $F^m(t=0) = [0:1:0]$ therefore deg $F^k[z:w:t] = d^k$ for all $k \leq m$ and deg $F^k[z:w:t] < d^k$ for all $k \geq m+1$.

Since $m$ varies in $\mathbb{N}$, there are infinitely many possibilities for the sequences of the degrees of iterates of rational self maps of $\mathbb{P}^2$ of any degree. □

## 4. $\mathcal{B}$-MAPS OF $\mathbb{P}^2$

In this section we show the existence of $\mathcal{B}$-linear and Fibonacci maps on $\mathbb{P}^2$. Furthermore we study their dynamical behavior.

***4.1 $\mathcal{B}$-linear maps.*** As established in the introduction:

***Definition 4.1.*** A $\mathcal{B}$-map will be called *linear*, if the growth of the degree of its iterates is non constant and affine.

***Theorem 4.2.*** *The family of $\mathcal{B}$-linear maps of $\mathbb{P}^2$ with an infinite number of degree lowering curves, is non-empty. In fact,*

$$\text{(1)} \qquad R : \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{via} \quad R[z:w:t] \mapsto [zt : z^2 + zt + wt : t^2 + zt]$$

*is such a $\mathcal{B}$-linear map.*

*In addition, there exists a dense open set $U \subset \mathbb{P}^2$, where the iteration of $R$ is well defined, and $U$ satisfies that:*

1. *$U$ is a parabolic basin, i.e., $R^m[z:w:t] \to [0:1:1]$, as $m \to \infty$. The convergence is uniform on compact subsets of $U$, and $[0:1:1] \in \partial U$.*
2. *$U \cong \mathbb{C}_w \times \mathbb{C}_t \setminus (\mathbb{Z}^- \cup \{0\})$, in particular, $U$ is not Kobayashi hyperbolic.*

Before proving the theorem, we will state the following dynamical features of $R$.

1. $\text{Fix}_R = \{[0:w:1] \mid w \in \mathbb{C}\} \cong \mathbb{C}$.
2. $\mathcal{I}_R = \{[0:1:0]\} = \{p_0\}$.

We observe that $R$ has infinitely many degree lowering curves. In fact, the degree lowering curves of $R$ are given by the set $\{(kz + t = 0) \mid k \in \mathbb{N} \cup \{0\}\}$.

The varieties $\{(kz + t = 0) \mid k \in \mathbb{N} \cup \{0\}\}$ converge in the Hausdorff metric of sets as $k \to \infty$ to $\mathrm{Fix}_R \cup \{p_0\}$, since $z = -t/k \to 0$ as $k \to \infty$.

We define $R^{-\infty}(p_0) := \bigcup_{k \geq 0}(kz + t = 0)$.

Observe that $R^{-\infty}(p_0)$ is not a closed set, since it does not contain all its limit points, namely the set $\mathrm{Fix}_R$.

Let us define

$$U := \mathbb{P}^2 \setminus \mathrm{Fix}_R \cup \Big( \bigcup_{k \geq 0} \{kz + t = 0\} \Big). \tag{2}$$

From what we have seen above, we can write $\mathbb{P}^2$ as the disjoint union of three sets: $\mathbb{P}^2 = \mathrm{Fix}_R \sqcup R^{-\infty}(p_0) \sqcup U$.

Each one of these sets is forward invariant for $R$, so we can study the dynamics on each piece separately.

***1. Dynamics on*** $\mathrm{Fix}_R$**.** The Jacobian matrix of the map (1) in $(t = 1)$ is given by

$$\mathcal{J}_R(0, w) = \begin{pmatrix} \dfrac{1}{(1+z)^2} & 0 \\ \dfrac{z^2 + 2z + 1 - w}{(1+z)^2} & \dfrac{1}{1+z} \end{pmatrix}_{(0,w)} = \begin{pmatrix} 1 & 0 \\ 1 - w & 1 \end{pmatrix},$$

so the point $[0 : 1 : 1]$ is the only fixed point where the Jacobian is the identity matrix. We show that in fact this is the only fixed point with an attracting basin. We are concerned with understanding the local dynamics near the other fixed points to gain insight into why they do not have a basin of attraction.

Trying to shed some light on this matter we will move an arbitrary fixed point $(0, a) \in \mathbb{C}^2$, to the origin under a change of coordinates. We expand $R$ in $(t = 1)$ in a Taylor series, and obtain:

$$R(z, w) = (z - z^2 + z^3 - z^4 + \cdots, (1-a)z + w + az^2 - zw + \cdots) \tag{3}$$

If $\{(z_k, w_k)\}_k$ is an orbit converging to $(0, 0)$ then, since the first coordinate of (3) depends only on $z$, we use the theory of parabolic dynamics in one dimension (see [2]) to show that $z_k \to 0$ along the positive real axis. In fact, asymptotically, $z_k \sim 1/k$. If we next consider the second coordinate, we note that $w_{k+1} \sim w_k + (1-a)z_k \sim w_k + (1-a)1/k$. Since $\sum_{k=1}^{\infty} 1/k$ is a divergent series, we can not expect $w_k \to 0$. This is what we consider to be the reason that $[0 : a : 1]$ does not have an attracting basin for $a \neq 1$. We show below that all points on a dense open set converge under iteration to the point $[0:1:1]$.

***2. Dynamics on*** $R^{-\infty}(p_0)$**.** Let us denote by $L_{-k} := \{kz + t = 0\}$, $k \in \mathbb{N} \cup \{0\}$, $L'_{-k} := L_{-k} \setminus \{p_0\} \cong \mathbb{C}$, $k \in \mathbb{N} \cup \{0\}$. With this notation we can write $R^{-\infty}(p_0) = \{p_0\} \sqcup (\bigcup_{k \geq 0} L'_{-k})$.

***Proposition 4.3.*** *Let* $R$ *be as in* (1). $R$ *is biholomorphic on* $L'_{-k}$, $k \geq 1$ *and* $R(L'_{-k}) = L'_{-(k-1)}$.

*Proof.* Parametrizing $L'_{-k}$ and computing $R : L'_{-k} \to L'_{-(k-1)}$ in this parametrization, we get $R[1 : \eta : -k] = [1 : \eta + (1 - 1/k) : 1 - k]$, so $R$ acts as a translation with respect to these parametrizations of $L'_{-k}$ and $L'_{-(k-1)}$, therefore it is biholomorphic. ◻

Since we are interested in studying the dynamics of $R$ near its points of indeterminacy, we need to define a suitable concept of blow up.

***Definition 4.4.*** Let $q \in \mathcal{I}_R$ and let $\mathcal{B}_c$ be the ball of radius $c$ about $q$. Write $\mathcal{W}_c := \overline{\{R(p) \mid p \in \mathcal{B}_c \setminus \mathcal{I}_R\}}$. The *blow up* of $q$ is then the set $\mathcal{W}_q := \bigcap_{c>o} \mathcal{W}_c$.

***Remark 4.5.*** In other words, the blow up $\mathcal{W}_q$ of the indeterminacy point $q$ will be the fiber over $q$ in the closure of the graph of $R|_{\mathbb{P}^2 \setminus \mathcal{I}_R}$ in $\mathbb{P}^2 \times \mathbb{P}^2$. With this convention, $q$ will not be in the preimage of the points in its blow up.

***Proposition 4.6.*** *The curve* $(z = t)$ *is the blow up of the point of indeterminacy,* $[0:1:0]$ *of* $R$*, given as in* (1).

*Proof.* Let $\{z_k\}_k$ be a sequence such that $z_k \neq 0$, $k \geq 1$, and $z_k \to 0$ as $k \to \infty$. Let $\alpha \in \mathbb{C}$. For large $k$, let $t_k = z_k^2/(\alpha z_k - 1)$, $k \geq 1$. We have that $[z_k : 1 : t_k] \to [0:1:0]$, $k \to \infty$, so $R[z_k : 1 : t_k] = [1 : \alpha + 1 : z_k/(\alpha z_k - 1) + 1]$. Then $R[z_k : 1 : t_k] \to [1 : \alpha + 1 : 1]$ as $k \to \infty$.

Now let $\{x_k\}_k$ be the sequence identically zero, and let $\{v_k\}_k$ be a sequence such that $v_k \neq 0$, $k \geq 1$, and $v_k \to 0$ as $k \to \infty$.

We have that $[x_k : 1 : v_k] = [0 : 1 : v_k] \to [0:1:0]$, so $R[0 : 1 : v_k] = [0 : 1 : v_k]$. Then $R[0:1:v_k] \to [0:1:0]$. Therefore the curve

$$(z = t) = \{[1 : \alpha + 1 : 1] \mid \alpha \in \mathbb{C}\} \cup \{[0:1:0]\}$$

is in the blow up of $[0:1:0]$.

Now we want to show that there are no other points in the blow up of $[0 : 1 : 0]$. Let us suppose that there is a point $q$ in the blow up of $[0:1:0]$ such that $q = [q_0 : q_1 : q_2] \notin (z = t)$. Then $q_0 \neq q_2$, and we can suppose that either $q_0 = 0$ or $q_0 = 1$.

Let us suppose first that $q_0 = 0$, then there exists $p_k := [z_k : 1 : t_k] \to [0 : 1 : 0]$ such that $q_k := R(p_k) \to q$. There exists $k_0 \in \mathbb{N}$ such that $t_k^2 + z_k t_k \neq 0$, for all $k \geq k_0$. On one hand, dividing by $t_k^2 + z_k t_k$, we get $R(p_k) \to [0 : q_1 : 1]$ as $k \to \infty$. On the other hand, since $R(p_k) = [z_k t_k : z_k^2 + z_k t_k + t_k : t_k^2 + z_k t_k]$, factoring out $t_k^2$, we get $[z_k/t_k : z_k^2/t_k^2 + z_k/t_k + 1/t_k : 1 + z_k/t_k] \to [0 : q_1 : 1]$ as $k \to \infty$. Therefore the sequences $\{z_k/t_k\}$ and $\{z_k^2/t_k^2 + z_k/t_k + 1/t_k\}$ converge, so $\{1/(t_k)\}$ converges. This is impossible, since $t_k \to 0$ as $k \to \infty$.

The case $q_0 = 1$ is analogous. ◻

***3. Dynamics on** $U$.* We study next the dynamics of $R$ on $U$. For this purpose we are going to consider the plane $(z = 1)$. With this new notation,

$$U = (z = 1) \setminus \bigcup_{k \geq 0} \{t = -k\} \cong \mathbb{C}^2_{w,t} \setminus \bigcup_{k \geq 0} \{t = -k\}.$$

The map $R$ now can be written as follows: $R[1 : w : t] = [1 : w + 1 + 1/t : t + 1]$, i.e., $(w, t) \overset{R}{\mapsto} (w + 1 + 1/t, t + 1)$. In general,

$$R^k(w, t) = \left( w + k + \frac{1}{t} + \frac{1}{t+1} + \cdots + \frac{1}{t + (k-1)}, t + k \right). \tag{4}$$

***Proposition 4.7.*** *All the points of $U$ converge under iteration to the line $\mathbb{P}^1 \cong (z = 0)$ at infinity, in the following sense:*

*In $\mathbb{P}^2$, for all neighborhoods $V$ of $\mathbb{P}^1 \cong (z = 0)$ and for all $K$ compact, $K \subset U$, there exists $k_0 = k_0(V, K)$ such that, for all $k \geq k_0$, $R^k(K) \subseteq V$.*

*Proof.* Without loss of generality, we can assume that

$$V = (z = 0) \cup (\mathbb{C}^2_{w,t} \setminus B(0, r))$$

for some large $r \geq 0$. Here $\mathbb{C}^2_{w,t}$ is denoting the dependence of $\mathbb{C}$ on $w$ and $t$.

Let us pick a compact set $K \subset U$. By equation (4), every time we iterate we are translating the set $K$ by one unit in the $t$-coordinate. Since $K$ is compact, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $R^k(K) \subseteq V$. ◻

We use the following standard estimate.

***Lemma 4.8.*** *For* $k = 1, 2, 3, \ldots$

(a) $\ln(k + 1) \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \leq 1 + \ln k$,

(b) $\lim_{k\to\infty} \dfrac{1+1/2+1/3+\cdots+1/k}{\ln k} = 1.$

***Remark 4.9.*** (b) is usually written as: $1+1/2+1/3+\cdots+1/k \sim \ln k$.

Let $U$ be as defined in (2), and let $R$ be as defined in (1). The following proposition completes the proof of Theorem (4.2).

***Proposition 4.10.***

(a) *The open set $U$ is a parabolic basin, i.e., $R^k[z:w:t] \to [0:1:1]$ as $k\to\infty$, for every $[z:w:t]\in U$. The convergence is uniform on compact subsets of $U$, and $[0:1:1]\in\partial U$.*

(b) *$U \cong \mathbb{C}_w \times \mathbb{C}_t \setminus (\mathbb{Z}^- \cup \{0\})$, in particular $U$ is not Kobayashi hyperbolic.*

*Proof.*

(b) By (2) it is clear that $U \cong \mathbb{C}\times\mathbb{C}\setminus(\mathbb{Z}^-\cup\{0\})$. $U$ is not Kobayashi hyperbolic since it contains $\mathbb{C}$-lines $\mathbb{C}\times\{p\}$, $p\in\mathbb{C}\setminus\mathbb{Z}^-\cup\{0\}$, (see [4]).

(a) Let us consider the compact set $K\subset U$. Pick a point $[1:w:t]\equiv(w,t)\in K$. By (4),

$$R^k[1:w:t] = \left[\frac{1}{t+k} : \frac{w+k+\sum_{j=0}^{k-1} 1/(t+j)}{t+k} : 1\right].$$

It is enough to show that $1/(t+k)\to 0$ and $\left(w+k+\sum_{j=0}^{k-1}1/(t+j)\right)/(t+k)\to 1$ uniformly on $K$. The fact that $1/(t+k)\to 0$ uniformly on $K$ is clear, since $t$ is bounded.

Let us prove next that

$$\frac{w+k+\sum_{j=0}^{k-1}1/(t+j)}{t+k} - 1 \to 0$$

uniformly on $K$.

$$\left\|\frac{w+k+\sum_{j=0}^{k-1}1/(t+j)}{t+k} - 1\right\| \le \frac{\|w-t\|}{\|t+k\|} + \frac{\left\|\sum_{j=0}^{k-1}1(t+j)\right\|}{\|t+k\|} \tag{5}$$

On $K$:

$$\frac{\|w-t\|}{\|t+k\|} \le \frac{\|w\|+\|t\|}{k-\|t\|} \le \frac{c_1(K)}{k-\|t\|} \to 0 \quad \text{as } k\to\infty. \tag{6}$$

By (5) and (6), we only need to show that

$$\frac{\left\|\sum_{j=0}^{k-1} 1/(t+j)\right\|}{\|t+k\|} \to 0 \quad \text{as } k \to \infty \text{ on } K.$$

Let us take $k_0 = \sup_{(w,t)\in K} 2\|t\|$.

$$\left\|\sum_{j=0}^{k-1} \frac{1}{t+j}\right\| \le \sum_{j=0}^{k_0-1} \left\|\frac{1}{t+j}\right\| + \sum_{j=k_0}^{k-1} \left\|\frac{1}{t+j}\right\|. \tag{7}$$

Now,

$$\frac{1}{\|t+j\|} \le \frac{1}{j-\|t\|} \le \frac{1}{\frac{1}{2}j} = \frac{2}{j}, \quad \text{for all } j \ge 2\|t\|. \tag{8}$$

So, by (7) and (8)

$$\left\|\sum_{j=0}^{k-1} \frac{1}{t+j}\right\| \le \sum_{j=0}^{k_0-1} \left\|\frac{1}{t+j}\right\| + \sum_{j=k_0}^{k-1} \frac{2}{j},$$

and hence, from Lemma 4.8,

$$\left\|\sum_{j=0}^{k-1} \frac{1}{t+j}\right\| \le 2(1+\ln k) + c_2(K). \tag{9}$$

By (8) and (9)

$$\frac{\left\|\sum_{j=0}^{k-1} 1/(t+j)\right\|}{\|t+k\|} \le \frac{4}{k}(1+\ln k + c_3(K)) \to 0 \quad \text{as } k \to \infty.$$

Therefore $R^k[z:w:t] \to [0:1:1]$ as $k \to \infty$, uniformly on $K$. □

We give here another example of a $\mathcal{B}$-linear birational map of $\mathbb{P}^2$, however in this case $R$ has only one degree lowering curve.

***Theorem 4.11.*** *The family of $\mathcal{B}$-linear maps of $\mathbb{P}^2$ with only one degree lowering variety is non-empty. In fact,*

$$R : \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{via} \quad R[z:w:t] \mapsto [zt : wz : t^2] \tag{10}$$

*is a $\mathcal{B}$-linear map. In addition,* *If* $|z| < 1$,

$$R^m[z : w : 1] \to [z : 0 : 1] \in \mathrm{Fix}_R \quad \text{as } m \to \infty,$$

*i.e., for all* $z_0$ *with* $|z_0| < 1$, $\mathbb{P}^1_{z_0} = [z_0 : w : 1]$ *satisfies that* $R^m(\mathbb{P}^1_{z_0}) \to [z_0 : 0 : 1]$.

*If* $|z| > 1$,

$$R^m[z : w : 1] \to [0 : 1 : 0] \in \mathcal{I}_R \quad \text{as } m \to \infty.$$

Before going into the proof of this theorem, we state the following dynamical features of $R$

1. $\mathrm{Fix}_R = \{[z : 0 : 1] \mid z \in \mathbb{C}\} \cup \{[1 : w : 1] \mid w \in \mathbb{C}\}$,
2. $\mathcal{I}_R = \{[0 : 1 : 0]\} \cup \{[1 : 0 : 0]\}$,
3. $R(z = 0) \in \mathrm{Fix}_R$,
4. $(t = 0)$ is the only degree lowering curve of this map.

***Remark 4.12.*** The point $[1 : 0 : 0] \in \mathcal{I}_R$ has no preimages.

The $m$-th iterate of the map (10), lifted to $\mathbb{C}^3$, is given by the formula

$$R^m(z, w, t) = t^{\alpha(m)}(zt^m, z^m w, t^{m+1}), \tag{11}$$

where $\alpha(m)$ is given by the equation $\alpha(m) = 2^m - (m + 1)$, $m \in \mathbb{N}$.

Before we study the dynamics of (10) on $\mathbb{P}^2 \setminus \mathcal{I}_R$, we will study its dynamics near the point of indeterminacy $[0 : 1 : 0]$.

***Proposition 4.13.*** *The curve* $(z = 0)$ *is the blow up of the point of indeterminacy,* $[0 : 1 : 0]$ *of* (10).

*Proof.* Similar to the proof of Proposition (4.6). □

Let us study now the dynamics of $R$ in $(t = 1)$. In this $\mathbb{C}^2$-plane our map can be written as $R[z : w : 1] = [z : zw : 1]$, since (11) the $m$-th iterate of $R$ in $(t = 1)$ will be given by

$$R^m[z : w : 1] = [z : z^m w : 1]. \tag{12}$$

The following remark completes the proof of Theorem (4.11).

***Remark 4.14.*** Observe that, by (12), If $|z| < 1$

$$R^m[z : w : 1] \to [z : 0 : 1] \in \mathrm{Fix}_R, \tag{13}$$

i.e., for all $z_0$ with $|z_0| < 1$, $\mathbb{P}^1_{z_0} = [z_0 : w : 1]$ satisfies that $R^m(\mathbb{P}^1_{z_0}) \to [z_0 : 0 : 1]$.

If $|z| > 1$

$$R^m[z : w : 1] \to [0 : 1 : 0] \in \mathcal{I}_R,$$

as we wanted to prove.

By looking at the eigenvalues of this map, we conclude that the fixed points $(z, 0)$ are semi-attracting when $|z| < 1$, and semi-repelling when $|z| > 1$. In the case of the point $(1, 0)$, which is precisely the point of intersection of the two lines of fixed points, both eigenvalues are equal to 1. The same thing happens if $z = -1$. By (12), in $(t = 1)$ the iterates of $(-1, w)$ will be jumping from $w$ to $-w$, depending on whether the iterate that we are considering is even or odd.

Let us consider now those points $z$ with $|z| = 1$, and $1 \neq z \neq -1$; then $z = \exp(i\theta)$. In this case, by (12),

$$R^m(z, w) = R^m(\exp(i\theta), \quad r\exp(i\Phi)) = (\exp(i\theta), r\exp(i(m\theta + \Phi))),$$

so for every $\theta$ fixed, $0 < \theta < 2\pi$, we will have in the plane $\mathbb{P}^1_{\exp(i\theta)} = [\exp(i\theta) : r\exp(i\Phi) : 1]$, $r \in \mathbb{R}$, and $0 \leq \Phi < 2\pi$, essentially a rotation of the points preserving the modulus of $w = r\exp(i\Phi)$.

### *4.2 $\mathcal{B}$-Fibonacci maps.* As stated before:

***Definition 4.15.*** A $\mathcal{B}$-map will be called *Fibonacci*, if the growth of the degrees of its iterates follows the Fibonacci sequence $\{C_m\}$, or if it follows a $d$-Fibonacci sequence, $\{dC_m\}$, i.e., a multiple of the Fibonacci sequence.

***Theorem 4.16.*** *The map*

$$R : \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{via} \quad R[z : w : t] \mapsto [zt : wt + z^2 : t^2 + wt] \tag{14}$$

*is a non birational $\mathcal{B}$-Fibonacci map. In addition, there exists an open set $U \subset \mathbb{P}^2$ where the iteration of $R$ is well defined, and*

$$R^m[z : w : t] \to [0 : 0 : 1] \quad \text{as } m \to \infty.$$

We give now some dynamical features of this map,

1. $\mathrm{Fix}_R = \{[0:0:1]\}$,
2. $\mathcal{I}_R = \{[0:1:0]\} := \{p_0\}$,
3. $R(t=0) = [0:1:0]$.

***Remark 4.17.*** *A computer experiment suggests that this map has infinitely many degree lowering curves.*

As before, we denote by $R^{-\infty}(p_0) := \{p_0\} \cup \left(\bigcup_{k\geq 0} R^{-k}(p_0)\right)$, where $p_0$ is the point of indeterminacy, $\mathcal{V} = \mathcal{V}_0 = (t=0)$ is the degree lowering curve, and $R^{-k}(p_0) = \mathcal{V}_{-k}$, $k \geq 1$, is denoting the $k$-th preimage of the degree lowering curve $\mathcal{V} = (t=0)$.

We will study now the dynamics of $R$ near the point of indeterminacy.

***Proposition 4.18.*** *The curve $(z=0)$ is the blow up of the point of indeterminacy $[0:1:0]$ of* (14).

*Proof.* Similar to the proof of Proposition (4.6). □

***Proposition 4.19.*** *The line $(z=0)$ is fixed. The points on the line $(z=0)$ converge to the fixed point $[0:0:1]$.*

*Proof.* The fact that the line $(z=0)$ is fixed will follow from the fact that the formula for the $m$-th iterate of $R$ in $(z=0)$ is given by $R^m(z=0) = [0:w:t+mw]$. This formula can be proved by induction.

Now if $w=0$, $R^m(z=0) = [0:0:1]$.

If $w \neq 0$, $t+(m+1)w \neq 0$, so dividing by it we have

$$R^m(z=0) = \left[0 : \frac{w}{t+(m+1)w} : 1\right] \to [0:0:1],$$

as we wanted to prove. □

We will study next the dynamics near the point $[0:0:1]$. In $(t=1)$, the Jacobian matrix at $(0,0)$ is given by

$$\mathcal{J}_R(0,0) = \begin{pmatrix} \dfrac{1}{w+1} & -\dfrac{z}{(w+1)^2} \\ \dfrac{2z}{w+1} & \dfrac{1-z^2}{(1+w)^2} \end{pmatrix}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We show below that an open set of $\mathbb{P}^2$ is attracted to this point.

The set, denoted by $\mathcal{P}\rceil\nabla_{k,R}$, of periodic points of period $k$ of the map given by (14) is nonempty, at least for $k=2$ and possibly $k=3$. In fact, $\mathcal{P}\rceil\nabla_{2,R} =$

$\{[2:2:-1]\} \cup \{[-2:2:-1]\}$, and using the computer we could approximate the periodic points of period three. We suspect that in general $\mathcal{P}\rceil\nabla_{k,R} \neq \emptyset$ for all $k \geq 3$.

***Remark 4.20.*** Before continuing, let us observe that the map (14) is not birational, in fact it is easy to show that generically it is 2 to 1.

As we showed in Proposition 4.19, the points on the line $(z = 0)$ converge under iteration to the point $[0 : 0 : 1]$. We study next the dynamics of $R$ on $U = (z = 1) \setminus \left(R^{-\infty}(p_0) \cup_{k\geq 2} \mathcal{P}\rceil\nabla_{k,R} \cup \mathrm{Fix}_R\right)$. On $U$, we can write $R$ as follows: $R[1 : w : t] = [1 : w + 1/t : t + w]$, i.e.,

$$(w, t) \overset{\mathrm{R}}{\mapsto} \left(w + \frac{1}{t},\, t + w\right). \tag{15}$$

***Proposition 4.21.*** *The $m$-th iterate of the map* (15) *is given by,*

$$R^m(w, t) = R^m(w_0, t_0) = (w_m, t_m), \tag{16}$$

*with $w_0 = w$, $t_0 = t$, and where $w_m = w_0 + 1/t_0 + 1/t_1 + 1/t_2 + \cdots + 1/t_{m-1}$, and $t_m = t_0 + w_0 + w_1 + \cdots + w_{m-1}$.*

*Proof.* It follows by induction on $m$. □

The following proposition proves Theorem (4.16).

***Proposition 4.22.*** *The $m$-th iterate of the map* (15) *behaves as follows:*

$$R^m[1 : w : t] \to [0 : 0 : 1] \quad \text{as } m \to \infty,$$

*uniformly on compact subsets of the open set* $V := \{(w, t) \in \mathbb{C}^2 \mid \mathrm{Re}\,(w)\mathrm{Re}\,(t) > 0\}$.

The proof of this Proposition will be given after the proof of Lemma (4.25).

***Remark 4.23.***

1. We will show that the convergence is uniformly tangential to the w-axis and uniform on compact subsets $W \subset V$ of the form $W := \{(w, t) \in \mathbb{C}^2 \mid w \in B_+,\ t \in B_+\}$, where, for $b$, $c \in \mathbb{R}^+ \setminus \{0\}$, $B_+ := A_+ \cap \{z \in \mathbb{C} \mid 0 < b \leq \mathrm{Re}\,(z) \leq c < \infty\} \subset \mathbb{C}$, and $A_+ := \{z : |\mathrm{Im}\,(z)| \leq a\mathrm{Re}\,(z),\ \mathrm{Re}\,(z) > 0\}$, $a > 0$.
2. Observe that $A_- := \{z : |\mathrm{Im}\,(z)| \leq a|\mathrm{Re}\,(z)|,\ \mathrm{Re}\,(z) < 0\}$, $a > 0$ and for $b_1, c_1 \in \mathbb{R}^- \setminus \{0\}$, $B_- := A_- \cap \{z \in \mathbb{C} \mid -\infty < b_1 \leq \mathrm{Re}\,(z) \leq c_1 < 0\} \subset \mathbb{C}$. This case is similar to the case described in (1), because of the symmetry $R(-w, -t) = -R(w, t)$.

Let $K \subset V$ be a compact. We can write $K$ as follows:

$$K := K_+ \cup K_-, \tag{17}$$

where $K_+ := K \cap V_+$, $K_- := K \cap V_-$, $V_+ := \{(w,t) \in \mathbb{C} \mid \text{Re}\,(w) > 0,\ \text{Re}\,(t) > 0\}$, and $V_- := \{(w,t) \in \mathbb{C} \mid \text{Re}\,(w) < 0,\ \text{Re}\,(t) < 0\}$. Since $V_+$ and $V_-$ are open, there exist $U_+$ and $U_-$, open sets, such that $K_+ \subset U_+$ and $K_- \subset U_-$.

Let us define

$$U := U_+ \cup U_-, \tag{18}$$

and

$$\hat{U}_+ := \hat{B}_+ \times \hat{B}_+, \tag{19}$$

where

$$\hat{B}_+ := A_+ \cap \{z \in \mathbb{C} \mid 0 < b \leq \text{Re}\,(z)\}. \tag{20}$$

Similarly, $\hat{B}_- := A_- \cap \{z \in \mathbb{C} \mid \text{Re}\,(z) \leq c_1 < 0\}$ and $\hat{U}_- := \hat{B}_- \times \hat{B}_-$.

We will define $\hat{U} := \hat{U}_+ \cup \hat{U}_-$.

Observe that $U \subset \hat{U}$, $U_+ \subset \hat{U}_+$, $B_+ \subset \hat{B}_+$, $B_- \subset \hat{B}_-$, $U_- \subset \hat{U}_-$, since $b$ and $c_1$ are arbitrarily chosen.

***Lemma 4.24.*** *The sets $\hat{U}_+$ and $\hat{U}_-$ are forward invariant, i.e., $R(\hat{U}_+) \subset \hat{U}_+$ and $R(\hat{U}_-) \subset \hat{U}_-$.*

*Proof.* We only will prove the lemma for $\hat{U}_+$, since the proof for $\hat{U}_-$ is similar. Let us take $(w,t) \in \hat{U}_+$. By (19), $w \in \hat{B}_+$ and $t \in \hat{B}_+$, and by (20), $w,\ t \in A_+$. Now from (1) in Remark (4.23), $t = r\exp(i\theta) \in A_+$, i.e.,

$$\frac{1}{t} = \frac{1}{r\exp(i\theta)} = \frac{\exp(-i\theta)}{r} \in A_+.$$

It is easy to see that $w,\ t \in A_+$ implies $w + t \in A_+$. Therefore $(w,t) \in A_+ \Rightarrow R(w,t) = (w + 1/t\ ,\ t + w) \in A_+$.

Also, since $\text{Re}\,(w + 1/t) \geq \text{Re}\,(w) \geq b > 0$ and $\text{Re}\,(w+t) \geq \text{Re}\,(t) \geq b > 0$ so $w + 1/t,\ t + w \in \hat{B}_+$, it follows that, for all $(w,t) \in \hat{U}_+$, $R(w,t) \in \hat{U}_+$.

Therefore, $\hat{U}_+$ is forward invariant. □

The following is immediate.

***Lemma 4.25.*** *On $\hat{B}_+$, the following inequalities are satisfied:*

1. $0 < b \leq \text{Re}\,(z) \leq |z| \leq k\,\text{Re}\,(z)$, *for some* $k \in \mathbb{R}^+ \setminus \{0\}$,
2. $|\text{Im}\,(z)| \leq a\,\text{Re}\,(z)$.

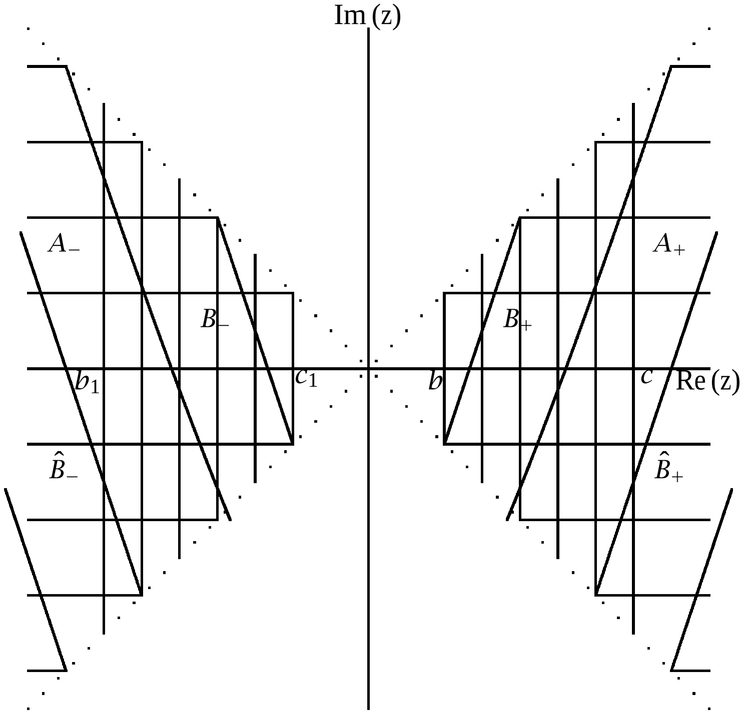


FIG 1. Regions $A_+, A_-, B_+, B_-, \hat{B}_+, \hat{B}_-$

***Proof of Proposition 4.22.***

Let us take $K \subset V$, $K$ compact, $V$ given as in Proposition(4.22). We will show uniform convergence on $U$, given by (18). Let us take a point $p_0 = (w_0, t_0) \in V$. Without loss of generality, $\mathrm{Re}\,(w_0) > 0$ and $\mathrm{Re}\,(t_0) > 0$. Let us take a set $B_+$ as described in Remark (4.23)-1, such that $(w_0, t_0) \in B_+ \times B_+ \subset U_+$ and $p_0 \in U_+$ implies $p_0 \in \hat{U}_+$ so that, from Lemma (4.24), $R^m(p_0) := p_m \in \hat{U}_+$, so $p_m = (w_m, t_m) \in \hat{U}_+$, and the inequalities in Lemma (4.25) are satisfied.

By Proposition (4.21) and from the fact that $\mathrm{Re}\,(t_j) > 0$ implies $\mathrm{Re}\,(1/t_j) > 0$, we have:

$$\mathrm{Re}\,(w_m) = \mathrm{Re}\left(w_0 + \frac{1}{t_0} + \cdots + \frac{1}{t_{m-1}}\right) \geq b, \quad \text{for all } m \in \mathbb{N}. \tag{21}$$

Again by Proposition (4.21) and (21)

$$\mathrm{Re}\,(t_m) = \mathrm{Re}\,(t_0 + w_0 + \cdots + w_{m-1}) \geq mb, \quad \text{for all } m \in \mathbb{N}. \tag{22}$$

Using the fact that $\mathrm{Re}\,(z) \le |z|$, by Proposition (4.21) and (22) we have:

$$\mathrm{Re}\,(w_m) \le |w_m| \le |w_0| + \frac{1}{|t_0|} + \cdots + \frac{1}{|t_{m-1}|}.$$

From Lemma (4.25), (22), and the fact that $(w_0, t_0) \in \hat{U}_+$,

$$\mathrm{Re}\,(w_m) \le d_0 + \frac{1}{\mathrm{Re}\,(t_0)} + \cdots + \frac{1}{\mathrm{Re}\,(t_{m-1})} = e + \frac{1}{b}\ln m,$$

where $d_0 = |w_0|$ and $e = d_0 + 1/b$. So there exists $m_1 \in \mathbb{N}$ such that $\mathrm{Re}\,(w_m) \le \ln m + 1/b \ln m$ for all $m \ge m_1$. Therefore,

$$\mathrm{Re}\,(w_m) \le \left(1 + \frac{1}{b}\right)\ln m = g \ln m \quad \text{for all } m \ge m_1, \tag{23}$$

with $g = 1 + 1/b$.

Now let us use Proposition (4.21) and (23) to get $\mathrm{Re}\,(t_m) \le e_1 + g\ln 1 + g\ln 2 + \cdots + g\ln(m-1)$, where $e_1 = \mathrm{Re}\,(t_0) + \mathrm{Re}\,(w_0)$, so $\mathrm{Re}\,(t_m) \le e_1 + g(\int_1^m \ln x\,dx) \le e_1 + gm\ln m$. There exists $m_2 \in \mathbb{N}$ such that

$$\mathrm{Re}\,(t_m) \le (1+g)m\ln m = e_2 m \ln m \quad \text{for all } m \ge m_2, \tag{24}$$

with $e_2 = 1 + g$.

From Proposition (4.21) and (24)

$$\begin{aligned}\mathrm{Re}\,(w_m) &\ge b + \mathrm{Re}\left(\frac{1}{t_0}\right) + \left(\frac{1}{e_2\ln 1}\right) + \cdots + \left(\frac{1}{(m-1)e_2\ln(m-1)}\right) \\ &= \ln\left(\frac{\ln(m-1)}{\ln 2}\right).\end{aligned} \tag{25}$$

From (22)-(25) we get the following estimates:

$$\ln\left(\frac{\ln(m-1)}{\ln 2}\right) \le \mathrm{Re}\,(w_m) \le g\ln m, \tag{26}$$

$$mb \le \mathrm{Re}\,(t_m) \le e_2 m \ln m. \tag{27}$$

Now Lemma (4.25) together with equations (26) and (27) imply $|t_m| \ge \mathrm{Re}\,(t_m) \ge mb$ and $|w_m| \le k\mathrm{Re}\,(w_m) \le kg\ln m$, for some $k \in \mathbb{R}^+ \setminus \{0\}$.

So since $p_0 = (w_0, t_0) \in U$, there exist constants $b > 0$ and $h := gk > 0$ (depending on the compact set $U$) such that, for all $m \ge \max\{m_1, m_2\}$, we have $|t_m| \ge mb$ and $|w_m| \le h\ln m$.

Hence $R^m[1 : w_0 : t_0] = [1/t_m : w_m/t_m : 1]$. Therefore $R^m[1 : w_0 : t_0] \to [0 : 0 : 1]$ as $m \to \infty$, since

$$\frac{1}{|t_m|} \le \frac{1}{mb} \to 0 \quad \text{and} \quad \frac{|w_m|}{|t_m|} \le \frac{h \ln m}{bm} \to 0 \quad \text{as } m \to \infty.$$

Let us observe that the convergence is tangential, i.e., $(1/t_m, w_m/t_m) \to (0,0)$ in $\mathbb{C}^2 = (z = 1)$ and the slope $w_m/t_m^2 = w_m > \ln(\ln(m-1)/\ln 2) \to \infty$, so the slope goes to infinity, i.e., the points converge to $[0 : 0 : 1]$ tangentially to the fixed point. ◻

We give now an example of a birational $\mathcal{B}$-Fibonacci map of $\mathbb{P}^2$.

***Theorem 4.26.*** *The family of $\mathcal{B}$-Fibonacci maps of $\mathbb{P}^2$ with only one degree lowering variety is non-empty. In fact,*

$$R : \mathbb{P}^2 \to \mathbb{P}^2 \quad \text{via} \quad [z : w : t] \mapsto [wt : wz : t^2] \tag{28}$$

*is a Fibonacci map.*

*In addition, if $U \subset (t = 1)$ is an open set where the iteration of $R$ is well defined, then we will have the following behavior of $R$ under iteration, here $\lambda = (1 + \sqrt{5})/2$.*

*If $|w^\lambda z| < 1$, $R^m[z : w : 1] \to [0 : 0 : 1]$ as $m \to \infty$,*
*If $|w^\lambda z| > 1$, $R^m[z : w : 1] \to [0 : 1 : 0]$ as $m \to \infty$,*
*If $|w^\lambda z| = 1$, then $R$ is chaotic.*

We give now some important dynamical features of this map.

1. $\mathrm{Fix}_R = \{[0 : 0 : 1]\} \cup \{[1 : 1 : 1]\}$.
2. $\mathcal{I}_R = \{[1 : 0 : 0]\} \cup \{[0 : 1 : 0]\}$.
3. $R(w = 0) = [0 : 0 : 1] \in \mathrm{Fix}_R$.
4. $R(t = 0) = [0 : 1 : 0] \in \mathcal{I}_R$.

***Remark 4.27.***

1. The point $[1 : 0 : 0] \in \mathcal{I}_R$ has no preimages.
2. $w^\lambda$ is multi-valued, however we are only interested in $|w|^\lambda$.

***Proposition 4.28.*** *The formula in $\mathbb{C}^3$ for the $m$-th iterate, $m \ge 2$ of the map given by equation (28) is given as follows:*

$$R^m(z, w, t) = t^{\alpha(m)}(w^{C_{m-1}} z^{C_{m-2}} t^{C_{m-1}}, w^{C_m} z^{C_{m-1}}, t^{C_{m+1}}), \tag{29}$$

*where*

$$\alpha(m) = 2\alpha(m-1) + C_{m-2}, \tag{30}$$

*with $\alpha(1) = 0$, $\alpha(2) = 1$, and $\{C_m\}$ the Fibonacci sequence.*

*Proof.* Use induction. □

Before we study the dynamics of this map on $\mathbb{P}^2 \setminus \mathcal{I}_R$, we will study its dynamics near the points of indeterminacy.

***Proposition 4.29.*** *The blow up of the point of indeterminacy* $[0 : 1 : 0]$ *of the map* $R$*, given by* (28)*, is the curve* $(t = 0)$*, and the blow up of the point of indeterminacy* $[1 : 0 : 0]$ *of the same map* $R$ *is the curve* $(z = 0)$.

*Proof.* Similar to the proof of Proposition (4.6). □

We will study next the dynamics of $R$ near the fixed points $[0 : 0 : 1]$ and $[1 : 1 : 1]$. By looking at the eigenvalues of this map in the plane $(t = 1)$, we realize that $[0 : 0 : 1]$ is an attractive fixed point and $[1 : 1 : 1]$ is a saddle.

***Remark 4.30.*** Let us observe that the set $\mathcal{P}\rceil\nabla_{m,R}$ of periodic points of period $m$ of $R$ is nonempty. In fact, in $(t = 1)$, (29) can be written in $\mathbb{C}^2$ as: $R^m(z, w) = (z^{C_{m-2}} w^{C_{m-1}}, z^{C_{m-1}} w^{C_m})$. So if $m$ is odd, finding the periodic points of period $m$ of $R$ is equivalent to looking for the $(C_m + C_{m-2})$-th roots of unity. If $m$ is even, we will be looking for the $(C_m + C_{m-2} - 2)$-th roots of unity.

We study next the dynamics of $R$ on

$$U = (t = 1) = \mathbb{P}^2 \setminus \Big(\{(t = 0)\} \cup \{[1 : 0 : 0]\} \cup \{[0 : 1 : 0]\} \cup \mathcal{P}\rceil\nabla_{m,R} \cup \{(w = 0)\} \cup \{(z = 0)\}\Big). \tag{31}$$

The following proposition proves the last statement of Theorem (4.26).

***Proposition 4.31.*** *For every* $(z, w) \in U$*,* $U$ *given as in* (31)
*If* $|w^\lambda z| < 1$*, then*

$$R^m[z : w : 1] \to [0 : 0 : 1]. \tag{32}$$

*If* $|w^\lambda z| > 1$*, then*

$$R^m[z : w : 1] \to [0 : 1 : 0]. \tag{33}$$

*If* $|w^\lambda z| = 1$*, then* $R$ *is chaotic.*

*Proof.* In (29) we established the general formula for the $m$-th iterate of $R$. Now, since we are studying the iteration of $R$ in $U$, we can write (29) in $(t = 1)$ as follows:

$$R^{m+1}[z : w : 1] = [z^{C_{m-1}} w^{C_m} : z^{C_m} w^{C_{m+1}} : 1]. \tag{34}$$

Let us prove the assertion in (32). Observe that (34) can be written as:

$$R^{m+1}[z:w:1] = [(zw^{C_m/C_{m-1}})^{C_{m-1}} : (zw^{C_{m+1}/C_m})^{C_m} : 1]. \tag{35}$$

Since $\lim_{m\to\infty}(C_m/C_{m-1}) = \lambda$,

$$\lim_{m\to\infty} q_m := \lim_{m\to\infty} zw^{C_m/C_{m-1}} = zw^{\lambda}, \tag{36}$$

where $\lambda = (1+\sqrt{5})/2$.

Now, by hypothesis $|zw^{\lambda}| < 1$, then by (36) there exists $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$, $|q_m| \leq c < 1$, so $\lim_{m\to\infty} |q_m|^{C_{m-1}} \leq \lim_{m\to\infty} c^{C_{m-1}}$.

Since $c < 1$ and $\lim_{m\to\infty} C_{m-1} = \infty$, we have

$$0 \leq \lim_{m\to\infty} |q_m|^{C_{m-1}} < \lim_{m\to\infty} c^{C_{m-1}} = 0. \tag{37}$$

Therefore by (37) and (35), $R^{m+1}[z:w:t] \to [0:0:1]$ as $m \to \infty$.

We prove next the assertion in (33). We write (34) as:

$$R^{m+1}[z:w:1] = \left[\frac{1}{(zw^{C_{m-1}/C_{m-2}})^{C_{m-2}}} : 1 : \frac{1}{(zw^{C_{m+1}/C_m})^{C_m}}\right]. \tag{38}$$

Let us observe that

$$\lim_{m\to\infty} q_{m-1} := \lim_{m\to\infty} zw^{C_{m-1}/C_{m-2}} = zw^{\lambda}, \tag{39}$$

where $\lambda = (1+\sqrt{5})/2$, as before.

Since by hypothesis $|w^{\lambda}z| > 1$, by (39) there exists $m_1 \in \mathbb{N}$ such that, for all $m \geq m_1$, $|1/q_{m-1}| \leq c_1 < 1$, so $\lim_{m\to\infty} |1/q_{m-1}|^{C_{m-2}} \leq c_1^{C_{m-2}} < 1$. Since $c_1 < 1$ and $\lim_{m\to\infty} C_m = \infty$, we have

$$0 \leq \lim_{m\to\infty} \left|\frac{1}{q_{m-1}}\right|^{C_{m-2}} \leq \lim_{m\to\infty} c_1^{C_{m-2}} = 0. \tag{40}$$

Therefore, by (40) and (38), $R^{m+1}[z:w:t] \to [0:1:0]$ as $m \to \infty$.

It remains only to prove the last assertion in Proposition (4.31). Let us consider now the case $|zw^{\lambda}| = 1$. In $(t = 1)$, we will define $\Sigma := \{(z,w) \in \mathbb{C}^2 : |zw^{\lambda}| = 1\}$, i.e., $\Sigma := \{(r_1 \exp(i\theta), r_2 \exp(i\Phi)) \mid r_1 r_2^{\lambda} = 1,\ 0 \leq \theta,\ \Phi \leq 2\pi\}$.

If $r_2 = x$,

$$\Sigma = \{(x^{-\lambda}\exp(i\theta), x\exp(i\Phi)) \mid x \in \mathbb{R}^+,\ 0 \leq \theta,\ \Phi \leq 2\pi\}. \tag{41}$$

***Lemma 4.32.*** *The set $\Sigma$ is $R$ invariant.*

*Proof.* Let $(z, w) \in \Sigma$.

In $(t = 1)$, (28) can be written as $R(z, w) = (w, zw)$, so $|w(zw)^\lambda| = |zw^{(\lambda+1)/\lambda}|^\lambda$.

Since $(\lambda+1)/\lambda = \lambda$, we have $|w(zw)^\lambda| = |zw^\lambda| = 1$ as we wanted to prove. So $\Sigma$ is $R$ invariant. □

We can think of $R|_\Sigma : \mathbb{R}^+ \times (S^1 \times S^1) \to \mathbb{R}^+ \times (S^1 \times S^1)$ as

$$R|_\Sigma = (f_\lambda|_{\mathbb{R}^+} \times g_{\theta,\Phi}|_{(S^1\times S^1)}), \tag{42}$$

where

$$f_\lambda : \mathbb{R}^+ \to \mathbb{R}^+ \text{ via } x \mapsto x^{1-\lambda}, \tag{43}$$

and

$$g_{\theta,\Phi} : (S^1 \times S^1) \to (S^1 \times S^1) \text{ via } (\theta, \Phi) \mapsto (\varphi, \theta + \varphi). \tag{44}$$

***Remark 4.33.*** By (34) and (41)

$$f_\lambda^m(x) = x^{C_m - \lambda C_{m-1}}, \tag{45}$$

and by (43)

$$f_\lambda^m(x) = x^{(1-\lambda)^m}. \tag{46}$$

So by (45) and (46)

$$(1-\lambda)^m = C_m - \lambda C_{m-1}.$$

***Dynamics on*** $R^+$**.** The fixed points of $f_\lambda$ are the points such that $x^{1-\lambda} = x$, i.e., the points that satisfy $x^{-\lambda} = 1$, therefore $x = 1$. Since $f_\lambda'(1) = 1 - \lambda < 1$, the fixed point $x = 1$ is attracting. Note that this could also be seen from (46) since $\lim_{m\to\infty} f_\lambda^m(x) = \lim_{m\to\infty} x^{(1-\lambda)^m} = 1$.

So, all the points in $\Sigma$ will converge under iteration of $R$ (given as in (42)) to the $\mathcal{R}^2$-torus sitting at $x = 1$. Now by looking at (44) we get the eigenvalues $x_1 = (1 + \sqrt{5})/2$ and $x_2 = (1 - \sqrt{5})/2$. Since it is a well known fact that a map on the torus with these eigenvalues is chaotic, our map $R|_\Sigma$ given as in (42) will be chaotic, as we wanted to prove. □

***Remark 4.34.*** *Let us observe that the set* $\{[z : w : 1] : |zw^\lambda| > 1\}$, *and the curve* $(t = 0)$ *converge under iteration to the point* $[0 : 1 : 0]$.

## 5. $\mathcal{B}$-MAPS OF $\mathbb{C}^3$

In this section we show that there are only a finite number of sequences of degrees of iterates for the polynomial automorphisms of $\mathbb{C}^3$ of degree 2. We base our work in the following theorem of Fornæss and Wu.

***Theorem 5.1.*** (see [5]) *If $H : \mathbb{C}^3 \to \mathbb{C}^3$ is a degree at most 2 polynomial self-map with nowhere vanishing Jacobian determinant, then $H$ is affinely conjugate to one of the following maps:*

1. *An affine automorphism;*
2. *An elementary polynomial automorphism*

$$\mathcal{E} = \{E(x,y,z) = (P(y,z)+ax\ ,\ Q(z)+by\ ,\ cz+p)\},$$

*where $P$ is a polynomial of $y$, $z$ of degree at most 2, $Q$ is a polynomial of $z$ of degree at most 2 and $abc \neq 0$. Note that $E(x,y,z)$ maps every hyperplane $z = k$ to a hyperplane $z = k'$ and maps every line $y = k_1$, $z = k_2$ to a line $y = k_1'$, $z = k_2'$;*

3. *One of the following:*

$$\begin{aligned}
\mathcal{H}_1 &= \{H_1(x,y,z) = (P(x,z)+ay\ ,\ Q(z)+x\ ,\ cz+p)\},\\
\mathcal{H}_2 &= \{H_2(x,y,z) = (P(y,z)+ax\ ,\ Q(y)+bz\ ,\ y)\},\\
\mathcal{H}_3 &= \{H_3(x,y,z) = (P(x,z)+ay\ ,\ Q(x)+z\ ,\ x)\},\\
\mathcal{H}_4 &= \{H_4(x,y,z) = (P(x,y)+az\ ,\ Q(y)+x\ ,\ y)\},\\
\mathcal{H}_5 &= \{H_5(x,y,z) = (P(x,y)+az\ ,\ Q(x)+by\ ,\ x)\},
\end{aligned}$$

*where $P$ and $Q$ are polynomials with $\max\{\deg(P),\deg(Q)\} = 2$ and $abc \neq 0$.*

The $m$-th iterate of $H$, $H^m(z,w,t)$ will be given by

$$H^m(x,y,z) = (H_{1,m}(x,y,z)\ ,\ H_{2,m}(x,y,z)\ ,\ H_{3,m}(x,y,z)),$$

where $H_{j,m}$, $j \in \{1,2,3\}$ denotes the $j$-th coordinate function of $H^m$ in $\mathbb{C}^3$.

***Definition 5.2.*** $\deg H^m = \max_{j=1,2,3}\{\deg H_{j,m}(x,y,z)\}$.

***Proposition 5.3.*** *$\mathcal{H}_2$ is a family of $\mathcal{A}$-maps. In fact, let*

$$H_2(x,y,z) = (dy^2+ez^2+fyz+gy+hz+j+ax\ ,\ ry^2+sy+t+bz\ ,\ y).$$

*Then*

1. $\deg H_2^m(x,y,z) = 2^m$ *for all $m \in \mathbb{N}$, if and only if $r \neq 0$,*
2. $\deg H_2^m(x,y,z) = 2$ *for all $m \in \mathbb{N}$, if and only if $r = 0$.*

*Proof.* The proof of each item follows by induction on $m$. □

**Proposition 5.4.** *Let* $H_1 \in \mathcal{H}_1$,

$$H_1(x,y,z) = (dx^2+ez^2+fxz+gx+hz+j+ay\ ,\ rz^2+sz+t+x\ ,\ cz+p).$$

*Then*

1. $\deg H_1^m(x,y,z) = 2$ *for all* $m \in \mathbb{N}$, *if and only if* $d = 0$, $f = 0$, *and* $e$ *and* $r$ *do not vanish at the same time,*
2. $\deg H_1^m(x,y,z) = 2^m$ *for all* $m \in \mathbb{N}$, *if and only if* $d \neq 0$,
3. $H_1(x,y,z)$ *is* $\mathcal{B}$*-linear if and only if* $d = 0$ *and* $f \neq 0$.

*Proof.* It follows by induction on $m$. □

We recall that the Fibonacci sequence $\{C_m\}$ satisfy the relationship

$$C_{m+2} = C_{m+1} + C_m, \quad m \in \mathbb{N} \cup \{0\},$$

with $C_0 = 1 = C_1$.

**Definition 5.5.** *A* $\mathcal{B}$*-map will be called flashing if the degrees of its iterates,* $d_m$ *grow every second time, i.e.,*

$$\begin{aligned} d_{2m} &= 2^{m+1}, \quad m \geq 1 \\ d_{2m+1} &= 2^{m+1}, \quad m \geq 1 \end{aligned}$$

**Proposition 5.6.** *Let* $H_3 \in \mathcal{H}_3$,

$$H_3(dx^2+ez^2+fxz+gx+hz+j+ay\ ,\ rx^2+sx+t+z\ ,\ x).$$

*Then*

1. $\deg H_3^m(x,y,z) = 2^m$ *for all* $m \in \mathbb{N}$, *if and only if* $d \neq 0$,
2. $\deg H_3^m(x,y,z) = C_{m+1}$ *for all* $m \in \mathbb{N}$, *if and only if* $d = 0$, $f \neq 0$, $r = 0$,
3. $\deg H_3^m(x,y,z) = 2C_m$ *for all* $m \in \mathbb{N}$, *if and only if* $d = 0$, $f \neq 0$, *and* $r \neq 0$,
4. $\deg H_3(x,y,z)$ *is a flashing map if and only if* $d = 0$, $f = 0$, *and* $e$ *and* $r$ *do not vanish at the same time.*

*Proof.* Induction on $m$. □

***Proposition 5.7.*** *Let $H_4 \in \mathcal{H}_4$,*

$$H_4(x,y,z) = (dx^2+ey^2+fxy+gx+hy+j+az\ ,\ ry^2+sy+t+x\ ,\ y).$$

*Then*

1. $\deg H_4^m(x,y,z) = 2^m$ *for all $m \in \mathbb{N}$, if and only if $d$ and $r$ do not vanish at the same time,*
2. $\deg H_4^m(x,y,z) = C_{m+1}$ *for all $m \in \mathbb{N}$, if and only if $d = 0$, $r = 0$, and $f \neq 0$,*
3. *The map $H_4(x,y,z)$ is flashing if and only if $d = 0$, $f = 0$, $r = 0$, and $e \neq 0$.*

*Proof.* It follows by induction on $m$. ◻

***Proposition 5.8.*** *Let $H_5 \in \mathcal{H}_5$,*

$$H_5(x,y,z) = (dx^2+ey^2+fxy+gx+hy+j+az\ ,\ rx^2+sx+t+by\ ,\ x).$$

*Then*

1. $\deg H_5^m(x,y,z) = 2$ *for all $m \in \mathbb{N}$, if and only if $d = 0$, $f = 0$, and one of the following two conditions holds:*
   (a) $e \neq 0$, $r = 0$, *and* $s = 0$,
   (b) $r \neq 0$, $e = 0$, *and* $h = 0$.
2. $\deg H_5^m(x,y,z) = 2^m$ *for all $m \in \mathbb{N}$, if and only if one of the following two conditions holds:*
   (a) $d \neq 0$,
   (b) $d = 0$, $r \neq 0$ *and $e$ and $f$ does not vanish at the same time.*
3. *$H_5(x,y,z)$ is $\mathcal{B}$-linear if and only if $d = 0$, $r = 0$, $s = 0$, and $f \neq 0$.*
4. $\deg H_5^m(x,y,z) = C_{m+1}$ *for all $m \in \mathbb{N}$, if and only if $d = 0$, $r = 0$, $s \neq 0$, and $f \neq 0$.*
5. *$H_5(x,y,z)$ is flashing if and only if $d = 0$, $f = 0$ and one of the following two conditions holds:*
   (a) $r \neq 0$, $e = 0$, *and* $h \neq 0$,
   (b) $e \neq 0$, $r = 0$, *and* $s \neq 0$.

*Proof.* It follows by induction on $m$. ◻

***Definition 5.9.*** *A $\mathcal{B}$-map will be called basic if after high enough iterates its degree remains constant.*

***Proposition 5.10.*** *Let $E \in \mathcal{E}$,*

$$E(x, y, z) = (dy^2+ez^2+fyz+gy+hz+j+ax\ , rz^2+sz+t+by\ , cz+p).$$

*Then,*

1. deg $E^m(x, y, z) = 2$ *for all* $m \in \mathbb{N}$, *if and only if one of the following two conditions holds:*
   (a) $d$ *and* $r$ *are not different from zero at the same time,*
   (b) $d = 0$, *and* $f$ *and* $r$ *are not different from zero at the same time.*
2. deg $E^m(x, y, z) = 3$ *for all* $m \in \mathbb{N}$, *if and only if* $d = 0$, $f \neq 0$, *and* $r \neq 0$.
3. deg $E^m(x, y, z) = 4$ *for all* $m \in \mathbb{N}$, *if and only if* $d \neq 0$ *and* $r \neq 0$.

*Proof.* It follows by induction on $m$. □

As we have seen in this section, the only possibilities for the $\mathcal{B}$-maps of $\mathbb{C}^3$ of degree two are to be linear, Fibonacci, flashing, or basic.

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