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GROWTH OF THE WEIL–PETERSSON INRADIUS OF MODULI SPACE

by Yunhui WU

Abstract. — In this paper we study the systole function along Weil–Petersson geodesics. We show that the square root of the systole function is uniformly Lipschitz on Teichmüller space endowed with the Weil–Petersson metric. As an application, we study the growth of the Weil–Petersson inradius of moduli space of Riemann surfaces of genus g with n punctures as a function of g and n. We show that the Weil–Petersson inradius is comparable to $\sqrt{\ln g}$ with respect to g, and is comparable to 1 with respect to n.

Moreover, we also study the asymptotic behavior, as g goes to infinity, of the Weil–Petersson volumes of geodesic balls of finite radii in Teichmüller space. We show that they behave like $o((\frac{1}{g})^{(3-\epsilon)g})$ as $g \to \infty$, where $\epsilon > 0$ is arbitrary.

Résumé. — Dans cet article, nous étudions la fonction systole le long des géodésiques de la métrique de Weil–Petersson. Nous montrons que la racine carrée de la systole est uniformément Lipschitz sur l'espace de Teichmüller muni de la métrique de Weil–Petersson. Comme application, nous étudions la croissance du rayon de la plus grande boule métrique inscrite dans l'espace des modules des surfaces de Riemann de genre g avec n piqûres en fonction de g et n. Nous montrons que ce rayon est comparable à $\sqrt{\ln g}$ par rapport à g, et comparable à 1 par rapport à n.

De plus, nous étudions aussi le comportement asymptotique, lorsque g tends vers l'infini, des volumes de Weil–Petersson des boules géodésiques de rayons finis dans l'espace Teichmüller. Nous montrons qu'ils se comportent comme $o((\frac{1}{g})^{(3-\epsilon)g})$ quand $g \to \infty$, où $\epsilon > 0$ est arbitraire.

1. Introduction

Let $S_{g,n}$ be a surface of genus g with n punctures with $3g + n \geqslant 4$, and $\mathrm{Teich}(S_{g,n})$ be Teichmüller space of $S_{g,n}$ endowed with the Weil–Petersson metric. The mapping class group $\mathrm{Mod}(S_{g,n})$ of $S_{g,n}$ acts on $\mathrm{Teich}(S_{g,n})$ by isometries. The moduli space $\mathcal{M}_{g,n}$ of $S_{g,n}$, endowed with the Weil–Petersson metric, is realized as the quotient $\mathrm{Teich}(S_{g,n})/\mathrm{Mod}(S_{g,n})$.

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The moduli space $\mathcal{M}_{g,n}$ is Kähler [1], incomplete [13, 50] and geodesically complete [54]. It has negative sectional curvature [47, 53], strongly negative curvature in the sense of Siu [44], dual Nakano negative curvature [30] and nonpositive definite Riemannian curvature operator [60]. The Weil–Petersson metric completion $\overline{\mathcal{M}}_{g,n}$ of moduli space $\mathcal{M}_{g,n}$, as a topological space, is the well-known Deligne–Mumford compactification of moduli space obtained by adding stable nodal curves [32]. One may refer to the book [58] for recent developments on the Weil–Petersson metric.

The asymptotic geometry of $\mathcal{M}_{g,n}$ as either g or n tends to infinity, has recently become quite active. For example, Brock–Bromberg [6] showed that the shortest Weil–Petersson closed geodesic in $\mathcal{M}_{g,0}$ is comparable to $\frac{1}{\sqrt{g}}$. Mirzakhani [34, 35, 36, 37] studied various aspects of the Weil–Petersson volume of $\mathcal{M}_{g,n}$ for large g. Together with M. Wolf [49], we studied the ℓ^p-norm ($1 \leqslant p \leqslant \infty$) of the Weil–Petersson curvature operator of $\mathcal{M}_{g,n}$ for large g. The Weil–Petersson curvature of $\mathcal{M}_{g,0}$ for large genus was studied in [61]. Cavendish–Parlier [12] studied the asymptotic behavior of the diameter $\operatorname{diam}(\mathcal{M}_{g,n})$ of $\mathcal{M}_{g,n}$. They showed that $\lim_{n\to\infty} \frac{\operatorname{diam}(\mathcal{M}_{g,n})}{\sqrt{n}}$ is a positive constant. They also showed that for large genus the ratio $\frac{\operatorname{diam}(\mathcal{M}_{g,n})}{\sqrt{g}}$ is bounded below by a positive constant and above by a constant multiple of $\ln g$. For the upper bound, they refined Brock's quasi-isometry of $\operatorname{Teich}(S_{g,n})$ to the pants graph [5]. As far as we know, the asymptotic behavior of $\operatorname{diam}(\mathcal{M}_{g,n})$ as g tends to infinity is still *open*. For other related topics, one may refer to [16, 22, 31, 39, 40, 45, 63] for more details.

Let $\partial\overline{\mathcal{M}}_{g,n}$ be the boundary of $\overline{\mathcal{M}}_{g,n}$, which consists of nodal surfaces. Let $\operatorname{dist}_{wp}(\,\cdot\,,\cdot\,)$ be the Weil–Petersson distance function. Define the *inradius* $\operatorname{InRad}(\mathcal{M}_{g,n})$ of $\mathcal{M}_{g,n}$ as

$$\operatorname{InRad}(\mathcal{M}_{g,n}) := \max_{X\in\mathcal{M}_{g,n}} \operatorname{dist}_{wp}(X, \partial\overline{\mathcal{M}}_{g,n}).$$

The inradius $\operatorname{InRad}(\mathcal{M}_{g,n})$ is the largest radius of geodesic balls (allowed to contain topology) in the interior of $\overline{\mathcal{M}}_{g,n}$. In this paper, one of our main goals is to study the asymptotic behavior of $\operatorname{InRad}(\mathcal{M}_{g,n})$ either as $g \to \infty$ or $n \to \infty$.

Notation. — In this paper, we use the notation

$$f_1 \asymp_t f_2$$

if there exists a universal constant $C > 0$, independent of t, such that

$$\frac{f_2}{C} \leqslant f_1 \leqslant Cf_2.$$

Our first result is

Theorem 1.1. — *For all $n \geqslant 0$ and $g \geqslant 2$, we have*

$$\mathrm{InRad}(\mathcal{M}_{g,n}) \asymp_g \sqrt{\ln g}.$$

We will show that as $g \to \infty$, the inradius $\mathrm{InRad}(\mathcal{M}_{g,n})$ is roughly realized by the family of surfaces constructed by Balacheff–Makover–Parlier in [3] (based on the work of Buser–Sarnak [11]), whose injectivity radii grow roughly as $\ln g$. We remark here that the method used in the proof of Theorem 1.1 also shows that $\mathrm{InRad}(\mathcal{M}_{g,[g^a]}) \asymp_g \sqrt{\ln g}$ for all $a \in (0,1)$. One can see Remark 5.4 for more details.

Our second result is

Theorem 1.2. — *For all $g \geqslant 0$ and $n \geqslant 4$, we have*

$$\mathrm{InRad}(\mathcal{M}_{g,n}) \asymp_n 1.$$

We remark that the method used in the proof of Theorem 1.2 also gives that $\mathrm{InRad}(\mathcal{M}_{[n^a],n}) \asymp_n 1$ for all $a \in (0,1)$. One can see Remark 5.6 for more details. We will give two different proofs for the lower bound in Theorem 1.2, one of which is by applying Theorem 1.3.

The difficult parts for Theorem 1.1 and 1.2 are the lower bounds, which rely on studying the systole function along Weil–Petersson geodesics.

For any $X \in \mathrm{Teich}(S_{g,n})$, we refer to the length of a shortest essential simple closed geodesic in X as the *systole* of X and denote it by $\ell_{sys}(X)$. The systole function $\ell_{\mathrm{sys}}(\,\cdot\,) : \mathrm{Teich}(S_{g,n}) \to \mathbb{R}^+$ is continuous, but not smooth as corners appear when it is realized by multiple essential isotopy classes of simple closed curves. However, it is a topological Morse function and its critical points can be characterized. One may refer to [2, 20, 42] for more details. The lower bounds in Theorems 1.1 and 1.2 will be established by using the following theorem, which gives a uniform lower bound for the Weil–Petersson distance in terms of systole functions.

Theorem 1.3. — *There exists a universal constant $K > 0$, independent of g and n, such that for all $X, Y \in \mathrm{Teich}(S_{g,n})$,*

$$\left|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}\right| \leqslant K \,\mathrm{dist}_{wp}(X,Y).$$

To the best of our knowledge, Theorem 1.3 is the first study of the systole function along Weil–Petersson geodesics, addressing a line of inquiry that Wolpert raised in [56, p. 274]: *determine the behaviors of the systole function along Weil–Petersson geodesics*. For the limits of relative systolic curves along a Weil–Petersson geodesic ray in Thurston's projective measured lamination space, one may see [8, 9, 10, 23] for more details.

The strategy for establishing Theorem 1.3 is to bound the Weil–Petersson norm of the gradient $\nabla\ell_{\alpha}^{\frac{1}{2}}(X)$ from above by a universal constant, independent of g and n, when α is an essential simple closed curve in X which realizes the systole of X. In order to do this, first by applying the real analyticity of the Weil–Petersson metric [1] and the convexity of geodesic length function along Weil–Petersson geodesics [54, 48], we make a thin-thick decomposition for the Weil–Petersson geodesic $\mathfrak{g}(X,Y) \subset \mathrm{Teich}(S_{g,n})$ connecting X and Y such that we can differentiate $\ell_{\mathrm{sys}}(\,\cdot\,)$ along the geodesic $\mathfrak{g}(X,Y)$ in some sense (see Lemma 3.5). Then, for the thin part of $\mathfrak{g}(X,Y)$ we use a result, due to Wolpert in [57] (see [57, Lemma 3.16] or Lemma 4.2), to get a uniform upper bound for the Weil–Petersson norm of the gradient $\nabla\ell_{\alpha}^{\frac{1}{2}}(X)$. For the thick part of $\mathfrak{g}(X,Y)$ (here the injectivity radius of some hyperbolic surface, which is a point on $\mathfrak{g}(X,Y)$, could be arbitrarily large [11]), we apply a special case of a formula of Riera [41] (see (4.2)) and some two-dimensional hyperbolic geometry theory to provide a uniform upper bound for the Weil–Petersson norm of the gradient $\nabla\ell_{\alpha}^{\frac{1}{2}}(X)$, where α realizes the systole of X (see Proposition 4.4). The step for the thick part almost takes up the entirety of Section 4. Then, Theorem 1.3 follows by integrating along the Weil–Petersson geodesic segment and the Cauchy–Schwartz inequality. See Section 4 for more details.

For any $\epsilon > 0$, let $\mathcal{M}_{g,n}^{\geqslant\epsilon}$ be the ϵ-thick part of moduli space. The Mumford compactness theorem tells that $\mathcal{M}_{g,n}^{\geqslant\epsilon}$ is compact. Denote by $\partial\mathcal{M}_{g,n}^{\geqslant\epsilon}$ the boundary of $\mathcal{M}_{g,n}^{\geqslant\epsilon}$, which consists of ϵ-thick surfaces whose injectivity radii are ϵ. It is clear that moduli space $\mathcal{M}_{g,n}$ is foliated by $\partial\mathcal{M}_{g,n}^{\geqslant\epsilon}$ for all $s > 0$. The following result bounds the Weil–Petersson distance between two leaves.

Theorem 1.4. — *There exists a universal constant $K' > 0$, independent of g and n, such that for any $s > t \geqslant 0$,*

$$\frac{\sqrt{s}-\sqrt{t}}{K'} \leqslant \mathrm{dist}_{wp}(\partial\mathcal{M}_{g,n}^{\geqslant s}, \partial\mathcal{M}_{g,n}^{\geqslant t}) \leqslant K'(\sqrt{s}-\sqrt{t}).$$

As stated above, the asymptotic behavior of the Weil–Petersson volume of $\mathcal{M}_{g,0}$ has been well studied as g tends to infinity. We are grateful to Maryam Mirzakhani for bringing the following interesting question to our attention.

Question 1.5. — *Fix a constant $R > 0$, are there any good upper bounds for the Weil–Petersson volume $\mathrm{Vol}_{wp}(B(X;R))$ as g tends to infinity? Here $B(X;R) = \{Y \in \mathrm{Teich}(S_{g,0});\ \mathrm{dist}_{wp}(Y,X) < R\}$ is the Weil–Petersson geodesic ball of radius R centered at X.*

The last part of this paper is to study Question 1.5. Let $S_g = S_{g,0}$ be the closed surface of genus g and $\mathrm{Teich}(S_g)$ be Teichmüller space endowed with the Weil–Petersson metric. Since the completion $\overline{\mathrm{Teich}(S_g)}$ of $\mathrm{Teich}(S_g)$ is not locally compact [55], it is well-known that the Weil–Petersson volume of a geodesic ball of finite radius blows up if this ball in $\overline{\mathrm{Teich}(S_g)}$ contains a boundary point (see Proposition 6.2 for more details). Thus, we need to assume that the Weil–Petersson geodesic balls in Question 1.5 stay away from the boundary of $\overline{\mathrm{Teich}(S_g)}$. For any positive constant r_0, we define

$$\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0} := \{X_g \in \mathrm{Teich}(S_g); \mathrm{dist}_{wp}(X_g; \partial\overline{\mathrm{Teich}(S_g)}) \geqslant r_0\}$$

where $\partial\overline{\mathrm{Teich}(S_g)}$ is the boundary of $\mathrm{Teich}(S_g)$. The space $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ is the subset in $\mathrm{Teich}(S_g)$ which is at least r_0-distance to the boundary. By applying Theorem 1.1 and Teo's [46] uniform lower bound for the Ricci curvature on the thick part of $\mathrm{Teich}(S_g)$, we will show that the Weil–Petersson volume of any Weil–Petersson geodesic ball in $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ rapidly decays to 0 as g tends to infinity. More precisely,

Theorem 1.6. — *For any $r_0 > 0$, then for any constant $\epsilon > 0$ we have*

$$\sup_{B(X_g;r_g)\subset\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}} \mathrm{Vol}_{wp}(B(X_g;r_g)) = o\left(\left(\frac{1}{g}\right)^{(3-\epsilon)g}\right)$$

where the supremum is taken over all the geodesic balls in $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ and $B(X_g;r_g) := \{Y_g \in \mathrm{Teich}(S_g);\ \mathrm{dist}_{wp}(Y_g, X_g) < r_g\}$.

Remark 1.7. — From Theorem 1.1 and Wolpert's upper bound for distance to strata (see Theorem 2.5), the largest radius of Weil–Petersson geodesic balls in $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ is comparable to $\sqrt{\ln g}$ as $g \to \infty$. In particular, Theorem 1.6 implies that for any constant $a \in (0, \frac{1}{2})$,

$$\lim_{g\to\infty} \inf_{X_g\in\mathrm{Teich}(S_g)} \mathrm{Vol}_{wp}(B(X_g;(\ln g)^a)) = 0.$$

A direct consequence of Theorem 1.6 is the following result.

Corollary 1.8. — *Fix a constant $R > 0$. Then there exists a constant $\epsilon(R) > 0$, only depending on R, such that for any $\epsilon > 0$,*

$$\sup_{X_g\subset\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant\epsilon(R)}} \mathrm{Vol}_{wp}(B(X_g;R)) = o\left(\left(\frac{1}{g}\right)^{(3-\epsilon)g}\right).$$

In particular, $\lim_{g\to\infty}\sup_{X_g\subset\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant\epsilon(R)}} \mathrm{Vol}(B(X_g;R)) = 0$.

The corollary above answers Question 1.5 at least following a certain interpretation.

Plan of the paper. Section 2 provides some necessary background and the basic properties on two-dimensional hyperbolic geometry and the Weil–Petersson metric. In Section 3 we will show that the systole function is piecewise real analytic along Weil–Petersson geodesics, which will be applied to prove Theorem 1.3. We will prove Theorem 1.3 in Section 4. In Section 5 we will prove Theorem 1.4 and apply Theorem 1.3 to prove Theorem 1.1 and 1.2. In Section 6 we will establish Theorem 1.6 and Corollary 1.8.

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2. Notations and Preliminaries

In this section we will set up the notations and provide some necessary background on two-dimensional hyperbolic geometry, Teichmüller theory and the Weil–Petersson metric.

2.1. Hyperbolic upper half plane

Let $\mathbb{H}$ be the upper half plane endowed with the hyperbolic metric $\rho(z)|\mathrm{d}z|^2$ where

$$\rho(z) = \frac{1}{(\mathrm{Im}(z))^2}.$$

A geodesic line in $\mathbb{H}$ is either a vertical line or an upper semi-circle centered at some point on the real axis. For $z = (r, \theta) \in \mathbb{H}$ given in polar coordinate where $\theta \in (0,\pi)$, the hyperbolic distance between z and the imaginary axis $\mathbf{i}\mathbb{R}^+$ is

$$\mathrm{dist}_{\mathbb{H}}(z,\mathbf{i}\mathbb{R}^+) = \ln|\csc\theta + |\cot\theta||. \tag{2.1}$$

Thus,

$$e^{-2\,\mathrm{dist}_{\mathbb{H}}(z,\mathbf{i}\mathbb{R}^+)} \leqslant \sin^2\theta = \frac{\mathrm{Im}^2(z)}{|z|^2} \leqslant 4e^{-2\,\mathrm{dist}_{\mathbb{H}}(z,\mathbf{i}\mathbb{R}^+)}. \tag{2.2}$$

It is known that any eigenfunction with positive eigenvalue of the hyperbolic Laplacian of $\mathbb{H}$ satisfies the mean value property [15, Corollary 1.3]. For $z=(r,\theta)\in\mathbb{H}$ given in polar coordinate, the function

$$u(\theta) = 1 - \theta\cot\theta$$

is a positive 2-eigenfunction. Thus, u satisfies the mean value property. It is not hard to see that $\min\{u(\theta), u(\pi-\theta)\}$ also satisfies the mean value property. Since $\min\{u(\theta), u(\pi-\theta)\}$ is comparable to $\sin^2\theta$, from inequality (2.2) we know that the function $e^{-2\,\mathrm{dist}_{\mathbb{H}}(z,\mathbf{i}\mathbb{R}^+)}$ satisfies the mean value property in $\mathbb{H}$. The following lemma is the simplest version of [57, Lemma 2.4].

Lemma 2.1. — *For any $r>0$ and $p\in\mathbb{H}$, there exists a positive constant $c(r)$, only depending on r, such that*

$$e^{-2\,\mathrm{dist}_{\mathbb{H}}(p,\mathbf{i}\mathbb{R}^+)} \leqslant c(r)\int_{B_{\mathbb{H}}(p;r)} e^{-2\,\mathrm{dist}_{\mathbb{H}}(z,\mathbf{i}\mathbb{R}^+)}\mathrm{d}A(z)$$

where $B_{\mathbb{H}}(p;r) = \{z\in\mathbb{H}; \mathrm{dist}_{\mathbb{H}}(p,z) < r\}$ is the hyperbolic geodesic ball of radius r centered at p and $\mathrm{d}A(z)$ is the hyperbolic area element.

2.2. Teichmüller space

Let $S_{g,n}$ be a surface of genus g with n punctures which satisfies that $3g-3+n>0$. Let M_{-1} be the space of Riemannian metrics on $S_{g,n}$ with constant curvatures -1, and $X=(S_{g,n},\sigma|\mathrm{d}z|^2)\in M_{-1}$. The group Diff_+, which is the group of orientation-preserving diffeomorphisms, acts by pull back on M_{-1}. In particular this holds for the normal subgroup Diff_0, the group of diffeomorphisms isotopic to the identity. The group $\mathrm{Mod}(S_{g,n}) := \mathrm{Diff}_+/\mathrm{Diff}_0$ is called the *mapping class group* of $S_{g,n}$.

The *Teichmüller space* $\mathcal{T}(S_{g,n})$ of $S_{g,n}$ is defined as

$$\mathcal{T}(S_{g,n}) := M_{-1}/\mathrm{Diff}_0\,.$$

The *moduli space* $\mathcal{M}(S_{g,n})$ of $S_{g,n}$ is defined as

$$\mathcal{M}(S_{g,n}) := \mathcal{T}(S_{g,n})/\mathrm{Mod}(S_{g,n}).$$

The Teichmüller space $\mathcal{T}(S_{g,n})$ is a real analytic manifold. Let α be an essential simple closed curve on $S_{g,n}$, then for any $X \in \mathrm{Teich}(S_{g,n})$, there exists a unique closed geodesic $[\alpha]$ in X which represents for α in the fundamental group of $S_{g,n}$. We denote by $\ell_\alpha(X)$ the length of $[\alpha]$ in X. In particular $\ell_\alpha(\cdot)$ defines a function on $\mathcal{T}(S_{g,n})$. The following property is well-known.

Lemma 2.2 ([27, Lemma 3.7]). — *The geodesic length function $\ell_\alpha(\cdot) : \mathcal{T}(S_{g,n}) \to \mathbb{R}^+$ is real-analytic.*

Let $X \in \mathcal{T}(S_{g,n})$ be a hyperbolic surface. The *systole* of X is the length of a shortest essential simple closed geodesic in X. We denote by $\ell_{sys}(X)$ the systole of X. It defines a continuous function $\ell_{\mathrm{sys}}(\cdot) : \mathcal{T}(S_{g,n}) \to \mathbb{R}^+$, which is called the *systole function*. In general, the systole function is clearly continuous and not smooth because of corners where there may exist multiple essential simple closed geodesics realizing the systole. This function is very useful in Teichmüller theory. Curves that realize the systole are often referred to *systolic curves*. One may refer to [2, 20, 42] for more details. In this paper we will study the behavior of this function along Weil–Petersson geodesics and apply these results to different problems.

Fixed a constant $\epsilon_0 > 0$. The ϵ_0-*thick part* of Teichmüller space of $S_{g,n}$, denoted by $\mathcal{T}(S_{g,n})^{\geqslant \epsilon_0}$, is defined as follows.

$$\mathcal{T}(S_{g,n})^{\geqslant \epsilon_0} := \{X \in \mathcal{T}(S_{g,n});\ \ell_{\mathrm{sys}}(X) \geqslant \epsilon_0\}.$$

The space $\mathcal{T}(S_{g,n})^{\geqslant \epsilon_0}$ is invariant by the mapping class group. The ϵ_0-*thick part* of moduli space of $S_{g,n}$, denoted by $\mathcal{M}(S_{g,n})^{\geqslant \epsilon_0}$, is defined by

$$\mathcal{M}(S_{g,n})^{\geqslant \epsilon_0} := \mathcal{T}(S_{g,n})^{\geqslant \epsilon_0} / \operatorname{Mod}(S_{g,n}).$$

It is known that $\mathcal{M}(S_{g,n})^{\geqslant \epsilon_0}$ is compact for all $\epsilon_0 > 0$, which is due to Mumford [38]. For more details on Teichmüller theory, one may refer to [26, 27].

2.3. Weil–Petersson metric

The real-analytic space $\mathcal{T}(S_{g,n})$ carries a natural complex structure. Let $X = (S_{g,n}, \sigma(z)|\mathrm{d}z|^2) \in \mathcal{T}_{g,n}$ be a point. The tangent space at X is identified with the space of harmonic Beltrami differentials on X which are forms of $\mu = \frac{\overline{\psi}}{\sigma}$ where ψ is a holomorphic quadratic differential on X. Let $\mathrm{d}A(z) = \sigma(z)\mathrm{d}x\mathrm{d}y$ be the volume form of $X = (S_{g,n}, \sigma(z)|\mathrm{d}z|^2)$ where $z = x + y\mathbf{i}$.

The *Weil–Petersson metric* is the Hermitian metric on $\mathcal{T}(S_{g,n})$ arising from the *Petersson scalar product*

$$\langle \varphi, \psi \rangle_{WP} = \int_X \frac{\varphi(z)}{\sigma(z)} \frac{\overline{\psi(z)}}{\sigma(z)} \mathrm{d}A(z)$$

via duality. We will concern ourselves primarily with its Riemannian part g_{WP}. We denote by $\mathrm{Teich}(S_{g,n})$ the Teichmüller space endowed with the Weil–Petersson metric. The mapping class group $\mathrm{Mod}(S_{g,n})$ acts properly discontinuously on $\mathrm{Teich}(S_{g,n})$ by isometries. Reversely, from Masur–Wolf [33] and Brock–Margalit [7] the whole isometry group of $\mathrm{Teich}(S_{g,n})$ is exactly the extended mapping class group except for some low complexity cases. The Weil–Petersson metric on Teichmüller space descends into a metric on moduli space. We denote by $\mathcal{M}_{g,n}$ moduli space $\mathcal{M}(S_{g,n})$ endowed with the Weil–Petersson metric.

The space $\mathrm{Teich}(S_{g,n})$ is incomplete [13, 50], negatively curved [47, 53] and uniquely geodesically convex [54]. The moduli space $\mathcal{M}_{g,n}$ is an orbifold with finite volume and finite diameter. One may refer to [27, 58] for more details on the Weil–Petersson metric. The following fundamental fact is due to Ahlfors [1], which will be used later.

Theorem 2.3 (Ahlfors). — *The space* $\mathrm{Teich}(S_{g,n})$ *is real-analytic Kähler.*

The following convexity theorem is due to Wolpert [54]. He used this result to give a new solution to the Nielsen Realization Problem which was first solved by Kerckhoff [29]. An alternative proof of this convexity theorem was given by Wolf [48], through using harmonic map theory.

Theorem 2.4 (Wolpert). — *For any essential simple closed curve* $\alpha \subset S_{g,n}$, *the length function* $\ell_\alpha : \mathrm{Teich}(S_{g,n}) \to \mathbb{R}^+$ *is strictly convex.*

2.4. Augmented Teichmüller space

The non-completeness of the Weil–Petersson metric corresponds to finite-length geodesics in $\mathrm{Teich}(S_{g,n})$ along which some essential simple closed curve pinches to zero. In [32] the completion $\overline{\mathrm{Teich}(S_{g,n})}$ of $\mathrm{Teich}(S_{g,n})$, called the *augmented Teichmüller space*, is described concretely by adding strata consisting of stratum $\mathcal{T}_\sigma$ defined by the vanishing of lengths

$$\ell_\alpha = 0$$

for each $\alpha \in \sigma$ where σ is a collection of mutually disjoint essential simple closed curves. The stratum $\mathcal{T}_\sigma$ are naturally products of lower dimensional

Teichmüller spaces corresponding to the nodal surfaces in $\mathcal{T}_\sigma$ [32]. The space $\overline{\mathrm{Teich}(S_{g,n})}$ is a complete CAT(0) space. It was shown in [14, 55, 62] that every stratum $\mathcal{T}_\sigma$ is totally geodesic in $\overline{\mathrm{Teich}(S_{g,n})}$. Since the completion $\overline{\mathcal{T}_\sigma}$ of $\mathcal{T}_\sigma$ is convex in $\overline{\mathrm{Teich}(S_{g,n})}$, by elementary CAT(0) geometry (see[4]) the nearest projection map

$$\pi_\sigma : \mathrm{Teich}(S_{g,n}) \to \overline{\mathcal{T}_\sigma}$$

is well-defined. Using Wolpert's theorem on the structure of the Alexandrov tangent cone at the boundary of $\overline{\mathrm{Teich}(S_{g,n})}$ (see [57, Theorem 4.18]) and the first variation formula for the distance function, one can show that for any $X \in \mathrm{Teich}(S_{g,n})$, the image $\pi_\sigma(X)$ is contained in $\mathcal{T}_\sigma$. One can see more details in [17, 59].

The following result of Wolpert (see [57, Section 4] for more details) will be used to prove the upper bounds in Theorems 1.1 and 1.2. Denote by $\mathrm{dist}_{wp}(\,\cdot\,,\cdot\,)$ the Weil–Petersson distance.

Theorem 2.5 (Wolpert). — *For any $X \in \mathrm{Teich}(S_{g,n})$, then we have*

$$\mathrm{dist}_{wp}(X, \pi_\sigma(X)) \leqslant \sqrt{2\pi \cdot \sum_{\alpha\in\sigma^0} \ell_\alpha(X)}.$$

It was shown by Masur [32] that the completion $\overline{\mathcal{M}}_{g,n}$ of moduli space $\mathcal{M}_{g,n}$ is homeomorphic to the Deligne–Mumford compactification of moduli space. Recall that the inradius $\mathrm{InRad}(\mathcal{M}_{g,n})$ of $\mathcal{M}_{g,n}$ is defined as $\max_{X\in\mathcal{M}_{g,n}} \mathrm{dist}_{wp}(X, \partial\overline{\mathcal{M}}_{g,n})$. The inradius $\mathrm{InRad}(\mathcal{M}_{g,n})$ is the largest radius of geodesic balls in the interior of $\overline{\mathcal{M}}_{g,n}$. Similarly, we also define the *inradius* $\mathrm{InRad}(\mathrm{Teich}(S_{g,n}))$ of $\mathrm{Teich}(S_{g,n})$ as

$$\mathrm{InRad}(\mathrm{Teich}(S_{g,n})) := \max_{X\in\mathrm{Teich}(S_{g,n})} \mathrm{dist}_{wp}(X, \partial\overline{\mathrm{Teich}(S_{g,n})})$$

where $\partial\overline{\mathrm{Teich}(S_{g,n})}$ is the boundary of $\overline{\mathrm{Teich}(S_{g,n})}$.

In this article we will study the asymptotic behaviors of $\mathrm{InRad}(\mathcal{M}_{g,n})$ and $\mathrm{InRad}(\mathrm{Teich}(S_{g,n}))$ either as g goes to infinity or as n goes to infinity.

3. The systole function is piecewise real analytic

As stated in Section 2, although the systole function $\ell_{\mathrm{sys}}(\,\cdot\,)$ is continuous over $\mathrm{Teich}(S_{g,n})$, it is not smooth. In this section we will provide two fundamental lemmas on the systole function $\ell_{\mathrm{sys}}(\,\cdot\,)$ along a Weil–Petersson geodesic such that we can take the derivative of the systole function along the Weil–Petersson geodesic, which are crucial in the proof of Theorem 1.3.

Before stating the results, we provides three basic claims on geodesic length functions. We always assume Weil–Petersson geodesics use arc-length parameters.

Claim 3.1. — *For any essential simple closed curve $\alpha \subset S_{g,n}$ and $\gamma : [0,s] \to \mathrm{Teich}(S_{g,n})$ be a Weil–Petersson geodesic where $s > 0$ is a constant. Then the geodesic length function $\ell_\alpha(\gamma(t)) : [0,s] \to \mathbb{R}^+$ is real-analytic on t.*

Proof of Claim 3.1. — From Lemma 2.3 we know that $\mathrm{Teich}(S_{g,n})$ is real-analytic. In particular, all the Christoffel symbols are real-analytic. Thus, the classical Cauchy–Kowalevski Theorem gives that the solution of the Weil–Petersson geodesic equation is real-analytic. That is, every Weil–Petersson geodesic is real-analytic. Then the claim follows from Lemma 2.2. □

Let $X \in \mathrm{Teich}(S_{g,n})$. We define the set $\mathrm{sys}(X)$ of systolic curves as

$$\mathrm{sys}(X) := \{\beta \subset S_{g,n};\ \ell_\beta(X) = \ell_{\mathrm{sys}}(X)\}.$$

It is clear that the set $\mathrm{sys}(X)$ is finite for all $X \in \mathrm{Teich}(S_{g,n})$.

Claim 3.2. — *Let $s > 0$ and $\gamma : [0,s] \to \mathrm{Teich}(S_{g,n})$ be a Weil–Petersson geodesic. Then the union $\cup_{0 \leqslant t \leqslant s} \mathrm{sys}(\gamma(t))$ is a finite set.*

Proof of Claim 3.2. — First we denote by $\mathrm{dist}_T(\,\cdot\,,\cdot\,)$ the Teichmüller distance. Since the image $\gamma([0,s])$ is a compact subset in $\mathrm{Teich}(S_{g,n})$, there exists a constant $K > 0$ such that the Teichmüller distance

$$\max_{t \in [0,s]} \mathrm{dist}_T(\gamma(0), \gamma(t)) \leqslant K$$

and

$$\max_{t \in [0,s]} \ell_{\mathrm{sys}}(\gamma(t)) \leqslant K.$$

By [51, Lemma 3.1] we know that for all $t \in [0,s]$ and $\beta(t) \in \mathrm{sys}(\gamma(t))$ we have $\ell_{\beta(t)}(\gamma(0)) \leqslant K \cdot e^{2K}$. That is, the union satisfies

$$\cup_{0 \leqslant t \leqslant s} \mathrm{sys}(\gamma(t)) \subset \{\beta \subset S_{g,n}; \ell_\beta(\gamma(0)) \leqslant K \cdot e^{2K}\}$$

which is a finite set. Then the claim follows. □

We do not know whether the cardinality of the union $\cup_{0 \leqslant t \leqslant s} \mathrm{sys}(\gamma(t))$ in the lemma above has any precise upper bound.

Claim 3.3. — *Let $s > 0$ be a constant, the curve $\gamma : [0,s] \to \mathrm{Teich}(S_{g,n})$ be a Weil–Petersson geodesic and $\alpha, \beta \in \mathrm{sys}(\gamma(0))$ be two distinct essential simple closed geodesics. Then either $\ell_\alpha(\gamma(t)) \equiv \ell_\beta(\gamma(t))$ over $[0,s]$ or there* *exists a constant* $0 < s_0 \leqslant s$ *such that either* $\ell_\alpha(\gamma(t)) < \ell_\beta(\gamma(t))$ *over* $(0, s_0)$ *or* $\ell_\beta(\gamma(t)) < \ell_\alpha(\gamma(t))$ *over* $(0, s_0)$.

Proof of Claim 3.3. — Since the image $\gamma([0, s])$ is contained in $\mathrm{Teich}(S_{g,n})$, we can extend the geodesic $\gamma([0, s])$ in both directions a little bit longer. That is, there exists a positive constant $\epsilon > 0$ such that $\gamma : (-\epsilon, s+\epsilon) \to \mathrm{Teich}(S_{g,n})$ is well-defined. By Claim 3.1 we know that both ℓ_α and ℓ_β are real-analytic along the Weil–Petersson geodesic $\gamma(-\epsilon, s+\epsilon)$. If all the derivatives $\ell_\alpha^{(k)}(\gamma(0)) = \ell_\beta^{(k)}(\gamma(0))$ for all $k \in \mathbb{N}^+$, then the Taylor expansions of ℓ_α and ℓ_β at $\gamma(0)$ tells that $\ell_\alpha(\gamma(t)) \equiv \ell_\beta(\gamma(t))$ over $[0, s]$. Otherwise, there exists a positive integer k_0 such that $\ell_\alpha^{(k)}(\gamma(0)) = \ell_\beta^{(k)}(\gamma(0))$ for all $0 \leqslant k \leqslant k_0 - 1$ and $\ell_\alpha^{(k_0)}(\gamma(0)) \neq \ell_\beta^{(k_0)}(\gamma(0))$. The Taylor expansions of ℓ_α and ℓ_β at $\gamma(0)$ clearly imply the later case of the claim. ☐

Now we are ready to state the first lemma, which will be applied to prove Proposition 4.3.

Lemma 3.4. — *Let* $X \neq Y \in \mathrm{Teich}(S_{g,n})$, $s = \mathrm{dist}_{wp}(X, Y) > 0$ *and* $\gamma : [0, s] \to \mathrm{Teich}(S_{g,n})$ *be the Weil–Petersson geodesic with* $\gamma(0) = X$ *and* $\gamma(s) = Y$. *Then there exist a positive integer* k, *a partition* $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = s$ *of the interval* $[0, s]$ *and a sequence of essential simple closed curves* $\{\alpha_i\}_{0 \leqslant i \leqslant k-1}$ *in* $S_{g,n}$ *such that for all* $0 \leqslant i \leqslant k-1$,

(1) $\alpha_i \neq \alpha_{i+1}$.
(2) $\ell_{\alpha_i}(\gamma(t)) = \ell_{\mathrm{sys}}(\gamma(t)), \quad \forall\, t_i \leqslant t \leqslant t_{i+1}$.

Proof. — First by Claim 3.2 one may assume that the union

$$\cup_{0 \leqslant t \leqslant s} \mathrm{sys}(\gamma(t)) = \{\beta_i\}_{1 \leqslant i \leqslant n'}$$

for some positive integer n' where $\beta_i \subset S_{g,n}$ is an essential simple closed curve for each $1 \leqslant i \leqslant n'$. Without loss of generality one may assume that $\mathrm{sys}(\gamma(0))$ consists of the first n_0 curves for some $0 < n_0 \leqslant n'$. That is

$$\mathrm{sys}(\gamma(0)) = \cup_{1 \leqslant i \leqslant n_0} \{\beta_i\}.$$

Thus, for all $1 \leqslant i \leqslant n_0$ and $n_0 + 1 \leqslant j \leqslant n'$ we have

$$\ell_{\beta i}(\gamma(0)) < \ell_{\beta j}(\gamma(0)).$$

By the inequality above and using Claim 3.3 finite number of steps (induction on n_0), there exist a positive constant $s_0 \leqslant s$ and an essential simple closed curve in the set of systolic curves $\mathrm{sys}(\gamma(0))$ of $\gamma(0)$, which is denoted by α_0, such that for all $1 \leqslant i \leqslant n'$ we have

$$\ell_{\alpha_0}(\gamma(t)) \leqslant \ell_{\beta_i}(\gamma(t)),\ \forall\, 0 \leqslant t \leqslant s_0.$$

Set

$$t_1 = \max\{t';\ \ell_{\alpha_0}(\gamma(t)) \leqslant \min_{1\leqslant i\leqslant n'} \ell_{\beta_i}(\gamma(t)),\ \forall\, 0 \leqslant t \leqslant t'\}.$$

In particular,

$$\ell_{\alpha_0}(\gamma(t)) = \ell_{\mathrm{sys}}(\gamma(t)),\ \forall\, 0 \leqslant t \leqslant t_1.$$

It is clear that

$$0 < s_0 \leqslant t_1 \leqslant s.$$

We may assume that $t_1 < s$; otherwise we are done.

Using the same argument above at $\gamma(t_1)$ there exist a positive constant t_2 with $t_1 < t_2 \leqslant s$ and an essential simple closed curve in $\mathrm{sys}(\gamma(t_1))$, which is denoted by α_1, such that

$$\ell_{\alpha_1}(\gamma(t)) \leqslant \min_{1\leqslant i\leqslant n'} \ell_{\beta_i}(\gamma(t)),\ \forall\, t_1 \leqslant t \leqslant t_2.$$

In particular,

$$\ell_{\alpha_1}(\gamma(t)) = \ell_{\mathrm{sys}}(\gamma(t)),\ \forall\, t_1 \leqslant t \leqslant t_2.$$

From the definition of t_1 we know that

$$\alpha_0 \neq \alpha_1.$$

Thus, from Claim 3.3 and the definition of t_1 we know that there exists a constant $r_1 > 0$ with $r_1 < t_2 - t_1$ such that

$$\ell_{\alpha_1}(\gamma(t)) < \ell_{\alpha_0}(\gamma(t)),\ \forall\, t_1 < t < t_1 + r_1.$$

Then the conclusion follows by a finite induction.

We argue by contradiction. If not, then there exist two infinite sequences of positive constants $\{t_i\}_{i\geqslant 1}$ with $t_i < t_{i+1} < s$, $\{r_i\}_{i\geqslant 1}$ with $0 < r_i < t_{1+i} - t_i$, and a sequence of essential simple closed curves

$$\{\alpha_i\}_{i\geqslant 1} \subset \cup_{0\leqslant t\leqslant s} \mathrm{sys}(\gamma(t)) = \{\beta_i\}_{1\leqslant i\leqslant n'}$$

such that for all $i \geqslant 1$,

$$\ell_{\alpha_i}(\gamma(t)) = \ell_{\mathrm{sys}}(\gamma(t)),\ \ \forall\, t_i \leqslant t \leqslant t_{i+1}. \tag{3.1}$$

$$\alpha_i \neq \alpha_{i-1}. \tag{3.2}$$

$$\ell_{\alpha_i}(\gamma(t)) < \ell_{\alpha_{i-1}}(\gamma(t)),\ \ \forall\, t_i < t < t_i + r_i. \tag{3.3}$$

Since $\{t_i\}$ is a bounded increasing sequence, we assume that $\lim_{i\to\infty} t_i = T$. It is clear that $0 < T \leqslant s$. Since $\{\alpha_i\}_{i\geqslant 1} \subset \cup_{0\leqslant t\leqslant s} \mathrm{sys}(\gamma(t)) = \{\beta_i\}_{1\leqslant i\leqslant n'}$ which is a finite set, there exist two essential simple closed curves $\alpha \neq \beta \in$

$\{\beta_i\}_{1\leqslant i\leqslant n'}$, a subsequence $\{t'_i\}_{i\geqslant 1}$ of $\{t_{2i}\}_{i\geqslant 1}$ and a subsequence $\{t''_i\}_{i\geqslant 1}$ of $\{t_{2i}+\frac{r_{2i}}{2}\}_{i\geqslant 1}$ such that for all $i\geqslant 1$,

$$t'_i < t''_i < t'_{i+1}. \tag{3.4}$$

$$\lim_{i\to\infty} t'_i = \lim_{i\to\infty} t''_i = T. \tag{3.5}$$

$$\ell_\alpha(\gamma(t'_i)) = \ell_{\mathrm{sys}}(\gamma(t'_i)). \tag{3.6}$$

$$\ell_\beta(\gamma(t''_i)) = \ell_{\mathrm{sys}}(\gamma(t''_i)). \tag{3.7}$$

Recall that t''_i is of form $t_{2i}+\frac{r_{2i}}{2}$, (3.3) tells us that

$$\ell_\beta(\gamma(t''_i)) = \ell_{\mathrm{sys}}(\gamma(t''_i)) < \ell_\alpha(\gamma(t''_i)). \tag{3.8}$$

Since geodesic length functions are continuous over $\mathrm{Teich}(S_{g,n})$,

$$\ell_\alpha(\gamma(T)) = \ell_\beta(\gamma(T)) = \ell_{\mathrm{sys}}(\gamma(T)).$$

Consider the Weil–Petersson geodesic $c:[0,T]\to \mathrm{Teich}(S_{g,n})$ which is defined as $c(t)=\gamma(T-t)$ for all $0\leqslant t\leqslant T$. We apply Claim 3.3 to c at $c(0)=\gamma(T)$. Then from inequality (3.8) and Claim 3.3 we know that there exists a constant $s'_0>0$ such that

$$\ell_\beta(c(t)) < \ell_\alpha(c(t)),\ \forall\, t\in(0,s'_0). \tag{3.9}$$

On the other hand, from (3.5) and (3.6) one may choose a number $\epsilon\in(0,s'_0)$ to be small enough such that

$$\ell_\alpha(c(\epsilon)) = \ell_\alpha(\gamma(T-\epsilon)) = \ell_{\mathrm{sys}}(\gamma(T-\epsilon)) = \ell_{\mathrm{sys}}(c(\epsilon)) \tag{3.10}$$

which contradicts inequality (3.9). □

For any $\epsilon_0>0$ we denote by $\mathrm{Teich}(S_{g,n})^{\geqslant\epsilon_0}$ the ϵ_0-thick part of Teichmüller space endowed with the Weil–Petersson metric. Let $\mathrm{Teich}(S_{g,n})^{>\epsilon_0}$ be the interior of $\mathrm{Teich}(S_{g,n})^{\geqslant\epsilon_0}$. The following lemma will be applied to prove Theorem 1.3.

Lemma 3.5. — *Fix a constant $\epsilon_0>0$. Let $X\neq Y\in\mathrm{Teich}(S_{g,n})$, $s=\mathrm{dist}_{wp}(X,Y)>0$ and $\gamma:[0,s]\to\mathrm{Teich}(S_{g,n})$ be the Weil–Petersson geodesic with $\gamma(0)=X$ and $\gamma(s)=Y$. Then there exist a positive integer k, a partition $0=t_0<t_1<\cdots<t_{k-1}<t_k=s$ of the interval $[0,s]$, a sequence of closed intervals $\{[a_i,b_i]\subseteq[t_i,t_{i+1}]\}_{0\leqslant i\leqslant k-1}$ and a sequence of essential simple closed curves $\{\alpha_i\}_{0\leqslant i\leqslant k-1}$ in $S_{g,n}$ such that for all $0\leqslant i\leqslant k-1$,*

(1) $\alpha_i\neq\alpha_{i+1}$.
(2) $\ell_{\alpha_i}(\gamma(t))=\ell_{\mathrm{sys}}(\gamma(t)),\quad \forall\, t_i\leqslant t\leqslant t_{i+1}$.
(3) $\gamma([0,s])\cap(\mathrm{Teich}(S_{g,n})-\mathrm{Teich}(S_{g,n})^{>\epsilon_0})=\cup_{0\leqslant i\leqslant k-1}\gamma([a_i,b_i])$.

Proof. — First we apply Lemma 3.4 to the Weil–Petersson geodesic $\gamma([0,s])$. Then there exist a positive integer k, a partition $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = s$ of the interval $[0,s]$ and a sequence of essential simple closed curves $\{\alpha_i\}_{0\leqslant i\leqslant k-1}$ in $S_{g,n}$ such that for all $0 \leqslant i \leqslant k-1$ we have

$$\ell_{\alpha_i}(\gamma(t)) = \ell_{\text{sys}}(\gamma(t)), \quad \forall\, t_i \leqslant t \leqslant t_{i+1}. \tag{3.11}$$

Thus, Part (1) and (2) follows.

We apply Theorem 2.4 to the geodesic length function

$$\ell_{\alpha_i}(\,\cdot\,) : \gamma([t_i, t_{i+1}]) \to \mathbb{R}^+$$

for all $0 \leqslant i \leqslant k-1$. Since $\ell_{\alpha_i}(\,\cdot\,)$ is strictly convex on $\gamma([t_i,t_{i+1}])$ and $\gamma([0,s]) \subset \mathrm{Teich}(S_{g,n})$, the maximal principle for a convex function gives that $\ell_{\alpha_i}^{-1}([0,\epsilon_0])$ is a closed connected subset in $\gamma([t_i,t_{i+1}])$, which is denoted by $\gamma([a_i,b_i])$ for some closed interval $[a_i,b_i] \subseteq [t_i,t_{i+1}]$ (note that $\gamma([a_i,b_i])$ may be just a single point or an empty set). Then Part (3) clearly follows from the choices of a_i and b_i. □

4. Uniformly Lipschitz

Recall that the systole function $\ell_{\text{sys}}(\,\cdot\,) : \mathrm{Teich}(S_{g,n}) \to \mathbb{R}^+$ is continuous and not smooth. The goal of this section is to prove Theorem 1.3 which says that the square root of the systole function is uniformly Lipschitz continuous along Weil–Petersson geodesics. The method in this section is influenced by [57]. For convenience we restate Theorem 1.3 here.

Theorem 4.1. — *There exists a universal constant $K > 0$, independent of g and n, such that for all $X, Y \in \mathrm{Teich}(S_{g,n})$,*

$$\left|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}\right| \leqslant K\,\mathrm{dist}_{wp}(X,Y).$$

We begin by outlining the idea of the proof.

For any Weil–Petersson geodesic $\mathfrak{g}(X,Y) \subset \mathrm{Teich}(S_{g,n})$ joining X and Y in $\mathrm{Teich}(S_{g,n})$, first we apply Lemma 3.5 to make a thick-thin decomposition for the geodesic $\mathfrak{g}(X,Y)$ such that both of the thick and thin parts are disjoint closed intervals with certain properties. Then we use different arguments for these two parts. For the thin part we will apply the following result due to Wolpert.

Lemma 4.2 ([57, Lemma 3.16]). — *There exists a universal constant $c > 0$, independent of g and n, such that for all $X \in \mathrm{Teich}(S_{g,n})$ and any essential simple closed curve $\alpha \subset S_{g,n}$,*

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) \leqslant c \cdot \left(\ell_\alpha(X) + \ell_\alpha^2(X) e^{\frac{\ell_\alpha(X)}{2}}\right).$$

Fix a constant $k_0 > 0$, the lemma above implies that for all essential simple closed curve $\alpha \subset S_{g,n}$ with $\ell_\alpha \leqslant k_0$,

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) \leqslant C(k_0)\ell_\alpha$$

where $C(k_0)$ is a constant only depending on k_0.

Recall that the length ℓ_α could be arbitrarily large for any essential simple closed curve $\alpha \subset S_{g,n}$ (Buser–Sarnak [11] constructed hyperbolic surfaces whose injectivity radii grow roughly as $\ln g$), actually for the thick part of the geodesic $\mathfrak{g}(X,Y)$, no matter how large the injectivity radius is, we will apply the following proposition, which is the main part of this section.

Proposition 4.3. — *Fix a constant $\epsilon_0 > 0$. Then there exists a positive constant $C(\epsilon_0)$, only depending on ϵ_0, such that for any $X, Y \in \mathrm{Teich}(S_{g,n})$ with the Weil–Petersson geodesic $\mathfrak{g}(X,Y) \subset \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$, we have*

$$\left| \sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)} \right| \leqslant C(\epsilon_0)\, \mathrm{dist}_{wp}(X,Y).$$

For any essential simple closed curve $\alpha \subset S_{g,n}$, the geodesic length function $\ell_\alpha(\,\cdot\,)$ is real-analytic over $\mathrm{Teich}(S_{g,n})$. Gardiner in [18, 19] provided formulas for the differentials of ℓ_α. Let $(X, \sigma(z)|\mathrm{d}z|^2) \in \mathrm{Teich}(S_{g,n})$ be a hyperbolic surface and Γ be its associated Fuchsian group. Since α is an essential simple closed curve, we may denote by A be the deck transformation on the upper half plane $\mathbb{H}$ corresponding to the simple closed geodesic $[\alpha] \subset X$. Consider the quadratic differential

$$\Theta_\alpha(z) = \sum_{E \in \langle A \rangle / \Gamma} \frac{E'(z)^2}{E(z)^2} \mathrm{d}z^2 \tag{4.1}$$

where $\langle A \rangle$ is the cyclic group generated by A.

Then the gradient $\nabla \ell_\alpha(\,\cdot\,)$ of the geodesic length function ℓ_α is

$$\nabla \ell_\alpha(X)(z) = \frac{2}{\pi} \frac{\overline{\Theta}_\alpha(z)}{\rho(z)|\mathrm{d}z|^2}$$

where $\rho(z)|\mathrm{d}z|^2$ is the hyperbolic metric on the upper half plane. The tangent vector $t_\alpha = \frac{\mathbf{i}}{2}\nabla \ell_\alpha$ is the infinitesimal Fenchel–Nielsen right twist deformation [52].

In [41] Riera provided a formula for the Weil–Petersson inner product of a pair of geodesic length gradients. Let $\alpha, \beta \subset X$ be two essential simple closed curves with $A, B \in \Gamma$ be its associated deck transformations with axes $\widetilde{\alpha}, \widetilde{\beta}$ on the upper half plane. Riera's formula [41, Theorem 2] says that

$$\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle_{wp}(X) = \frac{2}{\pi}\left(\ell_\alpha \delta_{\alpha\beta} + \sum_{E \in \langle A \rangle \backslash \Gamma / \langle B \rangle} \left(u \ln \left|\frac{u+1}{u-1}\right| - 2\right)\right)$$

for the Kronecker delta $\delta.$, where $u = u(\widetilde{\alpha}, E \circ \widetilde{\beta})$ is the cosine of the intersection angle if $\widetilde{\alpha}$ and $E \circ \widetilde{\beta}$ intersect and is otherwise $\cosh(\mathrm{dist}_{\mathbb{H}}(\widetilde{\alpha}, E \circ \widetilde{\beta}))$ where $\mathrm{dist}_{\mathbb{H}}(\widetilde{\alpha}, E \circ \widetilde{\beta})$ is the hyperbolic distance between the two geodesic lines. Riera's formula was applied in [57] to study Weil–Petersson gradient of simple closed curves of short lengths. In this paper we will use Riera's formula to study the systolic curves which may have large lengths.

In particular setting $\alpha = \beta$ in Riera's formula, then we have

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) = \frac{2}{\pi}\left(\ell_\alpha + \sum_{E \in \{\langle A \rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}} \left(u \ln \frac{u+1}{u-1} - 2\right)\right) \tag{4.2}$$

where $u = \cosh(\mathrm{dist}_{\mathbb{H}}(\widetilde{\alpha}, E \circ \widetilde{\alpha}))$ and the double-coset of the identity element is omitted from the sum. We can view the formula above as a function on essential simple closed curves in $S_{g,n}$. In this section, we will evaluate this function at $\alpha \in \mathrm{sys}(X)$ and make estimates to prove the following result, which is essential in the proof of Proposition 4.3.

Proposition 4.4. — *Fix a constant $\epsilon_0 > 0$. Then there exists a positive constant $D(\epsilon_0)$, only depending on ϵ_0, such that for any $X \in \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$ and any systolic curve $\alpha \in \mathrm{sys}(X)$ we have*

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) \leqslant D(\epsilon_0) \cdot \ell_\alpha(X).$$

Remark 4.5. — From Riera's formula it is clear that

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) \geqslant \ell_\alpha(X).$$

Thus, $\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X)$ is comparable to $\ell_\alpha(X)$ under the same conditions as in Proposition 4.4.

Before we prove Proposition 4.4, let's set up some notations and provide two lemmas.

As stated above, we let $X \in \mathrm{Teich}(S_{g,n})$ be a hyperbolic surface and $\alpha \subset X$ be an essential simple closed curve. Up to conjugacy, we may assume that the closed geodesic $[\alpha]$ corresponds to the deck transformation $A : z \to e^{\ell_\alpha} \cdot z$ with axis $\widetilde{\alpha} = \mathbf{i}\mathbb{R}^+$ which is the imaginary axis and the fundamental domain $\mathbb{A} = \{z \in \mathbb{H}; 1 \leqslant |z| \leqslant e^{\ell_\alpha}\}$. Let γ_1, γ_2 be two geodesic lines in $\mathbb{H}$. The distance $\mathrm{dist}_{\mathbb{H}}(\gamma_1, \gamma_2)$ is given by

$$\mathrm{dist}_{\mathbb{H}}(\gamma_1, \gamma_2) = \inf_{p \in \gamma_1} \mathrm{dist}_{\mathbb{H}}(p, \gamma_2).$$

The following lemma says that any two lifts of the closed geodesic $[\alpha]$ in the upper half plane are uniformly separated. More precisely,

Lemma 4.6. — *Fix a constant* $\epsilon_0 > 0$. *Then there exists a constant* $C_0(\epsilon_0) > 0$, *only depending on* ϵ_0, *such that for any* $X \in \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$, $\alpha \in \mathrm{sys}(X)$ *and all* $B \in \{\langle A\rangle\backslash\Gamma - \mathrm{id}\}$ *we have*

$$\mathrm{dist}_{\mathbb{H}}(\widetilde{\alpha}, B \circ \widetilde{\alpha}) \geqslant \frac{\epsilon_0}{4}.$$

Proof. — The proof follows from a standard argument in Riemannian geometry (the so-called closing lemma). Since $X \in \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$ and $\alpha \in \mathrm{sys}(X)$, for every point $m \in [\alpha]$, the closed geodesic in X representing α, we have the geodesic ball $B_X(m; \frac{\epsilon_0}{4}) \subset X$, of radius $\frac{\epsilon_0}{4}$ centered at m, is isometric to a hyperbolic geodesic ball of radius $\frac{\epsilon_0}{4}$ in $\mathbb{H}$. Since $[\alpha]$ is a systolic curve and $X \in \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$, the intersection $[\alpha] \cap B_X(m; \frac{\epsilon_0}{4})$ is a geodesic arc of length $\frac{\epsilon_0}{2}$ with the midpoint m.

Claim. — $\mathrm{dist}_{\mathbb{H}}(\widetilde{\alpha}, B \circ \widetilde{\alpha}) \geqslant C_0(\epsilon_0)$ *for all* $B \in \{\langle A\rangle\backslash\Gamma - \mathrm{id}\}$.

We argue by contradiction for the proof of the claim. Suppose it does not hold. Then we let $p \in \widetilde{\alpha}$ and $q \in B \circ \widetilde{\alpha}$ such that

$$\mathrm{dist}_{\mathbb{H}}(p, q) < \frac{\epsilon_0}{4}. \tag{4.3}$$

Let $B_{\mathbb{H}}(p; \frac{\epsilon_0}{4}) \subset \mathbb{H}$ be the geodesic ball centered at p of radius $\frac{\epsilon_0}{4}$. It is clear that the covering map

$$\pi : B_{\mathbb{H}}\left(p; \frac{\epsilon_0}{4}\right) \to X$$

is an isometric embedding. Thus,

$$\pi\left(B_{\mathbb{H}}\left(p; \frac{\epsilon_0}{4}\right) \cap \widetilde{\alpha}\right) = [\alpha] \cap B_X\left(\pi(p); \frac{\epsilon_0}{4}\right) \tag{4.4}$$

Since the two geodesic lines $\widetilde{\alpha}$ and $B \circ \widetilde{\alpha}$ are disjoint, by inequality (4.3) we know that $q \in B_{\mathbb{H}}(p; \frac{\epsilon_0}{4}) - B_{\mathbb{H}}(p; \frac{\epsilon_0}{4}) \cap \widetilde{\alpha}$. Since $q \in B \circ \widetilde{\alpha}$,

$$\pi(q) \in [\alpha] \cap B_X\left(\pi(p); \frac{\epsilon_0}{4}\right)$$

which, together with (4.4), implies that the covering map $\pi : B_{\mathbb{H}}(p; \frac{\epsilon_0}{4}) \to X$ is not injective, which is a contradiction. $\square$

Remark 4.7. — The condition $\alpha \in \mathrm{sys}(X)$ is essential in Lemma 4.6. Otherwise, the estimate above may fail if one think about that case that the intersection of $[\alpha]$ with a geodesic ball of small radius is not connected.

Recall that the axis $\widetilde{\alpha}$ of the closed geodesic $[\alpha] \subset X$ in the upper half plane is the imaginary axis $\mathbf{i}\mathbb{R}^+$. Let $B \in \{\langle A\rangle\backslash\Gamma/\langle A\rangle - \mathrm{id}\}$. It is clear that the two geodesic lines $B \circ (\mathbf{i}\mathbb{R}^+)$ and $\mathbf{i}\mathbb{R}^+$ are disjoint, and have disjoint boundary points at infinity. Since the distance function between two convex subsets in $\mathbb{H}$ is strictly convex (one may see [4, p. 176] in a more general setting), there exists a unique point $p_B \in B \circ (\mathbf{i}\mathbb{R}^+)$ such that

$$\mathrm{dist}_{\mathbb{H}}(p_B, \mathbf{i}\mathbb{R}^+) = \mathrm{dist}_{\mathbb{H}}(B \circ (\mathbf{i}\mathbb{R}^+), \mathbf{i}\mathbb{R}^+).$$

The goal of the following lemma is to study the position of the nearest projection point p_B in $\mathbb{H}$.

Lemma 4.8. — *Let $B \in \{\langle A\rangle\backslash\Gamma/\langle A\rangle - \mathrm{id}\}$. Then there exists a representative $B' \in \langle A\rangle\backslash\Gamma$ for B such that*

$$1 \leqslant r_{B'} \leqslant e^{\ell_\alpha}$$

where $p_{B'} = (r_{B'}, \theta_{B'})$ in polar coordinate be the nearest projection point on $B' \circ (\mathbf{i}\mathbb{R}^+)$ from $\mathbf{i}\mathbb{R}^+$.

Proof. — Recall that the fundamental domain of A, the deck transformation corresponding to $[\alpha]$, is $\mathbb{A} = \{z \in \mathbb{H}; 1 \leqslant |z| \leqslant e^{\ell_\alpha}\}$. For any $B \in \{\langle A\rangle\backslash\Gamma - \mathrm{id}\}$, the map $B : \mathbb{A} \to \mathbb{A}$ is biholomorphic. Let $p_B = (r_B, \theta_B)$ in polar coordinates be the nearest projection point on $B \circ (\mathbf{i}\mathbb{R}^+)$ from $\mathbf{i}\mathbb{R}^+$.

Case (1): $1 \leqslant r_B \leqslant e^{\ell_\alpha}$. — Then we are done by choosing $B' = B$.

Case (2): $0 < r_B < 1$ or $r_B > e^{\ell_\alpha}$. — First there exists an integer k such that

$$A^k \circ r_B \in \{(r, \theta) \in \mathbb{H}; 1 \leqslant r \leqslant e^{\ell_\alpha}\}.$$

Choose $B' = A^k \cdot B$. Then $B' = B \in \{\langle A\rangle\backslash\Gamma/\langle A\rangle - \mathrm{id}\}$ by the definition of double-cosets. Since $B' = A^k \cdot B$ and A^k acts on $\mathbf{i}\mathbb{R}^+$ by isometries,

$$\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, B' \circ (\mathbf{i}\mathbb{R}^+)) = \mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, A^k \circ r_B).$$

Let $p_{B'} = (r_{B'}, \theta_{B'})$ in polar coordinates be the nearest point projection on $B' \circ (\mathbf{i}\mathbb{R}^+)$ from $\mathbf{i}\mathbb{R}^+$. Then we have $1 \leqslant r_{B'} \leqslant e^{\ell_\alpha}$. □

Recall that in Riera's formula (see (4.2)) the function $(u \ln \frac{u+1}{u-1} - 2)$ satisfies

$$\lim_{u \to \infty} \frac{u \ln \frac{u+1}{u-1} - 2}{u^{-2}} = \frac{2}{3}.$$

From Lemma 4.8 we know that the quantity u in (4.2) satisfies

$$u \geqslant \cosh\left(\frac{\epsilon_0}{4}\right) > 1$$

provided that $X \in \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$ and $\alpha \in \mathrm{sys}(X)$. Thus, there exists a positive constant $C_2(\epsilon_0)$, depending only on ϵ_0, such that

$$\left(u \ln \frac{u+1}{u-1} - 2\right) \leqslant C_2(\epsilon_0) \cdot u^{-2}. \tag{4.5}$$

Now we are ready to prove Proposition 4.4.

Proof of Proposition 4.4. — We will apply (4.2) to finish the proof.

First from (4.2) and (4.5) we have

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) \leqslant \frac{2}{\pi}\left(\ell_\alpha + C_2(\epsilon_0) \sum_{B \in \{\langle A\rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}} e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, B \circ (\mathbf{i}\mathbb{R}^+))}\right).$$

Let $p_B \in B \circ (\mathbf{i}\mathbb{R}^+)$ such that

$$\mathrm{dist}_{\mathbb{H}}(p_B, \mathbf{i}\mathbb{R}^+) = \mathrm{dist}_{\mathbb{H}}(B \circ (\mathbf{i}\mathbb{R}^+), \mathbf{i}\mathbb{R}^+).$$

Then,

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp} \leqslant \frac{2}{\pi}\left(\ell_\alpha + C_2(\epsilon_0) \sum_{B \in \{\langle A\rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}} e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, p_B)}\right). \tag{4.6}$$

Lemma 2.1 implies that the function $e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, z)}$ has the mean value property. Set

$$r(\epsilon_0) = \frac{\epsilon_0}{8}.$$

Thus, from Lemma 2.1 we know that

$$e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, p_B)} \leqslant c(r(\epsilon_0)) \int_{B_{\mathbb{H}}(p_B; r(\epsilon_0))} e^{-2\,\mathrm{dist}_{\mathbb{H}}(z, \mathbf{i}\mathbb{R}^+)} \mathrm{d}A(z)$$

where $c(\,\cdot\,)$ is the constant in Lemma 2.1.

From our assumption that $X \in \mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$, Lemma 4.6 and the triangle inequality we know that the geodesic balls

$$\{B_{\mathbb{H}}(p_B; r(\epsilon_0))\}_{B \in \{\langle A\rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}}$$

are pairwise disjoint. Thus,

$$\begin{aligned}
&\sum_{B \in \{\langle A\rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}} e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, p_B)} \\
&\qquad \leqslant c(r(\epsilon_0)) \sum_{B \in \{\langle A\rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}} \int_{B_{\mathbb{H}}(p_B; r(\epsilon_0))} e^{-2\,\mathrm{dist}_{\mathbb{H}}(z, \mathbf{i}\mathbb{R}^+)} \mathrm{d}A(z) \\
&\qquad = c(r(\epsilon_0)) \int_{\cup_{B \in \{\langle A\rangle \backslash \Gamma / \langle A \rangle - \mathrm{id}\}} B_{\mathbb{H}}(p_B; r(\epsilon_0))} e^{-2\,\mathrm{dist}_{\mathbb{H}}(z, \mathbf{i}\mathbb{R}^+)} \mathrm{d}A(z).
\end{aligned}$$

Since $\ell_\alpha(X) \geqslant \epsilon_0$ and $r(\epsilon_0) \leqslant \frac{\epsilon_0}{4}$, from Lemma 4.8 we have that the union of the geodesic balls satisfy that

$$\cup_{B\in\{\langle A\rangle\backslash\Gamma/\langle A\rangle-\mathrm{id}\}} B_{\mathbb{H}}(p_B; r(\epsilon_0)) \subset \{(r,\theta)\in\mathbb{H}; e^{-\ell_\alpha} \leqslant r \leqslant e^{2\ell_\alpha}\}.$$

Thus,

$$\sum_{B\in\{\langle A\rangle\backslash\Gamma/\langle A\rangle-\mathrm{id}\}} e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, p_B)} \leqslant c(r(\epsilon_0)) \times \int_{\{(r,\theta)\in\mathbb{H}; e^{-\ell_\alpha}\leqslant r\leqslant e^{2\ell_\alpha}\}} e^{-2\,\mathrm{dist}_{\mathbb{H}}(z,\mathbf{i}\mathbb{R}^+)} \mathrm{d}A(z).$$

From inequality (2.2) we have

$$\begin{aligned}\sum_{B\in\{\langle A\rangle\backslash\Gamma/\langle A\rangle-\mathrm{id}\}} & e^{-2\,\mathrm{dist}_{\mathbb{H}}(\mathbf{i}\mathbb{R}^+, p_B)} \\ &\leqslant c(r(\epsilon_0)) \int_0^\pi \int_{e^{-\ell_\alpha}}^{e^{2\ell_\alpha}} \sin^2\theta \mathrm{d}A(z) \\ &= c(r(\epsilon_0)) \int_0^\pi \int_{e^{-\ell_\alpha}}^{e^{2\ell_\alpha}} \frac{\sin^2\theta}{r^2\sin^2\theta} r\mathrm{d}r\mathrm{d}\theta \\ &= c(r(\epsilon_0))\cdot 3\pi\cdot\ell_\alpha\end{aligned} \tag{4.7}$$

where in the first equality we apply $\mathrm{d}A(z) = \frac{|\mathrm{d}z|^2}{y^2} = \frac{r\mathrm{d}r\mathrm{d}\theta}{r^2\sin^2\theta}$.

Therefore, the conclusion follows from inequalities (4.6) and (4.7) by choosing

$$D(\epsilon_0) = \frac{2}{\pi}(1 + C_2(\epsilon_0)\cdot c(r(\epsilon_0))\cdot 3\pi). \qquad \square$$

Proof of Proposition 4.3. — Let $s = \mathrm{dist}_{wp}(X,Y) > 0$ and

$$\gamma : [0,s] \to \mathrm{Teich}(S_{g,n})^{\geqslant\epsilon_0}$$

be the geodesic $\mathfrak{g}(X,Y)$ with $\gamma(0) = X$ and $\gamma(s) = Y$. From Lemma 3.4 we know that there exist a positive integer k, a partition $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = s$ of the interval $[0,s]$ and a sequence of essential simple closed curves $\{\alpha_i\}_{0\leqslant i\leqslant k-1}$ in $S_{g,n}$ such that for all $0 \leqslant i \leqslant k-1$ we have

$$\ell_{\alpha_i}(\gamma(t)) = \ell_{\mathrm{sys}}(\gamma(t)), \quad \forall\, t_i \leqslant t \leqslant t_{i+1}.$$

Then,

$$\begin{aligned}\left|\sqrt{\ell_{\mathrm{sys}}(X)}-\sqrt{\ell_{\mathrm{sys}}(Y)}\right| &\leqslant \sum_{i=0}^{k-1}\left|\sqrt{\ell_{\mathrm{sys}}(\gamma(t_i))}-\sqrt{\ell_{\mathrm{sys}}(\gamma(t_{i+1}))}\right| \\ &= \sum_{i=0}^{k-1}\left|\sqrt{\ell_{\alpha_i}(\gamma(t_i))}-\sqrt{\ell_{\alpha_i}(\gamma(t_{i+1}))}\right| \\ &= \sum_{i=0}^{k-1}\left|\int_{t_i}^{i+1}\langle\nabla\ell_{\alpha_i}^{\frac{1}{2}}(\gamma(t)),\gamma'(t)\rangle_{wp}\mathrm{d}t\right| \\ &\leqslant \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\|\nabla\ell_{\alpha_i}^{\frac{1}{2}}(\gamma(t))\|_{wp}\mathrm{d}t\end{aligned}$$

where $||\cdot||_{wp}$ is the Weil–Petersson norm.

Since $\gamma([0,s])\subset\mathrm{Teich}(S_{g,n})^{\geqslant\epsilon_0}$, from Proposition 4.4 we have for all $0\leqslant i\leqslant(k-1)$ and $t_i\leqslant t\leqslant t_{i+1}$,

$$||\nabla\ell_{\alpha_i}^{\frac{1}{2}}(\gamma(t))||_{wp}\leqslant\sqrt{\frac{D(\epsilon_0)}{4}}.$$

Recall that $\mathrm{dist}_{wp}(X,Y)=s=t_k$ and $t_0=0$. Therefore, the two inequalities above yield that

$$\left|\sqrt{\ell_{\mathrm{sys}}(X)}-\sqrt{\ell_{\mathrm{sys}}(Y)}\right|\leqslant\frac{\sqrt{D(\epsilon_0)}}{2}\,\mathrm{dist}_{wp}(X,Y).$$

Then the conclusion follows by choosing $C(\epsilon_0)=\frac{\sqrt{D(\epsilon_0)}}{2}$. □

Remark 4.9. — It is not hard to see that the constant $C(\epsilon_0)\to\infty$ as $\epsilon_0\to 0$.

Before we prove Theorem 1.3, let us introduce the following result which is a direct consequence of Lemma 4.2.

Lemma 4.10. — *There exists a universal constant $c>0$, independent of g and n, such that for any $X\in\mathrm{Teich}(S_{g,n})$, and $\alpha\subset S_{g,n}$ which is an essential simple closed curve with $\ell_\alpha(X)\leqslant 1$, then the following holds*

$$\langle\nabla\ell_\alpha^{\frac{1}{2}},\nabla\ell_\alpha^{\frac{1}{2}}\rangle_{wp}(X)\leqslant c.$$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. — Let $X\neq Y\in\mathrm{Teich}(S_{g,n})$, $s=\mathrm{dist}_{wp}(X,Y)>0$ and $\gamma:[0,s]\to\mathrm{Teich}(S_{g,n})$ be the Weil–Petersson geodesic with $\gamma(0)=X$ and $\gamma(s)=Y$. We apply Lemma 3.5 to the geodesic $\gamma([0,s])$ with $\epsilon_0=1$. So there exist a positive integer k, a partition $0=t_0<t_1<$

$\cdots < t_{k-1} < t_k = s$ of the interval $[0,s]$, a sequence of closed intervals $\{[a_i,b_i] \subseteq [t_i,t_{i+1}]\}_{0\leqslant i\leqslant k-1}$ and a sequence of essential simple closed curves $\{\alpha_i\}_{0\leqslant i\leqslant k-1}$ in $S_{g,n}$ such that for all $0 \leqslant i \leqslant k-1$,

$$\ell_{\alpha_i}(\gamma(t)) = \ell_{\mathrm{sys}}(\gamma(t)), \quad \forall\, t_i \leqslant t \leqslant t_{i+1}. \tag{4.8}$$

$$\gamma([0,s]) \cap \ell_{\mathrm{sys}}^{-1}([0,1]) = \cup_{0\leqslant i\leqslant k-1}\gamma([a_i,b_i]). \tag{4.9}$$

Since $[a_i,b_i] \subseteq [t_i,t_{i+1}]$ for all $0 \leqslant i \leqslant k-1$, from (4.9) we know that for all $0 \leqslant i \leqslant k-1$,

$$\ell_{\mathrm{sys}}(\gamma(t)) \geqslant 1, \quad \forall\, t \in [t_i,a_i] \cup [b_i,t_{i+1}]. \tag{4.10}$$

Then,

$$\begin{aligned}
&\left|\sqrt{\ell_{\mathrm{sys}}(X)} - \sqrt{\ell_{\mathrm{sys}}(Y)}\right| \\
&= \left|\sqrt{\ell_{\mathrm{sys}}(\gamma(0))} - \sqrt{\ell_{\mathrm{sys}}(\gamma(s))}\right| \\
&\leqslant \sum_{i=0}^{k-1}\Bigg(\left|\sqrt{\ell_{\mathrm{sys}}(\gamma(t_i))} - \sqrt{\ell_{\mathrm{sys}}(\gamma(a_i))}\right| + \left|\sqrt{\ell_{\mathrm{sys}}(\gamma(a_i))} - \sqrt{\ell_{\mathrm{sys}}(\gamma(b_i))}\right| \\
&\qquad\qquad + \left|\sqrt{\ell_{\mathrm{sys}}(\gamma(b_i))} - \sqrt{\ell_{\mathrm{sys}}(\gamma(t_{i+1}))}\right|\Bigg).
\end{aligned}$$

From (4.10) and Proposition 4.3 we have

$$\begin{aligned}
&\left|\sqrt{\ell_{\mathrm{sys}}(X)} - \sqrt{\ell_{\mathrm{sys}}(Y)}\right| \\
&\leqslant \sum_{i=0}^{k-1}(C(1)\cdot|a_i - t_i| + C(1)\cdot|t_{i+1} - b_i|) \\
&\qquad\qquad + \sum_{i=0}^{k-1}\left|\sqrt{\ell_{\mathrm{sys}}(\gamma(a_i))} - \sqrt{\ell_{\mathrm{sys}}(\gamma(b_i))}\right| \\
&= \sum_{i=0}^{k-1} C(1)\cdot(t_{i+1} - t_i + a_i - b_i) + \sum_{i=0}^{k-1}\left|\sqrt{\ell_{\mathrm{sys}}(\gamma(a_i))} - \sqrt{\ell_{\mathrm{sys}}(\gamma(b_i))}\right| \\
&= \sum_{i=0}^{k-1} C(1)\cdot(t_{i+1} - t_i + a_i - b_i) + \sum_{i=0}^{k-1}\left|\sqrt{\ell_{\alpha_i}(\gamma(a_i))} - \sqrt{\ell_{\alpha_i}(\gamma(b_i))}\right|
\end{aligned} \tag{4.11}$$

where we apply (4.8) in the last step.

Using (4.8) and (4.9), we apply Lemma 4.10 to the geodesic segment $\gamma([a_i, b_i])$. Then for all $0 \leqslant i \leqslant k-1$,

$$\begin{aligned}(4.12)\quad \left|\sqrt{\ell_{\alpha_i}(\gamma(a_i))} - \sqrt{\ell_{\alpha_i}(\gamma(b_i))}\right| \\ &= \left|\int_{a_i}^{b_i} \langle \nabla \ell_{\alpha_i}^{\frac{1}{2}}(\gamma(t)), \gamma'(t)\rangle_{wp} \mathrm{d}t\right| \\ &\leqslant \int_{a_i}^{b_i} \|\nabla \ell_{\alpha_i}^{\frac{1}{2}}(\gamma(t))\|_{wp} \mathrm{d}t \leqslant \sqrt{c}\cdot (b_i - a_i)\end{aligned}$$

where $||\cdot||_{wp}$ is the Weil–Petersson norm.

Combine inequalities (4.11) and (4.12) we get

$$\begin{aligned}(4.13)\quad \left|\sqrt{\ell_{\text{sys}}(X)} - \sqrt{\ell_{\text{sys}}(Y)}\right| \\ &\leqslant \sum_{i=0}^{k-1} C(1)\cdot(t_{i+1} - t_i + a_i - b_i) + \sum_{i=0}^{k-1} \sqrt{c}\cdot(b_i - a_i) \\ &\leqslant \max\{C(1), \sqrt{c}\}\cdot(t_k - t_0) \\ &= \max\{C(1), \sqrt{c}\}\cdot \mathrm{dist}_{wp}(X, Y).\end{aligned}$$

Then the conclusion follows by choosing $K = \max\{C(1), \sqrt{c}\}$. □

Remark 4.11. — For the case $(g, n) = (1, 1)$ or $(0, 4)$, we let $\alpha, \beta \subset S_{g,n}$ be any two essential simple closed curves which fill the surface $S_{g,n}$. The strata $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are two single points. By [14, 55, 62] the Weil–Petersson geodesic I joining $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ is contained in $\mathrm{Teich}(S_{g,n})$ except the two end points. The Collar Lemma [28] implies that there exists at least one point $Z \in I$ such that $\ell_{\text{sys}}(Z) \geqslant 2 \operatorname{arcsinh} 1$. Then Theorem 1.3 gives that $\ell(I) = \mathrm{dist}_{WP}(Z, \mathcal{T}_\alpha) + \mathrm{dist}_{WP}(Z, \mathcal{T}_\beta) \geqslant \frac{2\sqrt{2 \operatorname{arcsinh} 1}}{K} > 0$. One can see [55, Corollary 22] for a more general statement, and see [6, Theorem 1.7] for a more explicit lower bound. Since the completion $\overline{\mathcal{M}_{g,n}}$ contains $\mathcal{M}_{0,2g+n}$ as a totally geodesic subspace, up to a uniform multiplicative constant the quantity $\sqrt{g}$ serves as a lower bound for the diameter $\mathrm{diam}(\mathcal{M}_{g,n})$ for large genus, as observed in [12, Proposition 5.1].

5. Proofs of Theorems 1.1, 1.2 and 1.4

In this section we will first prove Theorem 1.4 and then apply Theorem 1.3 to finish the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.4. — For the lower bound, by the Mumford compactness theorem we may assume that $X \in \partial\mathcal{M}_{g,n}^{\geqslant s}$ and $Y \in \partial\mathcal{M}_{g,n}^{\geqslant t}$ such that

$$\mathrm{dist}_{wp}(\partial\mathcal{M}_{g,n}^{\geqslant s}, \partial\mathcal{M}_{g,n}^{\geqslant t}) = \mathrm{dist}_{wp}(X, Y).$$

From Theorem 1.3 we know that $\mathrm{dist}_{wp}(X, Y) \geqslant K(\sqrt{s} - \sqrt{t})$. Thus,

$$\mathrm{dist}_{wp}(\partial\mathcal{M}_{g,n}^{\geqslant s}, \partial\mathcal{M}_{g,n}^{\geqslant t}) \geqslant K(\sqrt{s} - \sqrt{t}).$$

For the upper bound, for any $X \in \partial\mathcal{M}_{g,n}^{\geqslant s}$ we let $\alpha \subset X$ such that

$$\ell_{\mathrm{sys}}(X) = \ell_\alpha(X) = s.$$

Recall that (4.2) (Riera's formula) tells that

$$\langle \nabla\sqrt{2\pi\ell_\alpha}, \nabla\sqrt{2\pi\ell_\alpha}\rangle_{wp} > 1. \tag{5.1}$$

It follows from standard ODE theory that there exists a smooth curve γ of arc-length parameter r in $\mathcal{M}_{g,n}$ such that

$$\gamma(0) = X \text{ and } \gamma'(r) = -\frac{\nabla\sqrt{2\pi\ell_\alpha(\gamma(r))}}{||\nabla\sqrt{2\pi\ell_\alpha(\gamma(r))}||_{wp}}.$$

The length function ℓ_α is decreasing along γ because for $r_1 > r_2 > 0$,

$$\begin{aligned}\sqrt{2\pi\ell_\alpha(\gamma(r_1))} - \sqrt{2\pi\ell_\alpha(\gamma(r_2))} &= \int_{r_2}^{r_1} \langle \nabla\sqrt{2\pi\ell_\alpha(\gamma(r))}, \gamma'(t)\rangle_{wp} \mathrm{d}r \\ &= -\int_{r_2}^{r_1} \|\nabla\sqrt{2\pi\ell_\alpha(\gamma(r))}\|_{wp} \mathrm{d}r \\ &< 0.\end{aligned}$$

By the inequality above we know that the curve γ will go to the stratum whose pinching curve is α. Since $s > t \geqslant 0$ and $\ell_\alpha(\gamma(0)) = s$, we may assume that $r_0 > 0$ is a constant such that

$$\ell_\alpha(\gamma(r_0)) = t.$$

Then we have

$$\begin{aligned}\sqrt{2\pi s} - \sqrt{2\pi t} &= \sqrt{2\pi\ell_\alpha(\gamma(0))} - \sqrt{2\pi\ell_\alpha(\gamma(r_0))} \\ &= \int_{r_0}^{0} \langle \nabla\sqrt{2\pi\ell_\alpha(\gamma(r))}, \gamma'(t)\rangle_{wp} \mathrm{d}r \\ &= \int_0^{r_0} \|\nabla\sqrt{2\pi\ell_\alpha(\gamma(r))}\|_{wp} \mathrm{d}r \\ &\geqslant r_0 \qquad\qquad \text{(by (5.1))} \\ &\geqslant \mathrm{dist}_{wp}(X, \gamma(r_0))\end{aligned}$$

where the last inequality uses the fact that γ uses the arc-length parameter.

Since $\ell_\alpha(\gamma(r_0)) = t < s = \ell_\alpha(X)$, the Weil–Petersson geodesic joining X and $\gamma(r_0)$ will cross the leaf $\partial\mathcal{M}_{g,n}^{\geqslant t}$. Thus,

$$\mathrm{dist}_{wp}(X, \gamma(r_0)) \geqslant \mathrm{dist}_{wp}(X, \partial\mathcal{M}_{g,n}^{\geqslant t}).$$

Since $\ell_\alpha(X) = s$, the two inequalities above imply that

$$\begin{aligned}\mathrm{dist}_{wp}(\partial\mathcal{M}_{g,n}^{\geqslant s}, \partial\mathcal{M}_{g,n}^{\geqslant t}) &\leqslant \mathrm{dist}_{wp}(X, \partial\mathcal{M}_{g,n}^{\geqslant t}) \\ &\leqslant \sqrt{2\pi}(\sqrt{s} - \sqrt{t}).\end{aligned}$$

Then the conclusion follows by choosing

$$K' = \max\left\{\sqrt{2\pi}, \frac{1}{K}\right\}. \qquad \square$$

Remark 5.1. — The argument in the proof of Theorem 1.4 also gives that $\max_{X \in \partial\mathcal{M}_{g,n}^{\geqslant s}} \mathrm{dist}_{wp}(X, \partial\mathcal{M}_{g,n}^{\geqslant t})$ *is uniformly comparable to* $(\sqrt{s} - \sqrt{t})$.

Although Teichmüller space is non-compact, the systole function $\ell_{\mathrm{sys}}(\,\cdot\,) : \mathrm{Teich}(S_{g,n}) \to \mathbb{R}^+$ is bounded above by a constant depending on g and n. Follow [3] we define

$$\mathrm{sys}(g,n) := \sup_{X \in \mathrm{Teich}(S_{g,n})} \ell_{\mathrm{sys}}(X).$$

By Mumford's compactness theorem [38] this supremum is in fact a maximum. We list some bounds for $\mathrm{sys}(g,n)$ which will be useful in the proofs of Theorems 1.1 and 1.2. One can see [3] for more details on $\mathrm{sys}(g,n)$.

We always assume that $3g + n - 3 > 0$. Since the set of shortest closed geodesics of a maximal surface fills the surface, the Collar Lemma [28] gives that

$$\mathrm{sys}(g,n) \geqslant 2\,\mathrm{arcsinh}\,1. \tag{5.2}$$

Buser and Sarnak proved in [11] that there exists a universal constant $U > 0$ such that $\mathrm{sys}(g,0) \geqslant U \ln g$. And actually they also proved that there exists a subsequence $\{g_k\}_{k\geqslant 1}$ of $\{g\}_{g\geqslant 1}$ such that $\mathrm{sys}(g_k, 0) \geqslant \frac{4}{3}\ln g_k$. If we allow the surface to have punctures, based on Buser–Sarnak's work, Balacheff, Makover and Parlier [3, Proposition 2] proved the following lower bound which will be useful to prove Theorem 1.1.

$$\mathrm{sys}(g,n) \geqslant \min\left\{U \ln g, 2\,\mathrm{arccosh}\left(\frac{2(g-1)}{n} + 1\right)\right\}. \tag{5.3}$$

An interesting upper bound for $\mathrm{sys}(g,n)$ was provided by Schmutz in [43], which says that if $n \geqslant 2$, $\mathrm{sys}(g,n) \leqslant 4\,\mathrm{arccosh}\,(\frac{6g-6+3n}{n})$. If $g \geqslant 1$, Part 1 of [3, Theorem] tells that $\mathrm{sys}(g,0) < \mathrm{sys}(g,1) < \mathrm{sys}(g,2)$. Thus, these two results give that for all g, n with $3g+n-3>0$,

$$\mathrm{sys}(g,n) \leqslant \min\left\{4\,\mathrm{arccosh}\,(3(g+1)), 4\,\mathrm{arccosh}\left(\frac{6g-6+3n}{n}\right)\right\}. \tag{5.4}$$

Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. — For any $X \in \mathcal{M}_{g,n}$, we let $\alpha \subset X$ be a systolic curve, i.e., $\ell_\alpha(X) = \ell_{\mathrm{sys}}(X)$. Inequality (5.4) tells that if $g \geqslant 2$,

$$\ell_{\mathrm{sys}}(X) \leqslant 4\,\mathrm{arccosh}\,(4g). \tag{5.5}$$

Let $\mathcal{T}_\alpha \subset \overline{\mathcal{M}}_{g,n}$ be the stratum whose vanishing curve is α. Then we have for all $g \geqslant 2$,

$$\begin{aligned}
\mathrm{dist}_{wp}(X, \partial\overline{\mathcal{M}}_{g,n}) &\leqslant \mathrm{dist}_{wp}(X, \mathcal{T}_{\alpha_{g,n}}) \\
&\leqslant \sqrt{2\pi\ell_\alpha(X)}, \qquad \text{(by Theorem (2.5))} \\
&= \sqrt{2\pi\ell_{\mathrm{sys}}(X)} \\
&\leqslant \sqrt{2\pi \cdot 4\,\mathrm{arccosh}\,(4g)} \qquad \text{(by inequality (5.5))} \\
&< \sqrt{32\pi} \cdot \sqrt{\ln g}.
\end{aligned}$$

Since $X \in \mathcal{M}_{g,n}$ is arbitrary, we have

$$\mathrm{InRad}(\mathcal{M}_{g,n}) \leqslant \sqrt{32\pi} \cdot \sqrt{\ln g}.$$

For the lower bound, from inequality (5.3) one may choose a surface $Y \in \mathcal{M}_{g,n}$ such that

$$\ell_{\mathrm{sys}}(Y) \geqslant \min\left\{U \ln g, 2\,\mathrm{arccosh}\left(\frac{2(g-1)}{n}+1\right)\right\}. \tag{5.6}$$

Thus, there exists a constant $k(n)$, only depending on n, such that

$$\ell_{\mathrm{sys}}(Y) \geqslant k(n) \cdot \ln g.$$

We let $Z \in \partial\mathcal{M}_{g,n}$ such that

$$\mathrm{dist}_{wp}(Y, Z) = \mathrm{dist}_{wp}(Y, \partial\mathcal{M}_{g,n}).$$

Then we have

$$\begin{aligned}\mathrm{InRad}(\mathcal{M}_{g,n}) &\geqslant \mathrm{dist}_{wp}(Y, \partial\mathcal{M}_{g,n})\\ &= \mathrm{dist}_{wp}(Y, Z)\\ &\geqslant \frac{1}{K}\left|\sqrt{\ell_{\mathrm{sys}}(Y)} - \sqrt{\ell_{\mathrm{sys}}(Z)}\right| \qquad \text{(by Theorem 1.3)}\\ &= \frac{1}{K}\sqrt{\ell_{\mathrm{sys}}(Y)} \qquad (\text{because } \ell_{\mathrm{sys}}(Z) = 0)\\ &\geqslant \frac{\sqrt{k(n)}}{K}\sqrt{\ln g}\end{aligned}$$

where K is the universal constant from Theorem 1.3. □

Remark 5.2. — The proof of Theorem 1.1 also leads to the following result.

THEOREM 5.3. — *For all g, n with $g \geqslant 2$, then*

$$\mathrm{InRad}(\mathrm{Teich}(S_{g,n})) \asymp_g \sqrt{\ln g}.$$

Remark 5.4. — In the proof of the lower bound of Theorem 1.1, the quantity $2\,\mathrm{arccosh}\left(\frac{2(g-1)}{n} + 1\right)$ is applied. Observe that for any constant $a \in (0, 1)$, the quantity $2\,\mathrm{arccosh}\left(\frac{2(g-1)}{g^a} + 1\right)$ is comparable to $\ln g$ as g goes to infinity. So we also get that

$$\mathrm{InRad}(\mathcal{M}_{g,[g^a]}) \asymp_g \sqrt{\ln g}.$$

The proof of Theorem 1.2 is similar to the one of Theorem 1.1.

Proof of Theorem 1.2. — For any $X \in \mathcal{M}_{g,n}$, we let $\alpha \subset X$ be a systolic curve, i.e., $\ell_\alpha(X) = \ell_{\mathrm{sys}}(X)$. From inequality (5.4) we know that there exists a constant $d(g) > 0$, only depending on g, such that for all $n \geqslant 4$,

$$\mathrm{sys}(g, n) \leqslant d(g). \tag{5.7}$$

Let $\mathcal{T}_\alpha \subset \overline{\mathcal{M}}_{g,n}$ be the stratum whose vanishing curve is α. Then we have for all $n \geqslant 4$,

$$\begin{aligned}\mathrm{dist}_{wp}(X, \partial\overline{\mathcal{M}}_{g,n}) &\leqslant \mathrm{dist}_{wp}(X, \mathcal{T}_{\alpha_{g,n}})\\ &\leqslant \sqrt{2\pi\ell_\alpha(X)}, \qquad \text{(by Theorem (2.5))}\\ &= \sqrt{2\pi\ell_{\mathrm{sys}}(X)}\\ &\leqslant \sqrt{2\pi \cdot d(g)} \qquad \text{(by inequality (5.7))}.\end{aligned}$$

Since $X \in \mathcal{M}_{g,n}$ is arbitrary, we have

$$\mathrm{InRad}(\mathcal{M}_{g,n}) \leqslant \sqrt{2\pi \cdot d(g)}.$$

For the lower bound, we will give two different proofs: the first one will apply Theorem 1.3, and the other one will apply Lemma 4.10 instead of Theorem 1.3.

Method (1): we apply Theorem 1.3. — First from inequality (5.2) one may choose a surface $Y \in \mathcal{M}_{g,n}$ such that

$$\ell_{\mathrm{sys}}(Y) \geqslant 2 \operatorname{arcsinh} 1. \tag{5.8}$$

We let $Z \in \partial\mathcal{M}_{g,n}$ such that

$$\mathrm{dist}_{wp}(Y, Z) = \mathrm{dist}_{wp}(Y, \partial\mathcal{M}_{g,n}).$$

Then we have

$$\begin{aligned}
\mathrm{InRad}(\mathcal{M}_{g,n}) &\geqslant \mathrm{dist}_{wp}(Y, \partial\mathcal{M}_{g,n}) \\
&= \mathrm{dist}_{wp}(Y, Z) \\
&\geqslant \frac{1}{K}\left|\sqrt{\ell_{\mathrm{sys}}(Y)} - \sqrt{\ell_{\mathrm{sys}}(Z)}\right| \qquad \text{(by Theorem 1.3)} \\
&= \frac{1}{K}\sqrt{\ell_{\mathrm{sys}}(Y)} \qquad \text{(because } \ell_{\mathrm{sys}}(Z) = 0\text{)} \\
&\geqslant \frac{\sqrt{2 \operatorname{arcsinh} 1}}{K}
\end{aligned}$$

where K is the universal constant from Theorem 1.3.

Method (2): we apply Lemma 4.10 without using Theorem 1.3. — Similarly from inequality (5.2) one may choose a surface $Y \in \mathcal{M}_{g,n}$ such that

$$\ell_{\mathrm{sys}}(Y) \geqslant 2 \operatorname{arcsinh} 1. \tag{5.9}$$

We let $Z \in \partial\mathcal{M}_{g,n}$ such that

$$\mathrm{dist}_{wp}(Y, Z) = \mathrm{dist}_{wp}(Y, \partial\mathcal{M}_{g,n}).$$

Let $\alpha \subset S_{g,n}$ be a pinched curve on Z, i.e., $\ell_\alpha(Z) = 0$. Consider the shortest Weil–Petersson geodesic $\gamma : [0, s] \to \overline{\mathcal{M}}_{g,n}$ such that $\gamma(0) = Y$ and $\gamma(s) = Z$ where $s = \mathrm{dist}_{wp}(Y, Z)$. Since $\ell_\alpha(Z) = 0$, the constant

$$s_0 := \inf\{t_0 \in [0, s];\ \ell_\alpha(\gamma(t)) \leqslant 1,\ \forall\, t_0 \leqslant t \leqslant s\}$$

is well-defined. Since $2 \operatorname{arcsinh} 1 \geqslant 1$, from inequality (5.9) and the definition of s_0 we have

$$\ell_\alpha(\gamma(s_0)) = 1.$$

We apply Lemma 4.10 to the geodesic $\gamma([t_0, s))$. Then,

$$\begin{aligned} 1 &= \left| \sqrt{\ell_\alpha(\gamma(s_0))} - \sqrt{\ell_\alpha(\gamma(s))} \right| \\ &= \left| \int_{s_0}^{s} \langle \nabla \ell_\alpha^{\frac{1}{2}}(\gamma(t)), \gamma'(t) \rangle_{wp} \mathrm{d}t \right| \\ &\leqslant \int_{s_0}^{s} ||\nabla \ell_\alpha^{\frac{1}{2}}(\gamma(t))||_{wp} \mathrm{d}t. \end{aligned}$$

Since $\ell_\alpha(\gamma(t)) \leqslant 1$ for all $s_0 \leqslant t \leqslant s$, from Lemma 4.10 we have

$$1 \leqslant \sqrt{c} \cdot (s - s_0) \leqslant \sqrt{c} \cdot \mathrm{dist}_{wp}(Y, Z)$$

where c is the constant in Lemma 4.10.

Thus,

$$\begin{aligned} \mathrm{InRad}(\mathcal{M}_{g,n}) &\geqslant \mathrm{dist}_{wp}(Y, \partial \mathcal{M}_{g,n}) \\ &= \mathrm{dist}_{wp}(Y, Z) \\ &\geqslant \frac{1}{\sqrt{c}}. \end{aligned}$$

The positive lower bounds from the two methods above are different. But both of them are independent of g and n.

The proof is complete. □

Remark 5.5. — The proof of Theorem 1.2 also leads to

$$\mathrm{InRad}(\mathrm{Teich}(S_{g,n})) \asymp_n 1.$$

Remark 5.6. — In the proof above, the quantity $4 \operatorname{arccosh}\left(\frac{6g-6+3n}{n}\right)$ is applied to establish the upper bound. Observe that for any constant $a \in (0,1)$, $4 \operatorname{arccosh}\left(\frac{6n^a-6+3n}{n}\right)$ is comparable to 1 as n goes to infinity. Actually the proof of Theorem 1.2 also yields that

$$\mathrm{InRad}(\mathcal{M}_{[n^a],n}) \asymp_n 1.$$

6. Weil–Petersson volume for large genus

For simplicity, we will focus on Teichmüller space of closed surfaces endowed with the Weil–Petersson metric, which is denoted by $\mathrm{Teich}(S_g)$. The results in this section are still true for surfaces with punctures. The space $\mathrm{Teich}(S_g)$ is incomplete [13, 50], negatively curved [47, 53] and uniquely geodesically convex [54]. We will study the asymptotic behavior of the Weil–Petersson volumes of geodesic balls of finite radii in $\mathrm{Teich}(S_g)$ as the genus g goes to infinity. The main goal in this section is to prove Theorem 1.6.

The proof of Theorem 1.6 involves using Theorem 1.1 together with the following theorem due to Teo [46] on the Ricci curvature on the thick-part of the Teichmüller space. Let $\epsilon_0 > 0$. Recall that $\mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$ is the ϵ_0-thick part $\mathcal{T}(S_{g,n})^{\geqslant \epsilon_0}$ endowed with the Weil–Petersson metric.

Theorem 6.1 ([46, Proposition 3.3]). — *The Ricci curvature of* $\mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$ *is bounded from below by* $-C'(\epsilon_0)$ *where* $C'(\epsilon_0) > 0$ *is a constant which only depends on* ϵ_0.

The constant $C'(\epsilon_0)$ above roughly behaves like $\frac{2}{\pi\epsilon_0^2}$ as ϵ_0 goes to 0.

Huang [24] showed that the Weil–Petersson sectional curvature is not bounded below by any negative constant. For suitable choice of $\epsilon_0 > 0$, in [49] it was shown that the minimal Weil–Petersson sectional curvature over $\mathrm{Teich}(S_{g,n})^{\geqslant \epsilon_0}$ is comparable to -1 even as g goes to infinity. For the most recent developments on the Weil–Petersson curvature on the thick part of Teichmüller space, one may refer to [25, 49, 61].

Since the completion $\overline{\mathrm{Teich}(S_g)}$ of $\mathrm{Teich}(S_g)$ is not locally compact [55], the Weil–Petersson volume of a geodesic ball of finite radius in $\mathrm{Teich}(S_g)$ may blow up. The following result is well-known to experts. We provide it here for completeness.

Proposition 6.2. — *Let* $X_g \in \mathrm{Teich}(S_g)$. *Then, for any positive constant* r *with* $r > \mathrm{dist}_{wp}(X_g, \partial\overline{\mathrm{Teich}(S_g)})$ *the Weil–Petersson volume satisfies*

$$\mathrm{Vol}_{wp}(B(X_g; r)) = \infty$$

where $B(X_g; r) = \{Y \in \mathrm{Teich}(S_g);\ \mathrm{dist}_{wp}(Y, X_g) < r\}$.

Proof. — Let $s = \mathrm{dist}_{wp}(X_g, \partial\overline{\mathrm{Teich}(S_g)}) < r$ and $\gamma : [0, s] \to \overline{\mathrm{Teich}(S_g)}$ be the Weil–Petersson geodesic such that $\gamma(0) = X_g$ and $\gamma(s) \in \partial\overline{\mathrm{Teich}(S_g)}$. By results in [14, 55, 62] we know that the image satisfies

$$\gamma([0, s)) \subset \mathrm{Teich}(S_{g,n}).$$

Since $\gamma(s) \in \partial\overline{\mathrm{Teich}(S_g)}$, we may assume that $\gamma(s) \in \mathcal{T}_\sigma$ where $\mathcal{T}_\sigma$ is some stratum. Let $\tau_\sigma = \Pi_{\alpha\subset\sigma^0}\tau_\alpha$ be the Dehn-twist on the multi curves in σ^0. Take a number $0 < \epsilon < \frac{r-s}{2}$. Since the mapping class group acts properly discontinuously on $\mathrm{Teich}(S_g)$ [27], there exists a positive constant $\epsilon' < \epsilon$ such that the geodesic balls $\{\tau_\sigma^k \circ B(\gamma(s-\epsilon); \epsilon')\}_{k\geqslant 0}$ are pairwise disjoint. It is clear that $\tau_\sigma^k \circ \gamma(s) = \gamma(s)$ and $\tau_\sigma^k \circ B(\gamma(s-\epsilon); \epsilon') = B(\tau_\sigma^k \circ \gamma(s-\epsilon); \epsilon')$ for all $k \geqslant 0$. Then, for any $k \geqslant 0$ and $Z \in B(\tau_\sigma^k \circ \gamma(s-\epsilon); \epsilon')$, the triangle inequality tells that

$$\begin{aligned}\mathrm{dist}_{wp}(Z,X_g) &\leqslant \mathrm{dist}_{wp}(Z,\tau_\sigma^k\circ\gamma(s-\epsilon))\\ &\qquad + \mathrm{dist}_{wp}(\tau_\sigma^k\circ\gamma(s-\epsilon),\gamma(s)) + \mathrm{dist}_{wp}(\gamma(s),X_g)\\ &< \epsilon'+\epsilon+s\\ &< 2\epsilon+s\\ &< r.\end{aligned}$$

That is, for all $k\geqslant 0$, $\tau_\sigma^k\circ B(\gamma(s-\epsilon);\epsilon')\subset B(X_g;r)$. Since $\{\tau_\sigma^k\circ B(\gamma(s-\epsilon);\epsilon')\}_{k\geqslant 0}$ are pairwise disjoint, we have

$$\begin{aligned}\mathrm{Vol}_{wp}(B(X_g;r)) &\geqslant \mathrm{Vol}_{wp}(\cup_{k\geqslant 0}\tau_\sigma^k\circ B(\gamma(s-\epsilon);\epsilon'))\\ &= \sum_{k\geqslant 0}\mathrm{Vol}_{wp}(\tau_\sigma^k\circ B(\gamma(s-\epsilon);\epsilon'))\\ &= \infty\end{aligned}$$

where in the last step we use that fact τ_σ is an isometry on $\mathrm{Teich}(S_g)$. □

Let $\{X_g\}_{g\geqslant 2}$ be a sequence of points in Teichmüller space and $\{r_g\}_{g\geqslant 2}$ be a sequence of positive numbers. In this section we will study the asymptotic behavior of $\{\mathrm{Vol}_{wp}(B(X_g;r_g))\}_{g\geqslant 2}$ as g tends to infinity. In light of Proposition 6.2, we need to assume that the completions $\{\overline{B(X_g;r_g)}\}_{g\geqslant 2}\subset\overline{\mathrm{Teich}(S_g)}$ always do not intersect the boundary of Teichmüller space. For any $r_0>0$, we define $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ to be the subset in $\mathrm{Teich}(S_g)$ which is at least r_0-away from the boundary. More precisely,

$$\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0} := \{X_g\in\mathrm{Teich}(S_g);\mathrm{dist}_{wp}(X_g;\partial\overline{\mathrm{Teich}(S_g)}\geqslant r_0\}.$$

Theorems 1.1 and 2.5 tell that the largest radius of the geodesic ball in the set $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ is comparable to $\sqrt{\ln g}$ as g goes infinity.

Before we prove Theorem 1.6, we first provide a lemma which says that the set $\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ is contained in some thick part of Teichmüller space. More precisely,

Lemma 6.3. — *For any $r_0>0$, there exists a constant $\epsilon(r_0)$, only depending on r_0, such that*

$$\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}\subset\mathrm{Teich}(S_g)^{\geqslant\epsilon(r_0)}.$$

Proof. — The proof is a direct application of Theorem 2.5. For any $X_g\in\mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ we let $\alpha_g\subset X_g$ be an essential simple closed curve such that $\ell_{\alpha_g}(X_g)=\ell_{\mathrm{sys}}(X_g)$, and $\mathcal{T}_\alpha$ be the stratum in $\overline{\mathrm{Teich}(S_g)}$ whose vanishing curve is α. Then, by Theorem 2.5 we have

$$\begin{aligned} r_0 &\leqslant \operatorname{dist}_{wp}(X_g, \mathcal{T}_{\alpha_g}) \\ &\leqslant \sqrt{2\pi \ell_{\mathrm{sys}}(X_g)}. \end{aligned}$$

Thus,

$$X_g \in \operatorname{Teich}(S_g)^{\geqslant \frac{r_0^2}{4\pi^2}}.$$

Then the conclusion follows by choosing

$$\epsilon(r_0) = \frac{r_0^2}{4\pi^2}. \qquad \square$$

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. — Let $B(X_g; r_g) \subset \mathcal{U}(\operatorname{Teich}(S_g))^{\geqslant r_0}$ be an arbitrary geodesic ball where $X_g \in \operatorname{Teich}(S_g)$ and $r_g > 0$. Lemma 6.3 tells that there exists a constant $\epsilon(r_0)$, only depending on r_0, such that

$$B(X_g; r_g) \subset \operatorname{Teich}(S_g)^{\geqslant \epsilon(r_0)}.$$

By Teo's curvature bound (see Theorem 6.1) there exists a constant $C'(r_0) > 0$, only depending on r_0, such that the Ricci curvature satisfies

$$\begin{aligned} \operatorname{Ric}|_{B(X_g;r_g)} &\geqslant -C'(r_0) \\ &= (6g-7) \cdot \left(\frac{-C'(r_0)}{6g-7} \right). \end{aligned} \tag{6.1}$$

From the Gromov–Bishop Volume Comparison Theorem [21] we have

$$\operatorname{Vol}_{wp}(B(X_g; r_g)) \leqslant \operatorname{Vol}_{Euc}(\mathbb{S}^{6g-7}) \int_0^{r_g} \left(\frac{\sinh\left(\sqrt{\frac{C'(r_0)}{6g-7}} t \right)}{\sqrt{\frac{C'(r_0)}{6g-7}}} \right)^{6g-7} \mathrm{d}t \tag{6.2}$$

where $\operatorname{Vol}_{Euc}(\mathbb{S}^{6g-7})$ is the standard $(6g-7)$-dimensional volume of the unit sphere. By Stirling's formula we have

$$\begin{aligned} \frac{\operatorname{Vol}_{Euc}(\mathbb{S}^{6g-7})}{\left(\sqrt{\frac{C'(r_0)}{6g-7}} \right)^{6g-7}} &\leqslant \frac{2\pi^{\frac{6g-7}{2}}}{\Gamma(\frac{6g-7}{2})} \left(\frac{6g-7}{C'(r_0)} \right)^{\frac{6g-7}{2}} \\ &\leqslant 2\pi^{\frac{6g-7}{2}} \left(\frac{2}{6g-9} \right)^{3g-\frac{9}{2}} \left(\frac{6g-7}{C'(r_0)} \right)^{\frac{6g-7}{2}} \\ &\leqslant C^g \end{aligned}$$

for some constant $C > 0$. Thus,

$$\mathrm{Vol}_{wp}(B(X_g; r_g)) \leqslant C^g \int_0^{r_g} \left(\sinh\left(\sqrt{\frac{C'(r_0)}{6g-7}}t\right)\right)^{6g-7} \mathrm{d}t. \tag{6.3}$$

Since $B(X_g; r_g) \subset \mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$, Theorem 1.1 (or Remark 5.2) tells that $r_g \leqslant \sqrt{32\pi \ln g}$ for all $g \geqslant 2$. Note that $\lim_{g\to\infty} \frac{\ln g}{g} = 0$, thus one may assume that there exists a constant $D > 0$ such that

$$\sinh\left(\sqrt{\frac{C'(r_0)}{6g-7}}t\right) \leqslant \frac{Dt}{\sqrt{g}}, \quad \forall\, 0 \leqslant t \leqslant r_g.$$

Thus,

$$\mathrm{Vol}_{wp}(B(X_g; r_g)) \leqslant C^g \int_0^{r_g} \left(\frac{Dt}{\sqrt{g}}\right)^{6g-7} \mathrm{d}t. \tag{6.4}$$

Recall that $r_g \leqslant \sqrt{32\pi \ln g}$. A direct computation gives that

$$\mathrm{Vol}_{wp}(B(X_g; r_g)) \leqslant E^g \frac{(\ln g)^g}{g^{3g}} \tag{6.5}$$

for some constant $E > 0$. Observe that for any $\epsilon > 0$,

$$\lim_{g\to\infty} \frac{E^g \frac{(\ln g)^g}{g^{3g}}}{(\frac{1}{g})^{(3-\epsilon)g}} = 0.$$

Then, there exists a constant $F > 0$ such that

$$\mathrm{Vol}_{wp}(B(X_g; r_g)) \leqslant F \cdot \left(\frac{1}{g}\right)^{(3-\frac{\epsilon}{2})g}. \tag{6.6}$$

Since the geodesic ball $B(X_g; r_g) \subset \mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant r_0}$ is arbitrary, the conclusion follows. $\square$

Proof of Corollary 1.8. — Let $r_0 = 1$ in Theorem 1.6. For any fixed constant $R > 0$, by Theorem 1.6 it suffices to show that there exists a constant $\epsilon(R) > 0$ such that

$$B(X_g; R) \subset \mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant 1}, \ \ \forall\, X_g \in \mathcal{U}(\mathrm{Teich}(S_g))^{\geqslant \epsilon(R)}. \tag{6.7}$$

We choose $\epsilon(R) = R + 1$. The triangle inequality tells that for all $Y \in B(X_g; R)$,

$$\begin{aligned} \mathrm{dist}_{wp}(Y, \partial(\mathrm{Teich}(S_{g,n})) &\geqslant \mathrm{dist}_{wp}(X_g, \partial(\mathrm{Teich}(S_{g,n}))) - \mathrm{dist}_{wp}(Y, X_g) \\ &\geqslant \epsilon(R) - R \\ &= 1. \end{aligned}$$

Then (6.7) follows since $Y \in B(X_g; R)$ is arbitrary. $\square$

Remark 6.4. — Theorem 4.2 in [36] tells that the Weil–Petersson volume of moduli space $\mathcal{M}_g$ is concentrated in the thick part as the genus g tends to infinity, which blows up rapidly. Theorem 1.6 says that the Weil–Petersson volume of any Weil–Petersson geodesic ball in the thick part of moduli space will decay to 0 as g tends to infinity. It would be very *interesting* to study the asymptotic shape of $\mathcal{M}_g$ as g tends to infinity.

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