Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths [1,2]

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The proposition that limited natural resources provide a limit to growth and to the sustainable size of population is an old one. The natural resource that was the centre of the discussion in Malthus' day was land; more recently, some concern has been expressed over the limitations imposed by the supplies of oil, or more generally, energy sources, of phosphorus, and of other materials required for production. Those who predicted imminent doom in the nineteenth century were obviously wrong. Were they simply wrong about the immediacy of catastrophe, or did they leave out something fundamental from their calculations?

There are at least three economic forces offsetting the limitations imposed by natural resources: technical change, the substitution of man-made factors of production (capital) for natural resources, and returns to scale. This study is an attempt to determine more precisely under what conditions a sustainable level of *per capita* consumption is feasible, to characterize steady state paths in economies with natural resources, and to describe the optimal growth path of the economy, in particular to derive the optimal rate of extraction and the optimal savings rate in the presence of exhaustible natural resouces.

One of the interesting problems posed by the presence of exhaustible natural resources is that some of the basic concepts of growth theory, such as " steady state " and " natural rate of growth ", need to be re-examined. If, for instance, there are two unproduced factors, labour and natural resources, one of which is growing exponentially, the other of which is not growing at all, what is the " natural rate of growth "? In conventional economic discussions, the long-run growth rate of the economy is determined simply by the natural rate of growth and is independent of the savings rate. We shall show that in economies with natural resources, efficient growth paths which differ with respect to savings rate also differ, even asymptotically, with respect to the rate of growth.

The analysis of optimal growth paths presents certain technical difficulties, because there are two state variables (the stock of capital per man and the stock of natural resources per man) and two control variables (the rate of extraction of natural resources and the savings rate). Fortunately, by the appropriate choice of variables, the qualitative properties of the path can be completely described.

Optimal growth paths for economies with only capital or with just natural resources have been examined elsewhere. Typically, a country begins with little capital and hence, in the former models, optimal growth is characterized by increasing consumption *per capita*. On the other hand, natural resources act much like a capital good; since the stock

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of natural resources is largest initially and as one " consumes " the stock becomes smaller, not surprisingly consumption per man falls monotonically over time along the optimal path (if it exists). When there is both a capital good and a natural resource, it is not obvious what the qualitative properties of the optimal path will look like, e.g. whether consumption will be monotonic. What is of particular interest is that the choice among alternative efficient growth paths involves a choice about paths which differ in their rates of growth, even asymptotically. Paths which involve high rates of natural resource utilization (i.e. a high ratio of resource use per unit of time to stock) have permanently lower long run rates of growth.

The paper consist of three sections. In Section 1, we present the basic model. Section 2 analyses paths along which the rate of growth of consumption per man is constant. Section 3 analyses the optimal growth path of the economy.

1. THE BASIC MODEL

In most of our analyses we focus on the special, but, as we argue in the next section, central, case of an economy with a Cobb-Douglas technology of the form

$$Q = F(K, L, R, t) = K^{\alpha_1}L^{\alpha_2}R^{\alpha_3}e^{\lambda t}, \quad \alpha_1+\alpha_2+\alpha_3 = 1 \qquad \text{...(1)}$$

where R = rate of utilization of natural resources

L = supply of labour

λ = rate of technological progress, assumed to be constant

Q = aggregate output, which can be used either for investment or consumption.

Because of the assumption of a Cobb-Douglas technology, we do not need to specify whether technical change is labour, resource, or capital augmenting. For our purposes, nothing is gained by assuming different sectors have different production functions. Hence we write:

$$Q = C+\dot{K} \qquad \text{...(2)}$$

where C is consumption

and $\dot{K}$ is net investment.

As usual, we either can think of Q as net output, or we can explicitly assume that there is no depreciation. The necessary modifications for exponential depreciation are straightforward.

We assume population grows at the constant rate n:

$$\frac{\dot{L}}{L} = n. \qquad \text{...(3)}$$

Differentiating (1) logarithmically, we obtain (letting $g_Q = \dot{Q}/Q$, $g_k = \dot{K}/K$, etc.)

$$g_Q = \alpha_1 g_K + \alpha_2 n + \alpha_3 g_R + \lambda. \qquad \text{...(4)}$$

The crucial economic decisions concern the rate of growth of capital and the rate of change of the input of natural resources. Since resources are finite, resource inputs must eventually be declining, but the question is, at what rate? Further insight may be had by considering the basic *efficiency condition*:

$$F_K = \frac{d \ln F_R}{dt}. \qquad \text{...(5)}$$

The return to capital must be the same as the rate of change of the marginal product of the natural resource,[1, 2] or, in our model

$$\alpha_1\beta = g_Q - g_R \qquad \text{...(6)}$$

where $\beta = Q/K$, the output-capital ratio.

Letting $s = \dot{K}/Q$, the aggregate savings rate and

$x = 1-s$,

we obtain, *for any efficient path*

$$g_Q = \frac{\alpha_2 n+\lambda+\alpha_1\beta(s-\alpha_3)}{\alpha_1+\alpha_2} = \frac{\alpha_2 n+\lambda-\alpha_1\beta x}{\alpha_1+\alpha_2}+\alpha_1\beta, \qquad \text{...(7)}$$

$$g_R = \frac{\alpha_2 n+\lambda-\alpha_1\beta(1-s)}{\alpha_1+\alpha_2} = \frac{\alpha_2 n+\lambda-\alpha_1\beta x}{\alpha_1+\alpha_2}, \qquad \text{...(8)}$$

and

$$g_\beta = g_Q - g_K = \frac{\alpha_2 n+\lambda-\beta(s\alpha_2+\alpha_1\alpha_3)}{\alpha_1+\alpha_2} = \frac{\alpha_2 n+\lambda+\alpha_2\beta x}{\alpha_1+\alpha_2}-(1-\alpha_1)\beta. \qquad \text{...(9)}$$

Finally, it is convenient to focus our attention on the ratio of resource utilization, R, to the the stock of the resource, S. We define

$$\gamma = \frac{R}{S}$$

so

$$\frac{\dot{\gamma}}{\gamma} = g_R + \gamma. \qquad \text{...(10)}$$

Equations (6)-(10) will find repeated use in the subsequent analysis.

2. STEADY STATES

Long-term growth in models with only capital and labour has been so extensively discussed that we hardly need to think twice about what we mean by "balanced growth" or a "steady state"; we characterize a steady state by a constant capital-output ratio, a constant rate of growth of output, consumption, wages, etc. But with an exhaustible resource, we must reconsider what it is we mean by a "steady state". I shall consider here the asymptotic states of paths for which consumption is growing exponentially. The results of Section 3 and the analyses of [7] provide some justification for why we should be particularly interested in such paths.

Since

$$C = xQ$$

if C is to grow exponentially at rate $\bar{g}_C$, then

$$\bar{g}_C = g_x + g_Q$$

or (using (7))

$$g_x = \bar{g}_C - \frac{\alpha_2 n+\lambda}{\alpha_1+\alpha_2} + \frac{\alpha_1\beta x}{\alpha_1+\alpha_2} - \alpha_1\beta. \qquad \text{...(11)}$$

[1] This is just the familiar efficiency condition for growth with several capital goods, as developed, e.g. in Dorfman-Samuelson-Solow.

[2] In [7] we observe that this is equivalent to the equilibrium condition for competitive asset markets that the return to holding capital, F_K, be equal to the return to holding a stock of the natural resources, which is just the capital gain on the stock.

Equations (9) and (11) provide a complete characterization of such paths in terms of the variables x and β. Alternatively, we can characterize the paths in terms of β and βx:

$$g_{\beta x} = \bar{g}_C + \beta x - \beta. \qquad \ldots(12)$$

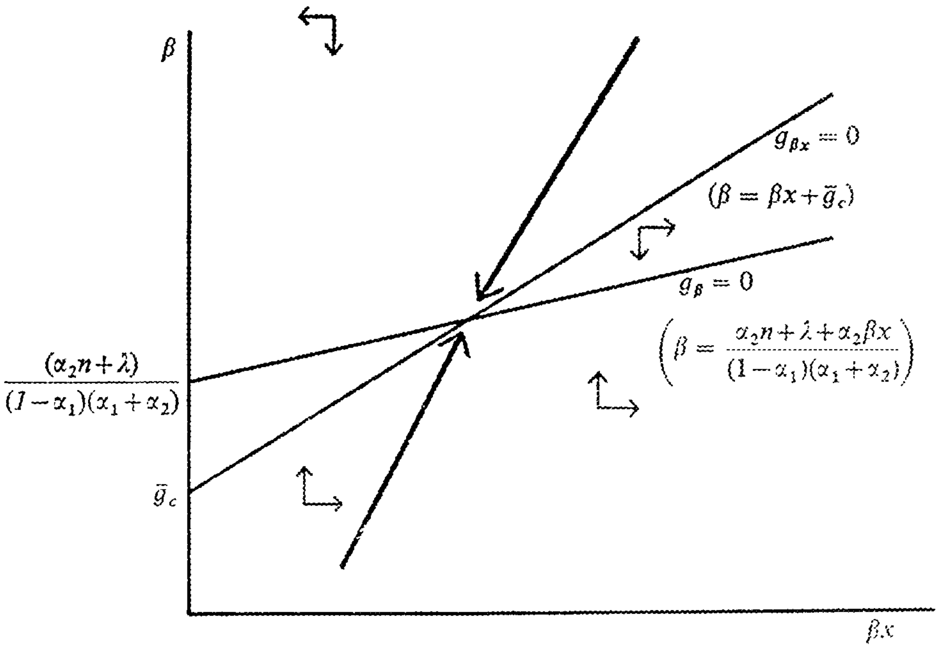

FIGURE 1

In Figure 1 we draw the phase diagram in $(\beta x, \beta)$ space. Provided

$$\bar{g}_C < \frac{\alpha_2 n + \lambda}{(1-\alpha_1)(\alpha_1+\alpha_2)} \qquad \ldots(13)$$

there is a unique value of (β^*, x^*) such that $g_\beta = g_x = 0$. It is immediately apparent that (β^*, x^*) is a saddle point, and no path not converging to (β^*, x^*) is feasible; eventually either x exceeds one (those diverging to the right) or $\beta x \to 0$, in which case, from (8), there exists a finite T after which $g_R > 0$, which is clearly not feasible. The fact that $\beta \to \beta^*$, $x \to x^*$, implies, from (8) that $g_R \to$constant and, from (10) $\gamma^* = -g_R$.

Moreover, since (β^*, x^*) is a saddlepoint, we can easily solve for the (unique) savings rate corresponding to any value of the output capital ratio, or

$$\beta x = \Psi(\beta), \; \Psi' > 0.$$

substituting into (8) and the result into (10), and using (9) to obtain the (β, γ) phase diagram (figure 2), showing that the unique equilibrium (β^*, γ^*) is a saddlepoint, and that convergence to the equilibrium is monotonic.[1]

Proposition 1. *Any path for which consumption grows at a constant rate must asymptotically have a constant savings rate, a constant rate of change of input of the natural resource and a constant resource flow-stock ratio.*

[1] This establishes that any path converging to the saddle-point equilibrium uses a finite amount of resources. Our supply of resources gives us our " boundary " values, i.e. it enables us to establish (see figure 2)

$$(\gamma(0), \beta(0), x(0)).$$

We can characterize the different steady state values as a function of the rate of growth consumption:

$$s^* = \frac{\alpha_1\alpha_3 g_C}{\lambda - \alpha_2(g_C - n)} \qquad \text{or} \qquad g_C = \frac{s^*(\lambda + \alpha_2 n)}{\alpha_1\alpha_3 + \alpha_2 s^*}$$

$$\beta^* = \frac{\lambda - \alpha_2(g_C - n)}{\alpha_1\alpha_3} \qquad \beta^* = \frac{g_C}{s^*} = \frac{\lambda + \alpha_2 n}{\alpha_1\alpha_3 + \alpha_2 s^*} \qquad \ldots(14)$$

$$-\gamma^* = \frac{g_C(1-\alpha_1) - (\alpha_2 n + \lambda)}{\alpha_3} \qquad -\gamma^* = \frac{(\alpha_2 n + \lambda)(s^* - \alpha_1)}{\alpha_1\alpha_3 + \alpha_2 s^*}$$

Straightforward differentiation of (14) yields

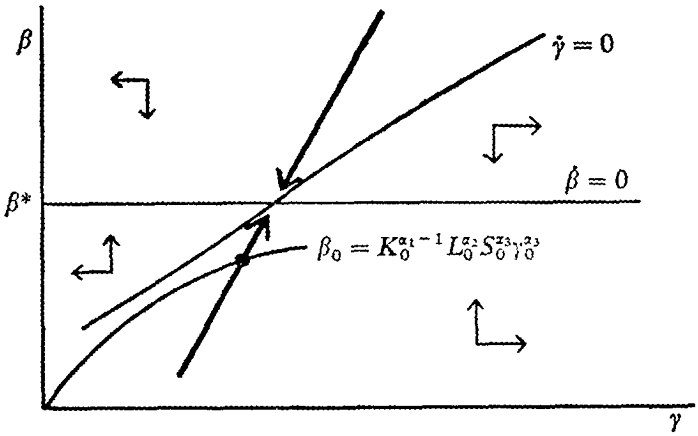

FIGURE 2

Proposition 2. *An increase in the savings rate increases the growth rate, increases the asymptotic capital output ratio, and is associated with a lower rate of resource utilization.*

The restrictions

$$0 \leqq s \leqq 1$$

and

$$g_R < 0$$

immediately imply that for steady, efficient growth

$$s < \alpha_1. \qquad \ldots(15)$$

Proposition 3. *Steady efficient growth implies an asymptotic savings rate smaller than the share of capital.*

Using this result with (14), we observe that

$$g_C < \frac{\alpha_2 n + \lambda}{1 - \alpha_1}. \qquad \ldots(16)$$

Hence, if there is to be sustained growth in *per capita* consumption

$$\lambda > \alpha_3 n. \qquad \ldots(17)$$

Proposition 4. *If the rate of population growth is positive, necessary and sufficient condition for sustaining a constant level of consumption per capita is that the ratio of the rate of technical change, γ, to the rate of population growth must be greater than or equal to the share of natural resources.*[1, 2]

Since $Q = K^{\alpha_1}L^{\alpha_2}R^{\alpha_3}e^{\lambda t} = K^{\alpha_1}L^{\alpha_2}(Re^{(\lambda/\alpha_3)t})^{\alpha_3}$, λ/α_3 is, effectively, the rate of resource augmenting technical progress. We require that the rate of resource augmenting technical progress exceed the population growth rate.

These results should be contrasted with the corresponding results without natural resources. There, as we noted in our introduction, the rate of growth was independent of the savings rate; different savings rates were associated with different levels of (asymptotic) *per capita* income. Here, there is a unique savings rate associated with each rate of growth; increases in savings do lead to permanently higher growth rates. On the other hand, as the results of the next section will make clear, the intertemporal trade-offs are very much the same; growth paths with higher savings rates entail lower consumption today but higher consumption at some future date.

2.1. *A Special Case.* There is one special case for which our previous analysis must be modified: when $\lambda = n = g_C = 0$, equations (7), (9) and (11) become:

$$g_Q = -\frac{\alpha_1\beta x}{\alpha_1+\alpha_2} + \alpha_1\beta \qquad \ldots(7')$$

$$g_x = \frac{\alpha_1\beta x}{\alpha_1+\alpha_2} - \alpha_1\beta \qquad \ldots(11')$$

$$g_\beta = \frac{\alpha_2\beta x}{\alpha_1+\alpha_2} + (1-a_1)\beta. \qquad \ldots(9')$$

[1] Our analysis establishes that if $g_c = n$ and the economy was efficient, $\lambda > \alpha_3 n$. But clearly if $g_c = n$ is not feasible for efficient paths, it is not feasible. The same result can be established directly as follows: Let

$$\overline{g_Q} \equiv \lim_{T\to\infty} \int_\tau^T \frac{g_Q(v)}{T-\tau}\, dv$$

the average value of g_Q. Similarly define $\overline{s\beta}$, $\overline{g_\beta}$. Then from (4). since $\overline{g_R} < 0$

$$\overline{g_Q} < \alpha_2 n + \lambda + \alpha_1\overline{s\beta} \qquad \ldots(18)$$

and from (9)

$$\overline{g_\beta} < \alpha_2 n + \lambda - (1-\alpha_1)\overline{s\beta}.$$

Since $s \leqq 1$, if

$$\beta < \frac{\alpha_2 n + \lambda}{(\alpha_1+\alpha_2)(1-\alpha_1)}, \quad g_\beta > 0.$$

Hence

$$\lim_{t\to\infty} \beta \geqq \frac{\alpha_2 n + \lambda}{(1-\alpha_1)(\alpha_1+\alpha_2)}.$$

Thus $\overline{g_\beta} \geqq 0$ and

$$\overline{s\beta} < \frac{\alpha_2 n + \lambda}{1-\alpha_1}.$$

Hence, using (18)

$$\overline{g_Q} < \frac{\alpha_2 n + \lambda}{1-\alpha_1}.$$

If

$$\lambda < \alpha_3 n,\ \overline{g_Q} - n < \frac{\lambda - \alpha_3 n}{1-\alpha_1} < 0.$$

[2] The above argument establishes necessity. Sufficiency is trivial. Let s and g_R be constant at

$$s = \alpha_1\alpha_3 n/\lambda < \alpha_1,\ g_R = \alpha_3 n - \lambda/\alpha_3 < 0; \text{ let } \beta_0 = \lambda/\alpha_1\alpha_3,$$

i.e.

$$R_0 = \left(\frac{\lambda}{\alpha_1\alpha_3} K_0^{1-\alpha_1} L_0^{-\alpha_2}\right)^{1/\alpha_3}.$$

Clearly $g_Q = n$, $g_\beta = 0$ and consumption *per capita* is constant.

The unique path which is feasible has a constant savings rate:

$$x^* = \alpha_1 + \alpha_2 = 1 - \alpha_3$$

If x is ever greater (smaller) than $1-\alpha_3$, it always is; since

$$\frac{d\beta/dt}{dx/dt} = \frac{\beta\left(\dfrac{\alpha_2 x}{\alpha_1+\alpha_2} - (1-\alpha_1)\right)}{\alpha_1 x\left(\dfrac{x}{\alpha_1+\alpha_2} - 1\right)} \to 0 \text{ as } \beta \to 0 \text{ if } x > 0;$$

the curves appear as in Figure 3. It is easy to show that if x is ever greater than $\alpha_1+\alpha_2$, in finite time, $x>1$, which is not possible. It is somewhat more difficult to show that paths with $x<\alpha_1+\alpha_2$ are not feasible; the proof is left to a footnote.[1]

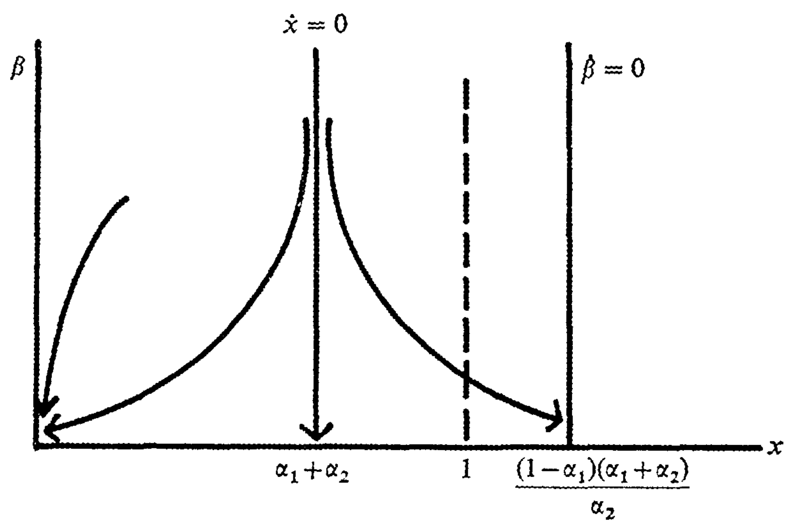

FIGURE 3

Proposition 5a. *If $\lambda = 0 = n$, there is, at most, one efficient path along which $g_C = 0$; the savings rate equals the share of natural resources.*

This characterizes the efficient path if it exists. We now show that

Proposition 5b. *A necessary and sufficient condition for a constant level of consumption with no technical change and no growth is that the share of natural resources (α_3) be less than the share of capital (α_1).*

(This theorem was originally stated in [6], and an alternative proof provided.)

[1] Consider the differential equations

$$g_{\beta x} = \beta x\left(1 - \frac{1}{x}\right) \qquad \text{...(12')}$$

$$g_R = -\frac{\alpha_1 \beta x}{\alpha_1+\alpha_2} \qquad \text{...(8')}$$

We consider the "best" path for which $s = \alpha_3$, i.e. the one which maximizes C, and just uses up all resources asymptotically. Denote this path by $[C^*, R^*(t), \beta^*(t)]$. We show that if $s>\alpha_3$, if consumption were to remain at C^*, resources would be more than exhausted. At $t = 0$, for $C = C^*$

$$\beta(0)x(0) = \beta^*(0)x^*(0); \text{ if } s>\alpha_3, \beta(0)>\beta^*(0)$$

and hence $R(0)>R^*(0)$. It immediately follows that $\beta(t)x(t)<\beta^*(t)x^*(t)$, all t, and $R(t)>R^*(t)$, all t. But

$$\int_0^\infty R^*(t)dt = S_0.$$

I—SYM

Our proof proceeds as follows: we consider paths along which s is constant. We show that a necessary and sufficient condition for $g_C = 0$ with constant s is that $\alpha_1 > \alpha_3$. Then we show that if a path with constant s and constant C is not feasible, no path with constant C is feasible.

We rewrite (3) as

$$g_Q = \alpha_1 g_K + \alpha_3 g_R = \alpha_1 s\beta + \alpha_3 g_R = 0. \quad \ldots(3')$$

Differentiating (3′) and using (9) we obtain

$$0 = \alpha_1 s\dot{\beta} + \alpha_3 \dot{g}_R = -\alpha_1 (s\beta)^2 + \alpha_3 \dot{g}_R.$$

Using (3′) we obtain

$$\dot{g}_R = \frac{g_R^2 \alpha_3}{\alpha_1}. \quad \ldots(19)$$

Let $\alpha_3/\alpha_1 = z$. Then integrating successively, we obtain

$$g_R = -\frac{1}{\kappa_1 + zt}$$

$$R = \kappa_2(\kappa_1 + zt)^{-1/z}.$$

This is feasible if and only if [1]

$$S_0 \geqq \int_0^\infty R(t)dt \equiv \kappa_2 \int_0^\infty (\kappa_1 + zt)^{-1/z} dt$$

i.e. $z < 1$, or,

$$\alpha_3 < \alpha_1$$

(otherwise the integral on the RHS diverges). If $z < 1$

$$S_0 = \frac{\kappa_2 \kappa_1^{(1-1/z)}}{1-z} \quad \ldots(20)$$

κ_1 and κ_2 are constants of integration chosen to satisfy the boundary value conditions (20) and

$$R_0 = \kappa_2(\kappa_1)^{-1/z}$$

$$g_{R_0} = -\frac{1}{\kappa_1} = -\frac{s}{z} K_0^{\alpha_1 - 1} R_0^{\alpha_3} L^{\alpha_2}.$$

This establishes that a constant consumption path with a constant savings rate is feasible if and only if $\alpha_1 > \alpha_3$.

Necessity. So far we have actually only established that a constant consumption-output path requires $\alpha_1 > \alpha_3$ if $s \leqq 1$ is constant.

But consider a path with $s = 1$. The above argument establishes that if $\alpha_3 > \alpha_1$, $\lim_{t\to\infty} Q(t) \to 0$. But clearly then $\lim_{t\to\infty} Q(t) = 0$ on any path with $s \leqq 1$. And since

$$C(t) \leqq Q(t),$$

$\lim_{t\to\infty} C(t) \to 0.$

2.2. *General Remarks.* This section has considered (efficient) paths along which consumption grows at a constant rate. The fact that there is a limited amount of natural resources

[1] The integral

$$\int_0^\infty (\kappa_1 + zt)^{-1/z} dt = \left. \frac{(\kappa_1 + zt)^{1-1/z}}{z-1} \right|_0^\infty$$

diverges if $z \geqq 1$.

and natural resources are necessary for production does not necessarily imply that the economy must eventually stagnate and then decline. Two offsetting forces have been identified: technical change and capital accumulation. Even with no technical change, capital accumulation can offset the effects of the declining inputs of natural resources, so long as capital is "more important" than natural resources, i.e. the share of capital is greater than that of natural resources. With technical change, at *any* positive rate, we can easily find paths along which aggregate output does not decline. For so long as the input of natural resources declines exponentially, no matter at how small a rate, provided the initial level of input is set correctly, we will just use up our resources. And the technical change can offset the effects of the slowly declining input of natural resources. To sustain a constant level of *per capita* consumption requires a more stringent condition on the rate of technical change (Proposition 4).

The Cobb-Douglas case has a number of special properties to which we must call attention. In a Cobb-Douglas production function, we need not distinguish between labour, capital, and resource augmenting technical progress. Consider for simplicity the case where population was constant. Then a sufficient condition for sustaining a constant *per capita* income with a general production function is that there be *resource augmenting technical change* at any positive rate (no matter how small).

Moreover, we do not require a unitary elasticity of substitution between each of the factors. Sustained levels of *per capita* consumption are also feasible if the production function is separable,

$$Q = F(\phi(K, R), L)$$

and there is an elasticity of substitution between K and R greater than unity, or equal to unity with the coefficient (exponent) on capital being greater than that on the natural resource.

Finally, we note that if there are returns to scale, the required rate of technical progress necessary to offset the effects of the decreased input of natural resources is smaller. With a Cobb-Douglas production function, letting $\sum_1^3 \alpha_i > 1$ we require [1]

$$(\alpha_1 + \alpha_2 - 1)n + \lambda > 0.$$

3. OPTIMAL ECONOMIC GROWTH

The previous section characterized the set of efficient paths along which consumption grew at a constant rate. We suggested that one reason we might be particularly interested in such paths is that asymptotically along a growth path which maximized the present discounted value of utility with a constant discount rate the rate of growth of consumption was a constant. In this section, we characterize more fully optimal growth paths for an economy with exhaustible natural resources.

From a technical point of view, the analysis involves two difficulties. First, there are two "state" variables, the capital stock and the stock of natural resources. Except under special circumstances (e.g. where the relevant functions are linear), the complete qualitative analysis of optimal control problems with more than one state variable appears to be a difficult problem. Secondly, as we noted earlier, the rate of decline of input of natural resources is a control variable; as a consequence, the rate of growth of consumption *per capita* is an endogenous variable. The conventional "trick" for analysing models with technical change is to convert the relevant variables into "intensive" units—per

[1] Letting $g_K = g_Q$ we have

$$g_Q = \alpha_1 g_K + \alpha_2 n + \lambda + \alpha_3 g_R = \frac{\alpha_2 n + \lambda}{1 - \alpha_1} + \alpha_3 g_R.$$

We require

$$\frac{\alpha_2 n + \lambda}{1 - \alpha_1} > n.$$

effective worker. Fortunately, it turns out that a similar trick works here, although one must be careful in choosing the appropriate deflator.

More formally, assume we wish to use a criterion such as

$$\max \int_0^\infty U(c)e^{-(\delta-n)t}dt \qquad \text{...(21)}$$

as the basis of our choice of growth paths, where c is *per capita* consumption and δ is the pure rate of time discount. Let

$$U(c) = c^v/v,\ v<1,\ \neq 0.$$

The special case $U(c) = \ln c$ corresponds to the case of $v = 0$. Then for an optimum to exist, we require

$$\delta > n + \frac{\lambda - \alpha_3 n}{\alpha_2 + \alpha_3} v. \qquad \text{...(22)}$$

Otherwise, using the analysis of Section 2, we can easily construct paths for which the integral diverges.

If a solution to the maximization problem exists, it is easy to show that it is unique. Thus, all we have to do is to exhibit a path satisfying the Euler equation.

Thus, let us define ρ, the asymptotic growth rate of *per capita* output, as (using (7))

$$\rho = g_Q^* - n = \frac{-\alpha_1 n + \lambda - \alpha_1 \beta^* x^*}{\alpha_1 + \alpha_2} + \alpha_1 \beta^*. \qquad \text{...(23)}$$

Let $y = ce^{-\rho t}$ = consumption per effective labourer

$k = \frac{K}{L} e^{-\rho t}$ = capital per effective labourer

$q = \frac{Q}{L} e^{-\rho t}$ = output per effective labourer.

Thus our maximization problem may be rewritten as

$$\max \int_0^\infty U\{q - (\dot{k} + (\rho + n)k))e^{\rho t}\}e^{-(\delta - n)t}dt \qquad \text{...(24)}$$

$$= \int_0^\infty \{q - (\dot{k} + (\rho + n)k)\}^v e^{(v\rho - (\delta - n))t}\ dt$$

subject to the constraint that

$$\int_0^\infty R dt = \int_0^\infty re^{(\rho+n)t}dt \leqq S_0 \qquad \text{...(24a)}$$

where r is resource utilization per effective labourer,

$$r = Re^{-(\rho+n)t}.$$

Let ϕ be the Lagrange multiplier associated with the constraint (24*a*). Then optimality requires (if r, or q, is non zero) [1]

$$-\frac{dU'e^{(v\rho - (\delta - n))t}}{dt} = U'e^{(v\rho - (\delta - n))t}(q_k - (n + \rho)) \qquad \text{...(25a)}$$

$$U'q_r e^{(v\rho - (\delta - n))t} = \phi e^{(\rho + n)t} \qquad \text{...(25b)}$$

[1] Where it is understood that

$$U' = (q - (\dot{k} + (\rho + n)k))^{v-1} = y^{v-1},\ q_k = \partial q/\partial k, \text{ etc.}$$

i.e.

$$\frac{\dot{y}}{y} = \frac{\alpha_1\beta-\delta}{1-v} - \rho \qquad \text{...(25a)}$$

and

$$\frac{\dot{q}_r}{q_r} = q_k. \qquad \text{...(25b)}$$

(25*b*) is simply the conventional efficiency condition and (25*a*) is the conventional Euler equation of optimal savings.

Thus the dynamics of the economy are described by the same set of differential equations as those presented earlier for efficient paths with constant rates of growth of consumption, except for the differential equation describing the rate of change of the consumption (savings) rate; for completeness, we rewrite the equations as [1]

$$g_x = g_y - (g_Q - n - \rho) = \frac{\alpha_1\beta-\delta}{1-v} - (g_Q - n) = \frac{\alpha_1 n - \lambda}{\alpha_1+\alpha_2} + \frac{\alpha_1\beta x}{\alpha_1+\alpha_2} + \frac{v\alpha_1\beta-\delta}{1-v} \qquad \text{...(26)}$$

$$g_\beta = g_Q - s\beta = \frac{\alpha_2 n+\lambda}{\alpha_1+\alpha_2} + \frac{\alpha_2\beta x}{\alpha_1+\alpha_2} - (1-\alpha_1)\beta \qquad \text{...(27)}$$

$$\frac{\dot{\gamma}}{\gamma} = \gamma + g_Q - \alpha_1\beta = \gamma + \frac{\alpha_2 n+\lambda-\alpha_1\beta x}{\alpha_1+\alpha_2} \qquad \text{...(28)}$$

Asymptotic Solution. We first solve for the steady state of the economy, i.e. let $g_x = g_\beta = g_\gamma = 0$. The easiest way to "solve" the equations (26)-(29) for the steady state values of x, β and γ is as follows:

Setting $g_x = g_\beta = 0$, we obtain from (26) and (27) [2]

$$\beta^* = \frac{\delta\alpha_2+\lambda(1-v)}{\alpha_1(1-\alpha_1-\alpha_3 v)} \qquad \text{...(29)}$$

$$s^* = 1-x^* = \frac{\alpha_1[(1-\alpha_1-\alpha_3 v)n+\lambda-\alpha_3\delta]}{\lambda(1-v)+\alpha_2\delta}. \qquad \text{...(30)}$$

Using these results in (26) and (28), we obtain

$$g_Q^* = \frac{\lambda-\alpha_3\delta}{1-\alpha_1-\alpha_3 v} + n = \rho^* + n \qquad \text{...(31)}$$

$$\gamma^* = \frac{\delta(1-\alpha_1)-v\lambda}{1-\alpha_1-\alpha_3 v} - n. \qquad \text{...(32)}$$

Notice that for a logarithmic utility function

$$\gamma = \delta - n, \qquad \text{...(32')}$$

the optimal rate of utilization is just the rate of discount minus the rate of population growth. Obviously, the higher the rate of discount, the faster we should use up our resources.

[1] Recalling $x = cL/Q = y/q$.

[2] We simply solve the linear equations

$$\begin{bmatrix} \frac{v\alpha_1}{1-v} & \frac{\alpha_1}{\alpha_1+\alpha_2} \\ -(1-\alpha_1) & \frac{\alpha_2}{\alpha_1+\alpha_2} \end{bmatrix} \begin{bmatrix} \beta \\ \beta x \end{bmatrix} = \begin{bmatrix} \frac{\lambda-\alpha_1 n}{\alpha_1+\alpha_2} + \frac{\delta}{1-v} \\ \frac{-(\alpha_2 n+\lambda)}{\alpha_1+\alpha_2} \end{bmatrix}.$$

Assuming a 5 per cent discount rate and a 1 per cent population growth rate, we should use 4 per cent of our stock of natural resources a year.

More generally, a higher rate of technical change leads to a higher or lower rate of extraction as the elasticity of marginal utility is greater or less than unity ($v \lesseqgtr 0$) and a higher elasticity of marginal utility $(1-v)$ leads to a higher or lower rate of extraction as the rate of resource-augmenting technical progress (γ/α_3) is larger or smaller than the rate of discount.

As expected, the rate of growth increases with the rate of technical progress, decreases with the pure rate of time discount and with the elasticity of marginal utility.

$$(\text{As } v \to -\infty,\ g_Q^* \to n.)$$

The savings rate decreases with the rate of discount, as expected.

Dynamics. We now turn to the characterization of the optimal trajectory. For simplicity, we shall go through the calculations only for the case of $v = 0$ (logarithmic utility function). First, we add (26) and (27) to obtain

$$\frac{\dot{\beta}}{\beta} + \frac{\dot{x}}{x} = n + \beta x - \delta - (1-\alpha_1)\beta.$$

In Figure 4 we have drawn the phase diagram for $(\beta x, \beta)$, $(\beta x, x)$ and (β, x) space.[1]

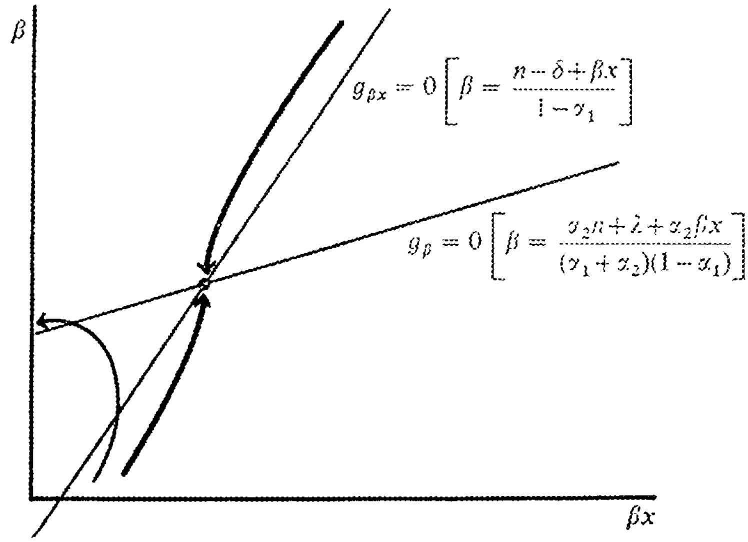

FIGURE 4*a*

The equilibrium is a saddlepoint, which means that if $x \to x^*$, then we can solve for the path

$$\beta x = \phi(\beta),\ \phi' > 0, \qquad \ldots(33a)$$

[1] The intercept of $g_{\beta x} = 0$ with the vertical axis is negative (since $\delta > n$, from (22)) and its slope is greater than unity; $g_\beta = 0$ has a positive intercept and a slope less than unity:

$$\frac{\alpha_2}{(1-\alpha_1)(\alpha_1+\alpha_2)} - 1 = -\alpha_1 + \alpha_1(\alpha_1+\alpha_2) = -\alpha_1\alpha_3 < 0,$$

or we can solve for x as a function of β

$$x = \psi(\beta),\ \psi' < 0. \qquad \text{...(33b)}$$

The optimal savings rate is a (decreasing) function only of the capital output ratio.

Substituting (33) into (27), we obtain

$$\frac{\dot{\beta}}{\beta} = \frac{\alpha_2 n + \lambda + \alpha_2 \beta\{\psi(\beta) - (1-\alpha)(\alpha_1 + \alpha_2)\}}{\alpha_1 + \alpha_2}. \qquad \text{...(34)}$$

Since $dx/d\beta < 0$, there is a unique value of β at which $\dot{\beta} = 0$. (This should also be clear for Figure 4.)

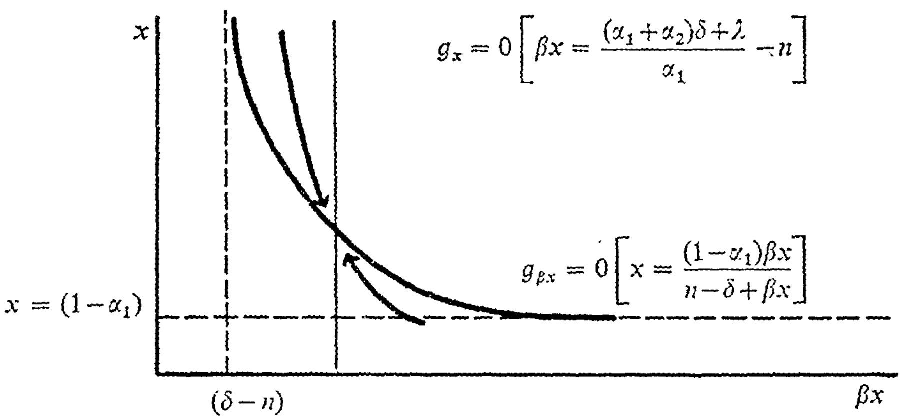

FIGURE 4b

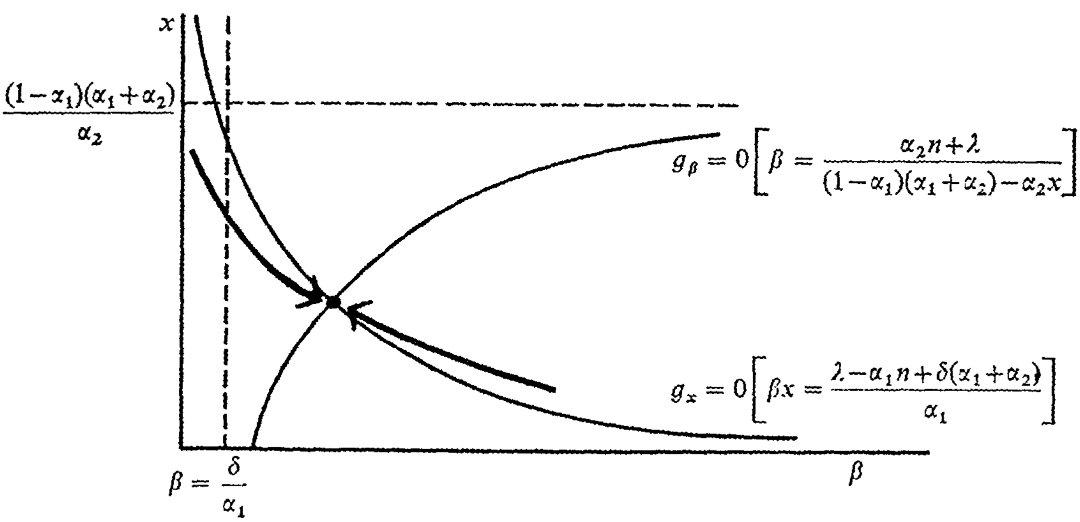

FIGURE 4c

In Figure 5 we have drawn the (γ, β) phase diagram

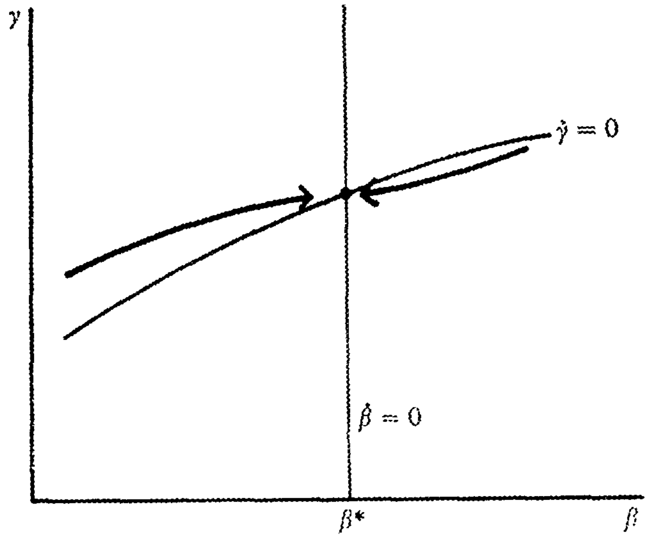

FIGURE 5

Again the equilibrium is a saddlepoint, which means we can solve for β as a function of γ, and from the production function, we have the boundary value condition

$$\beta_0 = K_0^{\alpha_1 - 1} L_0^{\alpha_2} S_0^{\alpha_3} \gamma_0^{\alpha_3}. \qquad \ldots(35)$$

We can thus establish

Proposition 6. *The path* $\{\beta(t), x(t), \gamma(t)\}$ *defined by* (26)-(28) *converging to the asypmtotic values given by* (29-31) *and with the initial boundary value condition* (35) *is the optimal trajectory of the economy. Along the optimal path, the interest rate, output capital ratio, savings rate, and rate of resource utilization fall* (*rise*) *monotonically.*

As we noted earlier, along the optimal trajectory asymptotically there is a constant rate of growth of consumption *per capita*:

$$\lim_{t \to \infty} \frac{\dot{c}}{c} \gtreqless 0 \quad \text{as} \quad \frac{\lambda}{\delta} \gtreqless \alpha_3$$

and does not depend on the value of v.

If $\dot{c}^*/c > 0$, for poor economies whose savings rate is monotonically decreasing, consumption increases monotonically. For very " capital-rich " economies, consumption *per capita* may initially fall (when $\beta < \beta^{**} = \delta/\alpha_1$) but then monotonically rises. (Similarly, if $\dot{c}^*/c < 0$, consumption *per capita* may initially rise for economies which are very " resource-rich ".)

4. CONCLUDING REMARKS

In this paper we have analysed a model of economic growth in which natural resources are exhaustible, in limited supply, and essential for production. If one views the simple model presented as a reasonable first approximation, not only is sustained growth in consumption *per capita* feasible, but the optimal rates of utilization of the resource for reasonable values of the parameters is of the order of magnitude observed for many natural resources. There seems to be no presumption that a situation in which there is a " thirty years " reserve of a natural resource is indicative of excessive consumption of the resource.

In the sequel to this paper, we investigate growth in competitive economies with natural resources.

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