Guaranteed Decentralized Pursuit-Evasion in the Plane with Multiple Pursuers

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***Abstract*— Pursuit-evasion games are an important problem in robotics and control, but games with many players are difficult to analyze and solve. This paper studies a game of multiple pursuers cooperating to capture a single evader in a bounded convex polytope in the plane. We present a decentralized control scheme based on the Voronoi partion of the game domain, where the pursuers jointly minimize the area of the evader's Voronoi cell. We prove that capturing the evader is guaranteed under this scheme regardless of the evader's actions, and show simulation results demonstrating the pursuit strategy.**

I. Introduction

Pursuit-evasion games describe a class of problems where one or more pursuers attempt to move within some distance of one or more evaders moving to escape them. Such games are relevant to a number of applications in robotics and control. With dynamics and multiple adversarial players, they are challenging to analyze and difficult to solve.

Early studies often focused on a single pursuer and evader in a continuous, unbounded state space. The problem is cast as a differential game, and optimal strategies are found for both players as saddle-points to a minimax optimization over the distance between them [1]. A large body of work based on this approach has since appeared, with particular solutions to specific problems [2], [3] and solutions for more general problems through numerical solutions of related Hamilton-Jacobi equations [4], [5].

Another significant body of work is on pursuit-evasion in bounded domains, especially for games played on graphs where the pursuers progressively reduce the states available to the evader [6], [7]. A closely related problem is search in finite domains, also known as visibility-based pursuit-evasion. One well-known approach is to decompose the continuous search space into a discrete graph, for which a plan is found that successively clears cells while preventing evaders from re-entering previously searched cells [8].

In all cases, managing multiple cooperating pursuers is a major challenge. Analytic solutions using differential games exist for some particular problems with small numbers of pursuers [9], [10], but more general solutions are limited by the curse of dimensionality. Limiting the planning horizon, for example by greedy optimization over one time-step of a cost function such as a distance metric [11] or entropy of a distribution [12], or via limited-depth tree search [8]. Others assume a known model for the evader's inputs and solve for pursuer trajectories in a receding-horizon framework [13], [14]. While these simplifications allow coordinated plans to be generated, they may lose guarantees of capture or completeness.

The pursuit-evasion problem we address consists of controlling a team of identical pursuers to capture a single evader with similar dynamics in a convex planar polytope. Both evader and pursuers are constrained to an identical maximum velocity, otherwise no assumptions are made about the evader's strategy or inputs. The set of all points that the evader can reach before any of the pursuers can be found through a Voronoi decomposition of the game domain [15].

Some previous work has looked at using Voronoi regions in a scenario where the evader's control law is known to the pursuers [16]. Voronoi decomposition has also been extensively used in distributed and decentralized control of multiple agents in surveillance and coverage tasks, where cooperative actions result from the influence of shared Voronoi boundaries [17].

We present a decentralized, guaranteed pursuit strategy where the pursuers cooperatively minimize the area of the evader's Voronoi cell by independently controlling each pursuer's shared Voronoi boundary with the evader. The pursuers compute control inputs independently given the boundaries of the evader's Voronoi cell, which is the only shared information. We show that this pursuit strategy results in guaranteed capture of the evader in finite time regardless of the strategy or inputs of the evader. Several simulations are also presented to illustrate some key characteristics of the strategy.

This paper unfolds as follows. Section II defines the pursuit-evasion problem in question, and Section III lays out the pursuit strategy. The proof of guaranteed capture is presented in Section IV, with simulation results in Section V and conclusions and future work in Section VI.

II. Problem Formulation

We consider a multi-player pursuit-evasion game involving N pursuers and a single evader, taking place in the interior of a polytope D in $\mathbb{R}^2$. The goal of the pursuers is to capture the evader by having at least one of the pursuers come within some distance r_c of the evader. Let $\mathbf{e} \in \mathbb{R}^2$ be the position of the evader and $\mathbf{p}_i \in \mathbb{R}^2$ be the position of pursuer i.

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Consider the equations of motion

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{u}_e,\ \mathbf{e}(0) = \mathbf{e}_0, \\ \dot{\mathbf{p}_i} &= \mathbf{u}_i,\ \mathbf{p}_i(0) = \mathbf{p}_{i,0},\ i = 1, ..., N, \end{aligned} \tag{1}$$

where $\mathbf{u}_e$ and $\mathbf{u}_i$ are the velocity control inputs of the evader and pursuers, respectively, and $\mathbf{e}_0,\ \mathbf{p}_{i,0} \in D$ are the initial evader and pursuer positions. The players are assumed to be subject to identical speed constraints, namely

$$||\mathbf{u}_e(t)||_2 \leq v_{max},\ ||\mathbf{u}_i(t)||_2 \leq v_{max},\ \forall t \geq 0, \tag{2}$$

for some maximum speed v_{max}. The motions of the evader and pursuers, as described by (1), are also constrained to lie within the region D:

$$\mathbf{e}(t) \in D,\ \mathbf{p}_i(t) \in D,\ \forall t \geq 0. \tag{3}$$

Any velocity input $\mathbf{u}_e(t)$ or $\mathbf{u}_i(t)$ which satisfies the constraints (2) and (3) is called an admissible input for the evader or pursuer i, respectively.

Now define the minimum separation distance between the evader and the pursuers at any given time t as

$$d_{min}(t) \triangleq \min_i ||\mathbf{p}_i(t) - \mathbf{e}(t)||_2.$$

Assuming a finite radius of capture $r_c > 0$, the capture condition for the pursuers is then given by

$$d_{min}(T) \leq r_c,\ \text{for some } T \geq 0. \tag{4}$$

In order to achieve this capture condition, each pursuer is allowed to select a pursuit strategy $\mu_i(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_N)$, based upon observations of the evader and pursuer positions at each time instant, resulting in the closed-loop system dynamics:

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{u}_e,\ \mathbf{e}(0) = \mathbf{e}_0, \\ \dot{\mathbf{p}_i} &= \mu_i(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_N),\ \mathbf{p}_i(0) = \mathbf{p}_{i,0},\ i = 1, ..., N \end{aligned} \tag{5}$$

Any strategy μ_i which satisfies the constraints (2) and (3) is referred to as an admissible pursuit strategy for pursuer i.

We can now give a precise statement of the problem for our multi-player pursuit-evasion game.

Problem 1: For any initial configuration $\mathbf{e}_0,\ \mathbf{p}_{i,0} \in D$ satisfying $d_{min}(0) > r_c$, find an admissible choice of pursuit strategy μ_i for each pursuer i such that, regardless of any admissible choice of evader input $\mathbf{u}_e$, the capture condition (4) is satisfied for some $T < \infty$.

It can be observed that by appropriate re-scaling of the dynamics (5), the polytope D, and the capture radius r_c, it is sufficient to consider this problem for the case where $v_{max} = 1$. Thus, for the rest of this paper, we will assume without loss of generality that $v_{max} = 1$ in (2).

III. Pursuit Strategy

The pursuit strategy we propose is based on the Voronoi partitions of D generated by the locations of the players. Roughly speaking, the strategy is designed so as to decrease the area of the evaders' Voronoi cell over time. Intuitively, as this area decreases towards zero, the capture condition will be satisfied. In this section, we will describe some mathematical properties of this pursuit strategy necessary for the proof of finite time capture.

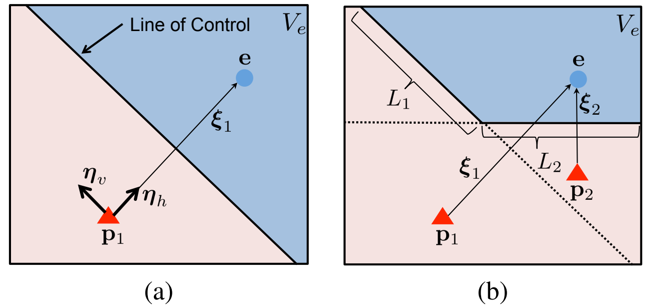


Fig. 1. **Illustrations showing the evader's Voronoi cell V_e (a) for a single pursuer and evader and (b) with an additional pursuer.**

Let $\mathcal{V}(D) = \{V_e, V_1, \ldots, V_N\}$ be the Voronoi partition of D generated by the points $\{\mathbf{e}, \mathbf{p}_1, \ldots, \mathbf{p}_N\}$:

$$\begin{aligned} V_e &= \{\mathbf{x} \in D : \|\mathbf{x} - \mathbf{e}\| \leq \|\mathbf{x} - \mathbf{p}_i\|, \forall i \leq N\}, \\ V_i &= \big\{\mathbf{x} \in D : \|\mathbf{x} - \mathbf{p}_i\| \leq \\ &\qquad \min\{\|\mathbf{x} - \mathbf{e}\|, \|\mathbf{x} - \mathbf{p}_j\|\}, \forall j \neq i\big\}, i \leq N \end{aligned}$$

Let $\mathcal{N}_e$ be the set of pursuer indices that are Voronoi neighbors of the evader. The edge shared by V_e and V_i, $i \in \mathcal{N}_e$ is called the *line of control* for pursuer i and is denoted by B_i, where L_i is the length of B_i (see Figure 1). We denote by A the area of the voronoi cell V_e containing the evader. This area can be calculated as

$$A(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_N) = \sum_{j=1}^{n_e} \left(x_j^e y_{j+1}^e - x_{j+1}^e y_j^e \right), \tag{6}$$

where $\{(x_j^e, y_j^e)\}_{j \leq n_e}$ is the set of vertices of V_e and $n_e + 1$ wraps around to the 1st vertex. It can be easily verified that the area A depends only on the locations of the neighboring pursuers and that this dependence is smooth whenever the pursuer locations are in D. The time derivative of A is given by

$$\frac{dA}{dt} = \frac{\partial A}{\partial \mathbf{e}} \dot{\mathbf{e}} + \sum_{i=1}^{N} \frac{\partial A}{\partial \mathbf{p}_i} \dot{\mathbf{p}}_i \tag{7}$$

Now consider a cooperative pursuit strategy that jointly minimizes $\frac{dA}{dt}$. According to (7), this joint objective can be decoupled into the individual objectives of minimizing $\frac{\partial A}{\partial \mathbf{p}_i} \dot{\mathbf{p}}_i$ for each pursuer i. Since A depends only on the Voronoi neighbors of the evader, we have $\frac{\partial A}{\partial p_i} = 0$ for all $i \notin \mathcal{N}_e$. Thus, for any pursuer i which is not a Voronoi neighbor of the evader, we simply set $\mathbf{u}_i = \frac{\mathbf{e} - \mathbf{p}_i}{||\mathbf{e} - \mathbf{p}_i||}$. On the other hand, for each pursuer $i \in \mathcal{N}_e$, the choice of pursuit strategy which minimizes (7) is given by:

$$\mathbf{u}_i^* = \mu_i^*(\mathbf{e}, \mathbf{p}_1, \ldots, \mathbf{p}_N) \triangleq \frac{-\frac{\partial A}{\partial \mathbf{p}_i}}{||\frac{\partial A}{\partial \mathbf{p}_i}||},$$

where we make use of the assumption that $v_{max} = 1$.

To enable analysis on the proposed pursuit strategy, we now derive an analytical expression for μ_i^*, $i \in \mathcal{N}_e$ using a particular choice of local coordinate system. First, let

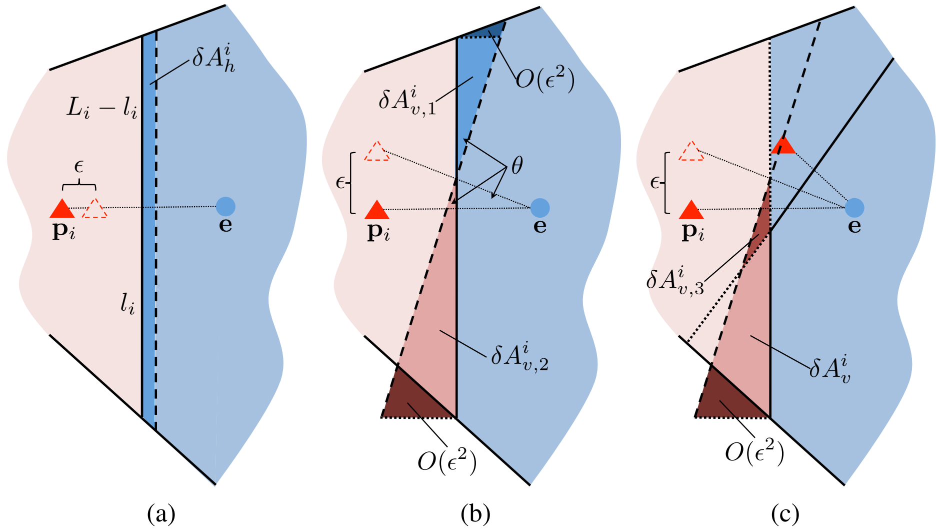


Fig. 2. **Variational change in area of the pursuer's Voronoi region with respect to (a) a perturbation toward the evader), (b) perturbation parallel to the line of control, and (c) when another pursuer is present and $\boldsymbol{\xi}_i$ no longer intersects B_i, as in Figure 1b.**

$\boldsymbol{\xi}_i(\mathbf{e}, \mathbf{p}) = \mathbf{e} - \mathbf{p}_i$ be the displacement vector pointing from the location of pursuer i towards the location of the evader. When there is no ambiguity, we will omit its arguments and denote this vector simply by $\boldsymbol{\xi}_i$. By the assumption that $d_{min}(0) > r_c$ and the fact that capture is achieved when $d_{min}(T) = r_c$, we have $||\boldsymbol{\xi}_i(\mathbf{e}(t), \mathbf{p}(t))|| \geq r_c$ for all $i \leq N$ and $t \in [0, T]$. Define $\boldsymbol{\eta}_h^i = \frac{\boldsymbol{\xi}_i}{\|\boldsymbol{\xi}\|}$ and let $\boldsymbol{\eta}_v^i \in \mathbb{R}^2$ be the unit vector orthogonal to $\boldsymbol{\eta}_h^i$, as shown in Figure 1a. The vectors $\{\boldsymbol{\eta}_h^i, \boldsymbol{\eta}_v^i\}$ define a local coordinate system that depends on the locations of $\mathbf{e}$ and $\mathbf{p}_i$. For any $\mathbf{x} \in \mathbb{R}^2$ and $(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_N)$ such that $\mathbf{x} + \mathbf{p}_i \in D$, define

$$A_i^+(\mathbf{x})|_{(\mathbf{e},\mathbf{p}_1,...,\mathbf{p}_N)} = A(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_i + \mathbf{x}, ...\mathbf{p}_N).$$

Denoting by $D_h^i A$ and $D_v^i A$ the directional derivatives of A along $\boldsymbol{\eta}_h^i$ and $\boldsymbol{\eta}_v^i$, we have

$$\begin{cases} D_h^i A|_{(\mathbf{e},\mathbf{p}_1,\ldots,\mathbf{p}_N)} = \lim_{\epsilon \to 0} \frac{A_i^+(\epsilon \cdot \boldsymbol{\eta}_h^i)|_{(\mathbf{e},\mathbf{p}_1,\ldots,\mathbf{p}_N)} - A}{\epsilon} \\ D_v^i A|_{(\mathbf{e},\mathbf{p}_1,\ldots,\mathbf{p}_N)} = \lim_{\epsilon \to 0} \frac{A_i^+(\epsilon \cdot \boldsymbol{\eta}_v^i)|_{(\mathbf{e},\mathbf{p}_1,\ldots,\mathbf{p}_N)} - A}{\epsilon}, \end{cases} \tag{8}$$

where we denote $A(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_N)$ by A for brevity. From this expression, we have that the partial derivative of A with respect to $\mathbf{p}_i$ is given by

$$\frac{\partial A}{\partial \mathbf{p}_i} = D_h^i A \cdot \boldsymbol{\eta}_h^i + D_v^i A \cdot \boldsymbol{\eta}_v^i \tag{9}$$

Lemma 1: For any $i \in \mathcal{N}_e$, we have

$$D_h^i A = -\frac{L_i}{2}$$
$$D_v^i A = \frac{l_i^2 - (L_i - l_i)^2}{2||\boldsymbol{\xi}_i||}$$

where L_i is the length of the line of control B_i and l_i is the length of the segment of B_i below the intersection of the displacement segment $\boldsymbol{\xi}_i$ with B_i, as shown in Figure 2.

Proof: **Perturbation along $\boldsymbol{\eta}_h^i$:** A perturbation ϵ in the pursuer's position toward the evader moves the line of control $\frac{\epsilon}{2}$ toward the evader, and generates a corresponding change in the area of the evader's Voronoi cell δA_h^i, as shown in Figure 2a. This change in area is

$$\delta A_h^i = -\frac{L_i \epsilon}{2} + O(\epsilon^2)$$

where the $O(\epsilon^2)$ term depends on the angle of intersection between B_i and the boundaries of the Voronoi cell V_e. From this expression, the directional derivative of A along η_h^i can be calculated as

$$D_h^i A = \lim_{\epsilon \to 0} \frac{\delta A_h^i}{\epsilon} = -\frac{L_i}{2}.$$

Perturbation along $\boldsymbol{\eta}_v^i$: There are two different scenarios for perturbation along $\boldsymbol{\eta}_v^i$, corresponding to the two pursuer configurations shown in Figure 1b. In one case, as for $\mathbf{p}_2$ in Figure 1b, $\boldsymbol{\xi}_i$ intersects B_i. A perturbation of ϵ, as shown in Figure 2b, will cause the evader's Voronoi cell to shrink above the new midline by $\delta A_{v,1}^i$ and grow below it by $\delta A_{v,2}^i$. Let $\delta A_v^i = \delta A_{v,2}^i - \delta A_{v,1}^i$, with

$$\delta A_{v,1}^i = \frac{1}{2}((L_i - l_i) - \frac{\epsilon}{2})^2 \tan\theta + O(\epsilon^2)$$
$$\delta A_{v,2}^i = \frac{1}{2}(l_i + \frac{\epsilon}{2})^2 \tan\theta + O(\epsilon^2)$$

where the terms $O(\epsilon^2)$ again depend on the angle of intersection between B_i and the boundaries of the Voronoi cell V_e. Thus, the resulting changes in area will be

$$\delta A_{v,1}^i = \frac{(L_i - l_i)^2 \epsilon}{2||\boldsymbol{\xi}_i||} + O(\epsilon^2)$$
$$\delta A_{v,2}^i = \frac{l_i^2 \epsilon}{2||\boldsymbol{\xi}_i||} + O(\epsilon^2)$$

which implies

$$D_v^i A = \lim_{\epsilon \to 0} \frac{\delta A_v^i}{\epsilon} = \frac{l_i^2 - (L_i - l_i)^2}{2||\boldsymbol{\xi}_i||}$$

The second case is that of $\mathbf{p}_1$ in Figure 1b, where $\boldsymbol{\xi}_i$ no longer intersects B_i due to the presence of other pursuers.

As shown in Figure 2c, we have

$$\delta A_v^i = \delta A_{v,2}^i + \delta A_{v,3}^i,$$

where $\delta A_{v,2}^i$ is calculated as before and

$$\delta A_{v,3}^i = \frac{1}{2}(l_i - L_i + \frac{\epsilon}{2})^2 \tan\theta + O(\epsilon^2).$$

Letting $\epsilon \to 0$ we again have

$$D_v^i A = \lim_{\epsilon \to 0} \frac{\delta A_v^i}{\epsilon} = \frac{l_i^2 - (l_i - L_i)^2}{2||\boldsymbol{\xi}_i||}.$$

■

With the above lemma, the proposed strategy μ_i^* can be rewritten in the local coordinate system as

$$\mu_i^* = -\left(\frac{\alpha_h^i}{\sqrt{|\alpha_h^i|^2 + |\alpha_v^i|^2}} \cdot \boldsymbol{\eta}_h^i + \frac{\alpha_v^i}{\sqrt{|\alpha_h^i|^2 + |\alpha_v^i|^2}} \boldsymbol{\eta}_v^i \right), \tag{10}$$

where α_h^i and α_v^i are given by

$$\alpha_h^i = -\frac{L_i}{2},\ \alpha_v^i = \frac{l_i^2 - (L_i - l_i)^2}{2||\boldsymbol{\xi}_i||}.$$

Lemma 2: It can be shown that under this choice of pursuit strategy, $\mathbf{u}_i$ always points toward the interior of D, thus satisfying the constraint (3). The proof is straightforward but requires some amount of algebra and is omitted.

IV. Proof of Guaranteed Capture

The goal of this section is to show that the proposed pursuit strategy $\{\mu_i^*\}_{i \le N}$ is guaranteed to capture the evader within finite time, regardless of any admissible evader input $\mathbf{u}_e$. It can be seen that if this holds for the case of a single pursuer ($N = 1$), then the conclusion also extends to the case of multiple pursuers ($N > 1$). Indeed, for the case of $N > 1$, one can choose any pursuer i which is a Voronoi neighbor of the evader and use the arguments for the case of $N = 1$ to show that the capture condition will be satisfied. Thus, we will focus on the proof for a single pursuer. Correspondingly, the notation from Section III will carry through without the indices i.

First, we make the observation that if A approaches zero, the evader's Voronoi cell approaches either a line or a point. Either of the two cases clearly implies $\|\mathbf{e} - \mathbf{p}\| \to 0$. Our strategy here is then to show that, under the proposed pursuit strategy μ^* and any admissible evader control input $\mathbf{u}_e$, the area A is guaranteed to monotonically decrease until the capture condition is met.

In terms of preliminaries, we have by Lemma 1 and equation (9) that

$$\frac{\partial A}{\partial \mathbf{p}} = \alpha_h \boldsymbol{\eta}_h + \alpha_v \boldsymbol{\eta}_v.$$

It can be also verified in a similar manner as the proof of Lemma 1 that the partial derivative $\frac{\partial A}{\partial e}$ in the local coordinate system is given by

$$\frac{\partial A}{\partial \mathbf{e}} = \alpha_h \boldsymbol{\eta}_h - \alpha_v \boldsymbol{\eta}_v. \tag{11}$$

Also recall that the variable L in the statement of Lemma 1 depends on the spatial locations of the evader and the pursuer, as well as the geometry of the region D. For our proof, we will need the following definitions of parameters $l_{\min}$ and $l_{\max}$, which depend solely on the geometry of D.

$$\begin{cases} l_{\min} = \inf_{\mathbf{e} \in D, \mathbf{p} \in D} L(\mathbf{e}, \mathbf{p}) \\ l_{\max} = \sup_{\mathbf{e} \in D, \mathbf{p} \in D} ||\mathbf{e} - \mathbf{p}||_2. \end{cases} \tag{12}$$

Since D is the interior of a bounded polygon, we have that $l_{\min} > 0$ and $l_{\max} < \infty$.

The following result shows that the area A is always non-increasing under the pursuit strategy μ^* for a single pursuer.

Lemma 3: Under the proposed pursuit strategy $\mu^*(\mathbf{e}, \mathbf{p})$, the area A satisfies $\frac{dA}{dt} \le 0$ for any admissible evader control input. Furthermore, $\frac{dA}{dt} = 0$ if and only if the evader follows the following strategy:

$$\nu^*(\mathbf{e}, \mathbf{p}) = \frac{\alpha_h \boldsymbol{\eta}_h - \alpha_v \boldsymbol{\eta}_v}{\sqrt{\alpha_h^2 + \alpha_v^2}}. \tag{13}$$

Proof: For an arbitrary $\mathbf{u}_e$ with $\|\mathbf{u}_e\| \le 1$, we have

$$\begin{aligned} \frac{dA}{dt} =& \frac{\partial A}{\partial p} \mu^*(\mathbf{e}, \mathbf{p}) + \frac{\partial A}{\partial e} \mathbf{u}_e \\ =& -\sqrt{\alpha_h^2 + \alpha_v^2} + (\alpha_h \boldsymbol{\eta}_h - \alpha_v \boldsymbol{\eta}_v)^T \mathbf{u}_e \le 0, \end{aligned}$$

where equality holds if and only if $\mathbf{u}_e(t) = \nu^*(\mathbf{e}(t), \mathbf{p}(t))$.

■

In order to prove that the capture condition is achieved in finite time, we will proceed to show that the distance between the pursuer and the evader is strictly decreasing whenever the area A is constant. For this, we define

$$z(\mathbf{e}, \mathbf{p}) = \|\xi(\mathbf{e}, \mathbf{p})\|^2 = (\mathbf{e} - \mathbf{p})^T (\mathbf{e} - \mathbf{p}).$$

Clearly, the variable z is the squared Euclidean distance between the evader and pursuer. From the preceding discussions, the range of z lies in $[r_c^2,\ l_{\max}^2]$. In the following result, we show that $\dot{z} < 0$ whenever $\dot{A} = 0$.

Lemma 4: If $\dot{A} = 0$, then under the pursuit strategy μ^*, the following holds:

$$\frac{dz}{dt} = -\frac{4z}{\sqrt{z + (l_a - l_b)^2}} \le \frac{-4r_c^2}{\sqrt{r_c^2 + l_{\max}^2}}.$$

Proof: By Lemma 3, $\dot{A}(t) = 0$ if and only if $\mathbf{u}_e(t) = \nu^*(\mathbf{e}(t), \mathbf{p}(t))$. Thus, if the pursuer input is selected according to the strategy μ^*, then whenever $\dot{A} = 0$, we have

$$\begin{aligned} \dot{z} =& 2(\mathbf{e} - \mathbf{p})^T (\dot{\mathbf{e}} - \dot{\mathbf{p}}) \\ =& 4\boldsymbol{\xi}^T \left(\frac{\alpha_h}{\sqrt{\alpha_h^2 + \alpha_v^2}} \boldsymbol{\eta}_h \right) \\ =& \frac{-2L\|\boldsymbol{\xi}\|}{\sqrt{\frac{L^2}{4} + \frac{(l^2 - (L-l)^2)^2}{4\|\boldsymbol{\xi}\|^2}}} \\ =& -\frac{4z}{\sqrt{z + (2l - L)^2}} \\ \le& \frac{-4r_c^2}{\sqrt{r_c^2 + l_{\max}^2}}, \end{aligned}$$

where the second equality follows from the fact that $\boldsymbol{\xi}^T \boldsymbol{\eta}_v = 0$, and the last inequality follows from the monotonicity of the function $\frac{4z}{\sqrt{z+(2l-L)^2}}$ for $z \geq 0$. ■

By this result, we have that z is strictly decreasing whenever the area A remains constant. However, there remains the possibility that z is increasing on time intervals where A is strictly decreasing. The question then becomes whether there exists a evader control that can keep z inside $[r_c^2,\ l_{\max}^2]$ while preventing A from reaching 0. In the following result, we will prove that this is not the case, by exploiting certain properties of the proposed pursuit strategy.

Lemma 5: Under the pursuit strategy μ^*, if $\dot{A} \geq -\beta$ for some positive constant $\beta > 0$, then $\dot{z} \leq -c(\beta)$, where the bound $c(\beta)$ is given by

$$c(\beta) = \frac{\sqrt{2} r_c^2}{l_{\max}} - \frac{4 l_{\max}}{l_{\min}} \beta.$$

Proof: Under strategy μ^*, the following identities hold

$$\begin{cases} \dot{A} = -\sqrt{\alpha_h^2 + \alpha_v^2} + (\alpha_h \boldsymbol{\eta}_h - \alpha_v \boldsymbol{\eta}_v)^T \mathbf{u}_e \\ \dot{z} = 2(\mathbf{e} - \mathbf{p})^T \mathbf{u}_e - \frac{2z}{\sqrt{z+(2l-L)^2}} \end{cases}$$

Now suppose $\dot{A} \geq -\beta$. Using the relations $\boldsymbol{\eta}_h = \frac{\mathbf{e}-\mathbf{p}}{\|\mathbf{e}-\mathbf{p}\|}$, $\alpha_h = -\frac{L}{2}$, and $\alpha_v \boldsymbol{\eta}_v^T \mathbf{u}_e \geq -|\alpha_v|$, we have

$$\begin{aligned} (\mathbf{e}-\mathbf{p})^T \mathbf{u}_e &\leq -\frac{2\|\mathbf{e}-\mathbf{p}\|}{L} \left[\sqrt{\alpha_h^2 + \alpha_v^2} - \beta - |\alpha_v| \right] \\ &\leq \frac{2\|\mathbf{e}-\mathbf{p}\|\beta}{L} \leq \frac{2 l_{\max}}{l_{\min}} \beta, \end{aligned}$$

which implies that

$$\dot{z} \leq \frac{4 l_{\max}}{l_{\min}} \beta - \frac{\sqrt{2} r_c^2}{l_{\max}}.$$

■

Notice that by the contrapositive of this lemma, we have $\dot{A} < -\beta$ whenever $\dot{z} > -c(\beta)$. For the rest of this section, it is assumed that the β parameter in Lemma 5 is chosen such that $c(\beta) > 0$. In the following, we combine the previous results in this section to show that under μ^*, the area A keeps decreasing until the capture condition is met.

Theorem 1: Under the pursuit strategy μ^*, if the capture condition has not been achieved before time $t > 0$, then

$$A(t) \leq A(0) + \frac{\beta(l_{\max}^2 - r_c^2)}{4 l_{\max} + c(\beta)} - \frac{\beta c(\beta)}{4 l_{\max} + c(\beta)} \cdot t,$$

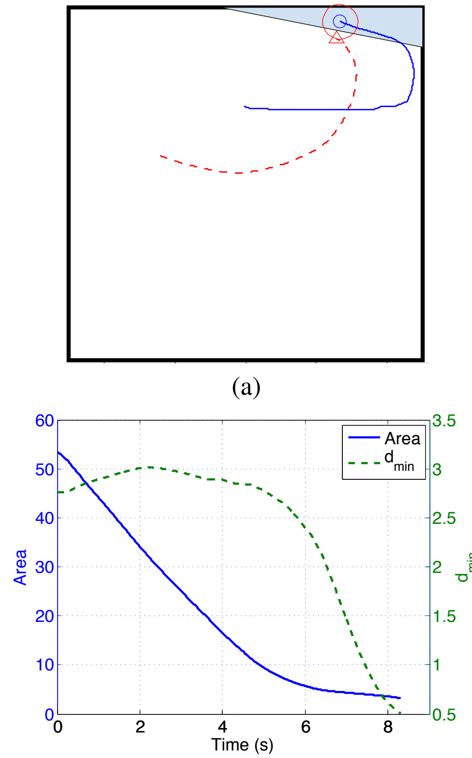
where $A(t) = A(\mathbf{e}(t), \mathbf{p}(t))$ denotes the area of the evader's Voronoi cell at time t.

Proof: Let $t > 0$ be an arbitrary time before the capture moment. Define

$$\begin{cases} \tau_-(t) = |\{s \in [0,t] : \dot{z}(s) \leq -c(\beta)\}| \\ \tau_+(t) = |\{s \in [0,t] : \dot{z}(s) > -c(\beta)\}|. \end{cases}$$

The operator $|\cdot|$ in the above equations specifically denotes the Lebesgue measure of a subset of $\mathbb{R}$. Clearly, $t = \tau_+(t) + \tau_-(t)$. Notice that, regardless of the choice of inputs by the evader and the pursuer, we always have

$$\dot{z} \leq 2\|\xi\|(\mathbf{u}_e - \mathbf{u}_p) \leq 4 l_{\max}.$$



(a)

(b)

Fig. 3. **Simulation results for a single pursuer (triangle, dashed line) and evader (circle, solid line), showing (a) the trajectories and (b) the area of V_e and d_{min} over time.**

Thus, the actual trajectory $z(t)$ must be bounded from above by $z(t) \leq Z^+(t)$, where

$$\begin{aligned} Z^+(t) &= l_{\max}^2 + 4 l_{\max} \cdot \tau_+(t) - c(\beta) \cdot \tau_-(t) \\ &= l_{\max}^2 + (4 l_{\max} + c(\beta)) \cdot \tau_+(t) - c(\beta) \cdot t \end{aligned}$$

Since $Z^+(t) \geq z(t) \geq r_c^2$, we have

$$\tau_+(t) \geq \frac{1}{4 l_{\max} + c(\beta)} \left[c(\beta) \cdot t + r_c^2 - l_{\max}^2 \right]. \quad (14)$$

By Lemma 5, A decreases at a rate faster than $-\beta$ whenever $\dot{z} > -c(\beta)$. This implies that $A(t) \leq A(0) - \beta \tau_+(t)$. Combining this with the inequality in (14), we have the statement of the theorem. ■

Remark 1: Although the area A may stay constant on certain time intervals, the upper bound of A will keep decreasing at a speed no slower than $\frac{\beta c(\beta)}{4 l_{\max} + c(\beta)}$. Thus the capture condition is guaranteed to be satisfied in finite time. Notice that depending on the evader's control inputs it may also be possible that capture is achieved before A becomes sufficiently small.

V. RESULTS

Simulation results are presented here for games involving different numbers of pursuers. The simulations are conducted in a 10 x 10 square, with a maximum speed of 1 for all players, and time steps of 0.01. In these simulations the trajectory of the evader is controlled by human input, and pursuers that do not have a line of control bordering on the

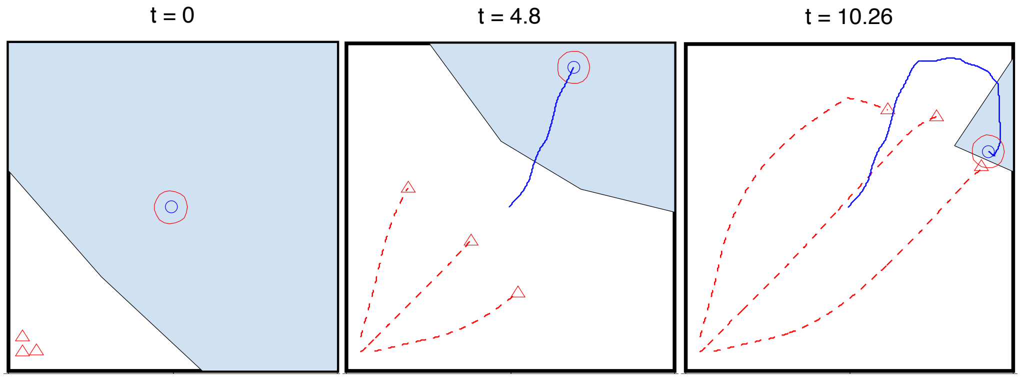


Fig. 4. **A scenario with three pursuers (triangles, dotted lines) highlighting the cooperation enforced by the area-minimizing pursuit strategy when the pursuers begin tightly spaced .**

evader Voronoi region are commanded to head straight for the evader.

Example trajectories for a game with a single pursuer are shown in Figures 3. The trade-off between area and distance is especially clear in this example. Note that initially the pursuer does not move directly toward the evader and thus the distance between the players does not decrease, but the area decreases very quickly. Near the end of the game the area decreases slowly while the distance decreases very quickly.

Figure 4 shows a simulated case with 3 pursuers, and highlights the cooperation in the area-minimization strategy. The pursuers begin closely grouped, but as they move the pursuers gradually separate to surround the evader. The cooperative behavior effectively contains the evader, limiting its movements until capture is achieved.

The simulations are conducted in Matlab on a Macbook Pro laptop, with computation per time-step of less than 1ms to calculate inputs for all the pursuers. Note that some small errors are introduced by discretization of the control scheme when distances between the evader and pursuers are comparable to the distance traveled by a player in a single time step. Reducing the time step alleviates the problem without eliminating it entirely, and increasing the capture radius also decreases the chance of this problem occurring. It is possible that some relationship can be found between step size, velocity, and the capture radius to formally guarantee this in a discrete-time situation.

VI. Conclusions & Future Work

In this work, we have presented a pursuit-evasion game with multiple pursuers and a single evader in bounded, convex polygons. We have proposed a decentralized pursuit strategy based on minimizing the area of the evader's Voronoi cell, and proven that this strategy guarantees capture of the evader in finite time. In the future, we hope to extend this work in several directions, including games with multiple evaders as well as in non-convex domains with obstacles.

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