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Guaranteed Transient Performance with $\mathcal{L}_1$ Adaptive Controller for Systems with Unknown Time-varying Parameters and Bounded Disturbances: Part I

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Abstract— **This paper presents a novel adaptive control methodology for uncertain systems with time-varying unknown parameters and time-varying bounded disturbances. The adaptive controller ensures uniformly bounded transient and asymptotic tracking for system's both signals, input and output, simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate. Simulations of a robotic arm with time-varying friction verify the theoretical findings.**

I. Introduction

This paper extends the $\mathcal{L}_1$ adaptive control architecture from [20], [21] to uncertain systems in the presence of unknown high-frequency gain, time-varying unknown parameters and time-varying bounded disturbances without restricting the rate of their variation. The redefined $\mathcal{L}_1$ adaptive control architecture achieves the desired transient performance for system's both signals, input and output, simultaneously and has guaranteed time-delay and gain margins.

Adaptive algorithms achieving arbitrarily improved transient performance in case of constant unknown parameters are given in [1]–[12], and for unknown time-varying parameters have been given in [13], [14]. While the results in [13], [14] improved upon [15]–[17], by extending the class of systems beyond the slow time-variation of the unknown parameters and guaranteeing performance improvement to arbitrary degree, they still did not provide means for regulating the frequency spectrum of the control signal during the transient. Following [18], subject to appropriate trajectory initialization, the following bound $||e||_\infty \leq V(t) \leq V(0) \leq \frac{\tilde{\theta}^2(0)}{\Gamma}$, where e is the tracking error, $\tilde{\theta}$ is the parametric error, $V(t)$ is the positive definite Lyapunov function with negative semidefinite derivative, implies that increasing the adaptation gain Γ leads to smaller tracking error for all $t \geq 0$, including the transient phase. However, large adaptive gain leads to high frequencies in the control signal, implying that the improvement in the transient tracking of the system output is achieved at the price of unacceptable high frequencies in the system input. One can observe from the open-loop transfer function analysis for a PI controller (which is a MRAC-structure controller for a linear system with constant disturbance) that increasing the adaptation gain leads to reduced phase-margin, and consequently reduced time-delay tolerance in input/output channels [19]. On the contrary, decreasing the adaptive gain leads to large deviations from the desired trajectory during the transient phase.

In [20], [21], we have developed a new architecture for control of uncertain systems, named $\mathcal{L}_1$ adaptive controller, which permits fast adaptation and yields guaranteed transient response for system's both signals, input and output, simultaneously, in addition to asymptotic tracking. In this paper, we extend the results of [20], [21] to systems in the presence of unknown high-frequency gain, time-varying unknown parameters and time-varying bounded disturbances without limiting the rate of their variation. By modifying the architecture correspondingly, we prove that the $\mathcal{L}_1$ adaptive controller ensures uniformly bounded transient response for system's both signals, input and output, simultaneously, in addition to stable tracking. The $\mathcal{L}_\infty$ norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation rate. In Part II of this paper, we quantify the stability margins of this controller [22].

The paper is organized as follows. Section II states some preliminary definitions, and Section III gives the problem formulation. In Section IV, the novel $\mathcal{L}_1$ adaptive control architecture is presented. Stability and uniform transient tracking bounds of the $\mathcal{L}_1$ adaptive controller are presented in Section V. In section VI, simulation results are presented, while Section VII concludes the paper.

II. Preliminaries

In this Section, we recall basic definitions and facts from linear systems theory, [25], [26].

Definition 1: For a signal $\xi(t),\ t \geq 0$, its truncated $\mathcal{L}_\infty$ norm and $\mathcal{L}_\infty$ norm are $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,..,n} (\sup_{0\leq\tau\leq t} |\xi_i(\tau)|)$, $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,..,n} (\sup_{\tau\geq 0} |\xi_i(\tau)|)$, where ξ_i is the i^{th} component of $\xi \in \mathbb{R}^n$.

Definition 2: The $\mathcal{L}_1$ gain of a stable proper SISO system is defined $||H(s)||_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$, where $h(t)$ is the impulse response of $H(s)$.

Definition 3: For a stable proper m input n output system $H(s)$ its $\mathcal{L}_1$ gain is defined as $\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,..,n}(\sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1})$, where $H_{ij}(s)$ is the corresponding entry of $H(s)$.

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Lemma 1: For a stable proper multi-input multi-output (MIMO) system $H(s)$ with input $r(t) \in \mathbb{R}^m$ and output $x(t) \in \mathbb{R}^n$, we have $\|x_t\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}, \quad \forall\, t \geq 0$.

Corollary 1: For a stable proper MIMO system $H(s)$, if the input $r(t) \in \mathbb{R}^m$ is bounded, then the output $x(t) \in \mathbb{R}^n$ is also bounded, and $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}$.

Lemma 2: For a cascaded system $H(s) = H_2(s)H_1(s)$, where $H_1(s)$ and $H_2(s)$ are stable proper systems, we have $\|H(s)\|_{\mathcal{L}_1} \leq \|H_2(s)\|_{\mathcal{L}_1} \|H_1(s)\|_{\mathcal{L}_1}$.

Theorem 1: ([25], Theorem 5.6)($\mathcal{L}_1$ **Small Gain Theorem**) The interconnected system $w_2(s) = \Delta(s)(w_1(s) - M(s)w_2(s))$ with input $w_1(t)$ and output $w_2(t)$ is stable if $\|M(s)\|_{\mathcal{L}_1} \|\Delta(s)\|_{\mathcal{L}_1} < 1$.

Consider a LTI system: $x(s) = (s\mathbb{I} - A)^{-1} b u(s)$, where $A \in \mathbb{R}^{n\times n}$ is Hurwitz, and assume that $(s\mathbb{I} - A)^{-1} b = n(s)/N_d(s)$ is strictly proper and stable, where $N_d(s) = \det(s\mathbb{I} - A)$, and $n(s)$ is a $n \times 1$ vector with its i^{th} element being a polynomial function $n_i(s) = \sum_{j=1}^{n} n_{ij} s^{j-1}$. The proofs of the next two lemmas can be found in [20], [21].

Lemma 3: If $(A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n)$ is controllable, the matrix N of entries n_{ij} is full rank.

Lemma 4: If (A, b) is controllable and $(s\mathbb{I} - A)^{-1} b$ is strictly proper and stable, there exists $c \in \mathbb{R}^n$ such that $c^\top (s\mathbb{I} - A)^{-1} b$ is minimum phase with relative degree one.

III. Problem Formulation

Consider the following system dynamics:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b\left(\omega u(t) + \theta^\top(t) x(t) + \sigma(t)\right), \\ y(t) &= c^\top x(t), \quad x(0) = x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $y \in \mathbb{R}$ is the regulated output, $b, c \in \mathbb{R}^n$ are known constant vectors, A_m is a known $n\times n$ matrix, $\omega \in \mathbb{R}$ is an unknown constant with known sign, $\theta(t) \in \mathbb{R}^n$ is a vector of time-varying unknown parameters, while $\sigma(t) \in \mathbb{R}$ is a time-varying disturbance. Without loss of generality, we assume that

$$\omega \in \Omega = [\omega_l,\ \omega_u],\, \theta(t) \in \Theta,\ |\sigma(t)| \leq \Delta, \quad t \geq 0, \tag{2}$$

where $\omega_u > \omega_l > 0$ are given bounds, Θ is known compact set, and $\Delta \in \mathbb{R}^+$ is a known (conservative) $\mathcal{L}_\infty$ bound of $\sigma(t)$. We further assume that $\theta(t)$ and $\sigma(t)$ are continuously differentiable and their derivatives are uniformly bounded:

$$\|\dot{\theta}(t)\|_2 \leq d_\theta < \infty, \quad |\dot{\sigma}(t)| \leq d_\sigma < \infty, \quad \forall\, t \geq 0, \tag{3}$$

where $\|\cdot\|_2$ denotes the 2-norm of a vector, while the numbers d_θ, d_σ can be arbitrarily large.

The control objective is to design a full-state feedback adaptive controller to ensure that $y(t)$ tracks a given bounded reference signal $r(t)$ *both in transient and steady state*, while all other error signals remain bounded.

IV. $\mathcal{L}_1$ Adaptive Controller

In this section, we develop a novel adaptive control architecture for the system in (1) that permits complete transient characterization for both $u(t)$ and $x(t)$. The elements of $\mathcal{L}_1$ adaptive controller are introduced next:

State Predictor: We consider the following state predictor:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_m \hat{x}(t) + b\left(\hat{\omega}(t) u(t) + \hat{\theta}^\top(t) x(t) + \hat{\sigma}(t)\right), \\ \hat{y}(t) &= c^\top \hat{x}(t), \quad \hat{x}(0) = x_0, \end{aligned} \tag{4}$$

which has the same structure as the system in (1). The only difference is that the unknown parameters $\omega, \theta(t), \sigma(t)$ are replaced by their adaptive estimates $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$ that are governed by the following adaptation laws.

Adaptive Laws: Adaptive estimates are given by:

$$\dot{\hat{\theta}}(t) = \Gamma_\theta \mathrm{Proj}(\hat{\theta}(t), -x(t)\tilde{x}^\top(t)Pb),\ \hat{\theta}(0) = \hat{\theta}_0, \tag{5}$$

$$\dot{\hat{\sigma}}(t) = \Gamma_\sigma \mathrm{Proj}(\hat{\sigma}(t), -\tilde{x}^\top(t)Pb), \qquad \hat{\sigma}(0) = \hat{\sigma}_0, \tag{6}$$

$$\dot{\hat{\omega}}(t) = \Gamma_\omega \mathrm{Proj}(\hat{\omega}(t), -\tilde{x}^\top(t)Pbu(t)), \hat{\omega}(0) = \hat{\omega}_0, \tag{7}$$

where $\tilde{x}(t) = \hat{x}(t) - x(t)$, $\Gamma_\theta = \Gamma_c \mathbb{I}_{n\times n} \in \mathbb{R}^{n\times n}$, $\Gamma_\sigma = \Gamma_\omega = \Gamma_c > 0$, and $P = P^\top > 0$ is the solution of the algebraic equation $A_m^\top P + PA_m = -Q$, $Q > 0$.

Control Law: The control signal is generated through gain feedback of the following system:

$$\chi(s) = D(s) r_u(s),\, u(s) = -k\chi(s), \tag{8}$$

where $k > 0$ is a feedback gain, $r_u(s)$ is the Laplace transformation of $r_u(t) = \hat{\omega}(t)u(t) + \bar{r}(t)$, $\bar{r}(t) = \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t) - k_g r(t)$, $k_g = -1/(c^\top A_m^{-1} b)$, while $D(s)$ is any transfer function that leads to strictly proper stable

$$C(s) = \omega k D(s)/(1 + \omega k D(s)) \tag{9}$$

with low-pass gain $C(0) = 1$. One simple choice is $D(s) = 1/s$, which yields a first order strictly proper $C(s)$ in the following form: $C(s) = \omega k/(s + \omega k)$. Further, let

$$L = \max_{\theta(t)\in\Theta} \sum_{i=1}^{n} |\theta_i(t)|, \tag{10}$$

where $\theta_i(t)$ is the i^{th} element of $\theta(t)$, Θ is the compact set defined in (2).

The $\mathcal{L}_1$ adaptive controller consists of (4), (5)-(7), (8) subject to the following $\mathcal{L}_1$-gain bound:

$\mathcal{L}_1$-gain stability requirement: Design $D(s)$ to ensure that

$$\|G(s)\|_{\mathcal{L}_1} L < 1, \quad G(s) = (sI - A_m)^{-1} b(1 - C(s)). \tag{11}$$

In case of constant $\theta(t)$ and $\sigma(t)$, the stability requirement of the $\mathcal{L}_1$ adaptive controller can be simplified. For the specific choice of $D(s) = 1/s$, the stability requirement of $\mathcal{L}_1$ adaptive controller is reduced to

$$A_g = \begin{bmatrix} A_m + b\theta^\top & b\omega \\ -k\theta^\top & -k\omega \end{bmatrix} \tag{12}$$

being Hurwitz for all $\theta \in \Theta$, $\omega \in \Omega$.

V. Analysis of $\mathcal{L}_1$ Adaptive Controller

A. Closed-loop Reference System

We now consider the following closed-loop LTI reference system with its control signal and system response being defined as follows:

$$\dot{x}_{ref}(t) = A_m x_{ref}(t) + b\left(\omega u_{ref}(t) + \theta^\top(t) x_{ref}(t) + \sigma(t)\right), \quad (13)$$

$$u_{ref}(s) = C(s)\bar{r}_{ref}(s)/\omega, \quad x_{ref}(0) = x_0, \quad (14)$$

$$y_{ref}(t) = c^\top x_{ref}(t), \quad (15)$$

where $\bar{r}_{ref}(s)$ is the Laplace transformation of the signal $\bar{r}_{ref}(t) = -\theta^\top(t)x_{ref}(t) - \sigma(t) + k_g r(t)$.

Lemma 5: If $D(s)$ verifies the condition in (11), the closed-loop reference system in (13)-(15) is stable.

Proof. Let $H(s) = (s\mathbb{I} - A_m)^{-1}b$. It follows from (13)-(15) that $x_{ref}(s) = G(s)r_1(s) + H(s)C(s)k_g r(s)$, where $r_1(s)$ is the Laplace transformation of $r_1(t) = \theta^\top(t)x_{ref}(t) + \sigma(t)$ subject to the following bound: $\|r_1\|_{\mathcal{L}_\infty} \le L\|x_{ref}\|_{\mathcal{L}_\infty} + \|\sigma\|_{\mathcal{L}_\infty}$. Since $D(s)$ verifies the condition in (11), then Theorem 1 ensures that the closed-loop system in (13)-(15) is stable. □

Lemma 6: If $\theta(t)$ is constant, and $D(s) = 1/s$, then the closed-loop reference system in (13)-(15) is stable *iff* the matrix A_g in (12) is Hurwitz.

Proof. In case of constant $\theta(t)$, the state space form of the closed-loop system in (13)-(15) is given by: $\dot{x}_{ref}(t) = A_m x_{ref}(t) + b\left(\omega u_{ref}(t) + \theta^\top x_{ref}(t) + \sigma(t)\right)$, $\dot{u}_{ref}(t) = -\omega k u_{ref}(t) + k\left(-\theta^\top x_{ref}(t) - \sigma(t) + k_g r(t)\right)$. Letting $\zeta(t) = [x_{ref}(t)\ u_{ref}(t)]^\top$, it can be rewritten as $\dot{\zeta}(t) = A_g\zeta(t) + [b\sigma(t) \quad kk_g r(t) - k\sigma(t)]^\top$, which is stable iff A_g is Hurwitz. □

B. Transient and Steady State Performance

To prove uniform transient and steady state tracking between the closed-loop adaptive system with $\mathcal{L}_1$ adaptive controller (1), (4), (5)-(7), (8) and the reference system in (13)-(15), we first need to quantify the prediction error performance that is used in the adaptive law.

Lemma 7: For the system in (1) and the $\mathcal{L}_1$ adaptive controller in (4), (5)-(7) and (8), the prediction error between the system state and the predictor is bounded $\|\tilde{x}\|_{\mathcal{L}_\infty} \le \sqrt{\frac{\theta_m}{\lambda_{\min}(P)\Gamma_c}}$, where $\theta_m \triangleq \max_{\theta\in\Theta}\sum_{i=1}^n 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2 + 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\left(\max_{\theta\in\Theta}\|\theta\|d_\theta + d_\sigma\Delta\right)$.

A proof is given in Appendix. We further notice that this bound is proportional to the rate of variation of uncertainties and is inverse proportional to the adaptation gain.

Recalling that $H(s) = (s\mathbb{I} - A_m)^{-1}b$, it follows from Lemma 4 that there exists $c_o \in \mathbb{R}^n$ s.t.

$$c_o^\top H(s) = N_n(s)/N_d(s), \quad (16)$$

where $\deg(N_d(s)) - \deg(N_n(s)) = 1$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials. The next two theorems are in charge for both transient and steady-state performance of $\mathcal{L}_1$ adaptive controller and are proved in Appendix.

Theorem 2: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (4), (5)-(7) and (8) subject to (11), we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \le \gamma_1, \quad \|u - u_{ref}\|_{\mathcal{L}_\infty} \le \gamma_2, \quad (17)$$

where $\gamma_1 = \frac{\|C(s)\|_{\mathcal{L}_1}}{1-\|H(s)(1-C(s))\|_{\mathcal{L}_1}L}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$, $\gamma_2 = \left\|\frac{C(s)}{\omega}\right\|_{\mathcal{L}_1} L\gamma_1 + \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$.

Theorem 3: For the closed-loop system in (1) with $\mathcal{L}_1$ adaptive controller defined via (4), (5)-(7) and (8), subject to (12), if $\theta(t)$ is (unknown) constant and $D(s) = \frac{1}{s}$, we have:

$$\|x - x_{ref}\|_{\mathcal{L}_\infty} \le \gamma_3, \quad \|u - u_{ref}\|_{\mathcal{L}_\infty} \le \gamma_4, \quad (18)$$

where $\gamma_3 = \left\|H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$, $H_g(s) = (s\mathbb{I} - A_g)\begin{bmatrix} b \\ 0 \end{bmatrix}$, $\gamma_4 = \left\|\frac{C(s)}{\omega}\theta^\top\right\|_{\mathcal{L}_1}\gamma_3 + \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$.

Corollary 2: Given the system in (1) and the $\mathcal{L}_1$ adaptive controller defined via (4), (5)-(7) and (8) subject to (11), for all $t \ge 0$ we have: $\lim_{\Gamma_c\to\infty}(x(t) - x_{ref}(t)) = 0$, $\lim_{\Gamma_c\to\infty}(u(t) - u_{ref}(t)) = 0$.

Thus, the tracking error between $x(t)$ and $x_{ref}(t)$, as well between $u(t)$ and $u_{ref}(t)$, is uniformly bounded by a constant inverse proportional to Γ_c. This implies that during the transient phase one can achieve arbitrarily close tracking performance for both signals simultaneously by increasing the adaptation rate Γ_c.

Remark 1: We notice that the above analysis assumes zero trajectory initialization error, i.e. $\hat{x}_0 = x_0$, which is in the spirit of the methods for transient performance improvement in [18]. In [23], we have proved that non-zero trajectory initialization error leads only to an exponentially decaying term in both system state and control signal, without affecting the performance throughout.

C. Asymptotic Convergence

Since the bounds in (17) are uniform for all $t \ge 0$, they are in charge for both transient and steady state performance. In case of constant θ and σ one can prove in addition the following asymptotic result.

Lemma 8: Given the system in (1) with constant θ, σ and $\mathcal{L}_1$ adaptive controller defined via (4), (5)-(7) and (8) subject to (11), we have: $\lim_{t\to\infty}\tilde{x}(t) = 0$.

Proof: It follows from Lemmas 5 and 7, and Theorem 2 that both $x(t)$ and $\hat{x}(t)$ in $\mathcal{L}_1$ adaptive controller are bounded for bounded reference inputs. The adaptive laws in (5)-(7) ensure that the estimates $\hat{\theta}(t)$, $\hat{\omega}(t)$, $\hat{\sigma}(t)$ are also bounded. Hence, it can be checked easily from (21) that $\dot{\tilde{x}}(t)$ is bounded, and it follows from Barbalat's lemma that $\lim_{t\to\infty}\tilde{x}(t) = 0$. □

D. Design Guidelines

We note that the control law $u_{ref}(t)$ in the closed-loop reference system, which is used in the analysis of $\mathcal{L}_\infty$ norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 2 ensures that the $\mathcal{L}_1$

adaptive controller approximates $u_{ref}(t)$ both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with *desired* specifications. We notice that the following *ideal* control signal

$$u_{ideal}(t) = \frac{k_g r(t) - \theta^\top(t) x_{ideal}(t) - \sigma(t)}{\omega} \qquad (19)$$

is the one that leads to desired system response:

$$\dot{x}_{ideal}(t) = A_m x_{ideal}(t) + b k_g r(t), y_{ideal}(t) = c^\top x_{ideal}(t) \qquad (20)$$

by cancelling the uncertainties exactly. In the closed-loop reference system (13)-(15), $u_{ideal}(t)$ is further low-pass filtered by $C(s)$ to have guaranteed low-frequency range. Thus, the reference system in (13)-(15) has a different response as compared to (20) achieved with (19). In [21], specific design guidelines are suggested for selection of $C(s)$ to ensure that in case of constant θ and σ the response of $x_{ref}(t)$ and $u_{ref}(t)$ can be made as close as possible to (20).

VI. Simulations

Consider the dynamics of a single-link robot arm rotating on a vertical plane:

$$I\ddot{q}(t) + \frac{Mgl\cos q(t)}{2} + F(t)\dot{q}(t) + F_1(t)q(t) + \bar{\sigma}(t) = u(t)\,,$$

where $q(t)$ and $\dot{q}(t)$ are measured angular position and velocity, respectively, $u(t)$ is the input torque, I is the unknown moment of inertia, M is the unknown mass, l is the unknown length, $F(t)$ is an unknown time-varying friction coefficient, $F_1(t)$ is position dependent external torque, and $\bar{\sigma}(t)$ is unknown bounded disturbance. The control objective is to design $u(t)$ to achieve tracking of bounded reference input $r(t)$ by $q(t)$. Let $x = [q\ \dot{q}]^\top$. The system dynamics can be presented in the state-space form as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\Big(\frac{u(t)}{I} + \frac{Mgl\cos(x_1(t))}{2I} + \frac{\bar{\sigma}(t)}{I} \\ &\quad + \frac{F_1(t)}{I}x_1(t) + \frac{F(t)}{I}x_2(t)\Big), \quad x(0) = x_0\,, \\ y(t) &= c^\top x(t)\,, \end{aligned}$$

where x_0 is the initial condition, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

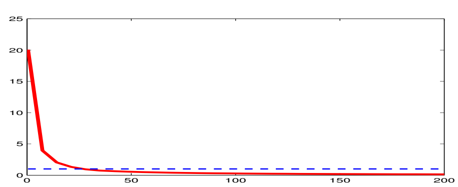
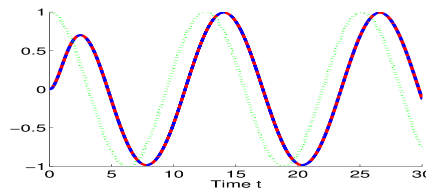
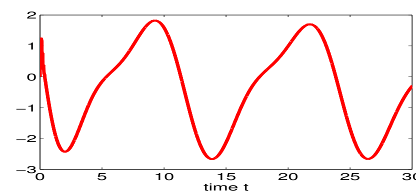


Fig. 1. $\|G(s)\|_{\mathcal{L}_1} L$ with respect to ωk.

The system can be further put into the form: $\dot{x}(t) = A_m x(t) + b(\omega u(t) + \theta^\top(t)x(t) + \sigma(t))$, $y(t) = c^\top x(t)$, where $\omega = 1/I$ is the unknown control effectiveness, $\theta(t) = [1 + \frac{F_1(t)}{I}\ \ 1.4 + \frac{F(t)}{I}]^\top$, $\sigma(t) = \frac{Mgl\cos(x_1(t))}{2I} + \frac{\bar{\sigma}(t)}{I}$, and $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$. Let the unknown control effectiveness, time-varying parameters and disturbance be given by: $\omega = 1$, $\theta(t) = [2 + \cos(\pi t)\quad 2 + 0.3\sin(\pi t) + 0.2\cos(2t)]^\top$, $\sigma(t) = \sin(\pi t)$, so that the compact sets can be conservatively chosen as $\Omega = [0.5,\ 2]$, $\Theta = [-10,\ 10]$, $\Delta = [-10,\ 10]$. For implementation of the $\mathcal{L}_1$

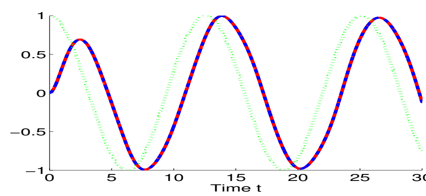


(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

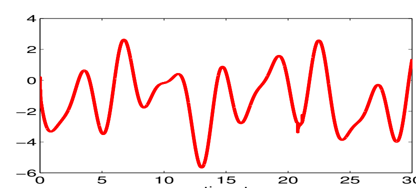


(b) Time-history of $u(t)$

Fig. 2. Performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \sin(\pi t)$



(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)

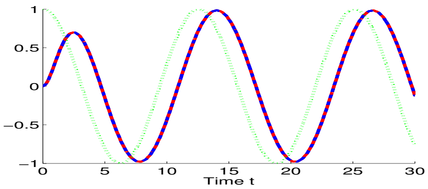


(b) Time-history of $u(t)$

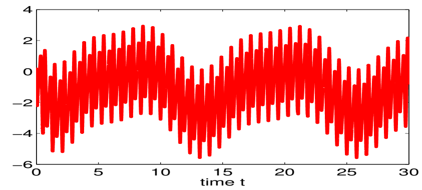
Fig. 3. Performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \cos(x_1(t)) + 2\sin(10t) + \cos(15t)$

adaptive controller (4), (5)-(7) and (8), we need to verify the $\mathcal{L}_1$ stability requirement in (11). Letting $D(s) = 1/s$, we have $G(s) = \frac{\omega k}{s+\omega k}H(s)$, $H(s) = [\frac{1}{s^2+1.4s+1}\ \ \frac{s}{s^2+1.4s+1}]^\top$. We can check that $L = 20$ in (10). In Fig. 1, we plot $\|G(s)\|_{\mathcal{L}_1} L$ as a function of ωk and compare it to 1. We notice that for $\omega k > 30$, we have $\|G(s)\|_{\mathcal{L}_1} L < 1$. Since $\omega > 0.5$, we set $k = 60$. We set the adaptive gain $\Gamma_c = 10000$.

The simulation results of $\mathcal{L}_1$ adaptive controller are shown



(a) $x_1(t)$ (solid), $\hat{x}_1(t)$ (dashed), and $r(t)$(dotted)



(b) Time-history of $u(t)$

Fig. 4. Performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \cos(x_1(t)) + 2\sin(100t) + \cos(150t)$

in Figures 2(a)-2(b) for the reference input $r = \cos(\pi t)$. Figures 3(a)-3(b) show the performance of $\mathcal{L}_1$ adaptive controller for $\sigma(t) = \cos(x_1(t)) + 2\sin(10t) + \cos(15t)$ without any retuning. Finally, we simulate much higher frequencies in the disturbance $\sigma(t) = \cos(x_1(t)) + 2\sin(100t) + \cos(150t)$ in 4(a)-4(b). We notice that $\mathcal{L}_1$ adaptive controller guarantees smooth and uniform transient performance in the presence of different unknown time-varying disturbances. The controller frequencies are exactly matched with the frequencies of the disturbance that it is supposed to cancel out. We also notice that $x_1(t)$ and $\hat{x}_1(t)$ are almost the same in Figs. 2(a), 3(a) and 4(a).

VII. Conclusion

A novel $\mathcal{L}_1$ adaptive control architecture is presented that has guaranteed transient response in addition to stable tracking for systems with time-varying unknown parameters and bounded disturbances. The control signal and the system response approximate the same signals of a closed-loop reference LTI system, which can be designed to achieve desired specifications. In Part II of this paper [22], we derive the stability margins of this $\mathcal{L}_1$ adaptive control architecture.

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Appendix

Proof of Lemma 7. Consider the candidate Lyapunov function: $V(\tilde{x}(t), \tilde{\theta}(t), \tilde{\omega}(t), \tilde{\sigma}(t)) = \tilde{x}^\top(t)P\tilde{x}(t) + \Gamma_c^{-1}\tilde{\theta}^\top(t)\tilde{\theta}(t) + \Gamma_c^{-1}\tilde{\omega}^2(t) + \Gamma_c^{-1}\tilde{\sigma}^2(t)$, where $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta(t)$, $\tilde{\sigma}(t) \triangleq \hat{\sigma}(t) - \sigma(t)$, $\tilde{\omega}(t) \triangleq \hat{\omega}(t) - \omega$. It follows from (1) and (4) that

$$\dot{\tilde{x}}(t) = A_m\tilde{x}(t) + b(\tilde{\omega}(t)u(t) + \tilde{\theta}^\top(t)x(t) + \tilde{\sigma}(t)), \quad \tilde{x}(0) = 0. \tag{21}$$

Using the projection based adaptation laws from (5)-(7), one has the following upper bound:

$$\dot{V}(t) \leq -\tilde{x}^\top(t)Q\tilde{x}(t) + 2\Gamma_c^{-1}\tilde{\theta}^\top(t)\dot{\theta}(t) + 2\Gamma_c^{-1}\tilde{\sigma}(t)\dot{\sigma}(t)\,. \tag{22}$$

The projection algorithm ensures that $\hat{\theta}(t) \in \Theta$, $\hat{\omega}(t) \in \Omega$, $\hat{\sigma}(t) \in \Delta$ for all $t \geq 0$, and therefore

$$\max_{t\geq 0}\left(\Gamma_c^{-1}\tilde{\theta}^\top(t)\tilde{\theta}(t) + \Gamma_c^{-1}\tilde{\omega}^2(t) + \Gamma_c^{-1}\tilde{\sigma}^2(t)\right) \leq \left(\max_{\theta\in\Theta}\sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2\right)/\Gamma_c \qquad (23)$$

for any $t \geq 0$. If $V(t) > \theta_m/\Gamma_c$ at some t, then it follows from (23) that $\tilde{x}^\top(t)P\tilde{x}(t) > 4\frac{\lambda_{\max}(P)}{\Gamma_c\lambda_{\min}(Q)}\left(\max_{\theta\in\Theta}\|\theta\|d_\theta + d_\sigma\Delta\right)$, and hence $\tilde{x}^\top(t)Q\tilde{x}(t) > \lambda_{\min}(Q)\tilde{x}^\top(t)P\tilde{x}(t)/\lambda_{\max}(P) > 4(\max_{\theta\in\Theta}\|\theta\|d_\theta + d_\sigma\Delta)/\Gamma_c$. The upper bounds in (3) along with the projection based adaptive laws lead to the following upper bound: $\tilde{\theta}^\top(t)\dot{\theta}(t) + \tilde{\sigma}(t)\dot{\sigma}(t) \leq 2\max_{\theta\in\Theta}\|\theta\|d_\theta + 2d_\sigma\Delta$. Hence, if $V(t) > \theta_m/\Gamma_c$, then from (22) we have

$$\dot{V}(t) < 0\,. \qquad (24)$$

Since we have set $\hat{x}(0) = x(0)$, we can verify that $V(0) \leq \left(\max_{\theta\in\Theta}\sum_{i=1}^{n} 4\theta_i^2 + 4\Delta^2 + 4\left(\omega_u - \omega_l\right)^2\right)/\Gamma_c < \theta_m/\Gamma_c$. It follows from (24) that $V(t) \leq \dfrac{\theta_m}{\Gamma_c}$ for any $t \geq 0$. Since $\lambda_{\min}(P)\|\tilde{x}(t)\|^2 \leq \tilde{x}^\top(t)P\tilde{x}(t) \leq V(t)$, then $||\tilde{x}(t)||^2 \leq \dfrac{\theta_m}{\lambda_{\min}(P)\Gamma_c}$, which concludes the proof. □

Proof of Theorem 2. Let $\tilde{r}(t) = \tilde{\omega}(t)u(t) + \tilde{\theta}^\top(t)x(t) + \tilde{\sigma}(t)$, $r_2(t) = \theta^\top(t)x(t) + \sigma(t)$. It follows from (8) that $\chi(s) = D(s)(\omega u(s) + r_2(s) - k_g r(s) + \tilde{r}(s))$, where $\tilde{r}(s)$ and $r_2(s)$ are the Laplace transformations of signals $\tilde{r}(t)$ and $r_2(t)$. Consequently

$$\chi(s) = \frac{D(s)}{1 + k\omega D(s)}(r_2(s) - k_g r(s) + \tilde{r}(s))\,,$$
$$u(s) = -\frac{kD(s)}{1 + k\omega D(s)}(r_2(s) - k_g r(s) + \tilde{r}(s))\,.$$

Using the definition of $C(s)$ from (9), we can write

$$\omega u(s) = -C(s)(r_2(s) - k_g r(s) + \tilde{r}(s))\,, \qquad (25)$$

and the system in (1) consequently takes the form:

$$x(s) = H(s)\left((1 - C(s))r_2(s) + C(s)k_g r(s) - C(s)\tilde{r}(s)\right). \qquad (26)$$

It follows from (13)-(15) that $x_{ref}(s) = H(s)\left((1 - C(s))r_1(s) + C(s)k_g r(s)\right)$, where $r_1(s)$ is the Laplace transformation of the signal $r_1(t)$. Let $e(t) = x(t) - x_{ref}(t)$. Then, using (26), one gets

$$e(s) = H(s)\left((1 - C(s))r_3(s) - C(s)\tilde{r}(s)\right), \quad e(0) = 0\,, \qquad (27)$$

where $r_3(s)$ is the Laplace transformation of the signal

$$r_3(t) = \theta^\top(t)e(t)\,. \qquad (28)$$

Lemma 7 gives the following upper bound:

$$\|e_t\|_{\mathcal{L}_\infty} \leq \|H(s)(1 - C(s))\|_{\mathcal{L}_1}\|r_{3_t}\|_{\mathcal{L}_\infty} + \|r_{4_t}\|_{\mathcal{L}_\infty}\,, \qquad (29)$$

where $r_4(t)$ is the signal with its Laplace transformation being $r_4(s) = C(s)H(s)\tilde{r}(s)$. From the relationship in (21) we have $\tilde{x}(s) = H(s)\tilde{r}(s)$, which leads to $r_4(s) = C(s)\tilde{x}(s)$, and hence $\|r_{4_t}\|_{\mathcal{L}_\infty} \leq \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. Using the definition of L in (10), one can verify easily that $\|(\theta^\top e)_t\|_{\mathcal{L}_\infty} \leq L\|e_t\|_{\mathcal{L}_\infty}$, and from (28) we have that $\|r_{3_t}\|_{\mathcal{L}_\infty} \leq L\|e_t\|_{\mathcal{L}_\infty}$. From (29) we have $\|e_t\|_{\mathcal{L}_\infty} \leq \|H(s)(1 - C(s))\|_{\mathcal{L}_1}L\|e_t\|_{\mathcal{L}_\infty} + \|C(s)\|_{\mathcal{L}_1}\|\tilde{x}_t\|_{\mathcal{L}_\infty}$. The upper bound from Lemma 7 and the $\mathcal{L}_1$-gain requirement from (11) lead to the following upper bound $\|e_t\|_{\mathcal{L}_\infty} \leq \frac{\|C(s)\|_{\mathcal{L}_1}}{1-\|H(s)(1-C(s))\|_{\mathcal{L}_1}L}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$, which holds uniformly for all $t \geq 0$ and therefore leads to the first bound in (17).

To prove the second bound in (17), we notice that from (15) and (25) one can derive $u(s) - u_{ref}(s) = -\frac{C(s)}{\omega}\theta^\top(t)(x(s) - x_{ref}(s)) - r_5(s)$, where $r_5(s) = \frac{C(s)}{\omega}\tilde{r}(s)$. Therefore, it follows from Lemma 7 that

$$\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq (L/\omega)\|C(s)\|_{\mathcal{L}_1}\|x - x_{ref}\|_{\mathcal{L}_\infty} + \|r_5\|_{\mathcal{L}_\infty}\,. \qquad (30)$$

We have $r_5(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top H(s)\tilde{r}(s) = \frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\tilde{x}(s)$, where c_o is introduced in (16). Using the polynomials from (16), we can write that $\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)} = \frac{C(s)}{\omega}\frac{N_d(s)}{N_n(s)}$. Since $C(s)$ is stable and strictly proper, the complete system $C(s)\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Hence, we have $\|r_5\|_{\mathcal{L}_\infty} \leq \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}$. Lemma 7 consequently leads to the upper bound: $\|r_5\|_{\mathcal{L}_\infty} \leq \left\|\frac{C(s)}{\omega}\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$, which, when substituted into (30), leads to the second bound in (17). □

Proof of Theorem 3: Let $\zeta(s) = -C(s)\theta^\top e(s)/\omega$. With this notation, (27) can be written as $e(s) = H(s)(\theta^\top e(s) + \omega\zeta(s) - C(s)\tilde{r}(s))$ and further put into state space form as:

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\zeta}(t) \end{bmatrix} = A_g \begin{bmatrix} e(t) \\ \zeta(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} r_6(t)\,, \qquad (31)$$

where $r_6(t)$ is the signal with its Laplace transformation $r_6(s) = -C(s)\tilde{r}(s)$. Let $x_\zeta(t) = [e^\top(t)\ \zeta(t)]^\top$. Since A_g is Hurwitz, then $H_g(s)$ is stable and strictly proper. It follows from (31) that $x_\zeta(s) = -H_g(s)C(s)\tilde{r}(s)$. Therefore, we have $x_\zeta(s) = -H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top H(s)\tilde{r}(s) = -H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top\tilde{x}(s)$, where c_o is introduced in (16). It follows from (16) that $H_g(s)C(s)\frac{1}{c_o^\top H(s)} = H_g(s)C(s)\frac{N_d(s)}{N_n(s)}$. Since both $H_g(s)$ and $C(s)$ are stable and strictly proper, the complete system $H_g(s)C(s)\frac{1}{c_o^\top H(s)}$ is proper and stable, which implies that its $\mathcal{L}_1$ gain exists and is finite. Hence, we have $\|x_\zeta\|_{\mathcal{L}_\infty} \leq \left\|H_g(s)C(s)\frac{1}{c_o^\top H(s)}c_o^\top\right\|_{\mathcal{L}_1}\|\tilde{x}\|_{\mathcal{L}_\infty}$.

The proof of (18) is similar to the proof of (17). □