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Guided wave attenuation in laterally varying media

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SUMMARY

Coupled mode techniques for guided wave propagation are extended to 2-D stochastic heterogeneity superimposed on a stratified medium. This approach requires the variations to be smoothly varying and of modest size (less than ±2 per cent). By averaging over an ensemble of statistically similar models, coupled equations for the modal energy transport can be generated. The intermode coupling depends on the horizontal correlation functions for the heterogeneity in the crust and mantle, and the integrated effect of the vertical variations in velocity and the modal eigenfunctions.

For a particular stochastic model, the attenuation of a single mode as a function of distance can be calculated as a superposition of intrinsic attenuation and scattering loss by energy transfer to other modes of propagation. These statistical estimates of attenuation can be compared with observations of regional phases travelling over a variety of paths in a single region. For *Lg* and *Sn* phases, intermode scattering may represent up to 30 per cent of the apparent loss.

Key words: attenuation, guided wave propagation, regional seismic phases, stochastic heterogeneity.

INTRODUCTION

Detailed studied of the structure of the crust and upper mantle show lateral heterogeneity on a wide variety of scales superimposed on a basic stratification with depth. Guided seismic waves travelling nearly horizontally through such structures are particularly vulnerable to scattering due to the presence of heterogeneity.

A convenient way of representing such guided waves is in terms of a superposition of surface wave modes whose eigenfunctions are largely confined to the waveguide. Such a description has been used by Malin (1980) and Wang & Hermann (1988) in simulations of the coda of local earthquakes. For longer distance propagation, Malischewsky (1987) has summarized the techniques for handling the interactions of surface waves with vertical interfaces or sharp transitions. These methods relate the amplitudes of the surface wave modes in the sturcture on one side of the boundary to those on the other side. In a similar way, for continuous heterogeneity the wavefield can be described in terms of surface wave modes if allowance is made for energy transfer between modes (Kennett 1984; Maupin 1988). These coupled mode methods have generally been formulated for deterministic velocity structures and much less work has been done on a stochastic description of guided wave propagation even though this has been extensively developed for body waves (see e.g. Aki & Richards 1980, chapter 13; Hudson 1982).

Here we present an adaptation of the coupled mode procedure to a 2-D stochastic heterogeneous medium, and generate coupled equations for the modal energy transport averaged over an ensemble of statistically similar models. The approach is based on techniques developed for fibre optics (Marcuse 1974), but is extended to multiple waveguides e.g. the crust and upper mantle for regional seismic phases. For a specified stochastic model of the velocity heterogeneity we are able to estimate for each mode the contribution to the modal loss factor ${}_sQ_m(\omega)$ induced by scattering into other modes of propagation. Such estimates can be compared with observations of guided waves over a variety of paths in a single geographic region, as e.g. the observations of Nuttli (1980) for *Sn* and *Lg* phases propagating across Iran.

COUPLED MODE EQUATIONS

Kennett (1984) has shown how the displacement and traction fields for guided seismic waves in 2-D laterally varying media can be represented as a sum of modal eigenfunctions with horizontally varying coefficients. When the variations in seismic parameters do not show a systematic trend horizontally, the expansion may be made in terms of the modes for a fixed reference structure. Maupin & Kennett (1987) have shown how the effect of inclined interfaces can be included in the scheme by modifying the matrix elements in the differential equations for coupling between the modal coefficients. When the structure has systematic horizontal variations a more effective representation is to work in terms of the local modes of the structure (Maupin 1988). Although the nature of the coupling coefficients differ, the two styles of modal representation lead to sets of coupled equations with similar structure.

The coupled mode equations give a representation in terms of a particular velocity structure. However, in considering different classes of heterogeneity it can be advantageous to adopt a stochastic treatment for the heterogeneity structure. We will consider the case where the heterogeneity has no more than ±2 per cent variation in seismic parameters from those of a reference model. This provides a good representation of many situations in the crust and uppermost mantle and can be well treated using the coupled mode approach with a fixed set of reference modes (Kennett 1989).

At each point in the 2-D varying medium we represent the displacement field $\mathbf{w}(x, z)$ as a sum of vertically varying modal eigenfunctions with coefficients which depend only on horizontal position (Kennett 1984). Thus

$$\mathbf{w}(x, z) = \sum_r c_r(x) \exp(ik_r x) \mathbf{w}_r^{\mathrm{o}}(k_r, z), \tag{1}$$

where $\mathbf{w}_r^{\mathrm{o}}$ is the eigenfunction for the rth mode with horizontal wavenumber k_r, and the sum is taken over both forward- and backward-travelling modes. The total number of modes must be chosen large enough to include all the propagation processes of interest (see Maupin & Kennett 1987). The eigenfunctions for the modes are normalized so that they have the same horizontal energy transport.

The horizontal evolution of the modal coefficients c_r is described by a set of ordinary differential equations

$$\frac{\partial}{\partial x} c_r(x) = \sum_s K_{rs}(x) \exp[i(k_s - k_r)x] c_s(x), \tag{2}$$

where the coupling matrix $\mathbf{K}$ depends on the deviations of the properties of the actual medium from those of the reference. Using the concise notation of Maupin & Kennett (1987), which is suitable for anisotropic media

$$\begin{aligned} K_{rs} = i\int_0^\infty dz[&\bar{\mathbf{t}}_{1r}^{\mathrm{o}} \cdot \Delta \mathbf{C}_{11}^{-1} \mathbf{t}_{1s}^{\mathrm{o}} + \Delta\rho\omega^2 \bar{\mathbf{w}}_r^{\mathrm{o}} \cdot \mathbf{w}_s^{\mathrm{o}} \\ &- \partial_z \bar{\mathbf{w}}_r^{\mathrm{o}} \cdot \Delta \mathbf{Q}_{33} \partial_z \mathbf{w}_s^{\mathrm{o}} + \partial_z \bar{\mathbf{w}}_r^{\mathrm{o}} \cdot \Delta(\mathbf{C}_{31}\mathbf{C}_{11}^{-1}) \mathbf{t}_{1s}^{\mathrm{o}} \\ &- \bar{\mathbf{t}}_{1r}^{\mathrm{o}} \cdot \Delta(\mathbf{C}_{11}^{-1}\mathbf{C}_{13}) \partial_z \mathbf{w}_s^{\mathrm{o}}] \\ &+ \sum_n h_n (\bar{\mathbf{w}}_r^{\mathrm{o}} \cdot \mathbf{t}_{1s}^{\mathrm{o}})_n, \end{aligned} \tag{3}$$

where $\mathbf{t}_{1r}^{\mathrm{o}}$ is the horizontal traction derived from $\mathbf{w}_r^{\mathrm{o}}$ and we have written $\bar{\mathbf{w}}_r^{\mathrm{o}} = \mathbf{w}_r^{\mathrm{o}}(-k_r, z)$. In terms of the elastic modulus tensor c_{kilj}

$$(\mathbf{C}_{ij})_{kl} = c_{kilj}, \qquad \mathbf{Q}_{33} = \mathbf{C}_{33} - \mathbf{C}_{31}\mathbf{C}_{11}^{-1}\mathbf{C}_{13}.$$

The interface terms depend on the slope of the interface and the jumps in the horizontal traction $\mathbf{t}_{1r}^{\mathrm{o}}$ across the interface.

The set of equations (2) include the possibility of both reflection and transmission. However, direct calculations using the coupled mode equations show that reflected waves can be neglected without appreciable error, provided that deviations in the seismic parameters from the reference model are not too large (less than ±2 per cent) and the scale of variation is not rapid compared with the horizontal wavelengths involved. This neglect of reflected waves can also be justified by using first-order Born scattering results in the wavenumber domain (Kennett 1972). For a heterogeneity pattern with wavenumber spectrum $f(k)$, the scattering between two wavenumbers k_a and k_b is proportional to $f(k_a - k_b)$. For reflected waves k_b is negative and so the difference $k_a - k_b$ will be large. Thus scattering into reflected waves will only be noticeable if $f(k)$ has significant amplitude for large k, i.e. if the heterogeneity itself varies on very small horizontal scales (or is discontinuous).

We will therefore restrict attention to transmitted waves and so consider the limited set of coupled equations

$$\frac{\partial}{\partial x} c_r(x) = \sum_{s=0}^{N} K_{rs}(x) \exp[i(k_s - k_r)x] c_s(x). \tag{4}$$

With the restrictions we have placed on the nature of the heterogeneity, a good approximation for the modal coefficients after passage through a small horizontal distance h is given by

$$c_r(x+h) = c_r(x) + \sum_{s=0}^{N} c_s(x) \int_x^{x+h} dq K_{rs}(q) \exp[i(k_s - k_r)q], \tag{5}$$

the first term in a systematic expansion in terms of the coupling coefficients K_{rs}; higher order interactions can be neglected since we have assumed weak heterogeneity. We are thus able to treat the set $\{c_r\}$ as the amplitude distribution across the modes at the horizontal position x.

COUPLED POWER EQUATIONS

The set of coupled mode equations (2) can be solved for a particular heterogeneity model and then provides a detailed description of the phase and amplitude behaviour of all the modes at each point x in the waveguide. By combining this information for many frequencies we can, in principal, construct theoretical seismograms, but very substantial computational effort is required.

However, it is possible to obtain a measure of the energy redistribution between modes as a function of distance in propagating through a random heterogeneity by considering the average power over an ensemble of heterogeneity models. The resulting equations for the modal power contributions for each frequency are a coupled set of first-order differential equations with constant and symmetric coefficients.

We will adopt a physically based approach to the derivation of the coupled power equations following Marcuse (1974). The results we derive can be justified formally using the techniques of stochastic differential equations described in Kohler & Papanicolaou (1977).

We have adopted a normalization for modal eigenfunctions in which the horizontal energy transport is equal for each mode. As a result, for a particular heterogeneity model a measure of the power in an individual mode P_m is

$$P_m = |c_m|^2,$$

and from (4) satisfies the horizontal evolution equation

$$\frac{\partial}{\partial x} P_m = c_m^* \frac{\partial}{\partial x} c_m + c_m \frac{\partial}{\partial x} c_m^*$$

$$= c_m^* \sum_n K_{mn} c_n \exp[i(k_n - k_m)x] + c.c., \tag{7}$$

where $c.c.$ denotes the complex conjugate of the previous term. We now consider an ensemble average over a collection of statistically similar heterogeneity models, built according to the same prescription, but not identical. In particular we assume that the phases of any periodic components in the heterogeneity are randomly distributed across the members of the statistical ensemble.

The averaged power in each mode

$$S_m = \langle |c_m|^2 \rangle, \tag{8}$$

and its change with horizontal position will be governed by the ensemble average of equation (7):

$$\frac{\partial}{\partial x} S_m = \sum_n \langle K_{mn} c_m^* c_n \rangle \exp[i(k_n - k_m)x] + c.c. \tag{9}$$

We will assume that each of the elements defining the heterogeneity $\Delta\rho$, $\Delta \mathbf{C}_{11}^{-1}$, $\Delta \mathbf{Q}_{33}$, $\Delta(\mathbf{C}_{11}^{-1}\mathbf{C}_{13})$, $\Delta(\mathbf{C}_{31}\mathbf{C}_{11}^{-1})$ have similar statistical properties and the heterogeneity values at widely separated points are uncorrelated. The coupling coefficients will be described by a stationary random process with a finite correlation length D in the horizontal direction. We will assume the average heterogeneity level across the ensemble vanishes so that

$$\langle K_{mn}(x) \rangle = 0.$$

For the guided waves travelling in the direction of increasing x we anticipate that the modal amplitude $c_m(x)$ and the coupling matrix $\mathbf{K}(x)$ will be uncorrelated if

$$x' - x \gg D,$$

and then

$$\langle c_m(x')c_n(x')K_{mn}(x) \rangle = \langle c_m(x')c_n(x') \rangle \langle K_{mn}(x) \rangle$$

with similar behaviour for products of coupling coefficients. We can exploit this result in (9) if we express the modal field at x in terms of that at x' using (5). Under the assumption that third-order interaction terms of the type $\langle \mathbf{KKK} \rangle$ can be neglected because the heterogeneity is small, we find

$$\frac{\partial}{\partial x} S_m = \sum_{n,r} \Big\{ \langle c_m^*(x')c_r(x') \rangle \exp[i(k_n - k_m)x] \times \int_x^{x'} dq \langle K_{mn}(x)K_{nr}(q) \rangle \exp[i(k_r - k_n)x] + \langle c_n(x')c_r^*(x') \rangle \exp[i(k_n - k_m)x] \times \int_x^{x'} dq \langle K_{mn}(x)K_{mr}^*(q) \rangle \exp[-i(k_r - k_m)x] + c.c \Big\}. \tag{10}$$

Since we have assumed a short correlation length we can extend the lower limit of integration to $-\infty$, and so recast the integrals in the form of half-range Fourier transforms over the correlation between the coupling coefficients

$$\exp[i(k_n - k_m)x] \int_x^{x'} dq \langle K_{mn}(x)K_{mr}^*(q) \rangle \exp[i(k_m - k_r)q] = \exp[i(k_n - k_r)x] \times \int_0^\infty du \langle K_{mn}(x)K_{mr}^*(x-u) \rangle \exp[-i(k_m - k_r)u]. \tag{11}$$

The coupled power equations can therefore be written as

$$\frac{\partial}{\partial x} S_m = \sum_{n,r} \Big\{ \langle c_m^*(x')c_r(x') \rangle \exp[i(k_r - k_m)x] \times \int_0^\infty du \langle K_{mn}(x)K_{nr}(x-u) \rangle \exp[i(k_n - k_r)u] + \langle c_n(x')c_r^*(x') \rangle \exp[i(k_n - k_r)x] \times \int_0^\infty du \langle K_{mn}(x)K_{mr}^*(x-u) \rangle \exp[i(k_r - k_m)u] + c.c \Big\}. \tag{12}$$

We anticipate that the modal amplitudes will tend to have random phase so that

$$\langle c_n c_m^* \rangle = \langle |c_m|^2 \rangle \delta_{mn}$$

and this selection of diagonal elements will be reinforced by a situation akin to stationary phase; the main contribution from the right-hand side of (12) will arise from non-oscillatory terms in the integrals. Hence we are able to drop one summation to give

$$\frac{\partial}{\partial x} S_m = \sum_n \Big\{ s_m(x') \int_0^\infty du \langle K_{mn}(x)K_{nm}(x-u) \rangle \times \exp[i(k_n - k_m)u] \Big\} + s_n(x') \int_0^\infty du \langle K_{mn}(x)K_{mn}^*(x-u) \rangle \times \exp[i(k_n - k_m)u] + c.c. \Big\}. \tag{13}$$

We have assumed that the modal coefficients are slowly varying in going from x to x', and so S_m will be nearly constant over this interval. As a result, we can replace $S_m(x')$ by $S_m(x)$ on the right-hand side of (13). Further, we require that the total power should be independent of x in a lossless medium so that

$$K_{nm}(x) = -K_{mn}^*(x).$$

Thus

$$\frac{\partial}{\partial x} S_m = \int_{-\infty}^{\infty} du \{ \langle K_{mn}(x)K_{mn}^*(x-u) \rangle \times [S_n(x) - S_m(s)] \} \exp[i(k_n - k_m)u],$$

where we have used the fact that the term in braces is real to incorporate the complex conjugate term as an integral along the negative u axis. (Note that in this perfectly elastic situation, after propagation through a very large distance, the power in all the modes will equalize to the same level.)

Equation (14) covers the general case of weak heterogeneity but further simplifications can be made if we make specific assumptions about the form of the heterogeneity model. The near stratification within the Earth will impose different stochastic properties on the horizontal and vertical variations of the heterogeneity. The coupling coefficients K_{mn} are defined in terms of an integral over the full depth of the heterogeneity. After taking the ensemble average over the set of statistically similar models, the evolution equations for the average modal power will be dominated by the nature of the variations in the heterogeneity in the horizontal direction.

SEPARATION OF CRUST AND MANTLE HETEROGENEITY CONTRIBUTIONS

In general, we expect the heterogeneity in the crust to have a different character to that in the mantle. Since we have assumed weak heterogeneity, we are able to extract a common functional dependence from each of the coupling coefficients K_{mn} for the horizontal behaviour of the heterogeneity in each of the crust and mantle zones

$$K_{mn}(x) = f^{C}(x)K^{C}_{mn}(x) + f^{M}(x)K^{M}_{mn}(x). \tag{15}$$

$f^{C}(x)$ is the horizontal variation function for the crustal heterogeneity and $K^{C}_{mn}(x)$ is an integral restricted to the crust. Similarly $f^{M}(x)$ is the horizontal variation function for the mantle and $K^{M}_{mn}(x)$ is defined by an integral over the span of the mantle. We do not require a complete separation of the horizontal and vertical dependence of the velocity and density perturbations from the reference. The part of the variation unaccounted for by $f^{C}(x)$, $f^{M}(x)$ remains in the crust and mantle coupling coefficients $K^{C}_{mn}(x)$, $K^{M}_{mn}(x)$ which depend on the properties as a function of depth at the location x. When we take the average over an ensemble of statistically equivalent models, we remove this dependence on location. The ensemble average

$$\begin{aligned}\langle K_{mn}(x)K^{*}_{mn}(x-u)\rangle &= \langle f^{C}(x)f^{C}(x-u)\rangle |\hat{K}^{C}_{mn}|^2 + \langle f^{M}(x)f^{M}(x-u)\rangle\, |\hat{K}^{M}_{mn}|^2 \\ &= R^{C}(u)\, |\hat{K}^{C}_{mn}|^2 + R^{m}(u)\, |\hat{K}^{M}_{mn}|^2,\end{aligned} \tag{16}$$

in terms of the autocorrelation functions R^{C}, R^{M} of the horizontal variation of the crust and mantle heterogeneity e.g. $R^{C}(u) = \langle f^{C}(x)f^{C}(x-u)\rangle$. $\hat{K}^{C}_{mn}$, $\hat{K}^{M}_{mn}$ represent ensemble averages over the vertical variations. We have made the not unreasonable assumption that the horizontal variations in the crust and mantle heterogeneity are uncorrelated.

With the expression (16) for the heterogeneity contribution the evolution equations for the averaged modal power can be recast as

$$\frac{\partial}{\partial x}S_m = \sum_n [S_n(x) - S_m(x)]\big[|\hat{K}^{C}_{mn}|^2 \langle |F^{C}(k_n - k_m)|^2\rangle + |\hat{K}^{M}_{mn}|^2 \langle |F^{M}(k_n - k_m)|^2\rangle\big] \tag{17}$$

where $\langle |F^{C}|^2\rangle$, $\langle |F^{M}|^2\rangle$ are the power spectra of the crust and mantle correlation functions e.g.

$$\langle |F^{C}(k_n - k_m)|^2\rangle = \int_{-\infty}^{\infty} du\, R^{C}(u) \exp\,[i(k_n - k_m)u]. \tag{18}$$

We have so far ignored the possibility of intrinsic attenuation of the modes as they propagate through the medium, but it is easy to compensate for the effect of Q by introducing a simple power loss term into (17). Thus for an attenuative medium

$$\frac{\partial}{\partial x}S_m = -k_m Q_m^{-1} S_m(x) + \sum_{n=0}^{N} H_{mn}[S_n(x) - S_m(x)], \tag{19}$$

where the power coupling coefficients

$$H_{mn} = |\hat{K}^{C}_{mn}|^2 \langle |F^{C}(k_n - k_m)|^2\rangle + |\hat{K}^{M}_{mn}|^2 \langle |F^{M}(k_n - k_m)|^2\rangle \tag{20}$$

and Q_m^{-1} is the intrinsic loss factor for the mth mode. The spatial loss factor Q_m^{-1} is related to the temporal loss factor ${}^{t}Q_m^{-1}$ for the same mode by the ratio of the group velocity ($\mathbf{U}_m$) and phase velocity ($\mathbf{c}_m$) for the mode

$${}^{t}Q_m^{-1} = (\mathbf{U}_m/\mathbf{c}_m)Q_m^{-1}).$$

The set of coupled equations (19) can be readily solved numerically for any given power distribution. Similar equations can be derived for other configurations of heterogeneity; the differences will lie in the form of the power coupling coefficients as e.g. where the mantle has to be sub-divided into different zones.

For small perturbations of an interface

$$K_{mn}(x) = f^{I}(x)K^{I}_{mn}(x), \tag{21}$$

where $f^{I}(x)$ is the shape of the varying interface. The coupling integral K^{I}_{mn} will be confined to a depth band representing the extent of the interface variations, and will depend on the contrast in seismic properties across the interface. In this case, the equivalent to (16) is

$$\begin{aligned}\langle K_{mn}(x)K^{*}_{mn}(x-u)\rangle &= \langle f^{I}(x)f^{I}(x-u)\rangle |\hat{K}^{I}_{mn}|^2 \\ &= R^{I}(u)\, |\hat{K}^{I}_{mn}|^2,\end{aligned} \tag{22}$$

where R^{I} is the autocorrelation function of the interface shape and $\hat{K}^{I}_{mn}$ the ensemble average over the coupling terms. The subsequent analysis will parallel equations (17–19) and the corresponding power coupling coefficients

$$H_{mn} = |\hat{K}^{I}_{mn}|^2 \langle F^{I}(k_n - k_m)|^2\rangle. \tag{23}$$

For interfaces with considerable contrast, e.g. the crust–mantle boundary, the effect of random topography can be comparable to that of velocity perturbations. The present approach is not suitable for situations with a systematic variation in the depth of an interface.

ATTENUATION VIA INTERMODE COUPLING

We can rewrite the coupled power equations (19) in a form which emphasizes the behaviour of the mth mode

$$\frac{\partial}{\partial x}S_m = -(k_m Q_m^{-1} + b_m)S_m(x) + {\sum_{n=0}^{N}}' H_{mn}S_n(x), \tag{24}$$

where Σ' denotes a summation omitting $m = n$, and

$$b_m = {\sum_{n=0}^{N}}' H_{mn}$$

i.e. a sum over one column of the matrix of power coupling coupling coefficients **H** excluding the diagonal element. Note that coupling a mode to itself will not affect the power distribution with the natural consequence of excluding the diagonal element.

For a given initial power distribution $S_{0m}(x_0)$, an equivalent integral equation to (24) is

$$S_m(x) = \exp\left[-(k_m Q_m^{-1} + b_m)(x - x_0)\right] S_{0m}(x_0) + \sum_{n=0}^{N}{}' \int_{x_0}^{x} dv \exp\left[-(k_m Q_m^{-1} + b_m)(x - v)\right] H_{mn} S_n(v). \quad (25)$$

As we have assumed relatively weak heterogeneity, the dominant behaviour can be seen from the first-order approximate solution

$$S_m(x) = \exp\left[-a_m(x - x_0)\right] S_{0m}(x_0) + \sum_{n=0}^{N}{}' \int_{x_0}^{x} dv \exp\left[-a_m(x - v)\right] H_{mn} \exp\left[-a_m(v - x_0)\right] S_n(x_0), \quad (26)$$

where we have introduced the effective decay rate with distance for energy in the mth mode

$$a_m = k_m Q_m^{-1} + b_m = k_m Q_m^{-1} + \sum_{n}{}' H_{mn}. \quad (27)$$

If we now consider an initial distribution $S_{0m}(x_0)$ concentrated solely in the mth mode, we see from (25) and (26) that the main part of the behaviour of the mth mode with distance will be decay as $\exp[-a_m(x - x_0)]$. There will be limited retransfer of energy to the mth mode by secondary scattering from other modes.

We can therefore characterize the energy decay rate for the mth mode by a_m and regard the intermode coupling as introducing a 'scattering' attenuation term for the mth mode

$$ {}_sQ_m^{-1} = k_m^{-1} b_m = k_m^{-1} \sum_{n}{}' \left[|\hat{K}_{mn}^{C}|^2 \langle |F^{C}(k_n - k_m)|^2 \rangle + |\hat{K}_{mn}^{M}|^2 \langle |F^{M}(k_n - k_m)|^2 \rangle\right], \quad (28)$$

where we have reinstated the explicit form (20) for the power coupling coefficients between the modes. Equation (26) represents the first term in an iterative solution in terms of **H** and, since all elements of H_{mn} are positive, the full solution will give a slightly larger value for $S_m(x)$ than estimated from (26). The loss factor ${}_sQ_m^{-1}$ defined by (28) is therefore a little too small, but gives a very useful indication of the behaviour. Equation (28) represents the spatial loss factor; the corresponding temporal loss factor for the mth mode will be $(\mathbf{U}_m/\mathbf{c}_m)\,{}_sQ_m^{-1}$.

For regional seismic phases, the scattering attenuation will be significantly different for the mantle phases than the crustally guided waves. The dominant term in (25) for Lg type modes will be $|\hat{K}_{mn}^{C}|^2 \langle F^{C}(k_n - k_m)|^2 \rangle$ and since we anticipate the largest heterogeneity to be concentrated in the crust ${}_sQ_m^{-1}$ can be significant. For the Sn modes the main contribution will be from the mantle heterogeneity.

When we wish to compare the behaviour of a set of modes at fixed frequency ω, it can be advantageous to work with the energy decay factors a_m, b_m since these terms are directly comparable between modes.

The expressions for the scattering attenuation for individual modes derived in this section are dependent on our assumption of 2-D heterogeneity. For 3-D heterogeneity, there is the additional complication that there is the likelihood of energy being scattered in directions other than forwards and backwards along the local direction of propagation. Once scattered out of a particular mode, energy is unlikely to return to it in the case of 3-D heterogeneity; thus attenuation will be stronger than the estimates based on our 2-D model.

However, when we consider a multimode seismic wavetrain, this directional scattering will work to reduce the reinforcement of modal amplitudes by cross-coupling. An average of the estimated scattering attenuation across a number of modes will therefore give a measure of the decay rate of the wavetrain with distance which can be compared with observations on attenuation. It should also be recalled that the derivation of the scattering attenuation depends on propagation occurring over distances which are long compared with the correlation length of the heterogeneity.

ESTIMATES OF SCATTERING ATTENUATION

The computational procedure which we have just derived enables us to generate a good estimate of the statistical loss factor for an individual mode as a combination of anelastic and scattering contributions. For direct comparison between different modes we can use the rate of energy decay with distance. Here we will apply our results to the regional phases Lg and Sn travelling through heterogeneous earth models.

From equation (25), calculation of the scattering loss factor requires us to specify the statistical nature of the horizontal heterogeneity spectrum in the crust and upper mantle as well as estimate the ensemble averages of the intermode coupling terms $\hat{\mathbf{K}}^{C}$, $\hat{\mathbf{K}}^{M}$ which depend on the modal eigenfunctions for the reference model. In Fig. 1 our reference model (ARANDA) is shown together with the vertical component of displacement for Rayleigh modes at 1 Hz. The modes clearly divide into two classes: first, those modes for which the displacement is largely restricted to the crust which will constitute the Lg wavetrain, and second, modes with little crustal displacement but significant energy transport in the mantle which represent the Sn phase. At 1 Hz the transition occurs at mode 11. The fundamental and first higher modes are confined to the sedimentary overburden and hardly interact with the other modes.

The ensemble averages $\hat{\mathbf{K}}^{C}$, $\hat{\mathbf{K}}^{M}$ require an average of depth integrals over a combination of heterogeneity and eigenfunction terms. We have estimated these averages by a Monte Carlo simulation; random velocity perturbations (up to 1 per cent) were applied to the velocities at the top and bottom of each of the layers in the reference model ARANDA and linear interpolation was used to calculate intermediate velocity values. Densities were varied in proportion to the velocities, and the depths of the interfaces were not varied. The coupling integrals (3) were then evaluated for both Love and Rayleigh waves for 300 different simulations of the vertical heterogeneity structure and averaged to give estimates of $\hat{\mathbf{K}}^{C}$, $\hat{\mathbf{K}}^{M}$. $\hat{\mathbf{K}}^{C}$ is calculated for an integral over the crust i.e. down to 30 km, and $\mathbf{K}^{M}$

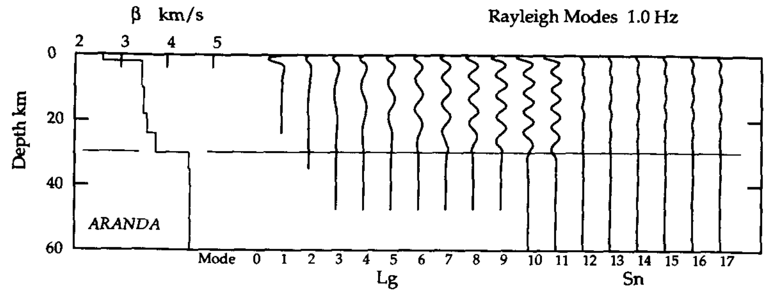

Figure 1. ARANDA reference model and the vertical component of displacement for the first 18 Rayleigh modes at 1 Hz.

over the mangle structure which is extended in the reference model to 200 km depth. This procedure gives quite stable estimates for $\hat{R}^C$ and $\hat{R}^M$.

The model adopted for the autocorrelation of the horizontal variation of the heterogeneity was exponential

$$R(u) = h^2 \exp(-|u|/D),$$

for a horizontal correlation length D. This form has the advantage that it can allow the existence of discontinuities in velocity gradients. The corresponding Fourier transform appearing in the loss factor terms is

$$\langle |F(k)|^2 \rangle = 2h^2/D[k^2 + (1/D)^2].$$

We have scaled the amplitude factor h to be unity for the ± 1 per cent heterogeneity assumed in the calculation of the vertical averages $\hat{R}^C$, $\hat{R}^M$.

The dependence of the loss factor on heterogeneity will be quadratic over the ranges for which the present theory is valid (less than ± 2 per cent). At higher levels, local multiple scattering will become important and the rate of increase of the loss factor will drop below quadratic.

In Fig. 2, the behaviour of the scattering loss factor is shown for three Rayleigh and Love modes at 1 Hz, with ± 1 per cent heterogeneity, as a function of the correlation length D_C for the crust. The mantle correlation length was fixed at 50 km. The modes were chosen to represent different aspects of the wavefield. At 1 Hz, mode 4 is sensitive to the upper middle crust and forms part of the onset of *Lg* with a group velocity of 3.45 km s^{-1}. Mode 9 has greater sensitivity to lower crustal properties and represents waves travelling near critical incidence on the crust–mantle interface; at 1 Hz it is close to an Airy Phase with a group velocity of 3.2 km s^{-1} and so represents the coda of *Lg*. Mode 14 has a group velocity around 4.45 km s^{-1} and forms part of the *Sn* phase.

The solid triangles in Fig. 2 indicate the calculations for 2-D structure for Rayleigh waves and the open diamonds indicate the corresponding values for Love waves. The behaviour is generally similar although there is more difference for the *Sn* type mode (14). For this slowness, the Rayleigh wave couples *P*-waves propagating in the near surface with *Sn* type behaviour for *S*; the influence of the *P*-wave velocity heterogeneity raises the loss factor for the mode.

For both of the crustal modes we see a tendency for the scattering loss factor to peak for correlation distances about three times the horizontal wavelength of the waves. For small-scale heterogeneity there will be considerable scattering, but loss and gain by intermodal interactions will tend to balance. For larger scale heterogeneity, scattering will be more infrequent and so loss will be reduced. The detailed behaviour depends on the mode and also the character prescribed for the vertical heterogeneity.

The loss factor for the mantle mode (14) is less sensitive to the character of the crustal heterogeneity. There is considerable mixing between mantle modes giving a significant scattering component. However, the intrinsic attenuation is likely to be low so that the overall loss factor will normally be less than for crustal propagation.

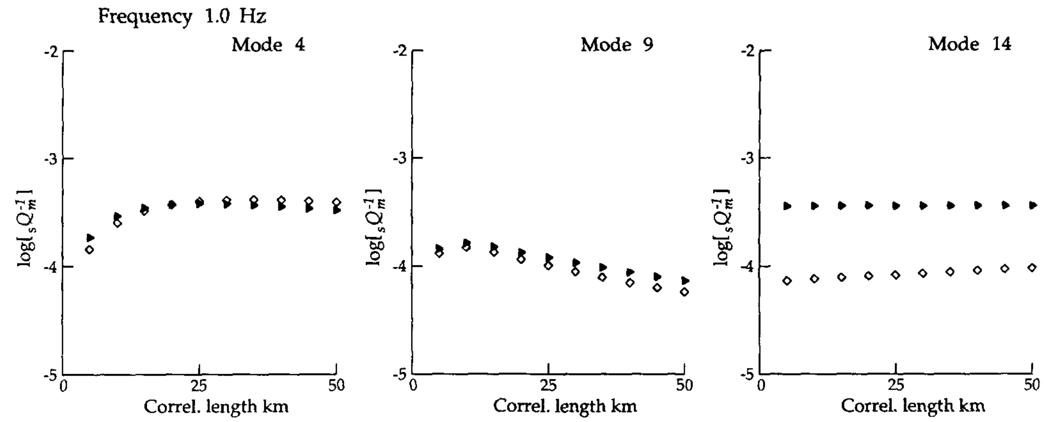

Figure 2. Variation of the scattering loss factor ${}_sQ_m^{-1}$ for different Rayleigh and Love modes at 1 Hz as a function of horizontal correlation length in the crust. Rayleigh waves are indicated by solid triangles and Love waves by open diamonds.

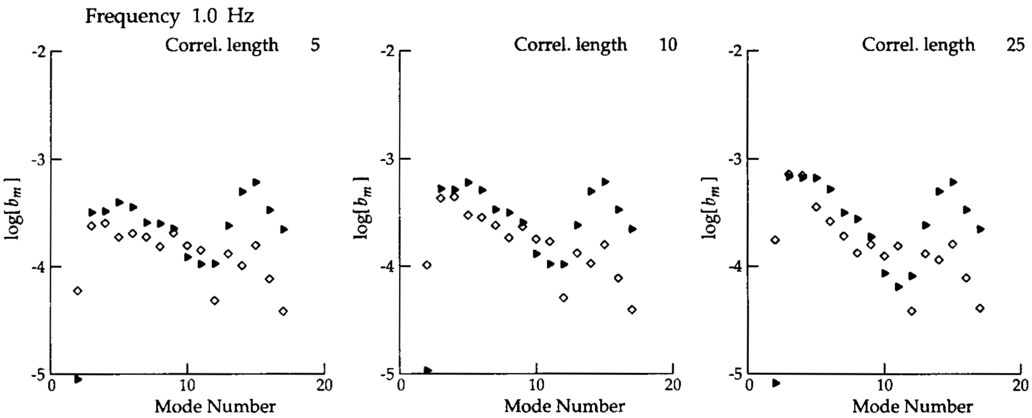

Figure 3. Variation of the energy loss with distance b_m with mode number for different horizontal correlation lengths in the crust. Rayleigh waves are indicated by solid triangles and Love waves by open diamonds.

In order to compare a number of different modes, it is preferable to work with the rate of energy loss with distance b_m rather than the scattering loss factors. In Fig. 3 we therefore display b_m as a function of mode number at 1 Hz, for both Love and Rayleigh waves and three different choices of crustal correlation length D_C. Mode 1 is confined to the sediments and has little interaction with other modes, so that its scattering attenuation is small. Attenuation for the other modes is of the same order of magnitude. As the horizontal correlation length shortens to less than 10 km, a reasonable value for crustal variation (Wu & Aki 1988), the decay rates of the various crustal modes tend to equalize which will give rise to a consistent decay rate for the *Lg* wavetrain.

The frequency dependence of the loss factor estimates is explicit through the dependence on ω and k in the coupling coefficients and correlation spectra, but also implicit through the shape of the eigenfunctions. For a crustally guided mode it is difficult to compare the Q^{-1} estimates at different frequencies because the character of the mode changes. However, we can compare modes with similar propagation characteistics at different frequencies and we find that our statisical estimates of ${}_sQ^{-1}$ increase with frequency. The increase is not a simple power law and this is an indication of saturation at higher frequencies (around 2 Hz).

As pointed out by Hudson (1982) it is often difficult to find a direct relation between the results of stochastic calculations and observable features of the siesmic wavefield. Fortunately, for guided wave attenuation we are able to find a very close correspondence between our estimate of energy loss with distance and observations of regional phase attenuation over a variety of paths within a single region (see e.g. Nuttli 1980; Herrmann 1980). Once the effects of dispersion and geometrical spreading are removed from the observations, the attenuation with distance is isolated and will include both intrinsic and scattering attenuation.

For a large number of different paths in Iran at 1 Hz, Nuttli (1980) found a broad spread of amplitude decay factors for both *Lg* and *Sn* with an average value around 0.0045 km^{-1}. To convert to energy loss coefficients we must multiply by a factor of two. Comparison with Fig. 3 shows that ± 1 per cent heterogeneity would account for about 10 per cent of the observed attenuation and this factor would rise to around 40 per cent for ± 2 per cent heterogeneity. The intrinsic attenuation is likely to be high in Iran, but the apparent Q of around 200 may well have a significant scattering component.

For the low-attenuation zone in the Eastern United States, Herrmann (1980) has deduced an amplitude decay rate of 0.0009 km^{-1}, with an apparent Q of 1500, for many paths to the station BLA (Blacksburg, Virginia). This would be equivalent to an energy decay rate of 0.0018 km^{-1}. Nearly a third of this loss could be accounted for by scattering due to ± 1 per cent heterogeneity, within a medium with low intrinsic loss ($Q_i \simeq 2250$).

These results show that our estimate of the scattering loss for guided waves fit well with observations of guided wave attenuation. If, then, we have a measure of the heterogeneity structure and intrinsic attenuation for a region, we can use our statistical approach to estimate an average effective loss factor which can be used as a reference against which to compare observations on diffrent paths.

The seismic velocity distributions derived from tomographic inversion (e.g. Spakman 1989) often show higher levels of heterogeneity than have been assumed in the stochastic calculations (up to ± 5 per cent). However, such heterogeneity has a horizontal scale length typically greater than 200 km and so will not be a significant contributor to scattering for higher frequency waves (1 Hz and above). The estimates of scattering attenuation due to small-scale variations in seismic properties will therefore give a good guide to the behaviour in the presence of variability on much longer horizontal scales.

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