Gumbel fluctuations for cover times in the discrete torus

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David Belius

Abstract This work proves that the fluctuations of the cover time of simple random walk in the discrete torus of dimension at least three with large side-length are governed by the Gumbel extreme value distribution. This result was conjectured for example in Aldous and Fill (Reversible Markov chains and random walks on graphs, in preparation). We also derive some corollaries which qualitatively describe "how" covering happens. In addition, we develop a new and stronger coupling of the model of random interlacements, introduced by Sznitman (Ann Math (2) 171(3):2039–2087, 2010), and random walk in the torus. This coupling is used to prove the cover time result and is also of independent interest.

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1 Introduction

In this article we prove that if C_N is the cover time of the discrete torus of side-length N and dimension $d \geq 3$ then $C_N/(g(0)N^d) - \log N^d$ (where $g(\cdot)$ is the $\mathbb{Z}^d$ Green function) tends in law to the Gumbel distribution as $N \to \infty$, or in other words, that the fluctuations of the cover time are governed by the Gumbel distribution. This result was conjectured e.g. in [3, Chapter 7, p. 23]. We also construct a new stronger coupling of random walk in the torus and the model of random interlacements, thus improving a result from [25]. The coupling is of independent interest and is also used as a tool to prove the cover time result.

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Cover times of finite graphs by simple random walk have been studied extensively, see e.g. [1–3,8,9,15]. One important case is the cover time of the discrete torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d$. Let P be the canonical law of continuous time simple random walk in this graph, starting from the uniform distribution, and let the canonical process be denoted by $(Y_t)_{t \geq 0}$. The cover time of the discrete torus is the first time $Y.$ has visited every vertex of the graph, or in other words

$$C_N = \max_{x \in \mathbb{T}_N} H_x, \tag{1.1}$$

where H_x denotes the entrance time of the vertex $x \in \mathbb{T}_N$. For $d \geq 3$ it is known that $E[C_N] \sim g(0)N^d \log N^d$, as $N \to \infty$, and that C_N concentrates in the sense that $\frac{C_N}{g(0)N^d \log N^d} \to 1$ in probability, as $N \to \infty$. However previously it was only conjectured that the finer scaling result

$$\frac{C_N}{g(0)N^d} - \log N^d \overset{\text{law}}{\longrightarrow} G \quad \text{as } N \to \infty, \quad \text{for } d \geq 3, \tag{1.2}$$

holds, where G denotes the standard Gumbel distribution, with cumulative distribution function $F(z) = e^{-e^{-z}}$ (see e.g. Chapter 7, pp. 22–23, [3]). In this article we prove (1.2). In fact our result, Theorem 3.1, proves more, namely that the (appropriately defined) cover time of any "large" subset of $\mathbb{T}_N$ satisfies a similar relation. (For $d = 1, 2$, the asymptotic behaviour of $E[C_N]$ is different; see [9] for $d = 2$, the case $d = 1$ is an exercise in classical probability theory. The concentration result $C_N/E[C_N] \to 1$ still holds for $d = 2$, but the nature of the fluctuations is unknown. For $d = 1$ one can show that C_N/N^2 converges in law to the time needed by Brownian motion to cover $\mathbb{R}/\mathbb{Z}$.)

Our second main result is a coupling of random walk in the discrete torus and random interlacements, which we now introduce. To do so we very briefly describe the model of random interlacements (see Sect. 2 for more details). It was introduced in [23] and helps to understand "the local picture" left by the trace of a simple random walk in (among other graphs) the discrete torus when $d \geq 3$. The random interlacement roughly speaking arises from a Poisson cloud of doubly infinite nearest-neighbour paths modulo time-shift in $\mathbb{Z}^d$, $d \geq 3$, with a parameter $u \geq 0$ that multiplies the intensity. The trace (vertices visited) of all the paths at some intensity u is a random subset of $\mathbb{Z}^d$ denoted by $\mathcal{I}^u$. Together the $\mathcal{I}^u$, for $u \geq 0$, form an increasing family $(\mathcal{I}^u)_{u \geq 0}$ of random subsets of $\mathbb{Z}^d$. We call the family $(\mathcal{I}^u)_{u \geq 0}$ a random interlacement and for fixed u we call the random set $\mathcal{I}^u$ a random interlacement at level u. Random interlacements are intimitaly related to random walk in the torus; intuitively speaking, the trace of $Y.$ in a "local box" of the torus, when run up to time uN^d, in some sense "looks like" $\mathcal{I}^u$ intersected with a box (see [23,26]).

Our coupling result is one way to make the previous sentence precise and can be formulated roughly as follows. Let $Y(0, t)$ denote the trace of $Y.$ up to time t (i.e. the set of vertices visited up to time t). For $n \geq 1$ pick vertices $x_1, \ldots, x_n \in \mathbb{T}_N$ and consider boxes $A_1, \ldots, A_n$, defined by $A_i = x_i + A$, $i = 1, \ldots, n$, where $A \subset \mathbb{Z}^d$ is a box centred at 0, of side-length such that the $A_1, \ldots, A_n$ are "well separated" and at most "mesoscopic". Then for a level $u > 0$ and a $\delta > 0$ (which may not be too small)

one can construct a coupling of random walk Y. with law P and independent random interlacements $(\mathcal{I}_1^v)_{v\geq 0}, \ldots, (\mathcal{I}_n^v)_{v\geq 0}$, such that "with high probability"

$$\mathcal{I}_i^{u(1-\delta)} \cap A \subset (Y(0, uN^d) - x_i) \cap A \subset \mathcal{I}_i^{u(1+\delta)} \cap A \quad \text{for } i = 1, \ldots, n. \quad (1.3)$$

The result is stated rigorously in Theorem 3.2. The case $n = 1$ (i.e. coupling for one box) and u, δ fixed (as $N \to \infty$) was obtained in [25] (and earlier a coupling of random walk in the so called discrete cylinder and random interlacements was constructed in [21,22]). In this paper we strengthen the result from [25] by coupling random interlacements with random walk in many separated boxes (the most important improvement), and by allowing δ to go to zero and u to go to zero or infinity. A similar improvement of the discrete cylinder coupling from [21,22] can be found in [6].

A coupling of random interlacements and random walk is a very powerful tool to study the trace of random walk. In this article we use the above coupling to study certain properties of the trace relevant to the cover time result (1.2). Similar couplings have also been used to study the percolative properties of the complement of the trace of random walk in terms of random interlacements; in this case the relevant properties of the trace studied with the coupling are very different (see [25] for the torus, and [21,22] for the discrete cylinder). We expect our coupling to find uses beyond the current cover time application. For more on this see Remark 4.6 (1).

We also prove two interesting corollaries of (1.2) (actually using the stronger subset version mentioned above) and (1.3). To formulate the first corollary we define the "point process of vertices covered last", a random measure on the Euclidean torus $(\mathbb{R}/\mathbb{Z})^d$, by

$$\mathcal{N}_N^z = \sum_{x\in\mathbb{T}_N} \delta_{x/N} 1_{\{H_x > g(0)N^d\{\log N^d + z\}\}}, \quad N \geq 1, z \in \mathbb{R}. \quad (1.4)$$

Note that $\mathcal{N}_N^z$ counts the vertices of $\mathbb{T}_N$ which have not yet been hit at time $g(0)N^d\{\log N^d + z\}$ (this is the "correct time-scale"; from (1.2) one sees that the probability that $\mathcal{N}_N^z$ is the zero measure is bounded away from zero and one). The result is then that (for $d \geq 3$)

$$\mathcal{N}_N^z \xrightarrow{\text{law}} \mathcal{N}^z \text{ as } N \to \infty, \text{ where } \mathcal{N}^z \text{ is a Poisson point process} \\ \text{on } (\mathbb{R}/\mathbb{Z})^d \text{ of intensity } e^{-z}\lambda, \text{ and } \lambda \text{ denotes Lebesgue measure.} \quad (1.5)$$

This is proven in Corollary 3.4. Intuitively speaking it means that the positions of the last vertices of the torus to be covered are approximately independent and uniform.

As a consequence of (1.5) we obtain Corollary 3.5 which says, intuitively speaking, that

the last few vertices of $\mathbb{T}_N$ to be covered are far apart, at distance of order N. (1.6)

Note that a priori it is not clear if the correct qualitative picture is that the random walk completes covering of the torus by "taking out several neighbouring vertices at once"

or if it "takes out the last few vertices one by one". Roughly speaking (1.6) implies that the latter is the case.

We now discuss the proofs of the above results. The result (1.5) is proven using Kallenberg's theorem, which allows one to verify convergence in law of certain point processes by checking only convergence of the intensity measure and convergence of the probability of "charging a set". The latter two quantities will be shown to converge using (1.2) (or rather the subset version of this statement) and the coupling (1.3). The result (1.6) follows from (1.5), using a calculation involving Palm measures and the fact that the points of the limit Poisson point processes $\mathcal{N}^z$ are "macroscopically separated".

We now turn to the proof of (1.2). The method builds on the ideas of the works [5,6], which contain the corresponding results for the so called cover levels of random interlacements ([5]) and the cover times of finite sets in the discrete cylinder ([6]). It is a well known "general principle" that entrance times of small sets in graphs often behave like exponential random variables. In the case of the torus the rescaled entrance time $\frac{1}{N^d} H_x$ of a vertex $x \in \mathbb{T}_N$ is approximately exponential with parameter $\frac{1}{g(0)}$ (this can, for instance, be proven with a good quantitative bound using the coupling (1.3), see Lemma 3.3). Thus, in view of (1.1), we see that the rescaled cover time $\frac{1}{N^d} C_N$ is the maximum of identically distributed exponential random variables with parameter $\frac{1}{g(0)}$. If the H_x, $x \in \mathbb{T}_N$, were also independent then standard extreme value theory (or a simple direct calculation of the distribution function of the maximum of i.i.d. exponentials) would give (1.2). But clearly the H_x, $x \in \mathbb{T}_N$, are not even approximately independent, (for example if x, $y \in \mathbb{T}_N$ are neighbouring vertices then H_x and H_y are highly dependent). There are theorems that give distributional results for the maxima of random fields with some sufficiently weak dependence (see [13,16]) but these do not apply to the random field $(H_x)_{x \in \mathbb{T}_N}$ because the dependence is too strong (using (1.3) one can show that correlation between $1_{\{H_x > uN^d\}}$ and $1_{\{H_y > uN^d\}}$ decays as $\frac{c(u)}{(d(x,y))^{d-2}}$, see (1.68) of [23]).

However for sets $F \subset \mathbb{T}_N$ that consist of isolated vertices $x_1, \ldots, x_n$, that are "well-separated", it turns out that using (1.3) one can show that $H_{x_1}, \ldots, H_{x_n}$, are approximately independent (see Lemma 4.3). By comparing to the independent case, we will therefore be able to show that for such F

$$\max_{x \in F} H_x \text{ has law close to } g(0)N^d\{\log |F| + G\}, \tag{1.7}$$

where G is a standard Gumbel random variable. In particular it will turn out that for such F, roughly speaking, the distribution of $\max_{x \in F} H_x$ essentially depends only on the cardinality of F.

To enable the proof of (1.2) we will introduce the set of "$(1 - \rho)-$late points" F_ρ defined as the vertices of $\mathbb{T}_N$ that have not been hit at time $t(\rho) = (1 - \rho)g(0)N^d \log N^d$, for a fixed but small $0 < \rho < 1$. Note that this is a $(1 - \rho)$ fraction of the "typical" cover time $g(0)N^d \log N^d$. By the Markov property $Y_{t(\rho)+}$. is a random walk, and using a mixing argument one can show that for our purposes it is basically independent from the random walk $(Y_t)_{0 \leq t \leq t(\rho)}$, so that the law of C_N is approximately the law of $t(\rho) + \max_{x \in F'} H_x$, where F' is a random set which is independent of the random walk Y, but has the law of F_ρ.

Furthermore we will be able to show, using (1.3), that "with high probability" F' (and F_ρ) consists of "well-separated" vertices, and that the cardinality of F' concentrates around its expected value, which is close to $|\mathbb{T}_N|^\rho = N^{d\rho}$. Thus as long as F' is "typical", in the sense that it is well-separated and has cardinality close to $N^{d\rho}$, we will "know" that $\max_{x \in F'} H_x$ has law close to $g(0)N^d\{\rho \log N^d + G\}$ (see (1.7)). Adding the deterministic time $t(\rho)$ the ρ will cancel and we will get that C_N has law close to $g(0)N^d\{\log N^d + G\}$, which is essentially speaking the claim in (1.2).

We turn finally to the proof of the coupling (1.3). It roughly speaking adapts the "poissonization" method used for the case $n = 1$ in [25] to the case $n > 1$, and combines it with a decoupling technique from [24].

The first step is to consider the *excursions* of $Y_\cdot$, that is the pieces of path $Y_{(R_k+\cdot)\wedge U_k, k=1,2,\ldots}$ where R_k and U_k are recursively defined, $U_0 = 0$, R_k is the first time $Y_\cdot$ enters $A_1 \cup \cdots \cup A_n$ after time U_{k-1} and U_k is the first time after R_k that the random walk has spent "a long time far away from $A_1 \cup \cdots \cup A_n$" (see (5.4)). By proving, using a mixing argument, that the distribution of Y_{U_k} is close to a certain probability distribution on $\mathbb{T}_N$ known as the quasistationary distribution, regardless of the value of Y_{R_k}, we will be able to couple the excursions $Y_{(R_k+\cdot)\wedge U_k, k=1,2,\ldots}$ with *independent* excursions $\tilde{Y}^1$, $\tilde{Y}^2$, ... that have the law of $Y_{\cdot \wedge U_1}$, when $Y_\cdot$ starts from the quasistationary distribution, such that "with high probability" the traces of $Y_{(R_k+\cdot)\wedge U_k}$ and $\tilde{Y}^k$ in $A_1 \cup \cdots \cup A_n$ coincide.

We will then collect a Poisson number of such independent excursions in a point process μ (in fact two different point processes, but for the purpose of this discussion let us ignore this) which will be such that the trace of μ in $A_1 \cup \cdots \cup A_n$ with high probability coincides with the trace of the random walk $Y_\cdot$ run up to time uN^d in that set. Because of the way we construct the point process μ, it will be a Poisson point process on the space of paths in $\mathbb{T}_N$ of a certain intensity related to the law of $\tilde{Y}^1$. This will complete the step that we refer to as "poissonization".

We will see that an "excursion" in the Poisson point process μ may visit several of the boxes $A_1, \ldots, A_n$, and, roughly speaking, "feels the geometry of the torus", since it may wind its way all around it before the time U. To deal with this we in essence split the excursions of μ into the pieces of excursion between successive returns to the set $A_1 \cup \cdots \cup A_n$ and successive departures from $B_1 \cup \cdots \cup B_n$, where the $B_i \supset A_i$ are larger boxes (still disjoint and at most mesoscopic), and use a decoupling technique to remove the dependence between pieces from the same excursion.

We then collect these, now independent, pieces of excursion into a point processes which we will be able to couple with a Poisson point processes ν (in fact two independent Poisson point processes) on the space of random paths in the torus, such that with high probability the trace of ν in $A_1 \cup \cdots \cup A_n$ coincides with the trace of μ (and therefore also with the trace of the random walk $Y_\cdot$ run up to time uN^d) in that set. The "excursions" of ν start in a box A_i and end upon leaving $B_i \supset A_i$ (which are disjoint), so they visit only one box and do not "feel the geometry of the torus" since B_i can be identified with a subset of $\mathbb{Z}^d$.

Also since ν is a Poisson point process we will see that we can split it into n independent Poisson point processes, one for each box A_i, such that roughly speaking the trace of ν_i in A_i coincides with high probability with that of $Y_\cdot$ (run up to time uN^d) in A_i. Thus we will have "decoupled" the boxes.

Now, as mentioned above, random interlacements are constructed from a "Poisson cloud" on a certain space on paths, and we will see that when restricted to a box A_i, a random interlacement has the law of the trace of a Poisson number of random walks. This will also basically be the law of trace of the ν_i, with the difference that the paths in ν_i do not return to A_i after leaving B_i, while for random interlacements a small but positive proportion the paths do return. By taking a small number of the paths from the ν_i, and "gluing them together" to form paths that do return to A_i after leaving B_i, we will be able to construct from the ν_i the random interlacements $(\mathcal{I}_i^v)_{v \geq 0}$ in (1.3).

We now describe how this article is organized. In Sect. 2 we introduce some notation and preliminary lemmas. In Sect. 3 we give the formal statements of the main theorems (1.2) and (1.3), and of corollaries (1.5) and (1.6). We also derive the corollaries from the main theorems. The proof of the cover time result (1.2), from (1.3), is contained in Sect. 4. The subsequent sections deal with the proof of (1.3). In Sect. 5 we introduce three further couplings and use them to construct the coupling (1.3). The first of the three, a coupling of the excursions $Y_{(R_k+\cdot) \wedge U_k}$ with the Poisson point process μ, is then constructed in Sects. 6 and 7. In Sect. 6 we define the quasistationary distribution and prove that the law of Y_{U_k} is close to it. In Sect. 7 we use this fact to construct the coupling of the excursions $Y_{(R_k+\cdot) \wedge U_k}$ with the Poisson point process μ. The second of the three couplings, a coupling of μ and the i.i.d. Poisson point processes $\nu_1, \ldots, \nu_n$, is constructed in Sect. 8. The third coupling, a coupling of a Poisson point process ν with the law of the ν_i and random interlacements, is constructed in Sect. 9. The appendix contains the proof of a certain lemma (Lemma 6.2) which is stated in Sect. 6.

We finish this section with a remark on constants. Unnamed constants are represented by c, c', etc. Note that these may represent different constants in different formulas or even within the same formula. Named constants are denoted by c_4, c_5, . . . and have fixed values. All constants are understood to be positive and, unless stated otherwise, depend only on the dimension d. Dependence on e.g. a parameter α is denoted by $c(\alpha)$ or $c_4(\alpha)$.

2 Notation and preliminary lemmas

In this section we introduce basic notation and a few preliminary lemmas.

We write $[x]$ for the integer part of $x \in [0, \infty)$. The cardinality of a set U is denoted by $|U|$.

We denote the $d-$dimensional discrete torus of side length $N \geq 3$ by $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d$ for $d \geq 1$. If $x \in \mathbb{Z}^d$ we write $|x|$ for the Euclidean norm of x and $|x|_\infty$ for the l_∞ norm of x. We take $d(\cdot, \cdot)$ to mean the distance on $\mathbb{T}_N$ induced by $|\cdot|$ and $d_\infty(\cdot, \cdot)$ to mean the distance induced by $|\cdot|_\infty$. The closed $l_\infty-$ball of radius $r \geq 0$ with centre x in $\mathbb{Z}^d$ or $\mathbb{T}_N$ is denoted by $B(x, r)$.

We define the inner and outer boundaries of a set $U \subset \mathbb{Z}^d$ or $U \subset \mathbb{T}_N$ by

$$\partial_i U = \{x \in U : d(x, U^c) = 1\}, \quad \partial_e U = \{x \in U^c : d(x, U) = 1\}. \tag{2.1}$$

For a set U we write $\Gamma(U)$ for the space of all cadlag piecewise constant functions from $[0, \infty)$ to U, with at most a finite number of jumps in any compact interval. When only

finitely many jumps occur for a function $w \in \Gamma(U)$ we set $w(\infty) = \lim_{t \to \infty} w(t)$. Usually we will work with $\Gamma(\mathbb{Z}^d)$ or $\Gamma(\mathbb{T}_N)$. We write Y_t for the canonical process on $\Gamma(\mathbb{Z}^d)$ or $\Gamma(\mathbb{T}_N)$. When $w \in \Gamma(\mathbb{Z}^d)$ or $w \in \Gamma(\mathbb{T}_N)$ we take $w(a, b)$ to mean the range $\{w(t) : t \in [a, b] \cap [0, \infty)\} \subset \mathbb{Z}^d$ or $\mathbb{T}_N$ (with this definition the range is empty if $b < 0$ or $a > b$). We let θ_t denote the canonical shift on $\Gamma(\mathbb{Z}^d)$ and $\Gamma(\mathbb{T}_N)$. The jump times of Y_t are defined by

$$\tau_0 = 0, \tau_1 = \inf\{t \geq 0 : Y_t \neq Y_0\} \quad \text{and} \quad \tau_n = \tau_1 \circ \theta_{\tau_{n-1}} + \tau_{n-1}, n \geq 2. \quad (2.2)$$

Due to the usual interpretation of the infimum of the empty set, $\tau_m = \infty$ for $m > n$ when only n jumps occur. For a set $U \subset \mathbb{Z}^d$ or $\mathbb{T}_N$ we define the entrance time, return time and exit time by

$$H_U = \inf\{t \geq 0 : Y_t \in U\}, \tilde{H}_U = \inf\{t \geq \tau_1 : Y_t \in U\}, T_U = \inf\{t \geq 0 : Y_t \notin U\}.$$
$$(2.3)$$

We let $P_x^{\mathbb{Z}^d}$ denote law on $\Gamma(\mathbb{Z}^d)$ of continuous time simple random in $\mathbb{Z}^d$ and let P_x denote the law on $\Gamma(\mathbb{T}_N)$ of continuous time simple random walk on $\mathbb{T}_N$ (so that τ_1 is an exponential random variable with parameter 1). If ν is a measure on $\mathbb{Z}^d$ we let $P_\nu^{\mathbb{Z}^d} = \sum_{x \in \mathbb{Z}^d} \nu(x) P_x^{\mathbb{Z}^d}$. We define P_ν analogously. Furthermore π denotes the uniform distribution on $\mathbb{T}_N$, and P denotes P_π, i.e. the law of simple random walk starting from the uniform distribution.

Essentially because the mixing time of the torus is of order N^2 (see Proposition 4.7, p. 50, and Theorem 5.5, p. 66 in [14]) we have that

for $N \geq 3, \lambda \geq 1, x \in \mathbb{T}_N$, a coupling $q(w, v), w, v \in \mathbb{T}_N$ exists for which the first marginal is $P_x[Y_{\lambda N^2} \in dw]$, the second is uniform, and $\sum_{w \neq v} q(w, v) \leq ce^{-c\lambda}$.
$$(2.4)$$

The Green function is defined by

$$g(x, y) = \int_0^\infty P_x^{\mathbb{Z}^d}[Y_t = y]dt, x, y \in \mathbb{Z}^d, \quad \text{and} \quad g(\cdot) = g(0, \cdot).$$

We have the following classical bounds on $g(x, y)$ (see Theorem 1.5.4, p. 31 in [12])

$$c|x - y|^{2-d} \leq g(x, y) \leq c'|x - y|^{2-d} \quad \text{for } x, y \in \mathbb{Z}^d, x \neq y, d \geq 3.$$
$$(2.5)$$

For $K \subset \mathbb{Z}^d$ we define the equilibrium measure e_K and the capacity $\text{cap}(K)$ by

$$e_K(x) = P_x^{\mathbb{Z}^d}[\tilde{H}_K = \infty]1_K(x) \quad \text{and} \quad \text{cap}(K) = \sum_{x \in \partial_i K} e_K(x). \quad (2.6)$$

It is well-known (see (2.16), Proposition 2.2.1 (a), p. 52-53 in [12]) that

$$cr^{d-2} \leq \text{cap}\ (B(0,r)) \leq c'r^{d-2} \quad \text{for } r \geq 1, d \geq 3. \tag{2.7}$$

The normalised equilibrium distribution $\frac{e_K(\cdot)}{\text{cap}(A)}$ can be thought of as the hitting distribution on K when "starting from infinity", and in this paper we will use that for all $K \subset B(0,r) \subset \mathbb{Z}^d, r \geq 1$, we have (see Theorem 2.1.3, Exercise 2.1.4 and (2.13) in [12])

$$c_1 \frac{e_K(y)}{\text{cap}(K)} \leq P_x[Y_{H_K} = y | H_K < \infty] \leq c_2 \frac{e_K(y)}{\text{cap}(K)} \quad \text{for all } y \in K, x \notin B(0, c_3 r). \tag{2.8}$$

If $K \subset U \subset \mathbb{Z}^d$, with K finite, we define the equilibrium measure and capacity of K relative to U by

$$e_{K,U}(x) = P_x^{\mathbb{Z}^d}[\tilde{H}_K > T_U]1_K(x) \quad \text{and} \quad \text{cap}_U(K) = \sum_{x \in \partial_i K} e_{K,U}(x). \tag{2.9}$$

We will need the following bounds on the probability of hitting sets in $\mathbb{Z}^d$ and $\mathbb{T}_N$.

Lemma 2.1 $(d \geq 3, N \geq 3)$

$$P_x^{\mathbb{Z}^d}[H_{B(0,r_1)} < \infty] \leq c(r_1/r_2)^{d-2} \quad \text{for } 1 \leq r_1 \leq r_2, x \notin B(0, r_2). \tag{2.10}$$

$$\sup_{x \notin B(0,r_2)} P_x[H_{B(0,r_1)} < N^{2+\lambda}] \leq c(\lambda)(r_1/r_2)^{d-2} \quad \text{for } 1 \leq r_1 \leq r_2 \leq N^{1-3\lambda}, \lambda > 0. \tag{2.11}$$

Proof (2.10) follows from Proposition 2.2.2, p. 53 in [12] and (2.7). To prove (2.11) we let $K = \cup_{y \in \mathbb{Z}^d, |y|_\infty \leq N^\lambda} B(yN, r_1) \subset \mathbb{Z}^d$ and note that by "unfolding the torus" we have

$$\sup_{z \notin B(0,r_2)} P_z[H_{B(0,r_1)} < N^{2+\lambda}] \leq \sup_{z \in B(0,\frac{N}{2}) \setminus B(0,r_2)} P_z^{\mathbb{Z}^d}[H_K < \infty]$$

$$+ P_0^{\mathbb{Z}^d}[T_{B(0,\frac{N^{1+\lambda}}{2})} \leq N^{2+\lambda}], \tag{2.12}$$

provided $N \geq c(\lambda)$. For any $z \in B(0, \frac{N}{2}) \setminus B(0, r_2)$

$$P_z^{\mathbb{Z}^d}[H_K < \infty] \leq \sum_{|y|_\infty \leq 1} P_z^{\mathbb{Z}^d}[H_{B(yN,r_1)} < \infty] + \sum_{1 < |y|_\infty \leq N^\lambda} P_z^{\mathbb{Z}^d}[H_{B(yN,r_1)} < \infty]$$

$$\overset{(2.10)}{\leq} c\,(r_1/r_2)^{d-2} + cN^{\lambda d}(r_1/N)^{d-2} \leq c(r_1/r_2)^{d-2},$$

since $N^{\lambda d}/N^{d-2} \overset{d \geq 3}{\leq} 1/N^{(1-3\lambda)(d-2)} \overset{r_2 \leq N^{1-3\lambda}}{\leq} 1/r_2^{d-2}$. Furthermore

$$P_0^{\mathbb{Z}^d}\left[T_{B(0,\frac{N^{1+\lambda}}{2})} \leq N^{2+\lambda}\right] \leq P_0^{\mathbb{Z}^d}\left[|Y_{\tau_n}| > \frac{N^{1+\lambda}}{2} \quad \text{for an } n \leq 2N^{2+\lambda}\right]$$
$$+ P_0^{\mathbb{Z}^d}\left[\tau_{[2N^{2+\lambda}]} \leq N^{2+\lambda}\right].$$

But by applying Azuma's inequality (Theorem 7.2.1, p. 99 in [4]) to the martingale Y_{τ_n} we get that the first probability on the right-hand side is bounded above by $cN^c e^{-cN^\lambda}$, and by a standard large deviations bound the second probability is bounded above by $e^{-cN^{2+\lambda}}$, so since $cN^c e^{-cN^\lambda} \leq (r_1/r_2)^{d-2}$ for $N \geq c(\lambda)$ we get (2.11). □

Using Lemma 2.1 we get the following bounds for equilibrium measures.

Lemma 2.2 ($d \geq 3$, $N \geq 3$)

$$e_K(x) \geq cr^{-1}, \quad \text{for } x \in \partial_i K, \text{ where } K = B(0,r), r \geq 1. \tag{2.13}$$

$$e_K(x) \leq e_{K,U}(x), \quad \text{for all } x \in K \subset U \subset B\left(0, \frac{N}{4}\right) \subset \mathbb{T}_N. \tag{2.14}$$

Furthermore if $K \subset B(0,r) \subset U = B(0,r^{1+\lambda}) \subset B(0,\frac{N}{4}) \subset \mathbb{T}_N$, $r \geq 1, \lambda > 0$, we have

$$e_{K,U}(x) \leq (1 + c(\lambda)r^{-\lambda})e_K(x), \quad \text{for all } x \in K. \tag{2.15}$$

Proof For a large enough constant c' we have $\inf_{x \in \partial_i B(0,c'r)} P_x^{\mathbb{Z}^d}[H_{B(0,r)} = \infty] \geq \frac{1}{2}$ (by (2.10)), so (2.13) follows from $\inf_{x \in \partial_i B(0,r)} P_x[\tilde{H}_K > T_{B(0,c'r)}] \geq cr^{-1}$ (which is a result of a one dimensional simple random walk estimate) and the strong Markov property. The second inequality (2.14) is obvious from (2.6) and (2.9). Finally for (2.15) note that $e_{K,U}(x) = e_K(x) + P_x^{\mathbb{Z}^d}[T_U < \tilde{H}_K < \infty]$ (by (2.6) and (2.9)), and $P_x^{\mathbb{Z}^d}[T_U < \tilde{H}_K < \infty] \leq P_x^{\mathbb{Z}^d}[T_U < \tilde{H}_K]\sup_{x \in \partial_e U} P_x^{\mathbb{Z}^d}[H_K < \infty] \leq cr^{-\lambda} P_x^{\mathbb{Z}^d}[T_U < \tilde{H}_K]$ (by (2.10)). Now $\inf_{x \in \partial_e U} P_x^{\mathbb{Z}^d}[H_K = \infty]$ is always positive and at least $\frac{1}{2}$ when $r \geq c(\lambda)$ (by (2.10)), so in fact $P_x^{\mathbb{Z}^d}[T_U < \tilde{H}_K < \infty] \leq c(\lambda)r^{-\lambda} P_x^{\mathbb{Z}^d}[T_U < \tilde{H}_K]\inf_{x \in \partial_e U} P_y^{\mathbb{Z}^d}[H_K = \infty] \leq c(\lambda)r^{-\lambda}e_K(x)$, so (2.15) follows. □

We now introduce some notation related to Poisson point processes. Let $\Gamma = \Gamma(\mathbb{T}_N)$ or $\Gamma = \Gamma(\mathbb{Z}^d)$. When μ is a Poisson point process on Γ^i, $i \geq 1$, we denote the trace of μ by

$$\mathcal{I}(\mu) = \bigcup_{(w_1,\ldots w_i) \in \text{Supp}(\mu)} \bigcup_{j=1}^i w_i(0,\infty) \subset \mathbb{Z}^d \quad \text{or} \quad \mathbb{T}_N. \tag{2.16}$$

We will mostly consider Poisson point processes μ on Γ where this simplifies to $\mathcal{I}(\mu) = \bigcup_{w \in \text{Supp}(\mu)} w(0,\infty)$, but in Sects. 8 and 9 we will also consider Poisson point processes μ on Γ^i, $i \geq 2$. If μ is a Poisson point process of labelled paths (that is a Poisson point process on $\Gamma \times [0,\infty)$, where Γ is as above) we denote the trace up to label u by

$$\mathcal{I}^u(\mu) = \mathcal{I}(\mu_u) \subset \mathbb{Z}^d \text{ or } \mathbb{T}_N, \text{ where } \mu_u(dw) = \mu(dw \times [0, u]). \quad (2.17)$$

Next let us recall some facts about random interlacements. They are, roughly speaking, defined as a Poisson point process on a certain space of labelled doubly-infinite trajectories modulo time-shift. The *random interlacement at level u*, or $\mathcal{I}^u \subset \mathbb{Z}^d$, is the trace of trajectories with labels up to $u \geq 0$. The family of random subsets $(\mathcal{I}^u)_{u \geq 0}$ is called a *random interlacement*. The rigorous definitions and construction that make this informal description precise can be found in Section 1 of [23] or Section 1 of [20]. In this article we will only use the facts (2.18)–(2.23) which now follow.

There exists a space $(\Omega_0, \mathcal{A}_0, Q_0)$ and a family $(\mathcal{I}^u)_{u \geq 0}$ of random subsets of $\mathbb{Z}^d$ such that $(\mathcal{I}^u \cap K)_{u \geq 0} \stackrel{\text{law}}{=} (\mathcal{I}^u(\mu_K) \cap K)_{u \geq 0}$ for all finite $K \subset \mathbb{Z}^d$, $\quad (2.18)$

where μ_K is a Poisson point process on $\Gamma(\mathbb{Z}^d) \times [0, \infty)$ of intensity $P_{e_K}^{\mathbb{Z}^d} \otimes \lambda$, $\quad (2.19)$

and λ denotes Lebesgue measure (see (0.5), (0.6), (0.7) in [23], also cf. (1.67) in [23]).

(From (2.18) and (2.19) we see that to "simulate" $\mathcal{I}^u \cap K$ for a finite $K \subset \mathbb{Z}^d$ one simply samples an integer $n \geq 0$ according to the Poisson distribution with parameter $u\text{cap}(K)$, picks n random starting points $Z_1, \ldots, Z_n$ according to the normalized equilibrium distribution $\frac{e_K(\cdot)}{\text{cap}(K)}$ and runs from each starting point Z_i an independent random walk, recording the trace the walks leave in the set K. The "full" random interlacement $\mathcal{I}^u$ can be seen, intuitively speaking, as a "globally consistent" version of the traces of the Poisson point processes μ_K.)

Random interlacements also satisfy (see above (0.5), (0.7), below (0.8) and (1.48) in [23])

the law of $\mathcal{I}^u$ under Q_0 is translation invariant for all $u \geq 0$, $\quad (2.20)$

$\mathcal{I}^u$ is increasing in the sense that $Q_0 -$ almost surely $\mathcal{I}^v \subset \mathcal{I}^u$ for $v \leq u$, $\quad (2.21)$

if $\mathcal{I}_1^u$ and $\mathcal{I}_2^v$ are independent with the laws under Q_0 of $\mathcal{I}^u$ and $\mathcal{I}^v$ $\quad (2.22)$ respectively, then $(\mathcal{I}_1^u, \mathcal{I}_1^u \cup \mathcal{I}_2^v)$ has the law of $(\mathcal{I}^u, \mathcal{I}^{u+v})$ under Q_0.

Finally (see (1.58) and (1.59) of [23])

$$Q_0[x \notin \mathcal{I}^u] = \exp\left(-\frac{u}{g(0)}\right) \text{ and } Q_0[x, y \notin \mathcal{I}^u] = \exp\left(-\frac{2u}{g(0) + g(x - y)}\right), x, y \in \mathbb{Z}^d. \quad (2.23)$$

(The properties (2.21)–(2.23) in fact also follow from (2.18) and (2.19)).

The following lemma, which follows from Lemma 1.5 from [6] (by letting a in (1.39) in [6] go to infinity, and using (2.23)), will be crucial in our proof of (1.2).

Lemma 2.3 ($d \geq 3$) *There is a constant $c_4 > 1$ such that or any $K \subset \mathbb{Z}^d$ with $0 \notin K$*

$$\sum_{v \in K} Q_0[0, v \notin \mathcal{I}^u] \leq |K| \left(Q_0[0 \notin \mathcal{I}^u] \right)^2 \{1 + cu\} + ce^{-c_4 \frac{u}{g(0)}}, \quad \text{for all } u \geq 0.$$

(2.24)

Finally we define the cover time C_F of a set $F \subset \mathbb{T}_N$ by

$$C_F \stackrel{\text{def}}{=} \max_{x \in F} H_x = \inf\{t \geq 0 : F \subset Y(0, t)\}.$$

(2.25)

Note that $C_N = C_{\mathbb{T}_N}$, cf. (1.1). For convenience we introduce the notation

$$u_F(z) = g(0)\{\log |F| + z\},$$

(2.26)

so that $\{\frac{C_N}{g(0)N^d} - \log N^d \leq z\} = \{C_N \leq u_{\mathbb{T}_N}(z)N^d\}$, cf. (1.2).

3 Gumbel fluctuations, coupling with random interlacements and corollaries

In this section we state our two main theorems and derive two corollaries. The first main result is Theorem 3.1, which, roughly speaking, says that the cover times of large subsets of the torus (for $d \geq 3$) have Gumbel fluctuations, (and implies (1.2) from the introduction). The second main result is Theorem 3.2 and it states the coupling of random walk in the torus and random interlacements (see (1.3) in the introduction). The proofs of the theorems are contained in the subsequent sections.

The first corollary is Corollary 3.4 which in essence says that the "point process of vertices covered last" (see (1.4)) converges in law to a Poisson point process as the side length of the torus goes to infinity (see (1.5)). The second corollary is Corollary 3.5 and roughly says that for any fixed $k \geq 1$ the last k vertices to be hit by the random walk are far apart, at distance of order N.

We now state our result about fluctuations of the cover time. Recall the notation from (2.25) and (2.26).

Theorem 3.1 ($d \geq 3, N \geq 3$) *For all $F \subset \mathbb{T}_N$ we have*

$$\sup_{z \in \mathbb{R}} \left| P[C_F \leq N^d u_F(z)] - \exp(-e^{-z}) \right| \leq c|F|^{-c}.$$

(3.1)

We will prove Theorem 3.1 in Sect. 4.

Next we will state the coupling result. For $n \geq 1$ and $x_1, \ldots, x_n \in \mathbb{T}_N$ we define the

separation s of the vertices $x_1, \ldots, x_n \in \mathbb{T}_N$, by $s = \begin{cases} N & \text{if } n = 1, \\ \min_{i \neq j} d_\infty(x_i, x_j) & \text{if } n > 1. \end{cases}$

$$(3.2)$$

For an arbitrarily small $\varepsilon > 0$ which does not depend on N we define the box

$$A = B(0, s^{1-\varepsilon}). \tag{3.3}$$

The result will couple the trace of random walk in the boxes $A + x_1, \ldots, A + x_n$ with independent random interlacements. Note that thanks to (3.2) and (3.3) the boxes are "far part" (in the sense that the distance between them, of order s, is "much larger" than their radius, which is $s^{1-\varepsilon}$) and "at most mesoscopic" (in that their radius, at most $N^{1-\varepsilon}$, is "much smaller" than the side-length N of the torus). Given a level $u > 0$ and a $\delta > 0$, with these parameters satisfying appropriate conditions, we will construct independent random interlacements $(\mathcal{I}_1^v)_{v \geq 0}, \ldots, (\mathcal{I}_n^v)_{v \geq 0}$ such that (recall the notation $Y(a, b)$ from above (2.2))

$$Q_1[\mathcal{I}_i^{u(1-\delta)} \cap A \subset (Y(0, uN^d) - x_i) \cap A \subset \mathcal{I}_i^{u(1+\delta)}] \geq 1 - c_5 u e^{-c s^{c_5}} \quad \text{for all } i.$$

$$(3.4)$$

Formally we have

Theorem 3.2 ($d \geq 3, N \geq 3$) *For $n \geq 1$ let $x_1, \ldots, x_n \in \mathbb{T}_N$ be distinct and have separation s (see (3.2)), and let $\varepsilon \in (0, 1)$. Further assume $u \geq s^{-c_5}, 1 \geq \delta \geq \frac{1}{c_5} s^{-c_5}$, $n \leq s^{c_5}$, where $c_5 = c_5(\varepsilon)$. We can then construct a space $(\Omega_1, \mathcal{A}_1, Q_1)$ with a random walk Y. with law P and independent random interlacements $(\mathcal{I}_i^v)_{v \geq 0}, i = 1, \ldots, n$, each with the law of $(\mathcal{I}^v)_{v \geq 0}$ under Q_0, such that (3.4) holds.*

Theorem 3.2 will be proved in Sect. 5.

In the proof of Corollary 3.4 we will need the following estimate on the probability of hitting a point, which is a straight-forward consequence of Theorem 3.2, (when $n = 1$).

Lemma 3.3 ($d \geq 3, N \geq 3$) *There exists a constant c_6 such that if $N^{-c_6} \leq u \leq N^{c_6}$ then for all $x \in \mathbb{T}_N$*

$$Q_0[0 \notin \mathcal{I}^u](1 - cN^{-c}) \leq P[x \notin Y(0, uN^d)] \leq Q_0[0 \notin \mathcal{I}^u](1 + cN^{-c}). \tag{3.5}$$

Proof We apply Theorem 3.2 with $n = 1$ (so that the separation s is N), $x_1 = 0$, $\varepsilon = \frac{1}{4}$ (say) and $\delta = c_5^{-1} N^{-c_5}$ (where $c_5 = c_5(\frac{1}{4})$ is the constant from Theorem 3.2). By choosing $c_6 \leq c_5$ we have $u \geq s^{-c_5}$, so that all the conditions of Theorem 3.2 are satisfied and the coupling Q_1 of Y. and random interlacements can be constructed.

Therefore it follows from (3.4) that

$$Q_0[0 \notin \mathcal{I}^{u(1+\delta)}] - cue^{-cN^{c_5}} \leq P[0 \notin Y(0, uN^d)] \leq Q_0[0 \notin \mathcal{I}^{u(1-\delta)}] + cue^{-cN^{c_5}}.$$
$$(3.6)$$

But we also have, if we pick $c_6 < c_5$, that

$$Q_0[0 \notin \mathcal{I}^{u(1-\delta)}] + cue^{-N^{c_5}} \overset{(2.23)}{=} Q_0[0 \notin \mathcal{I}^u] \left(e^{\frac{\delta u}{g(0)}} + cue^{\frac{u}{g(0)} - cN^{c_5}} \right)$$
$$\leq Q_0[0 \notin \mathcal{I}^u](1 + cN^{-c}),$$

since $cue^{\frac{u}{g(0)} - cN^{c_5}} \leq cN^c e^{-cN^c}$ and $\delta u \leq cN^{c_6 - c_5} \leq cN^{-c}$ (recall $u \leq N^{c_6}$, $\delta = cN^{-c_5}$ and $c_6 < c_5$). Similarly if $c_6 < c_5$ we have $Q_0[0 \notin \mathcal{I}^{u(1+\delta)}] - cue^{-cN^{c_5}} \geq Q_0[0 \notin \mathcal{I}^u](1 - cN^{-c})$, so (3.5) follows. □

We now state and prove the first corollary. The proof uses Theorem 3.1 and Theorem 3.2 (via Lemma 3.3). Recall the definition of $\mathcal{N}_N^z$ from (1.4).

Corollary 3.4 $(d \geq 3)$ (1.5) *holds.*

Proof By Kallenberg's Theorem (Proposition 3.22, p. 156 of [17]) the result follows from

$$\lim_{N \to \infty} P[\mathcal{N}_N^z(I) = 0] = \exp(-\lambda(I)e^{-z}) \quad \text{and} \tag{3.7}$$

$$\lim_{N \to \infty} E[\mathcal{N}_N^z(I)] = \lambda(I)e^{-z}, \tag{3.8}$$

for all I in the collection $\mathcal{J} = \{I : I \text{ a finite union of products of open intervals in } (\mathbb{R}/\mathbb{Z})^d, \lambda(I) > 0\}$. Note that

$$\lim_{N \to \infty} \frac{|NI \cap \mathbb{T}_N|}{|\mathbb{T}_N|} = \lambda(I) \quad \text{for all } I \in \mathcal{J}, \tag{3.9}$$

since $|NI \cap \mathbb{T}_N|/|\mathbb{T}_N|$ is the mass assigned to the set I by the measure $\sum_{x \in \mathbb{T}_N} N^{-d} \delta_{x/N}$ on $(\mathbb{R}/\mathbb{Z})^d$, which converges weakly to λ as $N \to \infty$ (note that I is a continuity set of λ, and see the Portmanteau theorem, Proposition 5.1, p. 9, [18]).

Fix an $I \in \mathcal{J}$. To check (3.7) note that $\{\mathcal{N}_N^z(I) = 0\} = \{C_{NI \cap \mathbb{T}_N} \leq N^d u_{\mathbb{T}_N}(z)\}$ (see (1.4), (2.25) and (2.26)). Since $u_{\mathbb{T}_N}(z) = u_{NI \cap \mathbb{T}_N}(z - \log \frac{|NI \cap \mathbb{T}_N|}{N^d})$ (see (2.26)) and $|NI \cap \mathbb{T}_N| \to \infty$ as $N \to \infty$ (recall $\lambda(I) > 0$) we get from (3.1) that

$$\left| P[\mathcal{N}_N^z(I) = 0] - \exp\left(-e^{-(z - \log \frac{|NI \cap \mathbb{T}_N|}{N^d})} \right) \right| \to 0 \text{ as } N \to \infty.$$

But by (3.9) we have $\exp(-e^{-(z - \log \frac{|NI \cap \mathbb{T}_N|}{N^d})}) \to \exp(-\lambda(I)e^{-z})$ as $N \to \infty$ so (3.7) follows.

To check (3.8) note that by (1.4) and (2.26)

$$E[\mathcal{N}_N^z(I)] = |NI \cap \mathbb{T}_N| P[0 \notin Y(0, N^d u_{\mathbb{T}_N}(z))]. \tag{3.10}$$

Using (3.5) with $u = u_{\mathbb{T}_N}(z)$ ($\in [N^{-c_6}, N^{c_6}]$ for $N \geq c(z)$) and $Q_0[0 \notin \mathcal{I}^{u_{\mathbb{T}_N}(z)}] = \frac{e^{-z}}{N^d}$ (see (2.23)) we get from (3.10) that

$$e^{-z} \lim_{N \to \infty} \frac{|NI \cap \mathbb{T}_N|}{N^d} (1 - cN^{-c}) \leq \lim_{N \to \infty} E[\mathcal{N}_N^z(I)]$$
$$\leq e^{-z} \lim_{N \to \infty} \frac{|NI \cap \mathbb{T}_N|}{N^d} (1 + cN^{-c}).$$

Now using (3.9) we find that (3.8) holds. Thus the proof of (1.5) is complete. □

See Remark 9.4 for a potential generalisation of Corollary 3.4. We now state the second corollary, which is a consequence of the first. The proof follows those of Corollary 2.3 of [6] and Proposition 2.8 of [5], so we omit some details.

Corollary 3.5 ($d \geq 3$) *Let $Z_1, \ldots, Z_{N^d}$, be the vertices of $\mathbb{T}_N$ ordered by entrance time,*

$$so \ that \ C_{\mathbb{T}_N} = H_{Z_1} > H_{Z_2} > \cdots > H_{Z_{N^d}}.$$

Then for any $k \geq 2$ and $0 < \delta \leq \frac{1}{4}$

$$\lim_{\delta \to 0} \limsup_{N \to \infty} P[\exists 1 \leq i < j \leq k \ with \ d(Z_i, Z_j) \leq \delta N] = 0, \tag{3.11}$$

or in other words "the last k vertices to be hit are far apart, at distance of order N".

Proof Note that for any N, δ, k and $z \in \mathbb{R}$ the probability in (3.11) is bounded above by

$$P[\exists x, y \in \text{Supp} \ (\mathcal{N}_N^z) \ with \ d(x, y) \leq \delta N] + P[\mathcal{N}_N^z((\mathbb{R}/\mathbb{Z})^d) < k]. \tag{3.12}$$

Furthermore the first probability in (3.12) is bounded above by $E[\mathcal{N}_N^z \otimes \mathcal{N}_N^z(g) - \mathcal{N}_N^z((\mathbb{R}/\mathbb{Z})^d)]$, where g is the function $(x, y) \to f(x - y)$, for $f : (\mathbb{R}/\mathbb{Z})^d \to [0, 1]$ continuous with $f(x) = 1$ when $d(0, x) \leq \delta$ and $f(x) = 0$ if $d(0, x) \geq 2\delta$. Thus using (1.5), one sees that the limsup in (3.11) is bounded above by

$$\mathbb{E}[\mathcal{N}^z \otimes \mathcal{N}^z(g) - \mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d)] + \mathbb{P}[\mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d) < k]. \tag{3.13}$$

Using a calculation involving Palm measures (we omit the details, cf. (2.19) of [6]) the first term in (3.13) can be shown to be equal to $(e^{-z} \int_{(\mathbb{R}/\mathbb{Z})^d} f(x)dx)^2$. Since $\int_{(\mathbb{R}/\mathbb{Z})^d} f(x)dx \leq c\delta^d$ one thus finds that for all $z \in \mathbb{R}$ the left-hand side of (3.11) is bounded above by $\mathbb{P}[\mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d) < k]$. But $\mathbb{P}[\mathcal{N}^z((\mathbb{R}/\mathbb{Z})^d) < k] \to 0$ as $z \to -\infty$, so (3.11) follows. □

4 Coupling gives Gumbel fluctuations

In this section we use the coupling result Theorem 3.2 to prove Theorem 3.1, i.e. to prove that cover times in $\mathbb{T}_N$ have Gumbel fluctuations. Essentially speaking we combine the method of the proofs of Theorem 0.1 in [5] and Theorem 0.1 in [6] with the coupling Theorem 3.2.

The first step is to prove that if $F \subset \mathbb{T}_N$ is smaller than N^c for some small exponent c (but still "large") and consists of isolated vertices separated by a distance at least $|F|^c$, then the cover time C_F is approximately distributed as $N^d\{g(0)\log|F| + G\}$, where G is a standard Gumbel random variable. We prove this (uniformly in the starting vertex of the random walk) in Lemma 4.1. To prove Lemma 4.1 we use a mixing argument to reduce to the case when random walk starts from the uniform distribution. This case is then proven in Lemma 4.4 using the fact that for vertices that are far apart, entrance times are approximately independent (which is proven in Lemma 4.3, using the coupling result Theorem 3.2), and from this a simple calculation will show that the cover time (which is then the maximum of almost i.i.d. random variables) has distribution close to the Gumbel distribution.

To get the Gumbel limit result for arbitrary $F \subset \mathbb{T}_N$ (e.g. for $F = \mathbb{T}_N$, where the entrance times of close vertices are far from being independent) we consider the set of $(1 - \rho)-$*late points* (using the terminology from [10]), which is the set of vertices that are not yet hit at a $1 - \rho$ fraction of the typical time it takes to cover F, or more formally

$$F_\rho = F \backslash Y(0, t(\rho)), \text{ where } t(\rho) = N^d(1 - \rho)g(0)\log|F| \text{ and } 0 < \rho < 1. \quad (4.1)$$

It turns out, roughly speaking, that if we use a fixed but small ρ then with high probability this set consists of isolated vertices that are at distance at least $|F|^c$ from one another, and that $|F_\rho|$ concentrates around $E[|F_\rho|]$, which we will see, is close to $|F|^\rho$. This is done in Lemma 4.2 by relating the probability of two vertices not being hit by random walk to the probability of two vertices not being in a random interlacement using the coupling (in Lemma 4.5), and using Lemma 2.3.

This will imply that with high probability the set F_ρ fulfils the conditions of Lemma 4.1 (i.e. is "smaller than N^c" and separated), so that, using the Markov property, we will be able to show that conditioned on F_ρ the time $C_{F_\rho} \circ \theta_{t(\rho)}$ has distribution close to $N^d g(0)\{\log|F_\rho| + G\} \approx N^d g(0)\{\rho\log|F| + G\}$, where G is a standard Gumbel random variable. Since, on the event that F_ρ is non-empty (which has probability close to one), $C_F = t(\rho) + C_{F_\rho} \circ \theta_{t(\rho)}$ we will be able to conclude that C_F has distribution close to $N^d g(0)\{\log|F| + G\}$, which is the claim of Theorem 3.1.

We start by stating Lemma 4.1, which says that the cover time has law close to the Gumbel distribution for "well-separated" sets that are not too large.

Lemma 4.1 ($d \geq 3$, $N \geq 3$) *There are constants c_7 and c_8 such that if $F \subset \mathbb{T}_N$ satisfies $2 \leq |F| \leq N^{c_7}$ and $d_\infty(x, y) \geq |F|^{c_8}$ for all $x, y \in F$, $x \neq y$, then*

$$\sup_{z \in \mathbb{R}, x \in \mathbb{T}_N} \left| P_x[F \subset Y(0, u_F(z)N^d)] - \exp(-e^{-z}) \right| \leq c|F|^{-c}. \quad (4.2)$$

(Recall that we have defined the range $Y(0, u_F(z)N^d)$ such that it is empty if $u_F(z)N^d < 0$.) We will prove Lemma 4.1 after proving Theorem 3.1. To prove Theorem 3.1 we must prove something like (4.2) for arbitrary $F \subset \mathbb{T}_N$. We do so by studying the set of late points F_ρ (recall (4.1)). We will show that "with high probability" it belongs to the collection $\mathcal{G}$ of "good subsets of F", where

$$\mathcal{G} = \{F' \subset F : \left||F'| - |F|^\rho\right| \leq |F|^{\frac{2}{3}\rho} \quad \text{and} \quad \inf_{x,y \in F', x \neq y} d_\infty(x, y) \geq |F|^{\frac{1}{2d}}\}, \quad (4.3)$$

or in other words that it has cardinality close to $|F|^\rho$ and is "well-separated". Formally:

Lemma 4.2 ($d \geq 3, N \geq 3$) *There exists a constant c_9 such that for $0 < \rho \leq c_9$ and $F \subset \mathbb{T}_N$ with $|F| \geq c(\rho)$*

$$P[F_\rho \notin \mathcal{G}] \leq c|F|^{-c(\rho)}. \quad (4.4)$$

Before proving Lemma 4.2 we use it to prove Theorem 3.1. Recall that $\{C_F \leq t\} \overset{(2.25)}{=} \{F \subset Y(0, t)\}$.

Proof of Theorem 3.1 By (4.4) we have for $0 < \rho \leq c_9$ and $|F| \geq c(\rho)$

$$\left| P[C_F \leq u_F(z)N^d] - P[F \subset Y(0, u_F(z)N^d), F_\rho \in \mathcal{G}] \right| \leq c|F|^{-c(\rho)}. \quad (4.5)$$

Also by the Markov property, if $|F| \geq c(\rho)$ (so that $\emptyset \notin \mathcal{G}$),

$$P[F \subset Y(0, u_F(z)N^d), F_\rho \in \mathcal{G}]$$
$$\overset{(3.1)}{=} \sum_{x \in \mathbb{T}_N, F' \in \mathcal{G}} P[F_\rho = F', Y_{t(\rho)} = x] P_x[F' \subset Y(0, u_F(z)N^d - t(\rho))]. \quad (4.6)$$

Set $h = \log \frac{|F'|}{|F|^\rho}$ so that $P_x[F' \subset Y(0, u_F(z)N^d - t(\rho))] = P_x[F' \subset Y(0, u_{F'}(z - h)N^d)]$ (see (2.26)).

Also fix $\rho = c \leq c_9$ small enough so that $2d\rho \leq c_7$ and $\frac{1}{4d\rho} \geq c_8$. Then (4.5) holds when $|F| \geq c$. Furthermore Lemma 4.1 applies to all $F' \in \mathcal{G}$ when $|F| \geq c$, since by (4.3) every $F' \in \mathcal{G}$ satisfies $|F'| \leq 2|F|^\rho \leq |F|^{2\rho}$, if $|F| \geq c = c(\rho)$, so that $2d\rho \leq c_7$ implies $|F'| \leq |F|^{2\rho} \leq N^{2d\rho} \leq N^{c_7}$ and $\frac{1}{4d\rho} \geq c_8$ implies $\inf_{x,y \in F', x \neq y} d_\infty(x, y) \geq |F|^{\frac{1}{2d}} \geq |F'|^{\frac{1}{4d\rho}} \geq |F'|^{c_8}$. So applying Lemma 4.1 with F' in place of F, we get that for all $|F| \geq c$, $x \in \mathbb{T}_N$ and $F' \in \mathcal{G}$ we have

$$\left| P_x[F' \subset Y(0, u_F(z)N^d - t(\rho))] - \exp(-e^{-(z-h)}) \right| \leq c|F|^{-c}. \quad (4.7)$$

But it is elementary that

$$\left| \exp(-e^{-(z-h)}) - \exp(-e^{-z}) \right| \leq c|h| \quad \text{for all } z, h \in \mathbb{R}, \quad (4.8)$$

and for the present h we have $|h| \leq \max(\log(1 + |F|^{-\frac{1}{3}\rho}), -\log(1 - |F|^{-\frac{1}{3}\rho})) \leq c|F|^{-\frac{1}{3}\rho}$ (see (4.3)) provided $|F| \geq c$, so in fact $|P_x[F' \subset Y(0, u_F(z)N^d - t(\rho))] - \exp(-e^{-z})| \leq c|F|^{-c}$ for all $F' \in \mathcal{G}$. Thus (4.6) implies that

$$|P[F \subset Y(0, u_F(z)N^d), F_\rho \in \mathcal{G}] - \exp(-e^{-z})P[F_\rho \in \mathcal{G}]| \leq c|F|^{-c}.$$

Combining this with (4.5) and one more application of (4.4), the claim (3.1) follows for $|F| \geq c$ (recall that ρ is itself a constant). But by adjusting constants (3.1) holds for all F, so the proof of Theorem 3.1 is complete. □

We now turn to the proof of Lemma 4.1. We will need the following lemma, which says that "distant vertices have almost independent entrance times".

Lemma 4.3 ($d \geq 3$, $N \geq 3$) *Let $x_1, \ldots, x_n \in \mathbb{T}_N$ and let s be the separation defined as in (3.2). There is a constant c_{10} such that if $n \leq s^{c_{10}}$ and $u \in [1, s^{c_{10}}]$ we have*

$$\left(Q_0[0 \in \mathcal{I}^u]\right)^n - cs^{-c} \leq P[x_1, \ldots, x_n \in Y(0, uN^d)] \leq \left(Q_0[0 \in \mathcal{I}^u]\right)^n + cs^{-c}. \tag{4.9}$$

Proof We apply Theorem 3.2 with $\varepsilon = \frac{1}{2}$ (say). We pick $c_{10} \leq \frac{c_5}{3}$, where $c_5 = c_5(\frac{1}{2})$ is the constant from Theorem 3.2, so that that $n \leq s^{c_5}$. Thus if $\delta = c_5^{-1}s^{-c_5}$ and $s \geq c$ (so that $u \geq 1 \geq s^{-c_5}$) all the conditions of Theorem 3.2 are satisfied. By (3.4) we have for $s \geq c$

$$Q_1[0 \in \mathcal{I}_i^{u(1-\delta)}\forall i] - cs^{-c} \leq P[x_1, \ldots, x_n \in Y(0, uN^d)]$$
$$\leq Q_1[0 \in \mathcal{I}_i^{u(1+\delta)}\forall i] + cs^{-c}, \tag{4.10}$$

where we also use that $cnue^{-cs^{c_5}} \leq cs^{-c}$, since $u, n \leq s^c$. By the independence of $\mathcal{I}_1, .., \mathcal{I}_n$,

$$
\begin{aligned}
Q_1[0 \in \mathcal{I}_i^{u(1+\delta)}\forall i] &= \left(Q_0[0 \in \mathcal{I}^{u(1+\delta)}]\right)^n \\
&\stackrel{(2.23)}{=} \left(Q_0[0 \in \mathcal{I}^u]\right)^n \left(\frac{1 - e^{-\frac{u(1+\delta)}{g(0)}}}{1 - e^{-\frac{u}{g(0)}}}\right)^n \\
&\leq \left(Q_0[0 \in \mathcal{I}^u]\right)^n (1 + cs^{-c}),
\end{aligned}
$$

where we use that

$$\frac{1 - e^{-\frac{u(1+\delta)}{g(0)}}}{1 - e^{-\frac{u}{g(0)}}} = 1 + e^{-\frac{u}{g(0)}} \frac{1 - e^{-\frac{u\delta}{g(0)}}}{1 - e^{-\frac{u}{g(0)}}} \leq 1 + c(1 - e^{-\frac{u\delta}{g(0)}}) \leq 1 + cs^{-2c_{10}}$$

(note $u \geq 1$ and $u\delta \leq cs^{c_{10}-c_5} = cs^{-2c_{10}}$) and $(1 + cs^{-2c_{10}})^n \leq 1 + cs^{-c}$ (note $n \leq s^{c_{10}}$). Similarly

$$Q_1[0 \in \mathcal{I}_i^{u(1-\delta)}\forall i] \geq \left(Q_0[0 \in \mathcal{I}^u]\right)^n (1 - cs^{-c}) \quad \text{if } s \geq c.$$

Applying these inequalities to the right- and left-hand sides of (4.10) yields (4.9) for $s \geq c$. But by adjusting constants in (4.9) the same holds for all $s \geq 1$. □

We will now use Lemma 4.3 to show (4.11), which is almost like our goal (4.2), but has random walk starting from the uniform distribution.

Lemma 4.4 ($d \geq 3, N \geq 3$) *There are constants c_{11} and c_{12}, such that for $F \subset \mathbb{T}_N$ satisfying $|F| \leq N^{c_{11}}$ and $d_\infty(x, y) \geq |F|^{c_{12}}$ for all $x, y \in F$, $x \neq y$, we have*

$$\sup_{z \in \mathbb{R}} \left| P[F \subset Y(0, u_F(z)N^d)] - \exp(-e^{-z}) \right| \leq c|F|^{-c}. \tag{4.11}$$

Proof The claim (4.11) follows from

$$\sup_{z \in [-\frac{\log|F|}{4}, \frac{\log|F|}{4}]} \left| P[F \subset Y(0, u_F(z)N^d)] - \exp(-e^{-z}) \right| \leq c|F|^{-c}, \tag{4.12}$$

since $P[F \subset Y(0, u_F(z))]$ is monotone in z, $\exp(-e^{-z}) \leq c|F|^{-c}$ when $z \leq -\frac{\log|F|}{4}$ and $\exp(-e^{-z}) \geq 1 - c|F|^{-c}$ when $z \geq \frac{\log|F|}{4}$. For the rest of the proof we therefore assume that $z \in [-\frac{\log|F|}{4}, \frac{\log|F|}{4}]$.

First, assume also that $|F| \geq c$. Let $F = \{x_1, \ldots, x_n\}$ so that $n = |F|$ and the separation s satisfies $s \geq |F|^{c_{12}}$. Also let $u = u_F(z)$. To be able to apply Lemma 4.3 we pick c_{12} large enough so that $c_{12}c_{10} \geq 1$, and thus $|F| \leq |F|^{c_{12}c_{10}} \leq s^{c_{10}}$, which implies $n \leq s^{c_{10}}$ and $1 \leq g(0)\frac{3}{4} \log|F| \leq u \leq g(0)\frac{5}{4} \log|F| \leq |F| \leq s^{c_{10}}$ since $z \in [-\frac{1}{4} \log|F|, \frac{1}{4} \log|F|]$ (recall that we assumed $|F| \geq c$). Thus by (4.9)

$$\left| P[F \subset Y(0, u_F(z)N^d)] - \left(Q_0[0 \in \mathcal{I}^{u_F(z)}] \right)^{|F|} \right| \leq s^{-c} \leq c|F|^{-c} \quad \text{for} \ |F| \geq c. \tag{4.13}$$

We have $(Q_0[0 \in \mathcal{I}^{u_F(z)}])^{|F|} = (1 - \frac{e^{-z}}{|F|})^{|F|}$ by (2.23). But it is elementary that

$$\exp(-e^{-z}) - c|F|^{-c} \leq \left(1 - \frac{e^{-z}}{|F|}\right)^{|F|} \leq \exp(-e^{-z}) \ \text{for} \ z \geq -\frac{1}{4} \log|F| \ \text{and} \ |F| \geq c,$$

(since $-\frac{e^{-z}}{|F|} - c|F|^{-\frac{1}{2}} \leq \log(1 - \frac{e^{-z}}{|F|}) \leq -\frac{e^{-z}}{|F|}$ using the Taylor expansion of $\log(1-x)$ and $e^{-z} \leq |F|^{\frac{1}{4}}$). Thus (4.12) follows for $|F| \geq c$. Finally by adjusting constants (4.12) holds for any F, so the proof of the lemma is complete. □

Now we use (4.11) to get the desired result (4.2) (where the random walk can start from any vertex). To do this we roughly speaking show that in time $|F|N^2$ the random walk will, with high probability, hit only the vertex in F closest to the starting point, if it hits any vertices at all. But it will turn out that in time $|F|N^2$ the random walk mixes so that what happens after this time is governed by (4.11), and from this (4.2) will follow.

Proof of Lemma 4.1 Fix $x \in \mathbb{T}_N$ and $z \in \mathbb{R}$. By (2.4) we can (on an extended probability space $(\Omega, \mathcal{A}, Q)$)) construct a coupling of $(Y_t)_{t \geq 0}$ with law P_x and a process $(Z_t)_{t \geq 0}$ with law P such that $(Y_{N^2|F|+t})_{t \geq 0}$ coincides with $(Z_t)_{t \geq 0}$ with probability at least $1 - ce^{-c|F|}$. Then if $z_- = z - \frac{1}{|F|}$ (so that $u_F(z_-)N^d \overset{(2.26),g(0)\geq 1}{\leq} u_F(z)N^d - \frac{N^d}{|F|} \overset{|F|\leq N^{c7}}{\leq} u_F(z)N^d - |F|N^2$, provided c_7 is chosen small enough) we have

$$Q[F \subset Z(0, u_F(z_-)N^d)] - ce^{-c|F|} \leq Q[F \subset Y(0, u_F(z)N^d)]$$
$$\leq Q[F \subset Y(0, |F|N^2) \cup Z(0, u_F(z)N^d)] + ce^{-c|F|}, \tag{4.14}$$

Now (possibly making c_7 smaller and c_8 larger) we see that Lemma 4.4 applies to the left-hand side of (4.14) (recall $Z.$ has law P) and we get $\exp(-e^{-z_-}) - c|F|^{-c} \leq Q[F \subset Y(0, u_F(z)N^d)]$, which together with (4.8) and $h = \frac{1}{|F|}$ implies

$$\exp(-e^{-z}) - c|F|^{-c} \leq Q[F \subset Y(0, u_F(z)N^d)] = P_x[F \subset Y(0, u_F(z)N^d)]. \tag{4.15}$$

It remains to bound the right-hand side of (4.14) from above. Let y denote a vertex of F of minimal distance from x and let $F' = F \setminus \{y\}$. We then have

$$Q[F \subset Y(0, |F|N^2) \cup Z(0, u_F(z)N^d)]$$
$$\leq Q[F' \subset Y(0, |F|N^2) \cup Z(0, u_F(z)N^d)],$$
$$\leq P_x[H_{F'} < |F|N^2] + Q[F' \subset Z(0, u_F(z)N^d)].$$

Now $P_x[H_{F'} < |F|N^2] \leq \sum_{v \in F'} P_x[H_v < N^{2+\frac{1}{10}}]$ (possibly decreasing c_7 so that $|F| \leq N^{\frac{1}{10}}$). Now by our assumption on F and choice of y we have $d(x, v) \geq \frac{1}{2}|F|^{c_8}$ for all $v \in F'$. Therefore using (2.11) with $\lambda = \frac{1}{10}, r_1 = 1$ and $r_2 = \frac{1}{2}|F|^{c_8}$ (possibly decreasing c_7 even more so that $r_2 \leq |F|^{c_8} \leq N^{c_7 c_8} \leq N^{1-3\lambda}$) we get $P_x[H_v < N^{2+\frac{1}{10}}] \leq c|F|^{-c_8}$ for all $v \in F'$ and thus $P_x[H_{F'} < |F|N^2] \leq c|F|^{1-c_8} \leq c|F|^{-c}$, since we may increase c_8 so that $c_8 > 1$. Letting $z_+ = z + \frac{2}{|F|}$ we have

$$u_F(z) \overset{(1.26)}{=} u_{F'}\left(z + \log \frac{|F|}{|F'|}\right) \overset{|F'|=|F|-1,|F|\geq 2}{\leq} u_{F'}(z_+),$$

so that

$$Q[F \subset Y(0, |F|N^2) \cup Z(0, u_F(z)N^d)] \leq Q[F' \subset Z(0, u_{F'}(z_+)N^d)] + c|F|^{-c}$$
$$= P[F' \subset Y(0, u_{F'}(z_+)N^d)] + c|F|^{-c}$$
$$\leq \exp(-e^{-z}) + c|F|^{-c}, \tag{4.16}$$

where the last inequality follows from Lemma 4.4 and (4.8) (similarly to above (4.15)). Together with (4.15) and (4.14) this implies (4.2). □

It still remains to prove Lemma 4.2, the other ingredient in the proof of (3.1). For the proof we will use the following bounds on the probability of not hitting two points, which is a consequence of the coupling.

Lemma 4.5 ($d \geq 3$, $N \geq 3$) *There exists a constant c_{13} such that for all $x, y \in \mathbb{T}_N$ we have, letting $v = d_\infty(x, y)$,*

$$P[x, y \notin Y(0, uN^d)] \leq (1 + cv^{-c_{13}})(Q_0[0 \notin \mathcal{I}^u])^2 \; if \; u \in [1, v^{c_{13}}], \tag{4.17}$$

$$P[x, y \notin Y(0, uN^d)] \leq (1 + cN^{-c_{13}})Q_0[0, x - y \notin \mathcal{I}^u] \; if \; v \leq N^{\frac{1}{2}}, u \in [1, N^{c_{13}}]. \tag{4.18}$$

Proof We start with (4.17). Let $n = 2$, $x_1 = x$, $x_2 = y$ (so that the separation s is v) and $\varepsilon = \frac{1}{2}$ (say). We have $u \geq s^{-c_5}$ and thus letting $\delta = c_5^{-1} s^{-c_5}$ it follows from Theorem 3.2 (see (3.4)) that, choosing $c_{13} < \frac{1}{2}c_5$ so that $u\delta \leq cs^{c_{13}}s^{-2c_{13}} = cs^{-c_{13}}$,

$$
\begin{aligned}
P[x, y \notin Y(0, uN^d)] &\leq \left(Q_0[0 \notin \mathcal{I}^{u(1-\delta)}] \right)^2 + cue^{-cs^{c_5}} \\
&\stackrel{(2.23)}{=} \left(Q_0[0 \notin \mathcal{I}^u] \right)^2 e^{2\frac{u\delta}{g(0)}} + cue^{-cs^{c_5}} \\
&\leq \left(Q_0[0 \notin \mathcal{I}^u] \right)^2 (1 + cs^{-c_{13}}) + cue^{-cs^{c_5}}.
\end{aligned}
\tag{4.19}
$$

But if c_{13} is chosen small enough $cue^{-cs^{c_5}} \leq ce^{-cs^{c_5}} \leq cs^{-c_{13}}e^{-cs^c} \leq cs^{-c_{13}}e^{-\frac{2u}{g(0)}} = cs^{-c_{13}}(Q_0[0 \notin \mathcal{I}^u])^2$, so (4.17) follows.

To prove (4.18) we let $x_1 = x$, $n = 1$ (so that the separation s is N). We further let $\varepsilon = \frac{1}{2}$ so that the box $A + x_1 = B(x_1, s^{1-\varepsilon})$ contains y, and note that $u \geq s^{-c_5} = N^{-c_5}$. Thus letting $\delta = c_5^{-1}s^{-c_5} = c_5^{-1}N^{-c_5}$ it follows from Theorem 3.2 that

$$P[x, y \notin Y(0, uN^d)] \leq Q_0[0, y - x \notin \mathcal{I}^{u(1-\delta)}] + cue^{-cN^{c_5}}.$$

Now similarly to above we find that the right-hand side is bounded above by $(1 + cN^{-c_{13}})Q_0[0, y - x \notin \mathcal{I}^u]$ (provided c_{13} is chosen small enough), so (4.18) follows. $\qquad\square$

We are now in a position to prove Lemma 4.2. We will show that $E[|F_\rho|]$ is close to $|F|^\rho$, so that proving that the probability of $F_\rho \notin \mathcal{G}$ is small reduces (via Chebyshev's inequality) to bounding the variance of $\mathrm{Var}\,[|F_\rho|]$ from above and bounding the probability that $\inf_{x,y \in F_\rho, x \neq y} d_\infty(x, y)$ is small from above (recall (4.3) and (4.4)). But both $\mathrm{Var}\,[|F_\rho|]$ and the probability that $\inf_{x,y \in F_\rho, x \neq y} d_\infty(x, y)$ is small can be bounded above in terms of sums, over pairs x, y of vertices, of the probability $P[x, y \notin Y(0, t(\rho))]$, and these sums can be controlled by Lemma 2.3, via (4.17) and (4.18).

Proof of Lemma 4.2 Let

$$u = g(0)(1 - \rho) \log |F| \; \text{so that} \; t(\rho) = uN^d, \tag{4.20}$$

and record for the sequel that

$$Q_0[x \notin \mathcal{I}^u] \stackrel{(2.23),(4.20)}{=} |F|^{\rho-1} \quad \text{for all } x \in \mathbb{Z}^d. \tag{4.21}$$

By summing over $x \in F$ in (3.5) (note that $|F| \leq N^d$ so that we have

$$1 \stackrel{|F| \geq c(\rho)}{\leq} u \leq c \log N \stackrel{N \geq |F|^{1/d} \geq c}{\leq} N^{c_6} \tag{4.22}$$

by (4.20)) and using $|F| Q_0[0 \notin \mathcal{I}^u] \stackrel{(4.21)}{=} |F|^\rho$ we get

$$(1 - cN^{-c})|F|^\rho \leq E[|F_\rho|] \leq (1 + cN^{-c})|F|^\rho. \tag{4.23}$$

Therefore $\left| |F_\rho| - |F|^\rho \right| > |F|^{\frac{2}{3}\rho}$ implies

$$\left| |F_\rho| - E[|F_\rho|] \right| > |F|^{\frac{2}{3}\rho} - cN^{-c}|F|^\rho \stackrel{|F| \leq N^d}{\geq} |F|^{\frac{2}{3}\rho} - c|F|^{\rho-c} \stackrel{\rho \leq c, |F| \geq c}{\geq} \frac{|F|^{\frac{2}{3}\rho}}{2}.$$

Thus the Chebyshev inequality gives

$$P\left[\left| |F_\rho| - |F|^\rho \right| > |F|^{\frac{2}{3}\rho} \right] \leq P\left[\left| |F_\rho| - E[|F_\rho|] \right| > \frac{1}{2}|F|^{\frac{2}{3}\rho} \right] \leq 4 \frac{\mathrm{Var}[|F_\rho|]}{|F|^{\frac{4}{3}\rho}}.$$

Note that $\mathrm{Var}[|F_\rho|] = \sum_{x,y \in F} q_{x,y} \leq E[|F_\rho|] + \sum_{x \neq y \in F} q_{x,y}$, where $q_{x,y} = P[x, y \notin Y(0, uN^d)] - P[x \notin Y(0, uN^d)]P[y \notin Y(0, uN^d)]$. Therefore using (4.23), and splitting the sum between "far and close pairs of vertices", we get

$$P\left[\left| |F_\rho| - |F^\rho| \right| > |F|^{\frac{2}{3}\rho} \right] \leq c|F|^{-\frac{1}{3}\rho} + c \sum_{\{x,y\} \in V} P[x, y \notin Y(0, uN^d)]$$

$$+ c \sum_{\{x,y\} \in W} q_{x,y}, \tag{4.24}$$

where $V = \{\{x, y\} \subset F : 0 < d_\infty(x, y) \leq |F|^{\frac{1}{2d}}\}$ and $W = \{\{x, y\} \subset F : d_\infty(x, y) > |F|^{\frac{1}{2d}}\}$. Furthermore note that

$$P\left[\inf_{x,y \in F_\rho, x \neq y} d_\infty(x, y) < |F|^{\frac{1}{2d}} \right] \leq \sum_{\{x,y\} \in V} P[x, y \notin Y(0, uN^d)], \tag{4.25}$$

and thus by (4.3)

$$P[F_\rho \notin \mathcal{G}] \leq c|F|^{-c\rho} + c \sum_{\{x,y\} \in V} P[x, y \notin Y(0, uN^d)] + c \sum_{\{x,y\} \in W} q_{x,y}. \tag{4.26}$$

We seek to bound the sums $\sum_{\{x,y\}\in V} P[x, y \notin Y(0, uN^d)]$ and $\sum_{\{x,y\}\in W} q_{x,y}$. To this end note that if $\{x, y\} \in V$ then by (4.18) we have $P[x, y \notin Y(0, uN^d)] \leq c Q_0[0, y - x \notin \mathcal{I}^u]$, where (4.18) applies because $d_\infty(x, y) \leq |F|^{1/2d} \leq N^{1/2}$ (note $|F| \leq N^d$), and $1 \leq u \leq N^{c_{13}}$ (cf. (4.22)). Thus

$$\sum_{\{x,y\}\in V} P[x, y \notin Y(0, uN^d)] \leq c \sum_{\{x,y\}\in V} Q_0[0, y - x \notin \mathcal{I}^u]$$
$$\leq c \sum_{x\in F} \sum_{y\in K_x} Q_0[0, y - x \notin \mathcal{I}^u], \quad (4.27)$$

where $K_x = F \cap B(x, |F|^{\frac{1}{2d}})\setminus\{x\}$. Now using (2.24) on the inner sum of the right-hand side with $K_x - x$ in place of K, we get

$$\sum_{\{x,y\}\in V} P[x, y \notin Y(0, uN^d)] \overset{u\geq 1}{\leq} c \sum_{x\in F}\{|K_x|Q_0[0 \notin \mathcal{I}^u]^2 u + e^{-c_4 \frac{u}{g(0)}}\}$$
$$\overset{(4.20)}{\leq} c \sum_{x\in F}\{|K_x|Q_0[0 \notin \mathcal{I}^u]^2 \log|F| + |F|^{-c_4(1-\rho)}\}$$
$$\leq c|F|^{\frac{3}{2}} Q_0[0 \notin \mathcal{I}^u]^2 \log|F| + c|F|^{-c}, \quad (4.28)$$

where we have used that $\sum_{x\in F}|K_x| \leq \sum_{x\in F} c(|F|^{\frac{1}{2d}})^d = c|F|^{\frac{3}{2}}$, and choose c_9 small enough so that $c_4(1 - c_9) > 1$ (recall that $c_4 > 1$). Thus from (4.21) we have

$$\sum_{\{x,y\}\in V} P[x, y \notin Y(0, uN^d)] \leq c|F|^{2\rho-\frac{1}{2}} \log|F| + c|F|^{-c} \overset{\rho\leq c}{\leq} c|F|^{-c}. \quad (4.29)$$

We now turn to the sum $\sum_{\{x,y\}\in W} q_{x,y}$. Using (3.5) again we obtain

$$P[x \notin Y(0, uN^d)]P[y \notin Y(0, uN^d)] \geq (1 - cN^{-c})Q_0[0 \notin \mathcal{I}^u]^2. \quad (4.30)$$

Also by (4.17) we have that if x and y are such that $v = d_\infty(x, y) \geq |F|^{3c_{13}^{-1}\rho}$ then (similarly to (4.22) we have $1 \leq u \leq g(0) \log|F| \leq |F|^{3\rho} \leq v^{c_{13}}$, see (4.20) and note $|F| \geq c(\rho)$)

$$P[x, y \notin Y(0, uN^d)] \leq (1 + c|F|^{-3\rho})Q_0[0 \notin \mathcal{I}^u]^2. \quad (4.31)$$

Combining (4.30) and (4.31) we have

$$\sum_{x,y\in F, d_\infty(x,y)\geq|F|^{3c_{13}^{-1}\rho}} q_{x,y} \leq c(|F|^{-3\rho} + cN^{-c}) \sum_{x,y\in F} Q_0[0 \notin \mathcal{I}^u]^2 \leq c|F|^{-\rho},$$

$$(4.32)$$

since (possibly decreasing c_9)

$$N^{-c} \overset{|F| \leq N^d}{\leq} |F|^{-c} \leq |F|^{-3c_9} \leq |F|^{-3\rho} \quad \text{and} \quad \sum_{x,y \in F} Q_0[0 \notin \mathcal{I}^u]^2 \overset{(4.21)}{=} |F|^{2\rho}.$$

Possibly decreasing c_9 once again, we have that all $0 < \rho \leq c_9$ satisfy $3c_{13}^{-1}\rho \leq \frac{1}{2d}$. Then $|F|^{3c_{13}^{-1}\rho} \leq |F|^{\frac{1}{2d}}$ so that from the definition of W and (4.32)

$$\sum_{\{x,y\} \in W} q_{x,y} \leq c|F|^{-\rho}. \tag{4.33}$$

Now using (4.29) and (4.33) on the right-hand side of (4.26) gives (4.4). □

We have now completely reduced the proof of Theorem 3.1 to the coupling result Theorem 3.2. We end this section with a remark on the use of the coupling Theorem 3.2 as a general tool, and on the possibility of extending Theorem 3.1 to other families of graphs.

Remark 4.6 (1) As mentioned in the introduction, a coupling like Theorem 3.2 is a very powerful tool to study the trace of random walk. To prove the cover time result Theorem 3.1 we used the coupling to study certain properties of the trace of the random walk; namely the probabilities that points, pairs of points, and sets of "distant" points are contained in the trace (see Lemma 3.3, Lemma 4.5 and Lemma 4.3 respectively). When studying the percolative properties of the so called vacant set (the complement of the trace of random walk), similar couplings have been used, and there the properties of the trace studied are certain connnectivity properties of its complement (see e.g. (2.4), the display after (2.14) or (2.20)–(2.22) in [25], or (7.18) in [21]). The generality of the coupling Theorem 3.2 ensures that it can be used in the future to study further unrelated properties of the trace of random walk in the torus.

(2) The method used in this section to prove Gumbel fluctuations essentially consists of considering the set of "late points" (recall (4.1)) and proving that it concentrates and is separated (i.e. (4.2)). It has already been used to prove Gumbel fluctuations in related models in [5] and [6], and could potentially apply to prove Gumbel fluctuations for many families of graphs, as long as one can obtain good enough control of entrance times to replace (3.5), (4.5) and (4.9) (in a more general context the latter estimate may be difficult to obtain but could be replaced with an estimate on how close $H_{\{x_1,x_2,...,x_m\}}$ is to being exponential when $x_1, \ldots, x_m$ are "separated", since the cover time of a set $\{x_1, \ldots, x_m\}$ consisting of separated points is essentially the sum of m entrance times for sets consisting of $m, m-1, m-2, \ldots$ and finally 1 points; from this one can derive something similar to (4.2)). In a forthcoming work Roberto Oliveira and Alan Paula obtain such a generalization of Theorem 3.1. □

5 Coupling

We now turn to the proof of the coupling result Theorem 3.2. The proof has three main steps: first the trace of random walk in the union of the boxes $A + x_i, i = 1, \ldots, n,$

(recall (3.2) and (3.3)) is coupled with a certain Poisson point process on the space of trajectories $\Gamma(\mathbb{T}_N)$ (see below (2.1)). From this Poisson point process we then construct n independent Poisson point processes, one for each box $A + x_i$, which are coupled with the trace of random walk in the corresponding box. Lastly we construct from each of these Poisson processes a random interlacement which is coupled with the trace of the random walk in the corresponding box $A + x_i$. Essentially speaking the three steps are contained in the three Propositions 5.1, 5.2 and 5.4, which we state in this section and use to prove Theorem 3.2. The proofs of the propositions are postponed until the subsequent sections.

For the rest of the paper we assume that we are given centres of boxes

$$x_1, \ldots, x_n \in \mathbb{T}_N \text{ whose separation is } s \text{ (see (3.2)), and } \varepsilon \in (0, 1). \quad (5.1)$$

We also define the concentric boxes $B \subset C$ around A by

$$A \overset{(3.3)}{=} B(0, s^{1-\varepsilon}) \subset B = B(0, s^{1-\frac{\varepsilon}{2}}) \subset C = B(0, s^{1-\frac{\varepsilon}{4}}). \quad (5.2)$$

For convenience we introduce the notation

$$\bar{F} = \bigcup_{i=1}^{n} F_i \text{ where } F_i = F + x_i \quad \text{for any } F \subset \mathbb{T}_N. \quad (5.3)$$

Note that if $s \geq c(\varepsilon)$ then the C_i are disjoint. To state Proposition 5.1 we introduce U, the first time random walk spends a "long" time outside of $\bar{C}$ (roughly speaking long enough time to mix, see Proposition 6.1), defined by

$$U = \inf\{t \geq t^\star : Y(t - t^\star, t) \cap \bar{C} = \emptyset\} \quad \text{where } t^\star = N^{2+\frac{\varepsilon}{100}}. \quad (5.4)$$

We also define the intensity measure κ_1 on $\Gamma(\mathbb{T}_N)$ by (recall (2.6))

$$\kappa_1(dw) = P_e[Y_{\cdot \wedge U} \in dw] \quad \text{where } e(x) = \sum_{i=1}^{n} e_A(x - x_i). \quad (5.5)$$

For parameters $u \geq 0$ and $\delta > 0$ (satisfying suitable conditions), Proposition 5.1 constructs a coupling of Y. with two independent Poisson point processes μ_1 and μ_2 on $\Gamma(\mathbb{T}_N)$ of intensities $u(1 - \delta)\kappa_1$ and $2u\delta\kappa_1$ respectively such that (recall the notation from (2.16))

$$\left\{ \mathcal{I}(\mu_1) \cap \bar{A} \subset Y(0, uN^d) \cap \bar{A} \subset \mathcal{I}(\mu_1 + \mu_2) \cap \bar{A} \right\} \text{ with high probability. } \quad (5.6)$$

Proposition 5.2, the second ingredient in the proof of Theorem 3.2, couples Poisson processes like μ_1 and μ_2 with Poisson processes with intensity a multiple of

$$\kappa_2(dw) = P_e[Y_{\cdot \wedge T_{\bar{B}}} \in dw]. \quad (5.7)$$

More precisely if ν is a Poisson process of intensity $u\kappa_1$, $u > 0$, and $\delta > 0$ then (under appropriate conditions) Proposition 5.2 will construct Poisson point processes ν_1 and ν_2 of intensities $u(1 - \delta)\kappa_2$ and $2u\delta\kappa_2$ respectively such that

$$\left\{ \mathcal{I}(\nu_1) \cap \bar{A} \subset \mathcal{I}(\nu) \cap \bar{A} \right\} \text{ almost surely and} \tag{5.8}$$

$$\left\{ \mathcal{I}(\nu) \cap \bar{A} \subset \mathcal{I}(\nu_1 + \nu_2) \cap \bar{A} \right\} \text{ with high probability.} \tag{5.9}$$

Note that, in contrast to the situation for μ_1 and μ_2 from (5.6), each "excursion" in the support of ν_1 and ν_2 never returns to $\bar{A}$ after it has left $\bar{B}$. Under the law induced on it from the intensity measure κ_2, an excursion therefore, conditionally on its starting point, has the law of a random walk in $\mathbb{Z}^d$ stopped upon leaving B (up to translation). Furthermore it leaves a trace in only one of the boxes $A_1, \ldots, A_n$. This will allow us (in Corollary 5.3) to "split" the Poisson point processes ν_1 and ν_2 into independent Poisson point processes $\nu_1^i, \nu_2^i, i = 1, \ldots, n$, (on $\Gamma(\mathbb{Z}^d)$) such that the ν_1^i have intensity $u(1 - \delta)\kappa_3$ and the ν_2^i have intensity $2u\delta\kappa_3$, where

$$\kappa_3(dw) = P_{e_A}^{\mathbb{Z}^d}[Y_{.\wedge T_B} \in dw], \text{ and such that} \tag{5.10}$$

$$\left\{ \mathcal{I}(\nu_1^i) \cap A \subset (\mathcal{I}(\nu) - x_i) \cap A \text{ for all } i \right\} \text{ almost surely and} \tag{5.11}$$

$$\left\{ (\mathcal{I}(\nu) - x_i) \cap A \subset \mathcal{I}(\nu_1^i + \nu_2^i) \cap A \text{ for all } i \right\} \text{ with high probability.} \tag{5.12}$$

Proposition 5.4, the third ingredient in the proof of Theorem 3.2, constructs independent random subsets of $\mathbb{Z}^d$ that have the law of random interlacements intersected with A, from Poisson processes like ν_j^i. More precisely if η is a Poisson point process of intensity $u\kappa_3$, $u \geq 0$, and $\delta > 0$ then under appropriate conditions it constructs independent random sets $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbb{Z}^d$ such that $\mathcal{I}_1$ has the law of $\mathcal{I}^{u(1-\delta)} \cap A$ under Q_0, $\mathcal{I}_2$ has the law of $\mathcal{I}^{2\delta u} \cap A$ under Q_0, and

$$\{\mathcal{I}_1 \cap A \subset \mathcal{I}(\eta) \cap A \subset (\mathcal{I}_1 \cup \mathcal{I}_2) \cap A\} \text{ with high probability.} \tag{5.13}$$

But essentially speaking because of (2.22) we will be able to easily construct a random interlacement $(\mathcal{I}^u)_{u \geq 0}$ from such a pair $\mathcal{I}_1, \mathcal{I}_2$.

We now state the propositions. Recall the standing assumption (5.1).

Proposition 5.1 ($d \geq 3, N \geq 3, x_1, \ldots, x_n \in \mathbb{T}_N$) *If* $s \geq c(\varepsilon)$, $u \geq s^{-c(\varepsilon)}$, $\frac{1}{2} \geq \delta \geq cs^{-c(\varepsilon)}$ *and* $n \leq s^{c(\varepsilon)}$ *we can construct a coupling* $(\Omega_2, \mathcal{A}_2, Q_2)$ *of the random walk* Y. *with law* P *and independent Poisson point processes* μ_1 *and* μ_2 *on* $\Gamma(\mathbb{T}_N)$, *such that* μ_1 *has intensity* $u(1 - \delta)\kappa_1$, μ_2 *has intensity* $2u\delta\kappa_1$, *and* $Q_2[I_1] \geq 1 - cue^{-cs^{c(\varepsilon)}}$, *where* I_1 *is the event in* (5.6).

Proposition 5.1 will be proved in Sect. 7.

Proposition 5.2 ($d \geq 3, N \geq 3, x_1, \ldots, x_n \in \mathbb{T}_N$) *Assume* $s \geq c(\varepsilon)$ *and that* ν *is a Poisson point process on* $\Gamma(\mathbb{T}_N)$ *with intensity measure* $u\kappa_1$, $u \geq s^{-c(\varepsilon)}$, *constructed on some probability space* $(\Omega, \mathcal{A}, Q)$. *Then if* $1 \geq \delta \geq cs^{-c(\varepsilon)}$ *and* $n \leq s^{c(\varepsilon)}$ *we can extend the space to get independent Poisson point processes* ν_1, ν_2, *on* $\Gamma(\mathbb{T}_N)$ *such that*

ν_1 has intensity $u(1-\delta)\kappa_2$, ν_2 has intensity $2u\delta\kappa_2$, (5.8) holds and $Q[I_2] \geq 1 - ce^{-cs^{c(\varepsilon)}}$, where I_2 is the event in (5.9).

The proof of Proposition 5.2 is contained in Sect. 8. In the proof of Theorem 3.2 we will actually use the following corollary.

Corollary 5.3 *Under the conditions of Proposition* 5.2 *we can construct independent Poisson point processes* ν_1^i, ν_2^i, $i = 1, 2, \ldots, n$, *such that* ν_1^i *has intensity* $u(1 - \delta)\kappa_3$ *and* ν_2^i *has intensity* $2u\delta\kappa_3$ *for* $i = 1, \ldots, n$, (5.11) *holds and* $Q[I_3] \geq 1 - ce^{-cs^{c(\varepsilon)}}$, *where* I_3 *is the event in* (5.12).

Proof For $i = 1, \ldots, n$, and $j = 1, 2$, let ν_j^i be the image of $1_{\{Y_0 \in A_i\}}\nu_j$ under the map which sends $w(\cdot) \in \Gamma(B_i) \subset \Gamma(\mathbb{T}_N)$ to $w(\cdot) - x_i \in \Gamma(\mathbb{Z}^d)$ (recall that $\Gamma(B_i)$ for $B_i \subset \mathbb{T}_N$ denotes the set of paths in $\mathbb{T}_N$ that never leave B_i, and note that $B = B_i - x_i \subset \mathbb{T}_N$ may be identified with a subset of $\mathbb{Z}^d$, so that $w(\cdot) - x_i$ can be identified with an element of $\Gamma(\mathbb{Z}^d)$). Since the sets $\{Y_0 \in A_i\}$, $i = 1, \ldots, n$, are disjoint we have that ν_i^1, ν_i^2, $i = 1, \ldots, n$, are independent Poisson point processes of the required intensities (see the (5.5), (5.7) and (5.10)). Now (5.11) and the required bound on $Q[I_3]$ follows from Proposition 5.2 (see (5.8) and (5.9)), since $(\mathcal{I}(\nu_j) - x_i) \cap A = \mathcal{I}(\nu_j^i) \cap A$ for all i and j. □

We now state the proposition which couples processes like ν_j^i with random interlacements. Note that we will apply this proposition after "decoupling" the boxes $A_1, \ldots, A_n$, using Corollary 5.3, and that the statement of Proposition 5.4 therefore does not refer to these boxes or the centres $x_1, \ldots x_n$, except through their separation s (recall (3.2)), which goes into the definition of the radii of the boxes A and B (see (3.3) and (5.2)). The interpretation of s as separation is therefore irrelevant, and for the purposes of the following proposition it can be simply considered as a parameter that (together with ε) determines the radii of A and B.

Proposition 5.4 $(d \geq 3)$ *Let* η *be a Poisson point process on* $\Gamma(\mathbb{Z}^d)$ *with intensity measure* $u\kappa_3$, $u \geq 0$, *constructed on some probability space* $(\Omega, \mathcal{A}, Q)$. *If* $s \geq c(\varepsilon)$ *and* $1 \geq \delta \geq cs^{-c(\varepsilon)}$ *then we can construct a probability space* $(\Omega', \mathcal{A}', Q')$ *and (on the product space) independent* $\sigma(\eta) \times \mathcal{A}$-*measurable random sets* $\mathcal{I}_1, \mathcal{I}_2 \subset \mathbb{Z}^d$ *such that* $\mathcal{I}_1$ *has the law of* $\mathcal{I}^{u(1-\delta)} \cap A$ *under* Q_0, $\mathcal{I}_2$ *has the law of* $\mathcal{I}^{2u\delta} \cap A$ *under* Q_0, *and* $Q \otimes Q'[I_4] \geq 1 - ce^{-cs^{c(\varepsilon)}}$, *where* I_4 *is the event in* (5.13).

Proposition 5.4 will be proved in Sect. 9.

We are now ready to start the proof of Theorem 3.2. We will apply Proposition 5.1 to the random walk, then apply Corollary 5.3 to the resulting Poisson point processes, and finally apply Proposition 5.4 to the Poisson point processes resulting from Corollary 5.3. This gives us random subsets of $\mathbb{Z}^d$ from which we will construct random interlacements.

Proof of Theorem 3.2 Throughout the proof we decrease $c_5(\varepsilon)$ whenever necessary so that the conditions on u, δ and n needed for Proposition 5.1, Corollary 5.3 or Proposition 5.4 to hold are fulfilled. We first apply Proposition 5.1 with $\frac{\delta}{14}$ in place of δ to get the space $(\Omega_2, \mathcal{A}_2, Q_2)$ and independent Poisson point processes μ_1 and μ_2 such

that μ_1 has intensity $u(1 - \frac{\delta}{14})\kappa_1$, μ_2 has intensity $u\frac{\delta}{7}\kappa_1$ and

$$Q_2[\mathcal{I}(\mu_1) \cap \bar{A} \subset Y(0, uN^d) \cap \bar{A} \subset \mathcal{I}(\mu_1 + \mu_2) \cap \bar{A}] \geq 1 - cue^{-cs^{c(\varepsilon)}} \text{ for } s \geq c(\varepsilon).$$
(5.14)

Next we apply Corollary 5.3 once with μ_1 in place of ν, $u(1 - \frac{\delta}{14})$ in place of u and $\frac{\delta}{14}$ in place of δ and extend the space $(\Omega_2, \mathcal{A}_2, Q_2)$ to get the space $(\Omega_1, \mathcal{A}_1, Q_1)$ with Poisson point processes ν_1^i, ν_2^i of intensities $u(1 - \frac{\delta}{14})^2\kappa_3$ and $u(1 - \frac{\delta}{14})\frac{\delta}{7}\kappa_3$ such that $\nu_1^i, \nu_2^i, i = 1, \ldots, n, \mu_2$ are mutually independent and (for $s \geq c(\varepsilon)$)

$$Q_1[\mathcal{I}(\nu_1^i) \cap A \subset (\mathcal{I}(\mu_1) - x_i) \cap A \subset \mathcal{I}(\nu_1^i + \nu_2^i) \cap A \text{ for all } i] \geq 1 - ce^{-cs^{c(\varepsilon)}}.$$
(5.15)

For convenience we may "thicken" each ν_2^i so that they have intensity $u\frac{\delta}{7}\kappa_3$, while preserving the independence of $\nu_1^i, \nu_2^i, i = 1, \ldots, n, \mu_2$ and the validity (5.15) (by extending the space with independent Poisson point processes of intensities given by an appropriate multiple of κ_3, and adding them to the original ν_2^i). Repeating this extension but with μ_2 in place of μ, $u\frac{\delta}{7}$ in place of u and 1 in place of δ, we furthermore get processes ν_3^i of intensity $u\frac{2\delta}{7}\kappa_3$ (arising from the ν_2^i in the statement of Corollary 5.3, the ν_1^i in the statement of Corollary 5.3 are zero since $u(1 - \delta) = 0$) such that $\nu_1^i, \nu_2^i, \nu_3^i, i = 1, \ldots, n$ are mutually independent and

$$Q_1[(\mathcal{I}(\mu_2) - x_i) \cap A \subset \mathcal{I}(\nu_3^i) \cap A \text{ for all } i] \geq 1 - ce^{-cs^{c(\varepsilon)}} \quad \text{for } s \geq c(\varepsilon). \quad (5.16)$$

Now apply Proposition 5.4 with $u(1 - \frac{\delta}{14})^2$ in place of u, $\frac{\delta}{14}$ in place of δ, and ν_1^1 in place of μ and extend the space with mutually independent sets $\mathcal{I}_{1,1}, \mathcal{I}_{2,1}$ (independent of $\nu_1^j, \nu_2^i, \nu_3^i, j \geq 2, i \geq 1$) such that $\mathcal{I}_{1,1}$ has the law of $\mathcal{I}^{u(1 - \frac{\delta}{14})^3} \cap A$ under Q_0, $\mathcal{I}_{2,1}$ has the law of $\mathcal{I}^{u(1 - \frac{\delta}{14})^2\frac{\delta}{7}} \cap A$ under Q_0, and $Q_1[\mathcal{I}_{1,1} \cap A \subset \mathcal{I}(\nu_1^1) \cap A \subset (\mathcal{I}_{1,1} \cup \mathcal{I}_{2,1}) \cap A] \geq 1 - cue^{-cs^c}$ for $s \geq c(\varepsilon)$. Then apply Proposition 5.4 once again with $u\frac{3\delta}{7}$ in place of u, 1 in place of δ and $\nu_2^1 + \nu_3^1$ (which is a Poisson point process of intensity $u\frac{3\delta}{7}\kappa_3$) in place of η, to extend the space with a random set $\mathcal{I}_{3,1}$ (independent of $\mathcal{I}_{1,1}, \mathcal{I}_{2,1}, \nu_1^i, \nu_2^i, \nu_3^i, i \geq 2$) such that $\mathcal{I}_{3,1}$ has the law of $\mathcal{I}^{u\frac{6\delta}{7}}$ under Q_0, and such that $Q_1[\mathcal{I}(\nu_2^1 + \nu_3^1) \cap A \subset \mathcal{I}_{3,1} \cap A] \geq 1 - cue^{-cs^c}$ for $s \geq c(\varepsilon)$ (similarly to before, $\mathcal{I}_{3,1}$ arises from the $\mathcal{I}_2$ of the statement of Proposition 5.4, $\mathcal{I}_1$ is empty since $u(1 - \delta) = 0$). We can repeat this for $i = 2, 3, \ldots, n$, each time extending the space, to get mutually independent sets $\mathcal{I}_{1,i}, \mathcal{I}_{2,i}, \mathcal{I}_{3,i}, i = 1, \ldots, n$, such that for each $j = 1, 2, 3$ the $\mathcal{I}_{j,i}, i = 1, \ldots, n$, have the same law, and for all i and $s \geq c(\varepsilon)$

$$\begin{aligned} Q_1[\mathcal{I}_{1,i} \cap A &\subset \mathcal{I}(\nu_1^i) \cap A \subset (\mathcal{I}_{1,i} \cup \mathcal{I}_{2,i}) \cap A \\ \text{and} \quad \mathcal{I}(\nu_2^i + \nu_3^i) \cap A &\subset \mathcal{I}_{3,i} \cap A] \geq 1 - cue^{-cs^c}. \end{aligned}$$
(5.17)

By (5.14), (5.15), (5.16) and (5.17) we have for all i and (possibly decreasing c_5 and recalling that $u \geq s^{-c(\varepsilon)}$) that for $s \geq c(\varepsilon)$

$$Q_1[\mathcal{I}_{1,i} \cap A \subset (Y(0, uN^d) - x_i) \cap A \subset (\mathcal{I}_{1,i} \cup \mathcal{I}_{2,i} \cup \mathcal{I}_{3,i}) \cap A] \geq 1 - c_5^{-1} u e^{-c_5^{-1} s^{c_5}}.$$
(5.18)

It now only remains to construct "proper" random interlacements from $\mathcal{I}_{1,i}, \mathcal{I}_{2,i}, \mathcal{I}_{3,i}$, $i = 1, \ldots, n$.

By (2.22) the $\mathcal{I}_{2,i} \cup \mathcal{I}_{3,i}$ have the law of $\mathcal{I}^{u_2} \cap A$ under Q_0, where $u_2 = u(1 - \frac{\delta}{14})^2 \frac{\delta}{7} + u \frac{6\delta}{7}$. Once again by (2.22) the pair $(\mathcal{I}_{1,i} \cap A, (\mathcal{I}_{1,i} \cup \mathcal{I}_{2,i} \cup \mathcal{I}_{3,i}) \cap A)$ has the law of $(\mathcal{I}^{u_1} \cap A, \mathcal{I}^{u_1 + u_2} \cap A)$ under Q_0, where $u_1 = u(1 - \frac{\delta}{14})^3$. But this pair takes only finitely many values (so that the set of values that are taken with positive probability together have probability one), so we can, by "sampling from the conditional law (under Q_0) of $(\mathcal{I}^u)_{u \geq 0}$ given $(\mathcal{I}^{u_1} \cap A, \mathcal{I}^{u_1 + u_2} \cap A)$", construct for each $i = 1, \ldots, n$, a family $(\mathcal{I}_i^u)_{u \geq 0}$ with the law of $(\mathcal{I}^u)_{u \geq 0}$ under Q_0 such that $\mathcal{I}_i^{u(1-\delta)} \cap A \subset \mathcal{I}_i^{u_1} \cap A = \mathcal{I}_{1,i} \cap A$ (recall (2.21) and note $u(1 - \delta) \leq u(1 - \frac{3\delta}{14}) \leq u_1$) almost surely and $(\mathcal{I}_{1,i} \cup \mathcal{I}_{2,i} \cup \mathcal{I}_{3,i}) \cap A = \mathcal{I}_i^{u_1 + u_2} \cap A \subset \mathcal{I}_i^{u(1+\delta)}$ (note $u_1 + u_2 \leq u + u\frac{\delta}{7} + u\frac{6\delta}{7} = u(1 + \delta)$) almost surely, which combined with (5.18) implies (3.4) for $s \geq c(\varepsilon)$. But by adjusting constants (3.4) holds for all s, so the proof of Theorem 3.2 is complete. □

Theorem 3.2 (and therefore also Theorem 3.1) has now been reduced to Proposition 5.1, Proposition 5.2 and Proposition 5.4.

6 Quasistationary distribution

In this section we introduce the *quasistationary distribution*, which is a probability distribution on $\mathbb{T}_N \backslash \bar{C}$ (recall our standing assumption (5.1), (5.2) and (5.3)) denoted by $\sigma(\cdot)$ and which will be an essential tool when we prove (in Sect. 7) the coupling Proposition 5.1.

The main result is Proposition 6.1, which says that for all $x, y \in \mathbb{T}_N \backslash \bar{C}$ the probability $P_x[Y_t = y | H_{\bar{C}} > t^\star]$ (recall the definition of $t^\star$ from (5.4)) is very close to $\sigma(y)$ (and thus almost independent of x). The result will allow us to show, in Sect. 7, that regardless of where the random walk Y. starts, Y_U (where U was defined in (5.4)) is very close in distribution to the quasistationary distribution, and this in turn will let us "cut the random walk" Y. into almost independent excursions, each with law close to $P_\sigma[Y_{. \wedge U} \in dw]$ (cf. (5.5)). This will be the main step in constructing the Poisson processes μ_1 and μ_2 from the statement of Proposition 5.1.

At the end of this section we also give a result that says that the hitting distribution on $\partial_i \bar{A}$ when starting random walk from the quasistationary distribution is approximately the normalized sum of the equilibrium distributions on $A_1, A_2, \ldots, A_n$ (see (6.19)). This result will be used several times in the subsequent sections.

Let us now formally introduce the quasistationary distribution. We define the $(N^d - |\bar{C}|) \times (N^d - |\bar{C}|)$ matrix $(P^{\bar{C}})_{x,y \in \mathbb{T}_N \backslash \bar{C}} = \frac{1}{2d} 1_{\{x \sim y\}}$, where $x \sim y$ means that x and y share an edge in $\mathbb{T}_N$. When $s \geq c(\varepsilon)$, so that $\mathbb{T}_N \backslash \bar{C}$ is connected, the Perron–Frobenius theorem (Theorem 8.2, p. 151 in [19]) implies that this (real symmetric,

non-negative and irreducible) matrix has a unique largest eigenvalue $\lambda_1^{\bar{C}}$ with a non-negative normalized eigenvector v_1. We let $\lambda_2^{\bar{C}}$ denote the second largest eigenvalue of $P^{\bar{C}}$. The quasistationary distribution σ on $\mathbb{T}_N \backslash \bar{C}$ is then defined by

$$\sigma(x) = \frac{(v_1)_x}{v_1^T \mathbf{1}} \quad \text{for } x \in \mathbb{T}_N \backslash \bar{C}. \tag{6.1}$$

Since $\mathbb{T}_N \backslash \bar{C}$ is connected (when $s \geq c(\varepsilon)$) it holds that (see (6.6.3), p. 91 in [11])

$$\lim_{t \to \infty} P_x[Y_t = y | H_{\bar{C}} > t] = \sigma(y) \quad \text{for all } x, y \in \mathbb{T}_N \backslash \bar{C}. \tag{6.2}$$

Proposition 6.1, the main result of the section, is a quantitative version of (6.2), which we now state. Recall once again the assumption (5.1), and the definition of $t^\star$ in (5.4).

Proposition 6.1 $(d \geq 3, N \geq 3)$ *If* $n \leq s^{c(\varepsilon)}$ *and* $s \geq c(\varepsilon)$ *then*

$$\sup_{x, y \in \mathbb{T}_N \backslash \bar{C}} \left| P_x[Y_{t^\star} = y | H_{\bar{C}} > t^\star] - \sigma(y) \right| \leq c e^{-c N^{c(\varepsilon)}}. \tag{6.3}$$

To prove Proposition 6.1 we will express $P_x[Y_t = y | H_{\bar{C}} > t^\star]$ in terms of the matrix $P^{\bar{C}}$, and then use the spectral expansion of $P^{\bar{C}}$ to prove that $P_x[Y_t = y | H_{\bar{C}} > t^\star]$ is close to $\frac{(v_1)_y}{v_1^T \mathbf{1}}$. To control the error we will need an estimate of the spectral gap of $P^{\bar{C}}$, which we obtain in Lemma 6.3, and a lower bound on the minimum of $\sigma(\cdot)$, which we obtain in Lemma 6.4. This is the approach taken to prove Lemma 3.9 in [25], which is essentially the same result when $n = 1$ so that $\bar{C}$ consists of only one box. Since for us $\bar{C}$ consists of many boxes, bounding the minimum of $\sigma(\cdot)$ is harder, and achieving a good enough bound will consume most of our efforts in this section.

We prove Proposition 6.1 after introducing Lemma 6.3 and Lemma 6.4. To prove Lemma 6.3 we will need the following lemma, which roughly speaking says that $E[H_V] \approx \frac{N^d}{\text{cap}(V)}$ for appropriate sets $V \subset \mathbb{T}_N$.

Lemma 6.2 $(d \geq 3, N \geq 3)$ *For any (non-empty)* $V \subset \bar{C}$ *let* $V^i = (V \cap C_i) - x_i \subset \mathbb{Z}^d$, $i = 1, \ldots, n$. *Then if* $s \geq c(\varepsilon)$ *we have*

$$\frac{N^d}{E[H_V] \sum_{i=1}^n \text{cap}(V^i)} \leq 1 + c(\varepsilon) s^{-c(\varepsilon)}. \tag{6.4}$$

Furthermore if $V \subset \bar{B}$ *and* $n \leq s^{c(\varepsilon)}$ *then*

$$1 - c(\varepsilon) s^{-c(\varepsilon)} \leq \frac{N^d}{E[H_V] \sum_{i=1}^n \text{cap}(V^i)}, \quad \text{and} \tag{6.5}$$

$$(1 - c(\varepsilon) s^{-c(\varepsilon)}) E[H_V] \leq \inf_{x \notin \bar{C}} E_x[H_V] \leq \sup_{x \in \mathbb{T}_N} E_x[H_V] \leq (1 + c N^{-c(\varepsilon)}) E[H_V]. \tag{6.6}$$

The proof of Lemma 6.2 is contained in the appendix. We are now ready to prove Lemma 6.3 about the spectral gap of $P^{\bar{C}}$.

Lemma 6.3 $(d \geq 3, N \geq 3)$ If $n \leq s^{c(\varepsilon)}$ and $s \geq c(\varepsilon)$ we have

$$\lambda_1^{\bar{C}} - \lambda_2^{\bar{C}} \geq cN^{-2}. \tag{6.7}$$

Proof Lemma A.3 of [25] contains a proof for $n = 1$ (note that B in that lemma plays the role of $\bar{C}$ in this lemma). The proof for $n > 1$ is almost identical; one replaces B with $\bar{C}$ and the inequality $E[H_B] \geq c(\varepsilon)N^{2+\frac{\varepsilon(d-2)}{2}}$ with

$$E[H_{\bar{C}}] \overset{(2.7),(5.2),(6.4)}{\geq} \frac{c(\varepsilon)N^d}{s^{(1-\frac{\varepsilon}{4})(d-2)}n} \overset{s \leq N}{\geq} c(\varepsilon)s^{\frac{\varepsilon(d-2)}{8}} \frac{N^{2+\frac{\varepsilon(d-2)}{8}}}{n} \overset{s \geq c(\varepsilon), n \leq N^{\frac{\varepsilon(d-2)}{16}}}{\geq} cN^{2+\frac{\varepsilon(d-2)}{16}}. \tag{6.8}$$

We omit the details. □

The bound on the minimum of $\sigma(\cdot)$ comes from the following lemma.

Lemma 6.4 $(d \geq 3, N \geq 3)$ If $s \geq c(\varepsilon)$ we have

$$\inf_{x \in \mathbb{T}_N \setminus \bar{C}} \sigma(x) \geq N^{-cn}. \tag{6.9}$$

We will prove Lemma 6.4 after finishing the proof of Proposition 6.1.

Proof of Proposition 6.1 Note that $P_x[Y_{t^\star} = y, H_{\bar{C}} > t^\star] = \delta_x^T e^{-t^\star(I - P^{\bar{C}})} \delta_y$ for $x, y \in \mathbb{T}_N \setminus \bar{C}$, so

$$P_x[Y_{t^\star} = y | H_{\bar{C}} > t^\star] = \frac{\delta_x^T e^{-t^\star(I - P^{\bar{C}})} \delta_y}{\delta_x^T e^{-t^\star(I - P^{\bar{C}})} \mathbf{1}}, \tag{6.10}$$

where $\mathbf{1}$ denotes the vector $(1, \ldots, 1) \in \mathbb{R}^{N^d - |\bar{C}|}$. By the spectral theorem we have

$$e^{-t^\star(I - P^{\bar{C}})} = e^{-t^\star(1 - \lambda_1^{\bar{C}})} v_1 v_1^T + e^{-t^\star(1 - \lambda_2^{\bar{C}})} R,$$

where R is an operator onto the space orthogonal to v_1 with operator norm 1 (we use the Euclidean norm on $\mathbb{R}^{N^d - |\bar{C}|}$). We thus see from (6.10) that if we let $R' = \frac{e^{-t^\star(\lambda_1^{\bar{C}} - \lambda_2^{\bar{C}})}}{\sigma(x)(v_1^T \mathbf{1})^2} R$ then

$$
\begin{aligned}
P_x[Y_{t^\star} = y | H_{\bar{C}} > t^\star] &= \frac{(v_1)_x (v_1)_y + \delta_x^T e^{-t^\star(\lambda_1^{\bar{C}} - \lambda_2^{\bar{C}})} R \delta_y}{(v_1)_x (v_1^T \mathbf{1}) + \delta_x^T e^{-t^\star(\lambda_1^{\bar{C}} - \lambda_2^{\bar{C}})} R \mathbf{1}} \\
&\overset{(5.1)}{=} \frac{\sigma(y) + \delta_x^T R' \delta_y}{1 + \delta_x^T R' \mathbf{1}}.
\end{aligned}
\tag{6.11}
$$

Now since we require $n \leq s^{c(\varepsilon)}$ and $s \geq c(\varepsilon)$ both Lemma 6.3 and Lemma 6.4 hold. Therefore

$$|R'\delta_y|, |R'\mathbf{1}| \leq N^d \frac{e^{-t^\star(\lambda_1^{\bar{C}} - \lambda_2^{\bar{C}})}}{\sigma(x)|v_1^T\mathbf{1}|^2} \leq ce^{-cN^{c(\varepsilon)}},$$

since

$$e^{-t^\star(\lambda_1^{\bar{C}} - \lambda_2^{\bar{C}})} \overset{(4.4),(5.7)}{\leq} e^{-cN^{\frac{\varepsilon}{100}}}, \sigma(x) \overset{(5.9)}{\geq} N^{-cn} \overset{n \leq s^{\frac{\varepsilon}{200}} \leq N^{\frac{\varepsilon}{200}}}{\geq} e^{-cN^{\frac{\varepsilon}{200}} \log N},$$
$$|v_1^T\mathbf{1}| \geq |v_1|^2 = 1$$

and $N \geq c(\varepsilon)$ (since $s \geq c(\varepsilon)$). Thus (6.3) follows from (6.11) (using again that $N \geq c(\varepsilon)$). This completes the proof of Proposition 6.1. □

It still remains to prove Lemma 6.4. The proof will involve further concentric boxes D, E and F such that $A \subset B \subset C \subset D \subset E \subset F$ defined by

$$D = B(0, s^{1-\frac{\varepsilon}{8}}) \subset E = B(0, s^{1-\frac{\varepsilon}{16}}) \subset F = B(0, s^{1-\frac{\varepsilon}{32}}). \tag{6.12}$$

Proof of Lemma 6.4 Let y be the maximum of $\sigma(\cdot)$. Since $\sigma(\cdot)$ is a probability distribution we have $\sigma(y) \geq N^{-d}$. Also by reversibility we have for any $x \notin \bar{C}$ and $t \geq 0$ that $P_x[Y_t = y, H_{\bar{C}} > t] = P_y[Y_t = x, H_{\bar{C}} > t]$ and thus

$$P_y[Y_t = x | H_{\bar{C}} > t] = P_x[Y_t = y | H_{\bar{C}} > t] \frac{P_x[H_{\bar{C}} > t]}{P_y[H_{\bar{C}} > t]}. \tag{6.13}$$

Since by the Markov property $P_x[H_{\bar{C}} > t] \geq P_x[H_y < H_{\bar{C}}]P_y[H_{\bar{C}} > t]$ we see, by taking the limit $t \to \infty$ in (6.13) and using (6.2), that $\sigma(x) \geq \sigma(y)P_x[H_y < H_{\bar{C}}] \geq N^{-d}P_x[H_y < H_{\bar{C}}]$. To prove (6.9) it thus suffices to show that

$$P_x[H_y < H_{\bar{C}}] \geq N^{-cn} \quad \text{for all } x, y \notin \bar{C}. \tag{6.14}$$

For $i = 1, \ldots, n$, and $x \in D_i \backslash C_i$ (recall (5.3), (6.12)) it follows from a one-dimensional random walk estimate that $P_x[T_{D_i} < H_{\bar{C}}] \geq N^{-1}$, so that by the Markov property $P_x[H_y < H_{\bar{C}}] \geq N^{-1} \inf_{x' \in \partial_e D_i} P_{x'}[H_y < H_{\bar{C}}]$. If $x \notin \bar{D}$ and $y \in \bar{F}\backslash\bar{C}$ then we can use that by reversibility $P_x[H_y < H_{\bar{C}}] = P_y[H_x < H_{\bar{C}}]$ and another one-dimensional random walk estimate to show $P_x[H_y < H_{\bar{C}}] \geq N^{-1} \inf_{y' \notin \partial_e \bar{F}} P_x[H_{y'} < H_{\bar{C}}]$. To prove (6.14) and thus (6.9) it therefore suffices to show (recall $y \notin \bar{F}$)

$$P_x[H_y < H_{\bar{C}}] \geq N^{-cn} \quad \text{for all } x \notin \bar{D}, y \notin \bar{F}. \tag{6.15}$$

Now fix $y \notin \bar{F}$ and note that $P_x[H_y < H_{\bar{C}}] \geq cs^{-c}$ for all $x \in \partial_e(D + y)$, since $P_x[H_y < T_{E+y}] \geq cs^{-c}$ (e.g. by Proposition 1.5.9, p. 35 in [12]) and $(E + y) \cap \bar{C} = \emptyset$, since $s \geq c(\varepsilon)$. Therefore to prove (6.15) and thus (6.9) it suffices to show

$$P_{x_1}[H_y < H_{\bar{C}}] \geq N^{-cn}P_{x_2}[H_y < H_{\bar{C}}] \quad \text{for all } x_1, x_2 \notin \bar{D} \cup (y + D). \tag{6.16}$$

Consider the function $x \to P_x[H_y < H_{\bar{C}}]$. This function is non-negative and harmonic on $(\bar{C} \cup \{y\})^c$. Thus by the Harnack inequality (Theorem 1.7.2, p. 42 in [12]) we have, for any $z \in \mathbb{T}_N$ and $r \geq 0$ for which $B(z, 2(r + 1)) \cap (\bar{C} \cup \{y\}) = \emptyset$, that

$$\inf_{x \in B(z,r+1)} P_x[H_y < H_{\bar{C}}] \geq c \sup_{x \in B(z,r+1)} P_x[H_y < H_{\bar{C}}]. \tag{6.17}$$

To iterate this inequality we need the following lemma. □

Lemma 6.5 ($d \geq 3$, $N \geq 3$) *If* $s \geq c(\varepsilon)$ *one can cover* $(\bar{D} \cup (y + D))^c$ *by* $m \leq cn \log N$ *balls* $B(z_i, r_i)$, $i = 1, \ldots, m$, *that satisfy* $B(z_i, 2(r_i + 1)) \cap (\bar{C} \cup \{y\}) = \emptyset$.

Before proving Lemma 6.5 we use it to show (6.9). The balls $B(z_i, r_i + 1)$ "overlap" and cover the connected set $(\bar{D} \cup (y + D))^c$ (recall $s \geq c(\varepsilon)$), so for any $x_1, x_2 \in (\bar{D} \cup (y + D))^c$ we can find a "path" of at most $cn \log N$ balls $B(z, r + 1)$ satisfying (6.17), such that any two consecutive balls intersect, and such that x_1 is in the first ball and x_2 is in the last. Applying (6.17) at most $cn \log N$ times along these "paths" yields (6.16). This completes the proof of (6.9), so only the proof of Lemma 6.5 remains.

Proof of Lemma 6.5 By a standard argument $(B(0, \frac{N}{4}))^c$ can be covered by a bounded number of balls $B(z, r)$ that satisfy $B(z, 4r + 2) \cap B(0, \frac{N}{8}) = \emptyset$ (when $N \geq c$, which we may assume since we require $s \geq c$). Furthermore each of the annuli $B(0, 2^{l+1} s^{1-\frac{\varepsilon}{8}}) \backslash B(0, 2^l s^{1-\frac{\varepsilon}{8}})$, $l = 0, 1, 2, 3, \ldots$ can be covered by a bounded number of balls $B(z, r)$ that satisfy $B(z, 4r + 2)) \cap B(0, 2^{l-1} s^{1-\frac{\varepsilon}{8}}) = \emptyset$ (since $s^{1-\frac{\varepsilon}{8}} \geq c$). Now combining at most $c \log N$ coverings of annuli with the covering of $(B(0, \frac{N}{4}))^c$ we get that (provided $s \geq c(\varepsilon)$ so that $C \subset B(0, \frac{1}{2} s^{1-\frac{\varepsilon}{8}})$)

one can cover D^c by at most $c \log N$ balls $B(z, r)$ with $B(z, 4r + 2) \cap C = \emptyset$. (6.18)

We can now use (6.18) to get for each $x \in \{x_1, \ldots, x_n, y\}$ a covering $\mathcal{B}_x$ of $(x + D)^c$ consisting of at most $c \log N$ balls such that if $B(v, r) \in \mathcal{B}_x$ then $B(v, 4r + 2) \cap (x + C) = \emptyset$. We now combine the coverings by picking for every $x \notin \bar{D} \cup (y + D)$ a ball from $\mathcal{B}_{z(x)}$ that contains x, where $z(x)$ is a member of $\{x_1, \ldots, x_n, y\}$ of minimal d_∞ distance to x. This gives a covering of $(\bar{D} \cup (y + D))^c$ by at most $c(n + 1) \log N \leq cn \log N$ balls. Also if $x \notin \bar{D} \cup (y + D)$ and $x \in B(v, r) \in \mathcal{B}_{z(x)}$ then $B(v, 4r + 2) \cap (z(x) + C) = \emptyset$, which implies $d_\infty(v, z(x)) > s^{1-\frac{\varepsilon}{4}} + 4r + 2$. This in turn implies $d_\infty(v, \{x_1, \ldots, x_n, y\}) \geq d_\infty(x, z(x)) - d_\infty(v, x) \geq d_\infty(v, z(x)) - 2d_\infty(v, x) > s^{1-\frac{\varepsilon}{4}} + 2(r + 1)$ (using that $z(x)$ is of minimal distance to x and $d_\infty(x, v) \leq r$), so that $B(v, 2(r + 1)) \cap (\bar{C} \cup \{y + C\}) = \emptyset$. Thus we have the desired covering and the proof of Lemma 6.5 is complete. □
  This completes the proof of Lemma 6.4.

Finally we state and prove Lemma 6.6, which says that the hitting distribution on $\partial_i \bar{A}$ when starting from σ is approximately the normalized sum of the equilibrium distributions on $\partial_i A_1, \ldots, \partial_i A_n$.

Lemma 6.6 ($d \geq 3$, $N \geq 3$) If $s \geq c(\varepsilon)$ and $n \leq s^{c(\varepsilon)}$ then for all $i = 1, \ldots, n$,

$$\frac{e_A(x - x_i)}{n \, \mathrm{cap}(A)}(1 - cs^{-c(\varepsilon)}) \leq P_\sigma[Y_{H_{\bar{A}}} = x]$$

$$\leq \frac{e_A(x - x_i)}{n \, \mathrm{cap}(A)}(1 + cs^{-c(\varepsilon)}) \quad \text{for all} \ \ x \in \partial_i A_i. \tag{6.19}$$

Proof The proof is very similar to the proof of Lemma 3.10 in [25]. By redefining $t^\star$ and U from [25] to agree with our definition in (5.4), replacing A with $\bar{A}$, B with $\bar{C}$, and the application of Lemma 3.9 from [25] with an application of Proposition 6.1 (which is allowed since we assume $n \leq s^{c(\varepsilon)}$ and $s \geq c(\varepsilon)$) the argument leading up to (3.42) in [25] becomes a proof of

$$|P_x[\tilde{H}_{\bar{A}} > U] - P_\sigma[Y_{H_{\bar{A}}} = x] \sum_{y \in \partial_i \bar{A}} P_y[\tilde{H}_{\bar{A}} > U]| \leq ce^{-cN^{c(\varepsilon)}} \quad \text{for all} \ \ x \in \partial_i \bar{A}. \tag{6.20}$$

Furthermore note that by (2.9), (5.4) and the strong Markov property applied at time $T_{\bar{D}}$ we have $e_{A,D}(x - x_i) \inf_{x \notin \bar{D}} P_x[H_{\bar{C}} > U] \leq P_x[\tilde{H}_{\bar{A}} > U]$. But also

$$\sup_{x \notin \bar{D}} P_x[H_{\bar{C}} < U]$$

$$\overset{(5.4)}{=} \sup_{x \notin \bar{D}} P_x[H_{\bar{C}} < N^{2 + \frac{\varepsilon}{100}}] \overset{(2.11)}{\leq} nc(\varepsilon)s^{-c\varepsilon} \leq c(\varepsilon)s^{-c\varepsilon} \ \text{if} \ n \leq s^{c\varepsilon}, \tag{6.21}$$

and thus by (2.14) we have $(1 - cs^{-c(\varepsilon)})e_A(x - x_i) \leq P_x[\tilde{H}_{\bar{A}} > U]$ (recall $s \geq c(\varepsilon)$). Now

$$T_C \overset{(5.4)}{\leq} U \ \text{so} \ P_x[\tilde{H}_{\bar{A}} > U] \overset{(2.11)}{\leq} e_{A,C}(x - x_i),$$

and using (2.15) with $r = s^{1-\varepsilon}$ and λ such that $s^{(1-\varepsilon)(1+\lambda)} = s^{1 - \frac{\varepsilon}{4}}$ we get $e_{A,C}(x - x_i) \leq (1 + cs^{-c(\varepsilon)})e_A(x - x_i)$ (since $s \geq c(\varepsilon)$). Thus

$$(1 - cs^{-c(\varepsilon)})e_A(x - x_i) \leq P_x[\tilde{H}_{\bar{A}} > U] \leq e_A(x - x_i)(1 + cs^{-c(\varepsilon)}). \tag{6.22}$$

But plugging (6.22) into (6.20), and using (2.6) and (2.13) yields (6.19), cf. below (3.44) in [25]. We omit the details. □

7 Poissonization

In this section the goal is to construct the coupling of random walk in the torus with Poisson point processes of intensity a multiple of κ_1, i.e. prove Proposition 5.1. We recall the standing assumption (5.1).

First we define $R_k, k \geq 1$, the successive returns to $\bar{A}$ (see (5.3) for the notation) and $U_k, k \geq 0$, the successive "departures" from $\bar{C}$, by

$$U_0 = 0, U_k = U \circ \theta_{R_k} + R_k, k \geq 1, R_1 = H_{\bar{A}}, R_k = H_{\bar{A}} \circ \theta_{U_{k-1}} + U_{k-1}, k \geq 2. \tag{7.1}$$

We call the segments $(Y_{(R_k+\cdot)\wedge U_k})_{k\geq 1}$ the *excursions* of the random walk. The first step in the proof of Proposition 5.1 is to couple the random walk Y. when it starts from the *quasistationary* distribution with i.i.d. processes $\tilde{Y}^1, \tilde{Y}^2, \ldots$ with law $P_\sigma[Y_{\cdot \wedge U_1} \in dw]$, such that with high probability $Y(U_{i-1}, U_i) \cap \bar{C} = \tilde{Y}^i(0, \infty) \cap \bar{C}$. This is done in Lemma 7.1 using Proposition 6.1 from the previous section.

The second step in the proof of Proposition 5.1 is to relate the stopping times R_k, U_k, to deterministic times, roughly speaking showing that $U_{[un \, \text{cap}(A)]} \approx u N^d$. This is done in Lemma 7.4 using large deviation estimates.

The third step in the proof of Proposition 5.1 is to use the relation $U_{[un \, \text{cap}(A)]} \approx u N^d$ to modify the coupling from Lemma 7.1 so that, very roughly, $Y(0, u N^d) \cap \bar{C} \approx \cup_{i=1}^{[un \, \text{cap}(A)]} \tilde{Y}^i(0, \infty) \cap \bar{C}$ with high probability. This is done in Proposition 7.5, where we also use a mixing argument to ensure that the coupling, as opposed to that from Lemma 7.1, has Y. starting from the *uniform* distribution.

Finally at the end of the section we use Proposition 7.5 to prove Proposition 5.1 essentially by constructing a point process from $\tilde{Y}^1, \tilde{Y}^2, \ldots Y^J$, where J is a Poisson random variable. We will see that this gives rise to a Poisson point process which we can modify, using Lemma 6.6 to "change" the intensity from a multiple of $P_\sigma[Y_{\cdot \wedge U} \in dw]$ to a multiple of $P_e[Y_{\cdot \wedge U_1} \in dw]$ (i.e. of κ_1).

We now state and prove Lemma 7.1 which couples Y. under P_σ with i.i.d. excursions.

Lemma 7.1 $(d \geq 3, N \geq 3)$ *If $n \leq s^{c(\varepsilon)}$ and $s \geq c(\varepsilon)$ we can construct a coupling* $(\Omega_3, \mathcal{A}_3, Q_3)$ *of a process Y. with law P_σ and an i.i.d. sequence $\tilde{Y}^1, \tilde{Y}^2, \ldots$ with law* $P_\sigma[Y_{\cdot \wedge U_1} \in dw]$ *such that*

$$Q_3[Y(U_{i-1}, U_i) \cap \bar{C} \neq \tilde{Y}^i(0, \infty) \cap \bar{C}] \leq e^{-cN^{c(\varepsilon)}} \quad \text{for all } i \geq 1. \tag{7.2}$$

Proof For each i we define L_i, the last time that Y. leaves $\bar{C}$ before U_i, by

$$L_i = L \circ \theta_{U_{i-1}} + U_{i-1}, i \geq 1, \quad \text{where } L = \sup\{t \leq U_1 : Y_t \in \bar{C}\},$$

so that $0 = U_0 \leq L_1 \leq U_1 \leq L_2 \leq U_2 \leq \ldots$. Define for convenience

$$\mathcal{L}_i = Y_{L_i}, \mathcal{U}_i = Y_{U_i} \quad \text{and} \quad \hat{Y}^i_{\cdot} = Y_{(U_{i-1}+\cdot)\wedge L_i}, i \geq 1. \tag{7.3}$$

Note that $\mathcal{L}_i \in \partial_e \bar{C}$ almost surely since Y. is cadlag. Also note that the L_i are *not* stopping times. However, if we let $\sigma_y(z) = P_y[Y_{t^\star} = z | H_{\bar{C}} > t^\star]$ (cf. (6.3)) we have

Lemma 7.2 *For any $k \geq 1$, $x \in \mathbb{T}_N$, $y \in \partial_e \bar{C}$, $z \in \bar{C}^c$, and $E, F \subset \Gamma(\mathbb{T}_N)$ measurable*

$$P_x[Y_{\cdot \wedge L_k} \in E, \mathcal{L}_k = y, \mathcal{U}_k = z, Y_{U_k+\cdot} \in F] = P_x[Y_{\cdot \wedge L_k} \in E, \mathcal{L}_k = y]\sigma_y(z)P_z[F].$$
(7.4)

Proof Let $E' = E \cap \{w : w$ constant eventually, $w(\infty) = y\}$ (see below (2.1) for the notation) and $F' = F \cap \{w : w(0) = z\}$, so that the left-hand side of (7.4) equals

$$P_x[Y_{\cdot \wedge L_k} \in E', Y_{U_k+\cdot} \in F'] = \sum_{i \geq 1} P_x[Y_{\cdot \wedge \tau_i} \in E', L_k = \tau_i, Y_{\tau_i + t^\star + \cdot} \in F'], \quad (7.5)$$

(note that $U_i \overset{(5.4)}{=} L_i + t^\star$, and recall (2.2)). Now $G = \{Y_{\tau_{i-1}} \in \bar{C}, U_{k-1} \leq \tau_i \leq U_k\}$ is $\sigma(Y_{\cdot \wedge \tau_i})$−measurable, and $G \cap \{H_{\bar{C}} \circ \theta_{\tau_i} > t^\star\} = \{L_k = \tau_i\}$, so that by the strong Markov property applied at time τ_i and the definition of E' we have

$$\begin{aligned}
&P_x[Y_{\cdot \wedge \tau_i} \in E', L_k = \tau_i, Y_{\tau_i + t^\star + \cdot} \in F'] \\
&= P_x[\{Y_{\cdot \wedge \tau_i} \in E'\} \cap G]P_y[H_{\bar{C}} > t^\star, Y_{t^\star + \cdot} \in F'].
\end{aligned}$$

But $P_y[Y_{t^\star + \cdot} \in F', H_{\bar{C}} > t^\star] = P_y[H_{\bar{C}} > t^\star]\sigma_y(z)P_z[F]$ (by the Markov property applied at time $t^\star$ and the definitions of σ_y and F') so in fact

$$\begin{aligned}
&P_x[Y_{\cdot \wedge \tau_i} \in E', L_k = \tau_i, Y_{\tau_i + t^\star + \cdot} \in F'] \\
&= P_x[\{Y_{\cdot \wedge \tau_i} \in E'\} \cap G]P_y[H_{\bar{C}} > t^\star]\sigma_y(z)P_z[F] \\
&= P_x[\{Y_{\cdot \wedge \tau_i} \in E'\} \cap G \cap \{H_{\bar{C}} \circ \theta_{\tau_i} > t^\star\}]\sigma_y(z)P_z[F] \\
&= P_x[Y_{\cdot \wedge \tau_i} \in E', L_k = \tau_i]\sigma_y(z)P_z[F],
\end{aligned}$$

by an application of the strong Markov property and the definition of G. Plugging this into (7.5) and using the definition of E' gives (7.4). □

We now continue with the proof of Lemma 7.1. Because of (6.3) (together with Proposition 4.7, p. 50 in [14]) we can construct for each $y \in \bar{C}^c$ a measure $q_y(\cdot, \cdot)$ on $\bar{C}^c \times \bar{C}^c$ coupling σ and σ_y, such that

the first marginal is $\sigma(\cdot)$, the second is $\sigma_y(\cdot)$, and $\sum_{z \in \mathbb{T}_N} q_y(z, z) \geq 1 - ce^{-cN^{c(\varepsilon)}}.$
(7.6)

Let $q_y(\cdot | \cdot)$ denote the conditional distribution of the first argument given the second (note that $\sigma_y(z) > 0$ for all $z \in \bar{C}^c$, provided $s \geq c(\varepsilon)$ so that $\bar{C}$ consists of disjoint boxes).

We now construct $(\Omega_3, \mathcal{A}_3, Q_3)$ as a space with the following *mutually independent* families of random variables

$$Y. \text{ with law } P_\sigma, \tag{7.7}$$

$$(V_{y,z,i})_{y,z \in \bar{C}^c, i \geq 1} \text{ independent } \bar{C}^c - \text{valued, where } V_{y,z,i} \text{ has law } q_y(dw|z), \tag{7.8}$$

$$(Z^{v,i}_.)_{v \in \bar{C}^c, i \geq 1} \text{ independent } \Gamma(\mathbb{T}_N) - \text{valued, where } Z^{v,i}_. \text{ has law } P_v[\hat{Y}^1_{.\wedge L_1} \in dw], \tag{7.9}$$

(recall (7.3) for the definition of $\hat{Y}^1_{.\wedge L_1}$). We define on Ω_3 starting points of excursions Σ_i by

$$\Sigma_i = V_{\mathcal{L}_i, \mathcal{U}_i, i} \quad \text{for } i \geq 1. \tag{7.10}$$

We will see that the Σ_i are i.i.d. with law σ, but coincide with the $\mathcal{U}_i$ with high probability. Furthermore define the excursions $\bar{Y}^i$, with starting points Σ_{i-1} for $i \geq 2$, by

$$\bar{Y}^1_. = \hat{Y}^1_. \quad \text{and} \quad \bar{Y}^{i+1}_. = \begin{cases} \hat{Y}^{i+1}_. & \text{if } \mathcal{U}_i = \Sigma_i \\ Z^{\Sigma_i, i}_. & \text{if } \mathcal{U}_i \neq \Sigma_i \end{cases} \quad \text{for } i \geq 1. \tag{7.11}$$

(By a slight abuse of notation we view $\hat{Y}^i_., \mathcal{U}_i, \mathcal{L}_i$ and $Y.$ as being defined on Ω_3 as well as on $\Gamma(\mathbb{T}_N)$). We will see that the $\bar{Y}^i_., i \geq 1$, are i.i.d. with law $P_\sigma[Y_{.\wedge L_1} \in dw]$, essentially because the starting points Σ_i are i.i.d. with law σ. Furthermore we will see that $\bar{Y}^{i+1}_.$ coincides with $\hat{Y}^{i+1}_.$ with high probability, because Σ_i coincides with $\mathcal{U}_i$ (the starting point of $\hat{Y}^{i+1}_.$) with high probability. To this end note that for all $i \geq 2$

$$Q_3[\bar{Y}^i_. = \hat{Y}^i_.] \overset{(7.10),(7.11)}{=} Q_3[\mathcal{U}_{i-1} = V_{\mathcal{L}_{i-1}, \mathcal{U}_{i-1}, i-1}]$$
$$= \sum_{y,z \in \mathbb{T}_N} Q_3[\mathcal{L}_{i-1} = y, \mathcal{U}_{i-1} = z, V_{y,z,i-1} = z].$$

Now by independence and (7.8) we have $Q_3[\mathcal{L}_{i-1} = y, \mathcal{U}_{i-1} = z, V_{y,z,i-1} = z] = P_\sigma[\mathcal{L}_{i-1} = y, \mathcal{U}_{i-1} = z]q_y(z|z)$. Furthermore $P_\sigma[\mathcal{L}_{i-1} = y, \mathcal{U}_{i-1} = z] = P_\sigma[\mathcal{L}_{i-1} = y]\sigma_y(z)$ by (7.4) so

$$Q_3[\bar{Y}^i_. = \hat{Y}^i_.]$$
$$= \sum_{y \in \mathbb{T}_N} \{P_\sigma[\mathcal{L}_{i-1} = y] \sum_{z \in \mathbb{T}_N} \sigma_y(z)q_y(z|z)\} \overset{(7.6)}{\geq} 1 - ce^{-cN^{c(\varepsilon)}} \text{ for all } i \geq 1, \tag{7.12}$$

where we have used that $\sigma_y(z)q_y(z|z) = q_y(z,z)$ and that $\bar{Y}^1_. = \hat{Y}^1_.$ almost surely. The next lemma will be used to to show that the $\bar{Y}^i_.$ are i.i.d. with law $P_\sigma[\hat{Y}^1_{.\wedge L_1} \in dw]$.

Lemma 7.3 *For any measurable $E_1, \ldots, E_k \subset \Gamma(\mathbb{T}_N)$, let $F_k = \{\bar{Y}^i_. \in E_i, 1 \leq i \leq k\}$. Then for all $y, z \in \bar{C}^c$ and measurable $F \subset \Gamma(\mathbb{T}_N)$ we have*

$$Q_3[F_k \cap \{\mathcal{L}_k = y, \mathcal{U}_k = z, Y_{U_k+.} \in F\}] = Q_3[F_k \cap \{\mathcal{L}_k = y\}]\sigma_y(z)P_z[F], \tag{7.13}$$

Proof Essentially speaking (7.13) follows directly from (7.4) because the event F_k only depends on $Y_{.\wedge L_k}$, $Z^{v,i}_.$ and $V_{y,z,i}$ for $i \leq k$ (see (7.11)), where the $Z^{v,i}_.$ and $V_{y,z,i}$ are independent of $Y_{U_k+.}$ by construction. We omit the details (which involve "conditioning on $\mathcal{L}_i, \mathcal{U}_i, \Sigma_i, i \leq k - 1$", i.e. considering $Q_3[F_k \cap \{\mathcal{L}_k = y, \mathcal{U}_k = z, Y_{U_k+.} \in F\} \cap K(\bar{y}, \bar{z}, \bar{v})]$, where $K(\bar{y}, \bar{z}, \bar{v}) = \{(\mathcal{L}_i)_{i=1}^{k-1} = \bar{y}, (\mathcal{U}_i)_{i=1}^{k-1} = \bar{z}, (V_{\bar{y}_i, \bar{z}_i, i})_{i=1}^{k-1} = \bar{v}\}$ for vectors $\bar{y}, \bar{z}, \bar{v}$ in $(\bar{C}^c)^{k-1}$). $\square$

We now continue with the proof of Lemma 7.1 by showing that the $\bar{Y}^i_.$ are i.i.d with law $P_\sigma[Y_{.\wedge L_1} \in dw]$. For any measurable $E_1, \ldots, E_k, E_{k+1} \subset \Gamma(\mathbb{T}_N)$ let F_k be defined as in Lemma 7.3, let $F = \{Y_{.\wedge L_1} \in E_{k+1}\}$ and note that by (7.11), $Q_3[F_k \cap \{\bar{Y}^{k+1}_. \in E_{k+1}\}]$ equals

$$\sum_{y,z \in \bar{C}^c} Q_3[F_k \cap \{\mathcal{L}_k = y, \mathcal{U}_k = z, V_{y,z,k} = z, Y_{U_k+.} \in F\}] \tag{7.14}$$

$$+ \sum_{y,z,v \in \bar{C}^c, v \neq z} Q_3[F_k \cap \{\mathcal{L}_k = y, \mathcal{U}_k = z, V_{y,z,k} = v, Z^{v,k}_. \in F\}]. \tag{7.15}$$

By independence and (7.8) we have that the probability in (7.14) equals

$$Q_3[F_k \cap \{\mathcal{L}_k = y, \mathcal{U}_k = z, Y_{U_k+.} \in F\}]q_y(z|z)$$
$$\stackrel{(7.13)}{=} Q_3[F_k \cap \{\mathcal{L}_k = y\}]\sigma_y(z)P_z[F]q_y(z|z),$$

and similarly by independence, (7.8) and (7.9), the probability in (7.15) equals

$$Q_3[F_k \cap \{\mathcal{L}_k = y, \mathcal{U}_k = z\}]q_y(v|z)P_v[F]$$
$$\stackrel{(7.13)}{=} Q_3[F_k \cap \{\mathcal{L}_k = y\}]\sigma_y(z)q_y(v|z)P_v[F].$$

But $\sigma_y(z)P_z[F]q_y(z|z) = q_y(z, z)P_z[F]$ and $\sigma_y(z)q_y(v|z)P_v[F] = q_y(v, z)P_v[F]$, so

$$Q_3[F_k \cap \{\bar{Y}^{k+1}_. \in E_{k+1}\}] = \sum_{y,z,v \in \bar{C}^c} Q_3[F_k \cap \{\mathcal{L}_k = y\}]q_y(v, z)P_v[F]$$
$$= Q_3[F_k]P_\sigma[F] = Q_3[F_k]P_\sigma[Y_{.\wedge L_1} \in E_{k+1}],$$

where we have used (7.6) and the definition of F. But applying this recursively we get that for all $k \geq 1$ and measurable $E_1, \ldots, E_{k+1} \subset \Gamma(\mathbb{T}_N)$

$$Q_3[\bar{Y}^i_. \in E_i, 1 \leq i \leq k+1] = \prod_{i=1}^{k+1} P_\sigma[Y_{.\wedge L_1} \in E_i],$$

and thus the $\bar{Y}^i_., i \geq 1$, are i.i.d. with the same law as $Y_{.\wedge L_1}$ under P_σ.

We can now extend the space $(\Omega_3, \mathcal{A}_3, Q_3)$ by independently "appending a piece" with law $P_y[Y_{.\wedge t^\star} \in dw | H_{\bar{C}} > t^\star]$ to each $\bar{Y}^i$, conditionally on the event $\{\bar{Y}^i_\infty = y\}$ for every $y \in \partial_e \bar{C}$, to obtain i.i.d. processes $\tilde{Y}^1_., \tilde{Y}^2_., \ldots$ with law $P_\sigma[Y_{.\wedge U_1} \in dw]$ such that $\tilde{Y}^i_.(0, \infty) \cap \bar{C} = \bar{Y}^i_.(0, \infty) \cap \bar{C}$ almost surely. Then (7.2) is satisfied by (7.12) and since $\hat{Y}^i_.(0, \infty) \cap \bar{C} = Y(U_{i-1}, U_i) \cap \bar{C}$. This completes the proof of Lemma 7.1. □

The next step is to to relate the stopping times U_k to deterministic times.

Lemma 7.4 $(d \geq 3, N \geq 3)$ *If* $s \geq c(\varepsilon)$, $u \geq s^{-c(\varepsilon)}$, $\frac{1}{2} \geq \delta \geq cs^{-c(\varepsilon)}$ *and* $n \leq s^{c(\varepsilon)}$ *then*

$$P_\sigma[U_{[u(1+\delta)n \operatorname{cap}(A)]} \leq uN^d] \leq ce^{-s^{c(\varepsilon)}} \quad \text{and} \tag{7.16}$$

$$P_\sigma[U_{[u(1-\delta)n \operatorname{cap}(A)]} \geq uN^d] \leq ce^{-s^{c(\varepsilon)}}. \tag{7.17}$$

Proof Note that

$$U_k = \sum_{i=0}^{k-1} H_{\bar{A}} \circ \theta_{U_i} + \sum_{i=1}^{k} U \circ \theta_{R_i} \quad \text{for all } k \geq 0. \tag{7.18}$$

Let $k^+ = [u(1+\delta)n \operatorname{cap}(A)]$ and $k^- = [u(1-\delta)n \operatorname{cap}(A)]$. By (7.18) both (7.16) and (7.17) follow from (note that $\delta u N^c$, $\delta^2 un \operatorname{cap}(A) \overset{(2.7)}{\geq} s^{c(\varepsilon)}$ if $u, \delta \geq s^{-c(\varepsilon)}$)

$$P_\sigma\left[\sum_{i=0}^{k^+-1} H_{\bar{A}} \circ \theta_{U_i} \leq uN^d \right] \leq ce^{-c\delta^2 un \operatorname{cap}(A)}, \tag{7.19}$$

$$P_\sigma\left[\sum_{i=0}^{k^--1} H_{\bar{A}} \circ \theta_{U_i} \geq u(1-\frac{\delta}{2})N^d \right] \leq e^{-c\delta^2 un \operatorname{cap}(A)}, \tag{7.20}$$

$$P_\sigma\left[\sum_{i=1}^{k^-} U \circ \theta_{R_i} \geq \frac{\delta}{2}uN^d \right] \leq ce^{-c\delta uN^c}. \tag{7.21}$$

One shows (7.19), (7.20) and (7.21) using large deviations bounds. Since the proofs are very similar to those in Lemma 4.3 in [25] we omit the details. Let us simply state that to prove (7.19) one estimates the small exponential moments of $\delta \frac{\sum_{i=0}^{k^+-1} H_{\bar{A}} \circ \theta_{U_i}}{\inf_{x \notin \bar{C}} E_x[H_{\bar{A}}]}$

and to prove (7.20) one estimates the small exponential moments of $\delta \frac{\sum_{i=0}^{k^- - 1} H_{\bar{A}} \circ \theta_{U_i}}{\sup_{x \in \mathbb{T}_N} E_x[H_{\bar{A}}]}$.

This is possible because by the strong Markov property $E_\sigma[\exp(\lambda \sum_{i=0}^k H_{\bar{A}} \circ \theta_{U_k})] \leq (\sup_{x \notin \bar{C}} E_x[\exp(\lambda H_{\bar{A}})])^k$ for all $\lambda \in \mathbb{R}$, and by elementary bounds on the function $x \to e^x$ and Khasminskii's lemma (Lemma 3.8 in [25]) we have

$$\sup_{x \notin \bar{C}} E_x[\exp(\lambda H_{\bar{A}})] \leq 1 + \lambda \inf_{x \notin \bar{C}} E_x[H_{\bar{A}}] + c\lambda^2 (\sup_{x \in \mathbb{T}_N} E[H_{\bar{A}}])^2 \quad \text{for } \lambda < 0$$

(7.22)

$$\text{and} \sup_{x \notin \bar{C}} E_x[\exp(\lambda H_{\bar{A}})] \leq \sum_{m \geq 0} \lambda^m (\sup_{x \in \mathbb{T}_N} E_x[H_{\bar{A}}])^m \quad \text{for } \lambda > 0. \tag{7.23}$$

To show (7.19) one sets $\lambda = -\frac{c\delta}{\inf_{x \notin \bar{C}} E_x[H_{\bar{A}}]}$ in (7.22), for a small enough constant c, uses (6.6) to show that the term involving λ^2 is at most $c\delta^2$ (for a small enough constant c) and (6.4) and (6.6) to show that $\frac{N^d}{\inf_{x \notin \bar{C}} E_x[H_{\bar{A}}]} \leq (1 + c\delta)n\mathrm{cap}(A)$, for any small constant c, as long as we require $\delta \geq c(\varepsilon)s^{-c(\varepsilon)}$. To show (7.20) one sets $\lambda = \frac{c\delta}{\sup_{x \in \mathbb{T}_N} E_x[H_{\bar{A}}]} > 0$ in (7.23), for a small enough constant c, and uses (6.5) and (6.6) to show that $\frac{N^d}{\sup_{x \in \mathbb{T}_N} E_x[H_{\bar{A}}]} \geq (1 - c\delta)n\mathrm{cap}(A)$, for any small constant c, as long as we require $\delta \geq c(\varepsilon)s^{-c(\varepsilon)}$. (Note that (6.5) and (6.6) hold because we require $n \leq s^{c(\varepsilon)}$.)

To prove (7.21) one estimates $E_\sigma[\exp(\lambda \sum_{i=1}^{k^-} U \circ \theta_{R_i})]$ for $\lambda = (t^\star)^{-1}$, first by similarly bounding it above by $(\sup_{x \in \bar{C}} E_x[\exp(\lambda U)])^{k^-}$. By noting that if $U \leq T_{\bar{D}} + t^\star$ does not hold, then $H_{\bar{C}} \circ \theta_{T_{\bar{D}}} \leq t^\star$ and $U \leq U \circ \theta_{H_{\bar{C}}} \circ \theta_{T_{\bar{D}}} + T_{\bar{D}} + t^\star$ (recall (5.4) and (6.12)), we obtain the inequality

$$\sup_{x \in \bar{C}} E_x[\exp(\lambda U)]$$

$$\leq \sup_{x \in \bar{C}} E_x[\exp(\lambda(T_{\bar{D}} + t^\star))] \left(1 + \sup_{x \in \bar{C}} E_x[\exp(\lambda U)] \sup_{x \notin \bar{D}} P_x[H_{\bar{C}} < U] \right).$$

(7.24)

Using once again Khasminskii's lemma and the elementary $\sup_{x \in \mathbb{T}_N} E_x[T_{\bar{D}}] = \sup_{x \in \mathbb{T}_N} E_x[T_D] \leq cs^{2(1 - \frac{\varepsilon}{8})} \leq ct^\star$ (recall that D has radius $s^{1 - \frac{\varepsilon}{8}}$, $s \leq N$ and (5.4)) one obtains $\sup_{x \in \bar{C}} E_x[\exp(\lambda(T_{\bar{D}} + t^\star))] \leq e^{c\lambda t^\star} = c$, since $\lambda t^\star = 1$. Using this inequality together with (6.21) in (7.24), and rearranging terms, one obtains that $\sup_{x \in \bar{C}} E_x[\exp(\lambda U)] \leq e^{c\lambda t^\star}$ (provided $s \geq c(\varepsilon)$). To prove (7.21) using the exponential Chebyshev one must then check that $ct^\star k^- - \frac{\delta}{2}uN^d \leq -cu\delta N^d$ for $s \geq c(\varepsilon)$, which follows by noting that $\mathrm{cap}(A) \leq N^{(1-\varepsilon)(d-2)}$ (see (2.7)), using (5.4) and requiring $\delta \geq s^{-c(\varepsilon)}$ and $n \leq s^{c(\varepsilon)}$ for small enough exponents $c(\varepsilon)$. $\qquad \square$

We can now combine Lemma 7.1 and Lemma 7.4 to construct a coupling of a random walk Z. with law P and a sequence of i.i.d. excursions with law $P_\sigma[Y_{\cdot \wedge U_1} \in dw]$ such that, roughly speaking, $Z(0, uN^d) \cap \bar{A}$ coincides with the traces of the i.i.d. excursions.

Proposition 7.5 ($d \geq 3$, $N \geq 3$) *If $s \geq c(\varepsilon)$, $u \geq s^{-c(\varepsilon)}$, $\frac{1}{2} \geq \delta \geq cs^{-c(\varepsilon)}$ and $n \leq s^{c(\varepsilon)}$, then we can construct a coupling $(\Omega_4, \mathcal{A}_4, Q_4)$ of a process $Z_\cdot$ with law P and an i.i.d. sequence $\tilde{Y}^1_\cdot, \tilde{Y}^2_\cdot, \ldots$ with law $P_\sigma[Y_{\cdot \wedge U_1} \in dw]$ such that*

$$
Q_4 \left[\overset{[u(1-\delta)n\mathrm{cap}(A)]}{\underset{i=2}{\cup}} \tilde{Y}^i(0, \infty) \cap \bar{C} \subset Z(0, uN^d) \cap \bar{C} \subset \overset{[u(1+\delta)n\,\mathrm{cap}(A)]}{\underset{i=1}{\bigcup}} \tilde{Y}^i(0, \infty) \cap \bar{C} \right]
$$

$$
\geq 1 - c(\varepsilon)ue^{-cs^{c(\varepsilon)}}. \tag{7.25}
$$

(Note that the first union is over $i \geq 2$, see the remark after the proof.)

Proof We first use Lemma 7.1 to construct the space $(\Omega_3, \mathcal{A}_3, Q_3)$. We will now extend it to get $(\Omega_4, \mathcal{A}_4, Q_4)$. By (2.4) (with $\lambda = N^{-2}t^\star \overset{(5.4)}{=} N^{\frac{\varepsilon}{100}}$) and a standard coupling argument we can construct a process $Z_\cdot$ with law P such that $Z_\cdot$ agrees with $Y_{t^\star +\cdot}$ with probability at least $1 - ce^{-N^{c(\varepsilon)}}$, and in particular

$$
Q_4[Z(0, uN^d) = Y(t^\star, uN^d + t^\star)] \geq 1 - ce^{-cN^{c(\varepsilon)}}. \tag{7.26}
$$

Now letting $k^- = [u(1-\delta)n\mathrm{cap}(A)]$ and $k^+ = [u(1+\delta)n\mathrm{cap}(A)]$ we have

$$
Q_4[Y(U_1, U_{k^-}) \subset Y(t^\star, uN^d + t^\star) \subset Y(0, U_{k^+})] \geq 1 - ce^{-s^c}, \tag{7.27}
$$

since $U_1 \overset{(7.1)}{\geq} U \circ \theta_{R_1} \overset{(5.4)}{\geq} t^\star$, $Q_4[U_{k^-} \leq uN^d] \geq 1 - ce^{-s^c}$ by (7.17), and $uN^d + t^\star \overset{u,\delta \geq N^{-c}}{\leq} N^d u(1 + \frac{\delta}{4}) < U_{[u(1+\frac{\delta}{4})^2 n\,\mathrm{cap}(A)]} \leq U_{k^+}$ with probability at least $1 - ce^{-s^c}$ by (7.16) (applied with $u(1 + \frac{\delta}{4})$ in place of u and $\frac{\delta}{4}$ in place of δ). Finally by (7.2) we have

$$
Q_4 \left[\overset{k^-}{\underset{i=2}{\bigcup}} \tilde{Y}^i(0, \infty) \cap \bar{C} = Y(U_1, U_{k^-}) \cap \bar{C}, Y(0, U_{k^+}) \cap \bar{C} = \overset{k^+}{\underset{i=1}{\bigcup}} \tilde{Y}^i(0, \infty) \cap \bar{C} \right]
$$

$$
\geq 1 - cue^{-cs^{c(\varepsilon)}},
$$

where we have used that $k^+ \leq cun\mathrm{cap}(A) \overset{(2.7)}{\leq} cus^{c(\varepsilon)}s^{(1-\varepsilon)(d-2)} \leq cus^{c(\varepsilon)}$. Combining this with (7.26) and (7.27) now gives (7.25) (using also that $u \geq s^{-c(\varepsilon)}$). □

Note that to ensure that also the first excursion $\tilde{Y}^1_\cdot$ has law P_σ, we have generated the law P of $Z_\cdot$ by modifying $Y_{t^\star +\cdot}$ under P_σ, and getting the i.i.d. excursions $\tilde{Y}^i_\cdot$, $i \geq 1$, from $Y_\cdot$ via Lemma 7.1. The first union is (7.25) is over $i \geq 2$, since with this construction $\tilde{Y}^1_\cdot$ corresponds to a piece of $Y_\cdot$ that is not fully included in $Z_\cdot$.

We are ready to finish the proof of Proposition 5.1 using the previous Proposition 7.5.

Proof of Proposition 5.1 We apply Proposition 7.5 with $\frac{\delta}{4} \geq cs^{-c(\varepsilon)}$ in place of δ to construct the space $(\Omega_4, \mathcal{A}_4, Q_4)$ which we will extend to get $(\Omega_2, \mathcal{A}_2, Q_2)$. First of all we rename the process $Z.$ to $Y.$, so that we have from (7.25) that

$$Q_4\left[\bigcup_{i=2}^{[u(1-\frac{\delta}{4})n\,\mathrm{cap}(A)]} \tilde{Y}^i(0,\infty) \cap \bar{C} \subset Y(0, uN^d) \cap \bar{C} \subset \bigcup_{i=1}^{[u(1+\frac{\delta}{4})n\mathrm{cap}(A)]} \tilde{Y}^i(0,\infty) \cap \bar{C} \right]$$
$$\geq 1 - c(\varepsilon)ue^{-cs^{c(\varepsilon)}}, \tag{7.28}$$

where $\tilde{Y}_1, \tilde{Y}_2, \ldots$ are i.i.d. with law $P_\sigma[Y_{\cdot \wedge U_1} \in dw]$. We now add independent Poisson random variables J_1 and J_2 to the space, where J_1 has parameter $u(1-\frac{\delta}{2})n\mathrm{cap}(A)$, J_2 has parameter $u\delta n\mathrm{cap}(A)$ and define the following point processes on $\Gamma(\mathbb{T}_N)$

$$\mu_1' = \sum_{i=2}^{J_1+1} \delta_{\tilde{Y}^i_{H_{\bar{A}}+\cdot}} \quad \text{and} \quad \mu_2' = 1_{\{J_2 \neq 0\}}\delta_{\tilde{Y}^1_{H_{\bar{A}}+\cdot}} + \sum_{i=J_1+2}^{J_1+J_2} \delta_{\tilde{Y}^i_{H_{\bar{A}}+\cdot}}. \tag{7.29}$$

Then μ_1' and μ_2' are independent Poisson point processes such that μ_1' has intensity $u(1 - \frac{\delta}{2})n\mathrm{cap}(A)P_q[Y_{\cdot \wedge U} \in dw]$ and μ_2' has intensity $u\delta n\mathrm{cap}(A)P_q[Y_{\cdot \wedge U} \in dw]$, where q denotes the measure defined by $q(x) = P_\sigma[Y_{H_{\bar{A}}} = x]$. By a standard large deviations bound one can show that

$$Q_4\left[J_1 + 1 \leq \left[u\left(1 - \frac{\delta}{4}\right)n\mathrm{cap}(A) \right] \leq \left[u\left(1 + \frac{\delta}{4}\right)n\mathrm{cap}(A) \right] \leq J_1 + J_2 \right]$$
$$\geq 1 - ce^{-cun\mathrm{cap}(A)\delta^2},$$

so that from $u\mathrm{cap}(A)\delta^2 \overset{(2.7),u,\delta \geq s^{-(1-\varepsilon)\frac{d-2}{8}}}{\geq} s^{(1-\varepsilon)\frac{d-2}{2}}$, (7.28) and (7.29) it follows that

$$Q_4[\mathcal{I}(\mu_1') \cap \bar{A} \subset Y(0, uN^d) \cap \bar{A} \subset \mathcal{I}(\mu_1' + \mu_2') \cap \bar{A}] \geq 1 - cue^{-cs^{c(\varepsilon)}}. \tag{7.30}$$

By (6.19) we have the following inequality involving the intensity of μ_1'

$$u(1-\delta)\kappa_1 \overset{\delta \geq cs^{-c(\varepsilon)}}{\leq} u\left(1 - \frac{\delta}{2}\right)n\,\mathrm{cap}(A)P_q[Y_{\cdot \wedge U} \in dw] \overset{\delta \geq cs^{-c(\varepsilon)}}{\leq} u\left(1 - \frac{\delta}{3}\right)\kappa_1. \tag{7.31}$$

Using the lower bound we can thus construct, by means of a standard thinning procedure, a Poisson point process μ_1 of intensity $u(1-\delta)\kappa_1$, such that $\mu_1 \leq \mu_1'$ and μ_1 and $\mu_1' - \mu_1$ are independent (by placing each point $x \in \Gamma(\mathbb{T}_n)$ of μ_1' in μ_1 independently with probability $\frac{u(1-\delta)\kappa_1(x)}{u(1-\delta/2)n\mathrm{cap}(A)P_q[Y_{\cdot \wedge U}=x]} \in [0, 1]$, where we extend the space with the appropriate Bernoulli random variables). Using the upper bound we can furthermore thicken $\mu_1' - \mu_1$ to get a Poisson point process ν (independent of μ_1) of intensity

$u\frac{2}{3}\delta\kappa_1$ such that $\mu_1' \leq \mu_1 + \nu$ (by extending the space with with an independent Poisson point process of intensity $u(1-\delta/3)\kappa_1 - u(1-\delta/2)n\text{cap}(A)P_q[Y_{.\wedge U} \in dw] \geq 0$ and adding this process to $\mu_1' - \mu_1$ to form ν).

Once again by (6.19) we also have the following inequality involving the intensity of μ_2'

$$u\delta n \ \text{cap}(A)P_q[Y_{.\wedge U} \in dw] \overset{s\geq c(\varepsilon)}{\leq} u\frac{4}{3}\delta\kappa_1.$$

Thus we can (as above) thicken μ_2' to get a Poisson point process η of intensity $u\frac{4}{3}\delta\kappa_1$, such that $\mu_2' \leq \eta$ and μ_1, ν, η are independent. We then define $\mu_2 = \nu + \eta$, and see that μ_2 is a Poisson point process of intensity $u2\delta\kappa_1$ which is independent from μ_1, and $\mu_1 \leq \mu_1' \leq \mu_1' + \mu_2' \leq \mu_1 + \mu_2$. Thus it follows from (7.30) that the probability of the event in (5.6) is at least $1 - cue^{-s^{c(\varepsilon)}}$ and the proof of Proposition 5.1 is complete. □

To prove Theorem 3.2 it now remains to show Proposition 5.2 and Proposition 5.4.

8 From the torus to $\mathbb{Z}^d$, and decoupling boxes

In this section we prove Proposition 5.2 which dominates a Poisson point process ν of intensity a multiple of κ_1 (and whose "excursions" therefore roughly speaking "feel that they are in the torus", and may visit several of the boxes $A_1, \ldots, A_n$, see (5.4) and (5.5)), from above and below by Poisson point processes whose intensities are multiples of κ_2 (and whose "excursions" thus, conditionally on their starting point, "behave" like random walk in $\mathbb{Z}^d$ stopped upon leaving a box, and visit only a single box, see (5.7)). First we will (roughly speaking) take all excursions in ν that never return to $\bar{A}$ after leaving $\bar{B}$ (the great majority of all excursions), truncate them upon leaving $\bar{B}$, and collect them into a Poisson point process whose intensity can bounded from above and below by multiples of κ_2 (see Lemma 8.1). This will allow us to dominate this Poisson point process, from above and below, by Poisson point processes with intensities a multiple of κ_2. We will then, in Lemma 8.2, use an argument from the proof of Theorem 2.1 in [24] to dominate the excursions that do return to $\bar{A}$ after leaving $\bar{B}$ by a Poisson point process with intensity a small multiple of κ_2. That is, we will, essentially speaking, decouple the "successive visits to $\bar{A}$ (after leaving $\bar{B}$)" of a single excursion by dominating the hitting distribution on $\bar{A}$ when starting outside of $\bar{B}$ by a multiple of the measure e from (5.5) (see Lemma 8.3). We then collect a number of the successive visits of all the excursions (with high probability *all the successive visits* of *all the excursions*, and in addition a number of extra independent visits) into a Poisson point process of intensity a small multiple of κ_2. This is done in Lemma 8.2. Though the number of excursions that make several "successive returns" to $\bar{A}$ are small, dominating them with high enough probability (namely, stretched exponential in the separation s, see the statement of Proposition 5.2 and (8.4)), so that what happens in the individual boxes $A_1, A_2, \ldots, A_n$, is independent, is not straight-forward. Since Lemma 8.2 achieves this, it should be considered the heart of the proof of Proposition 5.2.

We recall the standing assumption (5.1). Define for $w \in \Gamma(\mathbb{T}_N)$ the successive returns $\hat{R}_k = \hat{R}_k(w)$ to $\bar{A}$ and departures $\hat{D}_k = \hat{D}_k(w)$ from $\bar{B}$ as follows

$$\hat{R}_1 = H_{\bar{A}}, \ \hat{R}_k = H_{\bar{A}} \circ \theta_{\hat{D}_{k-1}} + \hat{D}_{k-1}, k \geq 2, \ \hat{D}_k = T_{\bar{B}} \circ \theta_{\hat{R}_k} + \hat{R}_k, k \geq 1.$$
(8.1)

Note that $\hat{R}_k$ should not be confused with the R_k used in Sect. 7 (see (7.1)). To extract the "successive visits to $\bar{A}$" of an excursion we furthermore define for each $i \geq 1$ the map ϕ_i from $\Gamma(\mathbb{T}_N)$ into $\Gamma(\mathbb{T}_N)^i$ by

$$(\phi_i(w))_j = w((\hat{R}_j + \cdot) \wedge \hat{D}_j) \quad \text{for } j = 1, \dots, i, \ w \in \{\hat{R}_i < U < \hat{R}_{i+1}\} \subset \Gamma(\mathbb{T}_N), i \geq 1.$$
(8.2)

For each $i \geq 1$ we will apply this map to the Poisson point processes $1_{\{\hat{R}_i < U < \hat{R}_{i+1}\}} \nu$ to get a Poisson point processes μ_i of intensity $u\kappa_1^i$, where

$$\kappa_1^i = \phi_i \circ (1_{\{\hat{R}_i < U < \hat{R}_{i+1}\}} \kappa_1), i \geq 1.$$
(8.3)

To dominate the μ_1, the first of these Poisson point processes, which contains most excursions, from above and below by Poisson point processes of intensities that are multiples of κ_2 we will use the following inequality.

Lemma 8.1 $(d \geq 3, N \geq 3)$ *If $s \geq c(\varepsilon)$ and $n \leq s^{c(\varepsilon)}$ we have $(1 - cs^{-c(\varepsilon)})\kappa_2 \leq \kappa_1^1 \leq \kappa_2$.*

We postpone the proof of Lemma 8.1 until after the proof of Proposition 5.2. The Poisson point processes $\mu_2, \mu_3, \dots$ will contain the successive visits to $\bar{A}$ of the excursions that make $2, 3, \dots$ such visits, respectively. There will only be a few of these and using the following lemma we will be able to dominate them by a Poisson point process θ of intensity a small multiple of κ_2.

Lemma 8.2 $(d \geq 3, N \geq 3)$ *Let $(\Omega, \mathcal{A}, Q)$ be a probability space with independent Poisson point processes $\mu_2, \mu_3, \dots$ such that μ_i has intensity $u\kappa_1^i$, $u \geq 0$. Then if $1 \geq \delta \geq s^{-c(\varepsilon)}$, $n \leq s^{c(\varepsilon)}$ and $s \geq c(\varepsilon)$ we can construct a space $(\Omega', \mathcal{A}', Q')$ and, on the product space $\Omega \times \Omega'$, a $\sigma(\mu_i, i \geq 2) \times \mathcal{A}'-$measurable Poisson point process θ of intensity $u\delta\kappa_2$ such that (recalling the notation from (2.16))*

$$Q \otimes Q'[\cup_{i \geq 2} \mathcal{I}(\mu_i) \subset \mathcal{I}(\theta)] \geq 1 - ce^{-cu\delta cap(A)}.$$
(8.4)

We postpone the proof of Lemma 8.2 until later, and instead use it together with Lemma 8.1 to prove Proposition 5.2.

Proof of Proposition 5.2 We let

$$\mu_i = \phi_i(1_{\{\hat{R}_i < U < \hat{R}_{i+1}\}} \nu), i \geq 1, \text{ so that } \cup_{i \geq 1} \mathcal{I}(\mu_i) \cap \bar{A} \overset{(8.1),(8.2)}{=} \mathcal{I}(\nu) \cap \bar{A}. \quad (8.5)$$

Since the sets $\{\hat{R}_i < U < \hat{R}_{i+1}\}, i \geq 1$, are disjoint $\mu_i, i \geq 1$, are independent Poisson point processes on the respective spaces $\Gamma(\mathbb{T}_N)^i, i \geq 1$, and by (8.3) they have respective intensities $u\kappa_1^i, i \geq 1$. By Lemma 8.1 it follows, since we require $\delta \geq cs^{-c(\varepsilon)}$, that

$$u(1 - \delta)\kappa_2 \le u\kappa_1^1 \le u\kappa_2. \tag{8.6}$$

Now similarly to how we used (7.31) to construct the processes μ_1 and ν from μ', we now (extending our space appropriately) use (8.6) to construct processes ν_1 and ρ, such that $\nu_1, \rho, \mu_i, i \ge 2$, are independent, ν_1 has intensity $u(1-\delta)\kappa_2$, ρ has intensity $u\delta\kappa_2$ and

$$\nu_1 \le \mu_1 \le \nu_1 + \rho \text{ almost surely.} \tag{8.7}$$

Thus (5.8) holds, because $\mathcal{I}(\nu_1) \cap \bar{A} \overset{(8.7)}{\subset} \mathcal{I}(\mu_1) \cap \bar{A} \overset{(8.5)}{\subset} \mathcal{I}(\nu) \cap \bar{A}$, and since ν_1 has intensity $u(1-\delta)\kappa_2$ it now suffices to construct ν_2 appropriately.

To this end we apply Lemma 8.2, once again extending the space, to get a Poisson point process θ of intensity $u\delta\kappa_2$ such that ν_1, ρ and θ are independent and

$$Q[\cup_{i\ge2}\mathcal{I}(\mu_i) \subset \mathcal{I}(\theta)] \ge 1 - ce^{-cu\delta\,\mathrm{cap}(A)} \ge 1 - ce^{-cs^{c(\varepsilon)}}, \tag{8.8}$$

where we use that we require $u, \delta \ge s^{-c(1-\varepsilon)} = s^{-c(\varepsilon)}$ so that $u\delta\,\mathrm{cap}(A) \overset{(2.7),(5.2)}{\ge} u\delta s^{(1-\varepsilon)(d-2)} \ge s^{c(\varepsilon)}$. Now set $\nu_2 = \rho + \theta$ and note that ν_1 and ν_2 are independent, ν_2 has intensity $2u\delta\kappa_2$ and because of (8.5), (8.7) and (8.8) we have

$$Q[\mathcal{I}(\nu) \cap \bar{A} \subset \mathcal{I}(\nu_1 + \nu_2)] \ge 1 - ce^{-cs^{c(\varepsilon)}}.$$

Thus the proof of Proposition 5.2 is complete. $\qquad\square$

The proof of Proposition 5.2 has thus been reduced to Lemma 8.1 and Lemma 8.2. We now prove Lemma 8.1.

Proof of Lemma 8.1 Let $W \subset \Gamma(\mathbb{T}_N)$ be measurable. Then $\kappa_1^1(W) \overset{(5.5),(8.3)}{=}$ $P_e[Y_{\cdot \wedge T_{\bar{B}}} \in W, U < \hat{R}_2]$ so the upper bound follows directly from (5.7). Furthermore $\kappa_1^1(W) \ge \kappa_2(W) \inf_{x\in\partial_e\bar{B}} P_x[H_{\bar{A}} > T_{\bar{D}}] \inf_{x\in\partial_e\bar{D}} P_x[H_{\bar{A}} > U]$ (recall (6.12)) by the strong Markov property. But (if $s \ge c(\varepsilon)$) $\inf_{x\in\partial_e\bar{B}} P_x[H_{\bar{A}} > T_{\bar{D}}] = \inf_{x\in\partial_e B} P_x^{\mathbb{Z}^d}[H_A > T_D] \overset{(2.10)}{\ge} 1 - cs^{-c(\varepsilon)}$, and by (6.21) we have $\inf_{x\in\partial_e\bar{D}} P_x[H_{\bar{A}} > U] \ge 1 - c(\varepsilon)s^{-c(\varepsilon)}$, so the lower bound follows. $\qquad\square$

It thus only remains to prove Lemma 8.2 to complete the proof of Proposition 5.2. For this we will use the following bound on the intensities κ_1^i of the μ_i, (this corresponds to (2.33) in [24]).

Lemma 8.3 ($d \ge 3$, $N \ge 3$) *If $s \ge c(\varepsilon)$ and $n \le s^{c(\varepsilon)}$ then for all $i \ge 2$*

$$\kappa_1^i \le \tilde{\kappa}_1^i \text{ where } \tilde{\kappa}_1^i(d(w_1, \ldots, w_i)) = s^{-\frac{\varepsilon}{8}(i-1)}cap(A) \otimes_{k=1}^i P_{\bar{e}}[Y_{\cdot\wedge T_{\bar{B}}} \in dw_k], \tag{8.9}$$

and $\bar{e} = \frac{e}{n\,cap(A)}$ denotes the normalisation of the measure e from (5.5) (see (2.6)).

In the proof of Lemma 8.2 we will use Lemma 8.3 to dominate μ_i, $i \geq 2$, by Poisson point processes η_i of intensity $u\tilde{\kappa}_1^i$. Since $\tilde{\kappa}_1^i$ is proportional to a product measure the "points" of η_i will be vectors of *independent* excursions with law $P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in dw]$. Thus we will have "decoupled" the excursions and we will be able to use them (along with additional independent excursions) to construct the Poisson point process θ. We postpone the proof of Lemma 8.3 until after the proof of Lemma 8.2. In the proof of Lemma 8.2 we will use the following simple lemma about Poisson random variables.

Lemma 8.4 *Let N be a Poisson random variable of intensity $\lambda > 0$, and let N_i, $i \geq 2$, be independent Poisson random variables such that N_i has intensity at most λr^{i-1}. Then*

$$\mathbb{P}\Big[\sum_{i \geq 2} i N_i \leq N\Big] \geq 1 - ce^{-c\lambda}, \quad \text{if } 0 < r \leq c. \tag{8.10}$$

Proof This follows from the standard Chebyshev bounds $\mathbb{P}[N < \frac{\lambda}{2}] \leq \mathbb{E}[e^{-\frac{1}{2}N}]e^{\frac{\lambda}{4}} \leq e^{-\frac{\lambda}{8}}$ and

$$\mathbb{P}\Big[\sum_{l \geq 2} i N_i > \frac{\lambda}{2}\Big] \leq e^{-\frac{\lambda}{2}} \mathbb{E}[e^{\sum_{i \geq 2} i N_i}] \leq e^{-\frac{\lambda}{2} + \lambda \sum_{i \geq 2} r^{i-1}(e^i - 1)} \overset{re \leq c < 1}{\leq} e^{-\frac{\lambda}{4}} \quad \text{for all } \lambda > 0.$$

$\square$

We now prove Lemma 8.2. The proof corresponds roughly to (2.38)–(2.54) in [24].

Proof of Lemma 8.2 If we multiply (8.9) by u we get for each $i \geq 2$ an inequality for the intensity measure of μ_i. Because of this inequality we can "thicken" each μ_i, by constructing $(\Omega', \mathcal{A}', Q')$ with the appropriate random variables, to get (on $\Omega \times \Omega'$) $\sigma(\mu_i, i \geq 2) \times \mathcal{A}'-$measurable independent Poisson point processes

$$\eta_i, i \geq 2, \text{ on } \Gamma(\mathbb{T}_N)^i \text{ of intensities } u\tilde{\kappa}_1^i \text{ (respectively), such that } \mu_i \leq \eta_i, i \geq 2, \tag{8.11}$$

(analogously to below (7.31)). We note that if we let $N_i = \eta_i(\Gamma(\mathbb{T}_N)^i)$, $i \geq 2$, then (see (8.9))

$$N_i, i \geq 2, \text{ are independent and Poisson, where } N_i \text{ has intensity } us^{-\frac{\varepsilon}{8}(i-1)}\mathrm{cap}(A). \tag{8.12}$$

Now extend $(\Omega', \mathcal{A}', Q')$ to obtain $\sigma(\mu_i, i \geq 2) \times \mathcal{A}'-$measurable vectors v_j^i, $i \geq 2$, $j \geq 1$, such that v_j^i, $i \geq 2$, $j \geq 1$, N_i, $i \geq 1$, are independent,

$$v_j^i \text{ has law } \otimes_{k=1}^i P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in dw_k] \text{ (i.e. } u\tilde{\kappa}_1^i \text{ normalised) and } \eta_i = \sum_{j=1}^{N_i} \delta_{v_j^i}, i \geq 2, \tag{8.13}$$

(conditionally on η_i, we order the N_i points in the support of η_i according to say the time until the first jump of the first of the i paths that make up a point of η_i, and let $v_1^i, \ldots, v_{N_i}^i$, be a permutation of these points chosen uniformly at random; we then add i.i.d. vectors to form $v_j^i, j > N_i$). Define $\bar{N} = \sum_{i \geq 2} i N_i$ and construct on $(\Omega', \mathcal{A}', Q')$ a Poisson random variable N of intensity $u\delta n\mathrm{cap}(A)$, and trajectories $\tilde{w}_i, i \geq 1$, with law $P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in dw]$, such that $N, \tilde{w}_i, i \geq 1, v_j^i, i \geq 2, j \geq 1, N_i, i \geq 2$, are independent. Write $v_j^i = (w_{j,1}^i, \ldots, w_{j,i}^i)$ and let

$$
\theta = \begin{cases}
\sum_{i=2}^{\infty} \sum_{j=1}^{N_i} (\delta_{w_{j,1}^i} + \cdots + \delta_{w_{j,i}^i}) + \sum_{i=1}^{N-\bar{N}} \delta_{\tilde{w}_i} & \text{if } N \geq \bar{N}, \\
\sum_{i=1}^{N} \delta_{\tilde{w}_i} & \text{if } N < \bar{N}.
\end{cases} \tag{8.14}
$$

The number of points N of θ is a Poisson random variable, and conditionally on N the points of θ are i.i.d. with law $P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in dw]$ (see (8.13)), so that θ is (as claimed) a $\sigma(\mu_i, i \geq 2) \times \mathcal{A}'$−measurable Poisson point process of intensity $u\delta n\mathrm{cap}(A)P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in dw] = u\delta\kappa_2$ (see (5.7)).

It remains to show (8.4). We have $\cup_{i \geq 2}\mathcal{I}(\mu_i) \subset \cup_{i \geq 2}\mathcal{I}(\eta_i)$ (by (8.11)) and on the event $\{\bar{N} \leq N\}$ we have $\cup_{i \geq 2}\mathcal{I}(\eta_i) \subset \mathcal{I}(\theta)$ by (8.13) and (8.14). Thus by (8.10) with $\lambda = u\delta n\mathrm{cap}(A)$ and $r = \delta^{-1}s^{-\frac{\varepsilon}{8}} \leq c$ (we require $\delta \geq cs^{-\varepsilon/8}$) we get

$$
Q \otimes Q'[\cup_{i \geq 2}\mathcal{I}(\mu_i) \subset \mathcal{I}(\theta)] \geq Q \otimes Q'[\bar{N} \leq N] \geq 1 - ce^{-cu\delta\mathrm{cap}(A)}, \tag{8.15}
$$

(see (8.12) and note that $us^{-\frac{\varepsilon}{8}(i-1)}\mathrm{cap}(A) \leq \lambda\delta^{-1}s^{-\frac{\varepsilon}{8}(i-1)} \leq \lambda r^{i-1}$). Thus (8.4) holds. $\qquad \square$

It remains to prove Lemma 8.3. For the proof we will need the following upper bound on the probability of hitting $\bar{A}$ in a given point before U, from outside of $\bar{B}$.

Lemma 8.5 $(s \geq c(\varepsilon), n \leq s^{c(\varepsilon)})$ *For all* $x \in \partial_e\bar{B}$ *and* $y \in \partial_i\bar{A}$ *we have*

$$
h_x(y) \leq s^{-\frac{\varepsilon}{4}}\bar{e}(y) \text{ where } h_x(y) = P_x[H_{\bar{A}} < U, Y_{H_{\bar{A}}} = y]. \tag{8.16}
$$

Note that crucially $s^{-\frac{\varepsilon}{4}}\bar{e}(y)$ does not depend on the starting point x. In the proof of Lemma 8.3, which we now start, this is what will allow us to bound κ_1^i from above by an intensity that is proportional to a product measure (namely $\tilde{\kappa}_1^i$). The proof of Lemma 8.5 will follow after the proof of Lemma 8.3.

Proof of Lemma 8.3 Fix a $i \geq 2$. Let $W_1, \ldots, W_i \subset \Gamma(\mathbb{T}_N)$ be measurable. Then

$$
\begin{aligned}
\kappa_1^i(W_1 \times \cdots \times W_i) &\stackrel{(5.5),(8.3)}{=} P_e[\hat{R}_i < U < \hat{R}_{i+1}, Y_{(\hat{R}_j+\cdot)\wedge\hat{D}_j} \in W_j, 1 \leq j \leq i] \\
&\leq P_e[\hat{R}_i < U, Y_{(\hat{R}_j+\cdot)\wedge\hat{D}_j} \in W_j, 1 \leq j \leq i] \\
&\stackrel{\substack{(8.1),(8.17) \\ \text{Markov}}}{=} E_e[1_{\{\hat{R}_{i-1}<U, Y_{(\hat{R}_j+\cdot)\wedge\hat{D}_j}\in W_j, 1 \leq j \leq i-1\}} P_{h_{Y_{\hat{D}_{i-1}}}}[Y_{\cdot\wedge T_{\bar{B}}} \in W_j]], \\
&\stackrel{(8.16)}{\leq} s^{-\frac{\varepsilon}{4}} P_e[\hat{R}_{i-1} < U, Y_{(\hat{R}_j+\cdot)\wedge\hat{D}_j} \in W_j, 1 \leq j \leq i-1] P_{\bar{e}}[Y_{\cdot\wedge T_{\bar{B}}} \in W_i].
\end{aligned}
$$

Now iterating a similar inequality we get

$$\kappa_1^i(W_1 \times \cdots \times W_i) \leq s^{-\frac{\varepsilon}{4}(i-1)} P_e[Y_{\cdot \wedge T_{\bar{B}}} \in W_1] \prod_{j=2}^{i} P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in W_j]$$

$$\leq s^{-\frac{\varepsilon}{8}(i-1)} \mathrm{cap}(A) \prod_{j=1}^{i} P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in W_j] \overset{(8.9)}{=} \tilde{\kappa}_1^i(W_1 \times \cdots \times W_i),$$

where we have used that $s^{-\frac{\varepsilon}{4}(i-1)} P_e[Y_{\cdot \wedge T_{\bar{B}}} \in W_1] = s^{-\frac{\varepsilon}{4}(i-1)} n\, \mathrm{cap}(A) P_{\bar{e}}[Y_{\cdot \wedge T_{\bar{B}}} \in W_1]$ and $s^{-\frac{\varepsilon}{4}(i-1)} n \leq s^{-\frac{\varepsilon}{8}(i-1)}$ (we require $n \leq s^{\frac{\varepsilon}{8}}$).

Thus $\kappa_1^i(W) \leq \tilde{\kappa}_1^i(W)$ for all $W \in \Gamma(\mathbb{T}_N)^i$ that are products of measurable sets. This implies that $\kappa_1^i(W) \leq \tilde{\kappa}_1^i(W)$ for all $W \in \Gamma(\mathbb{T}_N)^i$ that are finite unions of such sets (W need not be a disjoint union, since "overlapping" unions of products of measurable sets may be turned into disjoint unions of such sets by further "subdividing" the "overlapping" sets). By a monotone class argument, this implies that $\kappa_1^i(W) \leq \tilde{\kappa}_1^i(W)$ for *all* measurable $W \in \Gamma(\mathbb{T}_N)^i$ (see Theorem 3.4, p. 39 in [7]), so (8.9) follows. □

Finally, we prove Lemma 8.5, using the Harnack inequality and (2.8).

Proof of Lemma 8.5 If $j \neq k$, $x \in \partial_e B_j$ and $y \in \partial_i A_k$ then by the Markov property

$$h_x(y) = P_x[H_{\bar{A}} < U, Y_{H_{\bar{A}}} = y] \leq \sup_{x \in \partial_e B_k} P_x[H_{\bar{A}} < U, Y_{H_{\bar{A}}} = y] = \sup_{x \in \partial_e B_k} h_x(y),$$
(8.17)

(provided $s \geq c(\varepsilon)$ so that B_j and B_k are disjoint), so without loss of generality we may assume $x \in \partial_e B_k$. We have (recall from (6.12) that $C \subset D = B(0, s^{1-\frac{\varepsilon}{8}})$)

$$h_x(y) \leq P_x[H_{\bar{A}} < T_{\bar{D}}, Y_{H_{\bar{A}}} = y] + P_x[T_{\bar{D}} < H_{\bar{A}} < U, Y_{H_{\bar{A}}} = y]. \quad (8.18)$$

By the strong Markov property, (6.21) and (8.17), we have for $s \geq c(\varepsilon)$

$$P_x[T_{\bar{D}} < H_{\bar{A}} < U, Y_{H_{\bar{A}}} = y] \leq \sup_{z \in \partial_e \bar{D}} P_z[H_{\bar{C}} < U] \sup_{z \in \partial_e \bar{B}} h_z(y) \leq c s^{-c(\varepsilon)} \sup_{z \in \partial_e B_k} h_z(y).$$
(8.19)

The function $z \to h_z(y)$ is non-negative harmonic on $C_k \backslash A_k$ (which can be identified with a subset of $\mathbb{Z}^d$) so by the Harnack inequality (Proposition 1.7.2, p. 42, [12]) and a standard covering argument we have $h_z(y) \leq c h_x(y)$ for all $z \in \partial_e B_k$. Applying this inequality to the right-hand side of (8.19), plugging the result into (8.18) and rearranging we find that

$$h_x(y) \leq c P_x[H_{\bar{A}} < T_{\bar{D}}, Y_{H_{\bar{A}}} = y] \leq c \sup_{z \in \partial_i B} P_z^{\mathbb{Z}^d}[H_A < \infty, Y_{H_A} = y] \quad \text{for } s \geq c(\varepsilon).$$
(8.20)

Thus using (2.8) with $K = A$, $r = s^{1-\varepsilon}$ (recall (3.3)) we have that if $s \geq c(\varepsilon)$ (so that $z \notin B(0, c_3 s^{1-\varepsilon})$ if $z \in \partial_i B$) then

$$h_x(y) \leq c_2 \frac{e_A(y)}{\mathrm{cap}(A)} \sup_{z \in \partial_i B} P_z^{\mathbb{Z}^d}[H_A < \infty] \stackrel{(2.10)}{\leq} cn\bar{e}(y)s^{-\frac{\varepsilon}{2}} \stackrel{n \leq s^{\frac{\varepsilon}{8}}, s \geq c(\varepsilon)}{\leq} \bar{e}(y)s^{-\frac{\varepsilon}{4}}.$$

□

Now all components used in the proof of Proposition 5.2 have been proved.

9 Coupling with random interlacements

In this section we prove Proposition 5.4. We use essentially the same techniques that were used to prove Proposition 5.2 in the previous section, but speaking very roughly we use them "in reverse" to reconstruct from the excursions in the Poisson point process η of intensity $u\kappa_3$, which all end upon leaving B, excursions with law $P_{e_A}^{\mathbb{Z}^d}$ (or rather, the successive visits to A after departures from B of such excursions). $P_{e_A}^{\mathbb{Z}^d}$ gives positive measure to excursions that return to A even after leaving B, and to construct such excursions we will in Lemma 9.2 take a "small number" of excursions from η and "glue them together", essentially reversing the argument from Lemma 8.2.

Let $\tilde{R}_1 \leq \tilde{D}_1 \leq \tilde{R}_2 \leq \tilde{D}_2 \leq \cdots$ on $\Gamma(\mathbb{Z}^d)$ denote the successive returns of Y. to A and successive departures from B,

$$\tilde{R}_1 = H_A, \ \tilde{R}_k = H_A \circ \theta_{\tilde{D}_{k-1}} + \tilde{D}_{k-1}, k \geq 2, \ \tilde{D}_k = T_B \circ \theta_{\tilde{R}_k} + \tilde{R}_k, k \geq 1.$$

These should not be confused with the $\hat{R}_k$, $\hat{D}_k$, which were defined on $\Gamma(\mathbb{T}_N)$ and used in Sect. 8 (see (8.1)), or the R_k from Sect. 7 (see (7.1)). Furthermore similarly to (8.2) define maps $\phi_i^{\mathbb{Z}^d}$, $i \geq 2$, from $\Gamma(\mathbb{Z}^d)$ to $\Gamma(\mathbb{Z}^d)^i$ extracting the excursions between A and B,

$$(\phi_i^{\mathbb{Z}^d}(w))_j = w((\tilde{R}_j + \cdot) \wedge \tilde{D}_j) \quad \text{for } j = 1, \ldots, i, \ w \in \{\tilde{R}_i < \infty = \tilde{R}_{i+1}\} \subset \Gamma(\mathbb{Z}^d), i \geq 1. \tag{9.1}$$

To construct the random set $\mathcal{I}_1$ from the statement of Proposition 5.4 we will construct Poisson point processes of intensities $u(1 - \delta)\kappa_4^i$, where (cf. (8.3))

$$\kappa_4^i = \phi_i \circ (1_{\{\tilde{R}_i < \infty = \tilde{R}_{i+1}\}} P_{e_A}^{\mathbb{Z}^d}), \quad i \geq 1. \tag{9.2}$$

This will be enough to construct $\mathcal{I}_1$ because if μ_i, $i \geq 1$, are i.i.d. Poisson point processes of intensity $u(1 - \delta)\kappa_4^i$ then by (2.18), (2.19), (9.1) and (9.2) (recalling the notation from (2.16))

$$\mathcal{I}^{u(1-\delta)} \cap A \stackrel{\text{law}}{=} \cup_{i \geq 1} \mathcal{I}(\mu_i) \cap A. \tag{9.3}$$

To construct a Poisson point process μ_1 of intensity $u(1 - \delta)\kappa_4^1$ we will, in the proof of Proposition 5.4 , "extract" a Poisson point process of intensity $u(1 - \delta)\kappa_3$ from η, and "thin" it to get μ_1. This will be possible because of the following inequality.

Lemma 9.1 ($N \geq 3, d \geq 3$) *If $s \geq c(\varepsilon)$ then $\kappa_4^1 \leq \kappa_3 \leq (1 + cs^{-c(\varepsilon)})\kappa_4^1$.*

Proof This is a consequence of (2.10), (5.10) and (9.2). The proof is a very similar to that of Lemma 8.1, so we omit it.   □

After constructing μ_1 in the proof of Proposition 5.4 we will take "what is left of η after thinning using Lemma 9.1", namely a Poisson point process θ of intensity $u\delta\kappa_3$, and "extract from it" the Poisson point processes $\mu_2, \mu_3, \ldots$ of respective intensities $u(1 - \delta)\kappa_4^i$. This will be done using the following lemma.

Lemma 9.2 ($d \geq 3, N \geq 3$) *Let $u \geq 0$, $\delta \geq cs^{-c(\varepsilon)}$ and $s \geq c(\varepsilon)$, and let $(\Omega, \mathcal{A}, Q)$ be a probability space with a Poisson point processes θ of intensity $u\delta\kappa_3$. Then we can construct a space $(\Omega', \mathcal{A}', Q')$ and, on the product space, independent $\sigma(\theta) \times \mathcal{A}'-$measurable Poisson point processes $\mu_i, i \geq 2$, and ρ_2 such that μ_i has intensity $u(1 - \delta)\kappa_4^i$, ρ_2 has intensity $u\frac{3\delta}{2}\kappa_3$ and*

$$Q \otimes Q' \left[\bigcup_{i \geq 2} \mathcal{I}(\mu_i) \subset \mathcal{I}(\theta) \subset \cup_{i \geq 2}\mathcal{I}(\mu_i) \bigcup \mathcal{I}(\rho_2) \right] \geq 1 - ce^{-cu\delta cap(A)}. \quad (9.4)$$

The "residual" Poisson point process ρ_2, as well as a "residual" Poisson point process left after "thinning" to obtain μ_1, will be used to construct $\mathcal{I}_2$. We postpone the proof of Lemma 9.2 until later, and instead use it to prove Proposition 5.4.

Proof of Proposition 5.4 We start by constructing $(\Omega', \mathcal{A}', Q')$ appropriately, to obtain, by "thinning" η, a $\sigma(\eta) \times \mathcal{A}'-$measurable Poisson point process θ on the product space of intensity $u\delta\kappa_3$, such that $\eta - \theta$ and θ are independent, and $\eta - \theta$ is a Poisson point process of intensity $u(1 - \delta)\kappa_3$ (similarly to below (7.31), note that of course $u\delta\kappa_3 \leq u\kappa_3$).

Since we require $\delta \geq cs^{-c(\varepsilon)}$ and $s \geq c(\varepsilon)$ we have $u(1-\delta)(1+s^{-c(\varepsilon)}) \leq u(1-\frac{\delta}{2})$, and thus $u(1 - \delta)\kappa_4^1 \leq u(s1 - \delta)\kappa_3 \leq u(1 - \frac{\delta}{2})\kappa_4^1$ by Lemma 9.1. But $u(1 - \frac{\delta}{2})\kappa_4^1 \leq u(1 - \delta)\kappa_4^1 + \frac{\delta}{2}\kappa_3$ since $\kappa_4^1 \leq \kappa_3$, so that

$$u(1 - \delta)\kappa_4^1 \leq u(1 - \delta)\kappa_3 \leq u(1 - \delta)\kappa_4^1 + u\frac{\delta}{2}\kappa_3.$$

Therefore we can, similarly to under (8.6), construct (extending $(\Omega', \mathcal{A}', Q')$ appropriately) $\sigma(\eta) \times \mathcal{A}'-$measurable Poisson point processes μ_1 and ρ_1 such that μ_1, ρ_1, θ are independent, μ_1 has intensity $u(1 - \delta)\kappa_4^1$, ρ_1 has intensity $u\frac{\delta}{2}\kappa_3$,

$$\mu_1 \leq \eta - \theta \leq \mu_1 + \rho_1, \text{ and thus } \mathcal{I}(\mu_1) \subset \mathcal{I}(\eta - \theta) \subset \mathcal{I}(\mu_1) \bigcup \mathcal{I}(\rho_1). \quad (9.5)$$

We then apply Lemma 9.2 (once again extending $(\Omega', \mathcal{A}', Q')$) to θ to get $\sigma(\eta) \times \mathcal{A}'$−measurable Poisson point processes $\mu_i, i \geq 2, \rho_2$ such that $\rho_1, \rho_2, \mu_i, i \geq 1$, are independent, μ_i has intensity $u(1 - \delta)\kappa_4^i$, ρ_2 has intensity $u\frac{3\delta}{2}\kappa_3$,

$$Q \otimes Q'\left[\bigcup_{i \geq 2} \mathcal{I}(\mu_i) \subset \mathcal{I}(\theta) \subset \cup_{i \geq 2}\mathcal{I}(\mu_i) \bigcup \mathcal{I}(\rho_2) \right] \geq 1 - ce^{-cu\delta cap(A)}. \quad (9.6)$$

Note that $\rho_1 + \rho_2$ is a Poisson point process of intensity $2u\delta\kappa_3$, and that the "points" of this process are pieces of random walk with law $\frac{1}{cap(A)}P_{e_A}[Y_{\cdot \wedge T_B} \in dw]$ (recall (2.6)). By constructing countably many independent random walks on $(\Omega', \mathcal{A}', Q')$, and "attaching" a different one to each piece of random walk in $\rho_1 + \rho_2$ we obtain a Poisson point process ρ_3 of intensity $2u\delta P_{e_A}$ (the "points" of ρ_3 have law $\frac{1}{cap(A)}P_{e_A}$, by the strong Markov property) such that

$$\mathcal{I}(\rho_1 + \rho_2) \subset \mathcal{I}(\rho_3) \text{ almost surely.} \quad (9.7)$$

Now let

$$\mathcal{I}_1 = \cup_{i \geq 1}\mathcal{I}(\mu_i) \cap A \quad \text{and} \quad \mathcal{I}_2 = \mathcal{I}(\rho_3) \cap A$$

and note that $\mathcal{I}_1$ has the law of $\mathcal{I}^{u(1-\delta)} \cap A$ under Q_0, by (9.3), $\mathcal{I}_2$ has the law of $\mathcal{I}^{2u\delta}$ under Q_0, by (2.18), $\mathcal{I}_1$ and $\mathcal{I}_2$ are $\sigma(\eta) \times \mathcal{A}'$−measurable and independent, and since $\mathcal{I}(\eta) \cap A = \mathcal{I}(\eta - \theta) \cap A \bigcup \mathcal{I}(\theta) \cap A$ we get from (9.5), (9.6) and (9.7) that

$$Q \otimes Q'[\mathcal{I}_1 \cap A \subset \mathcal{I}(\eta) \cap A \subset \mathcal{I}_1 \cup \mathcal{I}_2] \geq 1 - ce^{-cu\delta cap(A)} \geq 1 - ce^{-cs^{c(\varepsilon)}}, \quad (9.8)$$

were the second inequality holds because we require $u, \delta \geq s^{-c(1-\varepsilon)} = s^{-c(\varepsilon)}$, similarly to in (8.8). This completes the proof of Proposition 5.4. □

It remains to prove Lemma 9.2. In the proof we will extract from the Poisson point process θ of intensity $u\delta\kappa_3$ Poisson point processes of intensity $u(1 - \delta)\tilde{\kappa}_4^i$ (see (9.9) below). This will be possible because the "points" of a Poisson point process of intensity a multiple of $\tilde{\kappa}_4^i$ is an *i.i.d. vector* of excursions with law $P_{\tilde{e}_A}^{\mathbb{Z}^d}[Y_{\cdot \wedge T_B} \in dw]$ (see (9.9)), which is also the law of the single excursions that make up the points of θ (and because the number of "points" we need to construct the Poisson point processes of intensity $u(1 - \delta)\tilde{\kappa}_4^i$ will with high probability not exceed the number of points in θ). Once we have these Poisson point processes we will use the following lemma of intensity measures to "thin" them to obtain Poisson point processes of intensity $u(1 - \delta)\kappa_4^i$, and these will be the $\mu_2, \mu_3, ..$ from the statement of Lemma 9.2.

Lemma 9.3 ($N \geq 3, d \geq 3$) *If* $s \geq c(\varepsilon)$ *and* $i \geq 2$

$$\kappa_4^i \leq \tilde{\kappa}_4^i \text{ where } \tilde{\kappa}_4^i(d(w_1, \dots, w_i)) = s^{-\frac{\varepsilon}{4}(i-1)}cap(A) \otimes_{j=1}^i P_{\tilde{e}_A}^{\mathbb{Z}^d}[Y_{\cdot \wedge T_B} \in dw_j], \quad (9.9)$$

where $\bar{e}_A(\cdot) = \frac{e_A(\cdot)}{cap(A)}$ denotes the normalisation of the measure $e_A(\cdot)$, see (2.6), (and should not be confused with the measure $\bar{e}$ from (5.5) and the proof of Lemma 8.2).

Proof Similarly to how it (in the proof of Lemma 8.3) followed from (8.16) that $\kappa_3^i(W) \leq \tilde{\kappa}_3^i(W)$ for all $W \in \Gamma(\mathbb{T}_N)^i$ that are products of measurable sets, it follows from $\sup_{x \in \partial_e B} P_x^{\mathbb{Z}^d}[Y_{H_A} = y, Y_{H_A} < \infty] \leq cs^{-\frac{\varepsilon}{2}} \bar{e}_A(y) \leq s^{-\frac{\varepsilon}{4}} \bar{e}_A(y)$ (see (2.8) and (2.10) and recall $s \geq c(\varepsilon)$) that $\kappa_4^i(W) \leq \tilde{\kappa}_4^i(W)$ for all such W. But this implies (9.9) (by a monotone class argument, like at the end of the proof of Lemma 8.3). We omit the details. $\qquad \square$

We now prove Lemma 9.2.

Proof of Lemma 9.2 Note that

$$N \stackrel{\text{def}}{=} \theta(\Gamma(\mathbb{Z}^d)) \text{ is Poisson with intensity } u\delta cap(A). \tag{9.10}$$

Similarly to in the proof of Lemma 8.2 (see (8.13)) we can construct $(\Omega', \mathcal{A}', Q')$ appropriately to obtain i.i.d. $\sigma(\theta) \times \mathcal{A}'$−measurable trajectories w_i, $i \geq 1$, independent of N,

$$\text{such that } w_i \text{ has law } P_{\bar{e}}[Y_{\cdot \wedge T_B} \in dw], \text{ and } \theta = \sum_{i=1}^{N} \delta_{w_i}. \tag{9.11}$$

Extend $(\Omega', \mathcal{A}', Q')$ with independent Poisson random variables N_i, $i \geq 2$, of respective intensities $u(1-\delta)s^{-\frac{\varepsilon}{4}(i-1)}cap(A)$ (also independent of w_i, $i \geq 1$, and θ) and let

$$\eta_i = \sum_{j=1}^{N_i} \delta_{(w_{K_j^i}, w_{K_j^i+1}, \dots, w_{K_j^i+(i-1)})}, \quad \text{where } K_j^i = \sum_{k=1}^{i-1} kN_k + (j-1)i + 1 \text{ and} \tag{9.12}$$

$$\rho = \sum_{i=\bar{K}+1}^{\bar{K}+N} \delta_{w_i} \quad \text{where } \bar{K} = \sum_{k=1}^{\infty} kN_k. \tag{9.13}$$

The η_i "use" only $w_1, \dots, w_{\bar{K}}$, so on the event the event $\{\bar{K} \leq N\}$ we have $\cup_{i \geq 2} \mathcal{I}(\eta_i) \subset \mathcal{I}(\theta)$ (see (9.11)). Recalling (9.10) and that the N_i, $i \geq 2$, are independent Poisson random variables of intensity less than $us^{-\frac{\varepsilon}{4}(i-1)}cap(A) \leq u\delta s^{-\frac{\varepsilon}{8}(i-1)}cap(A) = \lambda r^{i-1}$ (we require $\delta \geq s^{-\frac{\varepsilon}{8}}$), where $\lambda = u\delta cap(A)$ and $r = s^{-\frac{\varepsilon}{8}} \leq c$ (we require $s \geq c(\varepsilon)$), we have by (8.10)

$$Q \otimes Q'[\cup_{i \geq 2} \mathcal{I}(\eta_i) \subset \mathcal{I}(\theta)] \geq Q \otimes Q'[\bar{K} \leq N] \geq 1 - ce^{-cu\delta cap(A)}. \tag{9.14}$$

Also, since the number of the w_i "used" by ρ is the same as the number "used by" θ

$$\mathcal{I}(\theta) \subset \cup_{i \geq 2} \mathcal{I}(\eta_i) \cup \mathcal{I}(\rho) \text{ almost surely.} \tag{9.15}$$

Furthermore, because they "use different w_i" and N, N_i, $i \geq 2$, w_i, $i \geq 1$, are independent ρ, η_i, $i \geq 2$ are independent $\sigma(\theta) \times \mathcal{A}'$−measurable point processes, where ρ has intensity $u\delta \mathrm{cap}(A) P_{\bar{e}_A}[Y_{.\wedge T_B} \in dw] = u\delta\kappa_3(dw)$ (see (5.10), (9.10) and (9.13)) and η_i has intensity $u(1-\delta)\tilde{\kappa}_4^i$ (see (9.9), (9.12) and recall that N_i has intensity $(1-\delta)s^{-\frac{\varepsilon}{4}(i-1)}\mathrm{cap}(A)$). By the inequality in (9.9) we have $u(1-\delta)\kappa_4^i \leq u(1-\delta)\tilde{\kappa}_4^i$. Together with the (very crude) bound $u(1-\delta)\tilde{\kappa}_4^i \leq u(1-\delta)\kappa_4^i + u\tilde{\kappa}_4^i$ this allows us to (similarly to under (8.6)) construct independent Poisson point processes μ_i, μ_i', $i \geq 1$, such that μ_i has intensity $u(1-\delta)\kappa_4^i$, μ_i' has intensity $u\tilde{\kappa}_4^i$ and

$$\mu_i \leq \eta_i \leq \mu_i + \mu_i' \quad \text{for } i \geq 2. \tag{9.16}$$

By (9.14) and (9.16) we have

$$Q \otimes Q'[\cup_{i\geq 2}\mathcal{I}(\mu_i) \subset \mathcal{I}(\theta)] \geq 1 - ce^{-cu\delta\,\mathrm{cap}(A)}. \tag{9.17}$$

Now $\mu_i'(\Gamma(\mathbb{Z}^d))$ are independent Poisson random variables of respective intensities $us^{-\frac{\varepsilon}{4}(i-1)}\mathrm{cap}(A) \leq \lambda r^{i-1}$, where $\lambda = u\frac{\delta}{2}\mathrm{cap}(A)$ and $r = s^{-\frac{\varepsilon}{8}}$ (we require $\delta \geq 2s^{-\frac{\varepsilon}{8}}$). Thus using (8.10) we see that $\sum_{i\geq 2} i \times \mu_i'(\Gamma(\mathbb{Z}^d)) \leq N'$ with probability at least $1 - ce^{-cu\delta\mathrm{cap}(A)}$, where N' is a Poisson random variable of intensity $u\frac{\delta}{2}\mathrm{cap}(A)$. Therefore we can, similarly how the θ from Lemma 8.2 (not to be confused with the θ here) was constructed (see (8.14)) construct a $\sigma(\theta) \times \mathcal{A}'$−measurable Poisson point process ρ' of intensity $u\frac{\delta}{2}\kappa_3$ such that μ_i, $i \geq 2$, ρ, ρ', are independent and

$$Q \otimes Q'[\cup_{i\geq 2}\mathcal{I}(\mu_i') \subset \mathcal{I}(\rho')] \geq 1 - ce^{-cu\delta\mathrm{cap}(A)}. \tag{9.18}$$

Now construct a $\sigma(\theta) \times \mathcal{A}'$−measurable Poisson point process

$$\rho_2 \text{ of intensity } u\frac{3\delta}{2}\kappa_3 \text{ by setting } \rho_2 = \rho + \rho'. \tag{9.19}$$

Note that μ_i, $i \geq 2$, ρ_2, are independent and and by (9.15), (9.16), (9.18) and (9.19)

$$Q \otimes Q'[\mathcal{I}(\theta) \subset \bigcup_{i\geq 2}\mathcal{I}(\mu_i) \bigcup \mathcal{I}(\rho_2)] \geq 1 - ce^{-cu\delta\mathrm{cap}(A)}.$$

Together with (9.17) this implies (9.4), so the proof of Lemma 9.2 is complete. □

Now all components used in the proof of Proposition 5.4 have been proved, and thus the last piece of the proof of Theorem 3.2 is complete. Since Theorem 3.1 was reduced to Theorem 3.2 in Sect. 4, also the last part of the proof of Theorem 3.1 is done. We finish with the following remark.

Remark 9.4 (1) As mentioned in Remark 4.6 (2), a generalisation of the cover time result Theorem 3.1 to other graphs may be possible. Roberto Oliveira and Alan Paula are working on such a generalisation.

(2) In Corollary 3.4 we proved that the point process $\mathcal{N}_N^z$ of points of $\mathbb{T}_N$ not hit at time $g(0)N^d\{\log N^d + z\}$, for $z \in \mathbb{R}$, converges in law to a Poisson point process. It is an open question whether this can be generalised to show that the point process $\mathcal{N}_N = \sum_{x \in \mathbb{T}_N} \delta_{(x/N, \frac{H_x}{g(0)N^d} - \log N^d)}$ on the space $(\mathbb{R}/\mathbb{Z})^d \times \mathbb{R}$, from which $\mathcal{N}_N^z$ can be recovered but which records also when the vertices are hit, converges in law to a Poisson point process of intensity $e^{-z}dxdz$, where dx denotes Lebesgue measure on $(\mathbb{R}/\mathbb{Z})^d$ and dz denotes Lebesgue measure on $\mathbb{R}$.

(3) It would be interesting to explicitly determine the dependence on ε of the constant $c_5 = c_5(\varepsilon)$ from the coupling Theorem 3.2. This constant arises as the minimum of a number of constants from sections 6–9, that appear in requirements on the level u, the number of boxes n, and the parameter δ. Several of the requirements cause us to choose $c_5(\varepsilon) \leq c\varepsilon$; see for instance (6.8), (6.21), and above (8.15) and (9.14). Several others cause us to choose $c_5(\varepsilon) \leq c(1 - \varepsilon)$; see for instance above (7.30) and below (8.8) and (9.8). It is plausible that c_5 can in fact be chosen to be $c\min(\varepsilon, 1 - \varepsilon)$, for a small constant c independent of ε.

Appendix

Here we prove Lemma 6.2. Recall (6.12).

Proof of Lemma 6.2 If $V \subset \bar{C}$ then by (3.22) and (3.23) of [25] with $A_1 = V$ and $A_2 = \bar{D}$ we have (recalling the definition from (2.9) and that the $D_1, .., D_n$ are disjoint when $s \geq c(\varepsilon)$)

$$\left(1 - c\sup_{x \notin \bar{D}}\left|\frac{E_x[H_V]}{E[H_V]} - 1\right|\right)\sum_{i=1}^{n}\mathrm{cap}_D(V^i) \leq \frac{N^d}{E[H_V]} \leq \frac{1}{(\pi(\mathbb{T}_N \backslash \bar{D}))^2}\sum_{i=1}^{n}\mathrm{cap}_D(V^i). \tag{10.1}$$

We have $(\pi(\mathbb{T}_N \backslash \bar{D}))^{-2} \overset{n \leq s^{c(\varepsilon)}, N \geq s \geq c(\varepsilon)}{\leq} 1 + cN^{-c(\varepsilon)}$, and by (2.15) (with $K = V^i$, $r = s^{1-\frac{\varepsilon}{4}}$ and λ such that $(1 - \frac{\varepsilon}{4})(1 + \lambda) = 1 - \frac{\varepsilon}{8}$ so that $U = D$), (2.6) and (2.9) we have $\mathrm{cap}_D(V^i) \leq (1 + c(\varepsilon)s^{-c(\varepsilon)})\mathrm{cap}(V^i)$, so (6.4) follows. Thanks to (2.14) we will obtain (6.5) from (10.1) once we have shown (6.6). It thus only remains to show (6.6).

So assume $V \subset \bar{B}$. By (2.4) and $\sup_{x,y \in \mathbb{T}_N} E_x[H_y] \leq cN^d$ (see e.g. (10.18), p. 133 in [14]) we have for all $x \in \mathbb{T}_N$ (recall from (5.4) that $t^\star = N^{2+c\varepsilon}$)

$$\left|E_x[E_{Y_{t^\star}}[H_V]] - E[H_V]\right| \leq ce^{-cN^{c\varepsilon}}\sup_{x \in \mathbb{T}_N}E_x[H_V] \overset{N \geq c(\varepsilon)}{\leq} ce^{-cN^{c\varepsilon}}. \tag{10.2}$$

Therefore for all $x \in \mathbb{T}_N$

$$E_x[H_V] \leq E_x[E_{Y_{t^\star}}[H_V]] + t^\star \overset{(10.2)}{\leq} E[H_V] + ct^\star \leq (1 + cN^{-c(\varepsilon)})E[H_V], \tag{10.3}$$

where we have used that (recall that the capacity is monotone, see e.g. Proposition 2.2.1, p.52 in [12])

$$ct^\star \overset{n \le s^{\frac{(d-2)\varepsilon}{4}}}{\le} \frac{N^d}{cns^{(1-\frac{\varepsilon}{2})(d-2)}} N^{-c(\varepsilon)} \overset{(2.7),(6.4), V \subset \bar{B}}{\le} E[H_V]c(\varepsilon)N^{-c(\varepsilon)}. \quad (10.4)$$

Thus the upper bound of (6.6) holds. The lower bound follows since for all $x \notin \bar{C}$

$$
\begin{aligned}
E_x[H_V] \quad &\ge \quad E_x[E_{Y_{t^\star}}[H_V]] - E_x[1_{\{H_V < t^\star\}} E_{Y_{t^\star}}[H_V]] \\
&\overset{(10.2), V \subset \bar{B}}{\ge} \quad E[H_V] - c - \sup_{y \notin \bar{C}} P_y[H_{\bar{B}} < t^\star] \sup_{y \in \mathbb{T}_N} E_y[H_V] \\
&\overset{(10.3),(10.4), n \le s^{c(\varepsilon)}}{\ge} \quad E[H_V](1 - c(\varepsilon)N^{-c(\varepsilon)} - c(\varepsilon)s^{-c(\varepsilon)}),
\end{aligned}
$$

where we have used that $\sup_{y \notin \bar{C}} P_y[H_{\bar{B}} < t^\star] \le c(\varepsilon)s^{-c(\varepsilon)}$ if $n \le s^{c(\varepsilon)}$, similarly to (6.21). $\qquad\square$

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