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Jean-Baptiste Leblond, Léo Morin

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Gurson's criterion and its derivation revisited

Jean-Baptiste Leblond
Institut Jean Le Rond d'Alembert, UMR 7190
Universite Pierre et Marie Curie (Paris VI)
Tour 65-55, 4 place Jussieu, 75005 Paris, France
Email: jbl@lmm.jussieu.fr

Léo Morin
Institut Jean Le Rond d'Alembert, UMR 7190
Universite Pierre et Marie Curie (Paris VI)
Tour 65-55, 4 place Jussieu, 75005 Paris, France
Email: leo.morin@ens-cachan.fr

This paper revisits Gurson's [1,2] classical limit-analysis of a hollow sphere made of some ideal-plastic von Mises material and subjected to conditions of homogeneous boundary strain rate (Mandel [3], Hill [4]). Special emphasis is placed on successive approximations of the overall dissipation, based on a Taylor expansion of one term appearing in the integral defining it. Gurson considered only the approximation based on the first-order expansion, leading to his well-known homogenized criterion; higher-order approximations are considered here. The most important result is that the correction brought by the second-order approximation to the first-order one is significant for the porosity rate, if not for the overall yield criterion. This bears notable consequences upon the prediction of ductile damage under certain conditions.

Keywords: Gurson's limit-analysis, hollow sphere, Taylor expansion, overall yield criterion, predicted porosity rate

Nomenclature

a Internal radius of the hollow sphere.
b External radius of the hollow sphere.
$\mathcal{C}$ Approximate domain of reversibility.
$\mathcal{C}^{(n)}$ n-th order approximation of $\mathcal{C}$.
$\mathcal{C}^{\text{exact}}$ Exact domain of reversibility.
$\mathbf{d}$ Local strain rate tensor.
d_{eq} Local von Mises equivalent strain rate.
$\mathbf{D}$ Overall strain rate tensor.
D_i i-th eigenvalue of $\mathbf{D}$.
D_m Mean part of $\mathbf{D}$.
$\mathbf{D}'$ Deviatoric part of $\mathbf{D}$.
D_{eq} Overall von Mises equivalent strain rate.
D_{III} Third invariant of $\mathbf{D}$.
E_{eq} Overall cumulated equivalent strain.
f Porosity.
$\mathbf{n}$ Unit vector collinear to the position-vector.
$\mathbf{r}$ Position-vector.
r Norm of $\mathbf{r}$.
$S(r)$ Spherical surface of radius r.
$\mathcal{S}$ Approximate yield surface.
$\mathcal{S}^{(n)}$ n-th order approximation of $\mathcal{S}$.
$\mathcal{S}^{\text{exact}}$ Exact yield surface.
T Triaxiality.
$T^{(n)}(\eta)$ n-th order Taylor expansion of $\sqrt{1+\eta}$.
$\mathbf{v}$ Trial velocity field.
$\mathbf{v}^A$ Gurson's first trial velocity field.
$\mathbf{v}^B$ Gurson's second trial velocity field.
ξ Normalized porosity rate.
η "Small" parameter (see text).
Π Approximate overall plastic dissipation.
$\Pi^{(n)}$ n-th order approximation of Π.
Π^{exact} Exact overall plastic dissipation.
σ_0 Yield stress in simple tension.
$\mathbf{\Sigma}$ Overall stress tensor.
Σ_i i-th eigenvalue of $\mathbf{\Sigma}$.
Σ_m Mean part of $\mathbf{\Sigma}$.
$\mathbf{\Sigma}'$ Deviatoric part of $\mathbf{\Sigma}$.
Σ_{eq} Overall von Mises equivalent stress.
ω Spherical void.
Ω Elementary cell (hollow sphere) considered.
$\partial\Omega$ External boundary of Ω.

1 Introduction

The most classical model of ductile rupture is due to Gurson [1, 2]. This model was based on an approximate limit-analysis of a hollow sphere (typical "unit cell" in a porous material) made of a rigid-ideal plastic material obeying the von Mises criterion, and subjected to conditions of homogeneous boundary strain rate (Mandel [3], Hill [4]). This limit-analysis stood as an extension of the earlier one of Rice and Tracey [5] of an infinite medium containing a spherical hole.

While the body of literature devoted to *applications* of Gurson's model is enormous, comparatively few papers have been devoted to the *foundations* of the model themselves. Among these, one may cite those of Garajeu [6], Monchiet *et al.* [7], Alves *et al.* [8] and Cazacu *et al.* [9]. Monchiet *et al.* questioned the relevance of the trial velocity fields used by Gurson themselves (thus paralleling, for a hollow sphere, Huang's [10] reconsideration of Rice and Tracey's [5] analysis of an infinite medium). In contrast, Garajeu, Alves *et al.* and Cazacu *et al.*, accepting these fields, questioned the accuracy of an approximation made by Gurson in order to get an explicit analytical expression of the overall plastic dissipation. More specifically, these authors showed that the integral expressing this dissipation could be calculated explicitly without any approximation in the specific case of an axisymmetric loading, and compared their exact result to Gurson's approximate one in this special case.

For an arbitrary 3D loading, the integral expressing the plastic dissipation is unfortunately no longer amenable to such an exact analytic calculation. But Gurson [1, 2] defined a procedure for approximate evaluation of this dissipation based on a Taylor expansion of a term appearing in the integral. He himself considered only the approximation based on the first-order expansion. The main purpose of this paper is to consider higher-order ones so as to examine the importance of the corrections brought. It will also, incidentally, be an occasion to revisit and complement some aspects of Gurson's treatment.

The paper is organized as follows:

* Section 2 briefly recalls the main elements of Gurson's treatment and especially his definition of a sequence of successive approximations of the overall plastic dissipation.
* In Section 3, after having established a few general properties of this sequence of approximations, we provide explicit analytical expressions of the second- and third-order approximations of the plastic dissipation.
* Section 4 examines the corrections brought by the second- and third-order approximations to the first-order one, for the *overall yield criterion*.
* In Section 5, the same job is done for the *predicted porosity rate* (connected to the normal to the yield criterion). The predictions of the successive approximations are also compared to reputedly exact results obtained through numerical limit-analysis of the hollow sphere.
* Finally Section 6 discusses the implications of the results found for the prediction of ductile rupture in various conditions.

2 Preliminaries

2.1 Limit-analysis of a hollow sphere

Gurson [1, 2] (see also the review of Benzerga and Leblond [11]) performed a limit-analysis of a hollow sphere Ω of internal radius a, external radius b, porosity $f \equiv a^3/b^3$, made of some rigid-ideal plastic material obeying von Mises's criterion with yield stress σ_0 in simple tension, and subjected to conditions of homogeneous boundary strain rate (Mandel [3], Hill [4]):

$$\mathbf{v}(\mathbf{r}) = \mathbf{D}.\mathbf{r} \quad \text{for } \mathbf{r} \in \partial\Omega \tag{1}$$

where $\mathbf{v}$ denotes the velocity, $\mathbf{r}$ the position-vector (originating from the center of the sphere), $\mathbf{D}$ the overall strain rate tensor and $\partial\Omega$ the external boundary of Ω.

Since for the general loading envisaged, the limit-analysis cannot be performed exactly, Gurson envisaged trial incompressible velocity fields of the form

$$\mathbf{v}(\mathbf{r}) = \mathbf{v}^A(\mathbf{r}) + \mathbf{v}^B(\mathbf{r})\ ,\ \mathbf{v}^A(\mathbf{r}) \equiv D_m \frac{b^3}{r^2}\,\mathbf{n}\ ,\ \mathbf{v}^B(\mathbf{r}) \equiv \mathbf{D}'.\mathbf{r} \tag{2}$$

where $r \equiv \parallel \mathbf{r} \parallel$, $\mathbf{n} \equiv \mathbf{r}/r$ and $D_m \equiv \frac{1}{3}\,\mathrm{tr}\,\mathbf{D}$ and $\mathbf{D}' \equiv \mathbf{D} - D_m \mathbf{1}$ are the mean and deviatoric parts of $\mathbf{D}$, respectively.

The approximate overall plastic dissipation Π associated to this family of trial velocity fields is defined by

$$\Pi(\mathbf{D}) \equiv \frac{1}{\frac{4}{3}\pi b^3} \int_{\Omega-\omega} \sigma_0\, d_{eq}(\mathbf{r})\, d\Omega. \tag{3}$$

In this expression ω denotes the void, and $d_{eq} \equiv \sqrt{\frac{2}{3}\,\mathbf{d} : \mathbf{d}}$ the von Mises equivalent strain rate corresponding to the strain rate tensor $\mathbf{d}$ associated to the velocity field $\mathbf{v}$ defined by Eqn. (2). Since the *true* dissipation Π^{exact} corresponding to the boundary conditions (1) envisaged is defined as the *infimum* of the right-hand side of Eqn. (3) over *all* incompressible velocity fields satisfying these conditions, Π necessarily obeys the inequality

$$\Pi(\mathbf{D}) \geq \Pi^{\text{exact}}(\mathbf{D}). \tag{4}$$

The approximate reversibility domain $\mathcal{C}$ defined by Π consists of those overall stress tensors $\mathbf{\Sigma}$ for which $\mathbf{\Sigma} : \mathbf{D} \leq \Pi(\mathbf{D})$ for every $\mathbf{D}$, and its boundary, that is the approximate yield surface $\mathcal{S}$, is given by the equation

$$\mathbf{\Sigma} = \frac{\partial \Pi}{\partial \mathbf{D}}(\mathbf{D}) \tag{5}$$

where the tensor $\mathbf{D}$ acts as a parameter. Since the *true* reversibility domain $\mathcal{C}^{\text{exact}}$ and yield surface $\mathcal{S}^{\text{exact}}$ are defined similarly to $\mathcal{C}$ and $\mathcal{S}$ but with Π^{exact} instead of Π, $\mathcal{C}$ necessarily contains $\mathcal{C}^{\text{exact}}$ by inequality (4), and therefore $\mathcal{S}$ *is necessarily exterior to* $\mathcal{S}^{\text{exact}}$.

Since the function $\Pi(\mathbf{D})$ is isotropic, the tensors $\mathbf{D}$ and $\mathbf{\Sigma}$ are diagonal in the same (orthonormal) basis and the eigenvalues Σ_1, Σ_2, Σ_3 of $\mathbf{\Sigma}$ are given by

$$\Sigma_i = \frac{\partial \Pi}{\partial D_i}(D_1, D_2, D_3) \quad (i = 1, 2, 3) \tag{6}$$

where Π is considered as a (symmetric) function of the eigenvalues D_1, D_2, D_3 of $\mathbf{D}$.

2.2 Successive approximations of the plastic dissipation and yield criterion

Even for the simple velocity fields defined by Eqn. (2), the integral in the right-hand side of Eqn. (3) cannot be calculated analytically, except in the special case of axisymmetric loadings (see Garajeu [6], Alves *et al.* [8] and Cazacu *et al.* [9]). Gurson [1, 2] therefore proposed a procedure for approximate calculation of Π, which is sketched hereafter.

By Eqn. (2), the local equivalent strain rate may be put, with obvious notations, in the following form:

$$\begin{aligned} d_{eq}(\mathbf{r}) &= \sqrt{\tfrac{2}{3}\,[\mathbf{d}^A(\mathbf{r})+\mathbf{d}^B(\mathbf{r})]:[\mathbf{d}^A(\mathbf{r})+\mathbf{d}^B(\mathbf{r})]} \\ &= \sqrt{\left[d^A_{eq}(\mathbf{r})\right]^2+\left[d^B_{eq}(\mathbf{r})\right]^2+\tfrac{4}{3}\mathbf{d}^A(\mathbf{r}):\mathbf{d}^B(\mathbf{r})} \\ &= \sqrt{\left[d^A_{eq}(\mathbf{r})\right]^2+\left[d^B_{eq}(\mathbf{r})\right]^2}\;\sqrt{1+\eta(\mathbf{r})} \end{aligned}$$

where

$$\eta(\mathbf{r}) \equiv \frac{\tfrac{4}{3}\mathbf{d}^A(\mathbf{r}):\mathbf{d}^B(\mathbf{r})}{\left[d^A_{eq}(\mathbf{r})\right]^2+\left[d^B_{eq}(\mathbf{r})\right]^2}. \qquad (7)$$

Equation (3) becomes, upon use of this expression of $d_{eq}(\mathbf{r})$ and calculation of $\left[d^A_{eq}(\mathbf{r})\right]^2$ and $\left[d^B_{eq}(\mathbf{r})\right]^2$:

$$\begin{aligned} \Pi(\mathbf{D}) &= \frac{\sigma_0}{\tfrac{4}{3}\pi b^3}\int_{\Omega-\omega}\sqrt{4D_m^2\frac{b^6}{r^6}+D_{eq}^2}\;\sqrt{1+\eta(\mathbf{r})}\;d\Omega \\ &= \frac{\sigma_0}{b^3}\int_{a^3}^{b^3}\sqrt{4D_m^2\frac{b^6}{r^6}+D_{eq}^2}\;\langle\sqrt{1+\eta(\mathbf{r})}\;\rangle_{S(r)}\,d(r^3) \end{aligned} \qquad (8)$$

where $D_{eq} \equiv \sqrt{\tfrac{2}{3}\mathbf{D}':\mathbf{D}'}$ is the overall von Mises equivalent strain rate and the symbol $\langle g(\mathbf{r})\rangle_{S(r)}$ denotes the average value of an arbitrary function $g(\mathbf{r})$ over the spherical surface $S(r)$ of radius r:

$$\langle g(\mathbf{r})\rangle_{S(r)} \equiv \frac{1}{4\pi r^2}\int_{S(r)} g(\mathbf{r})dS. \qquad (9)$$

Now

$$\left|\frac{2}{3}\mathbf{d}^A(\mathbf{r}):\mathbf{d}^B(\mathbf{r})\right| \le d^A_{eq}(\mathbf{r})d^B_{eq}(\mathbf{r}) \le \frac{1}{2}\left(\left[d^A_{eq}(\mathbf{r})\right]^2+\left[d^B_{eq}(\mathbf{r})\right]^2\right) \qquad (10)$$

where Eqn. $(10)_1$ is Cauchy-Schwartz's inequality and Eqn. $(10)_2$ results from the fact that $\left[d^A_{eq}(\mathbf{r})-d^B_{eq}(\mathbf{r})\right]^2 \ge 0$. It then follows from the definition (7) of η that

$$-1 \le \eta(\mathbf{r}) \le 1 \quad \text{for every } \mathbf{r}. \qquad (11)$$

This suggests, considering η as a "small" (!) parameter, to replace the expression $\sqrt{1+\eta(\mathbf{r})}$ in the integral of Eqn. (8) by $T^{(n)}(\eta(\mathbf{r}))$, where $T^{(n)}(\eta)$ denotes the n-th order Taylor expansion of $\sqrt{1+\eta}$ around the point $\eta = 0$. This leads to introducing a family of approximations $\Pi^{(n)}$ of Π defined by

$$\Pi^{(n)}(\mathbf{D}) \equiv \frac{\sigma_0}{b^3}\int_{a^3}^{b^3}\sqrt{4D_m^2\frac{b^6}{r^6}+D_{eq}^2}\;\langle T^{(n)}(\eta(\mathbf{r}))\rangle_{S(r)}\,d(r^3). \qquad (12)$$

Gurson in fact calculated only $\Pi^{(1)}$. At the first order, $\langle T^{(1)}(\eta(\mathbf{r}))\rangle_{S(r)} = \langle 1+\tfrac{1}{2}\eta(\mathbf{r})\rangle_{S(r)} = 1$, and calculation of the integral over r^3 yields

$$\begin{aligned} \Pi^{(1)}(\mathbf{D}) &\equiv \Pi^{\text{Gurson}}(\mathbf{D}) \\ &= \sigma_0\left[2D_m\,\text{argsinh}\left(\frac{2D_m x}{D_{eq}}\right)-\frac{\sqrt{4D_m^2x^2+D_{eq}^2}}{x}\right]_{x=1}^{1/f} \end{aligned} \qquad (13)$$

where $[g(x)]_{x=x_1}^{x_2} \equiv g(x_2)-g(x_1)$. It then follows from Eqn. (6) and the fact that $\Pi^{(1)}$ depends only on D_m and D_{eq} that the corresponding approximate yield surface $\mathcal{S}^{(1)} \equiv \mathcal{S}^{\text{Gurson}}$ is given by

$$\Sigma_i = \frac{\partial\Pi^{(1)}}{\partial D_m}\frac{\partial D_m}{\partial D_i}+\frac{\partial\Pi^{(1)}}{\partial D_{eq}}\frac{\partial D_{eq}}{\partial D_i} = \frac{1}{3}\frac{\partial\Pi^{(1)}}{\partial D_m}+\frac{2}{3}\frac{D_i'}{D_{eq}}\frac{\partial\Pi^{(1)}}{\partial D_{eq}} \qquad (14)$$

where $D_i' \equiv D_i - D_m$ denotes the i-th eigenvalue of $\mathbf{D}'$. Identifying the mean and deviatoric parts $\Sigma_m \equiv \frac{1}{3}\,\text{tr}\,\mathbf{\Sigma}$, $\mathbf{\Sigma}' \equiv \mathbf{\Sigma}-\Sigma_m\mathbf{1}$ of the tensor $\mathbf{\Sigma}$ in this expression, calculating the overall von Mises equivalent stress $\Sigma_{eq} \equiv \sqrt{\tfrac{3}{2}\mathbf{\Sigma}':\mathbf{\Sigma}'}$ from there, and eliminating the ratio D_m/D_{eq} between the expressions of Σ_m and Σ_{eq} found, one finally gets Gurson's classical homogenized criterion [1, 2]:

$$\frac{\Sigma_{eq}^2}{\sigma_0^2}+2f\cosh\left(\frac{3}{2}\frac{\Sigma_m}{\sigma_0}\right)-1-f^2 = 0. \qquad (15)$$

3 The second- and third-order approximations

3.1 General results

Before calculating the second- and third-order approximations of Π, it is instructive to study general properties of the sequence of successive approximations $\Pi^{(1)}$, $\Pi^{(2)}$, ..., $\Pi^{(n)}$, ...

It is shown in Appendix A that for every η in the interval $[-1,1]$ and every n,

$$T^{(2n-1)}(\eta) \ge T^{(2n+1)}(\eta) \;\text{ and }\; T^{(2n-1)}(\eta) \ge T^{(2n)}(\eta), \qquad (16)$$

and it immediately follows that

$$\Pi^{(2n-1)}(\mathbf{D}) \ge \Pi^{(2n+1)}(\mathbf{D}) \;\text{ and }\; \Pi^{(2n-1)}(\mathbf{D}) \ge \Pi^{(2n)}(\mathbf{D}). \qquad (17)$$

Furthermore, it is also shown in Appendix B that the sequence of approximations $\Pi^{(n)}(\mathbf{D})$ converges toward $\Pi(\mathbf{D})$

for every $\mathbf{D}$. (This property, although quite appealing, should probably not be just taken for granted, since Gurson's approximation procedure involves an expansion in powers of η which is not truly a "small" parameter, as is clear from Eqn. (11)!).

These properties bear the following consequences upon the sequences of approximate reversibility domains $\mathcal{C}^{(n)}$ and yield surfaces $\mathcal{S}^{(n)}$ corresponding to the sequence of approximations $\Pi^{(n)}$:

1. *The sequences of* odd *reversibility domains $\mathcal{C}^{(2n+1)}$ and yield surfaces $\mathcal{S}^{(2n+1)}$ are "decreasing", in the sense that $\mathcal{C}^{(2n+1)}$ is contained in $\mathcal{C}^{(2n-1)}$ and $\mathcal{S}^{(2n+1)}$ interior to $\mathcal{S}^{(2n-1)}$, and converge toward $\mathcal{C}$ and $\mathcal{S}$.*
2. *The sequences of* even *reversibility domains $\mathcal{C}^{(2n)}$ and yield surfaces $\mathcal{S}^{(2n)}$ also converge toward $\mathcal{C}$ and $\mathcal{S}$, $\mathcal{C}^{(2n)}$ being contained in $\mathcal{C}^{(2n-1)}$ and $\mathcal{S}^{(2n)}$ interior to $\mathcal{S}^{(2n-1)}$.*

(Note that in contrast, nothing can be said about the comparison of $\mathcal{C}^{(2n)}$ and $\mathcal{C}^{(2n+1)}$, $\mathcal{S}^{(2n)}$ and $\mathcal{S}^{(2n+1)}$, nor about that of $\mathcal{C}^{(2n)}$ and $\mathcal{C}$, $\mathcal{S}^{(2n)}$ and $\mathcal{S}$).

Point 1 here, combined with the properties mentioned in Subsection 2.1, implies in particular that the domains $\mathcal{C}$ and $\mathcal{C}^{\text{exact}}$ are contained in $\mathcal{C}^{(1)}$, and the surfaces $\mathcal{S}$ and $\mathcal{S}^{\text{exact}}$ interior to $\mathcal{S}^{(1)}$. These results can also be established in a somewhat more direct way using Cauchy-Schwartz's inequality, see Benzerga and Leblond [11].

3.2 Explicit second-order approximation

Although Gurson's work [1,2] has received considerable attention, no one seems to have calculated the second-order approximation $\Pi^{(2)}$.[1] Such a calculation is however perfectly feasible, as will now be seen.

The values of the strain rates $\mathbf{d}^A$, $\mathbf{d}^B$ corresponding to the velocity fields $\mathbf{v}^A$, $\mathbf{v}^B$ are easily deduced from the definition (2) of these fields:

$$\mathbf{d}^A(\mathbf{r}) = D_m \frac{b^3}{r^3}(\mathbf{1} - 3\,\mathbf{n}\otimes\mathbf{n}) \quad ; \quad \mathbf{d}^B(\mathbf{r}) = \mathbf{D}'. \tag{18}$$

It follows that

$$\begin{aligned}\mathbf{d}^A(\mathbf{r}) : \mathbf{d}^B(\mathbf{r}) &= -3D_m \frac{b^3}{r^3}\,\mathbf{n}.\mathbf{D}'.\mathbf{n} \\ &= -3D_m \frac{b^3}{r^3}\left(D'_1 n_1^2 + D'_2 n_2^2 + D'_3 n_3^2\right)\end{aligned}$$

where the vector $\mathbf{n}$ is expressed in the principal basis of $\mathbf{D}$, which in turns implies, by the definition (7) of η, that

$$\eta(\mathbf{r}) = -\frac{4D_m b^3/r^3}{4D_m^2 b^6/r^6 + D_{eq}^2}\left(D'_1 n_1^2 + D'_2 n_2^2 + D'_3 n_3^2\right). \tag{19}$$

[1] In his thesis [1], Gurson proposed an explicit *approximation* of the yield surface $\mathcal{S}^2$, but it was not clear whether the corresponding reversibility domain was even convex for all possible values of the parameters, and he discarded the proposal in his final paper [2].

One then sees that the calculation of the average value $\langle T^{(2)}(\eta(\mathbf{r}))\rangle_{S(r)} = \langle 1 + \frac{1}{2}\eta(\mathbf{r}) - \frac{1}{8}[\eta(\mathbf{r})]^2\rangle_{S(r)}$ just requires that of average values of the type $\langle n_i^2\rangle_{S(r)}$, $\langle n_i^4\rangle_{S(r)}$, $\langle n_i^2 n_j^2\rangle_{S(r)}$. The first two calculations are easily performed by noting that as a consequence of symmetries, $\langle n_1^2\rangle_{S(r)} = \langle n_2^2\rangle_{S(r)} = \langle n_3^2\rangle_{S(r)}$, $\langle n_1^4\rangle_{S(r)} = \langle n_2^4\rangle_{S(r)} = \langle n_3^4\rangle_{S(r)}$, and evaluating $\langle n_3^2\rangle_{S(r)}$ and $\langle n_3^4\rangle_{S(r)}$ using spherical coordinates. Also, the third calculation is reduced to the previous ones by noting that $\langle n_1^2 n_2^2\rangle_{S(r)} = \frac{1}{2}\langle n_1^2(n_2^2+n_3^2)\rangle_{S(r)} = \frac{1}{2}\langle n_1^2(1-n_1^2)\rangle_{S(r)} = \frac{1}{2}\langle n_1^2\rangle_{S(r)} - \frac{1}{2}\langle n_1^4\rangle_{S(r)}$.

The final result for $\langle T^{(2)}(\eta(\mathbf{r}))\rangle_{S(r)}$ reads

$$\langle T^{(2)}(\eta(\mathbf{r}))\rangle_{S(r)} = 1 - \frac{2}{5}\frac{D_m^2 D_{eq}^2\, b^6/r^6}{\left(4D_m^2 b^6/r^6 + D_{eq}^2\right)^2}. \tag{20}$$

Inserting this result into the definition (12) of $\Pi^{(2)}$, one finds that the integral is again calculable analytically; the final result reads

$$\Pi^{(2)}(\mathbf{D}) = \Pi^{(1)}(\mathbf{D}) - \frac{2}{5}\sigma_0\left[\frac{D_m^2 x}{\sqrt{4D_m^2 x^2 + D_{eq}^2}}\right]_{x=1}^{1/f} \tag{21}$$

where $\Pi^{(1)}$ is given by Eqn. (13).

Thus the approximate dissipation $\Pi^{(2)}$, just like $\Pi^{(1)}$, depends only on D_m and D_{eq}, so that the corresponding yield surface $\mathcal{S}^{(2)}$ is given by a formula similar to (14):

$$\Sigma_i = \frac{1}{3}\frac{\partial\Pi^{(2)}}{\partial D_m} + \frac{2}{3}\frac{D'_i}{D_{eq}}\frac{\partial\Pi^{(2)}}{\partial D_{eq}}. \tag{22}$$

3.3 Explicit third-order approximation

The calculation of $\Pi^{(3)}$ requires that of the average value $\langle T^{(3)}(\eta(\mathbf{r}))\rangle_{S(r)} = \langle 1 + \frac{1}{2}\eta(\mathbf{r}) - \frac{1}{8}[\eta(\mathbf{r})]^2 + \frac{1}{16}[\eta(\mathbf{r})]^3\rangle_{S(r)}$ and therefore, by Eqn. (19), of extra average values of the type $\langle n_i^6\rangle_{S(r)}$, $\langle n_i^4 n_j^2\rangle_{S(r)}$, $\langle n_i^2 n_j^2 n_k^2\rangle_{S(r)}$. Such calculations are feasible using the same methods as before. Again, the integration over r^3 can be done analytically and the final result for $\Pi^{(3)}$ reads

$$\Pi^{(3)}(\mathbf{D}) = \Pi^{(2)}(\mathbf{D}) + \frac{8}{315}\sigma_0\left[\frac{D_m D_{III}^3}{\left(4D_m^2 x^2 + D_{eq}^2\right)^{3/2}}\right]_{x=1}^{1/f} \tag{23}$$

where

$$D_{III} \equiv \left(D_1'^3 + D_2'^3 + D_3'^3\right)^{1/3} = \left[\mathrm{tr}\left(\mathbf{D}'^3\right)\right]^{1/3} \tag{24}$$

and $\Pi^{(2)}$ is given by Eqn. (21).

The major novelty here is that $\Pi^{(3)}$, unlike $\Pi^{(1)}$ and $\Pi^{(2)}$, *does not depend only on D_m and D_{eq} but also on the*

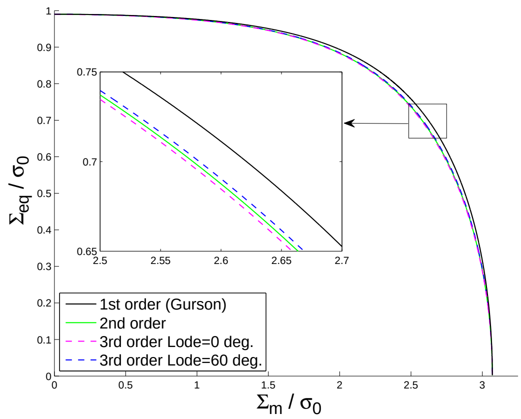

Fig. 1. Comparison of approximate criteria ($f = 0.01$)

third invariant D_{III} of $\mathbf{D}$. (The effect may however be anticipated to be small owing to the smallness of the coefficient $\frac{8}{315}$ in Eqn. (23)).

The third-order yield surface $\mathcal{S}^{(3)}$ is given by a formula analogous to those, (14) and (22), pertaining to the first- and second-order yield surfaces $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$, but slightly more complex because of the extra dependence of $\Pi^{(3)}$ upon D_{III}.

4 Comparison of the successive approximations of the criterion

Figure 1 shows the first-, second- and third-order approximate yield loci $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$, $\mathcal{S}^{(3)}$ in a plane $(\Sigma_m/\sigma_0, \Sigma_{eq}/\sigma_0)$ for a typical porosity of 10^{-2}. (The yield locus $\mathcal{S}$, corresponding to the plastic dissipation Π defined by Eqn. (3) with $\mathbf{v}(\mathbf{r})$ given by Eqn. (2) but without any further approximation, is not represented because it would be virtually indistinguishable from $\mathcal{S}^{(3)}$). Because of the dependence of $\Pi^{(3)}$ upon D_{III}, the third-order yield criterion has an extra dependence upon the third invariant of the overall stress tensor, or equivalently upon the Lode angle; hence $\mathcal{S}^{(3)}$ is represented for the two extreme values of this angle, 0° (axisymmetric load with major axial stress, $\Sigma_1 \geq \Sigma_2 = \Sigma_3$) and 60° (axisymmetric load with major lateral stress, $\Sigma_1 = \Sigma_2 \geq \Sigma_3$).

The following observations are in order:

1. *All yield surfaces are very close to each other*.
2. The distance between the second- and third-order surfaces $\mathcal{S}^{(2)}$, $\mathcal{S}^{(3)}$ is even smaller than that between the first- and second-order surfaces $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$.
3. The surfaces $\mathcal{S}^{(2)}$ and $\mathcal{S}^{(3)}$ are both interior to $\mathcal{S}^{(1)}$, as predicted.
4. The surface $\mathcal{S}^{(3)}$ is neither interior nor exterior to $\mathcal{S}^{(2)}$.

It thus appears that with regard to the yield criterion, Gurson's first-order approximation is sufficient, the corrections brought by higher-order ones being very small. (But this conclusion does not hold for the porosity rate, as will be seen in Section 5 below).

A final remark is that Gologanu *et al.* [12] have determined the *exact* yield surface $\mathcal{S}^{\text{exact}}$ of the hollow sphere considered through numerical minimization of the expression (3) of the plastic dissipation over a large space of trial velocity fields. The surface $\mathcal{S}^{\text{exact}}$ they obtained is not represented in Figure 1 because, just like the surface $\mathcal{S}$, it would be practically indistinguishable from $\mathcal{S}^{(3)}$.

5 Comparison of the successive approximations of the porosity rate and the exact one

The object of study of this Section is the "normalized porosity rate" ξ defined by

$$\xi \equiv \frac{1}{3(1-f)} \frac{df}{dE_{eq}} \tag{25}$$

where $E_{eq} \equiv \int_0^t D_{eq}(\tau)d\tau$ denotes the overall cumulated equivalent strain. This quantity is connected to the overall strain rate (and thus to the normal to the yield surface) since

$$\xi = \frac{\dot{f}}{3(1-f)D_{eq}} = \frac{D_m}{D_{eq}} \tag{26}$$

where use has been made of the equation $\dot{f} = 3(1-f)D_m$ resulting from matrix incompressibility.

5.1 Approximations of the porosity rate

At order $n = 1$ or 2, the plastic dissipation depends only on D_m and D_{eq}, and it results from Eqns. (14) or (22) that

$$\Sigma_m = \frac{\sigma_0}{3} \frac{\partial \Pi^{(n)}}{\partial D_m} \quad ; \quad \Sigma_{eq} = \sigma_0 \frac{\partial \Pi^{(n)}}{\partial D_{eq}},$$

which implies that the triaxiality $T \equiv \Sigma_m/\Sigma_{eq}$ is connected to D_m and D_{eq} through the relation

$$T = \frac{1}{3} \frac{\partial \Pi^{(n)}/\partial D_m}{\partial \Pi^{(n)}/\partial D_{eq}}. \tag{27}$$

Calculation of the derivatives of $\Pi^{(1)}$ and $\Pi^{(2)}$ using Eqns. (13) and (21) then yields the following relation connecting the triaxiality and the normalized porosity rate:

1. at order 1 (Gurson's prediction):

$$T = \frac{2}{3} \frac{\left[\, \operatorname{argsinh}(2\xi x) \,\right]_{x=1}^{1/f}}{\left[\, -\frac{1}{x}\sqrt{4\xi^2 x^2 + 1} \,\right]_{x=1}^{1/f}}; \tag{28}$$

2. at order 2:

$$T = \frac{2}{3} \frac{\left[\operatorname{argsinh}(2\xi x) - \frac{2}{5}\frac{\xi x(2\xi^2 x^2 + 1)}{(4\xi^2 x^2 + 1)^{3/2}}\right]_{x=1}^{1/f}}{\left[-\frac{1}{x}\sqrt{4\xi^2 x^2 + 1} + \frac{2}{5}\frac{\xi^2 x}{(4\xi^2 x^2 + 1)^{3/2}}\right]_{x=1}^{1/f}}. \tag{29}$$

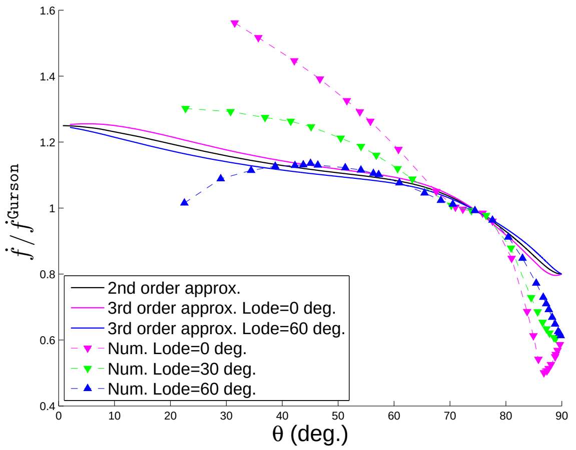

Fig. 2. Comparison of values of $\dot{f}/\dot{f}^{\text{Gurson}}$: second-order approximation and exact (numerical) values ($f = 0.01$)

At order 3, the relation between the triaxiality and the normalized porosity rate may be obtained in a similar way, but is more complex and depends on the third invariant of $\mathbf{\Sigma}$ because of the dependence of $\Pi^{(3)}$ upon D_{III}.

5.2 Comparison with the exact porosity rate

An important preliminary remark is that since all numerical results given below are for spherical cavities, they can only provide, in problems of ductile rupture involving important changes of the void shape, the *initial* porosity rate.[2]

Figure 2 shows, for a fixed porosity $f = 10^{-2}$, the porosity rate predicted by the second-order approximation (Eqn. (29)) normalized by Gurson's prediction (Eqn. (28)), $\xi/\xi^{\text{Gurson}} \equiv \dot{f}/\dot{f}^{\text{Gurson}}$, as a function of the angle $\theta \equiv \arctan T$. The figure also shows the predictions of the third-order approximation and, as a reference, the supposedly *exact* values of $\dot{f}/\dot{f}^{\text{Gurson}}$ deduced from some finite element limit-analyses of the hollow sphere considered, subjected to boundary conditions of type (1). (The details of the method used are presented in [13, 14] and are not repeated here). Except for those corresponding to the second-order approximation, the values of $\dot{f}/\dot{f}^{\text{Gurson}}$ are sensitive to the value of the Lode angle. Therefore three types of loading corresponding to Lode angles of 0° (axisymmetric load with major axial stress), 30° (pure shear with superposed hydrostatic tension) and 60° (axisymmetric load with major lateral stress) have been envisaged for the numerical values of $\dot{f}/\dot{f}^{\text{Gurson}}$. For those corresponding to the third-order approximation Lode angles of 0° and 60° have been considered sufficient to illustrate the results.

Several points are noteworthy here:

1. *The porosity rate resulting from the second-order approximation differs significantly from that predicted by Gurson's first-order approximation*, the ratio $\dot{f}/\dot{f}^{\text{Gurson}}$ amounting to about 1.25 for low triaxialities (values of θ close to 0°) versus about 0.8 for high ones (values of θ close to 90°).
2. The predictions of the third-order approximation differ very little from those of the second-order one.
3. The reputedly exact numerical results confirm the second-order approximation's predictions that the ratio $\dot{f}/\dot{f}^{\text{Gurson}}$ is a decreasing function of the triaxiality, this function being larger than unity for low T-values and lower than unity for large ones.
4. For small triaxialities, the numerical results exhibit a notable influence of Lode's angle upon the porosity rate, absent from the second-order predictions.[3] (The effect has been known for some time; see e.g. Gologanu [15]). The third-order approximation does incorporates such an influence, but unfortunately largely underestimates it.

6 Discussion

The results just presented have evidenced a lack of accuracy of the porosity rate predicted by Gurson's model. But this model is generally used in a slightly modified form, commonly referred to as the *GTN model* [16][4], in which the porosity is heuristically multiplied, in the expression of the yield function, by a parameter q slightly larger than unity (Tvergaard [17]). The question naturally arises of whether or not the deficiency just evidenced may be remedied by simply introducing such a parameter.

Figure 3 compares, for porosities of 10^{-3} and 10^{-2}, the values of the ratio $\dot{f}/\dot{f}^{\text{Gurson}}$ predicted by the second-order approximation and the GTN model, for a q-value of 1.25 ensuring coincidence of these values at low triaxialities. It is clear that the GTN model, once "calibrated" for such triaxialities, errs for larger ones by overestimating $\dot{f}/\dot{f}^{\text{Gurson}}$. (Note that the effect of q is not a trivial one, because this parameter does not only enter the expression of the porosity rate *explicitly*, but also *implicitly* through the values of the macroscopic equivalent and mean stresses, which depend upon it since they are tied through the q-dependent criterion).

In problems of *quasistatic* ductile rupture, however, the triaxiality is known to never exceed a value of about 3 in practice. Such triaxialities correspond to values of the angle θ not exceeding 70°, for which Fig. 3 makes it clear that use of the GTN model with a q-value of about 1.15 would provide an acceptable representation, on the average, of the porosity rates predicted by the second-order approximation. This means that the GTN model may safely be used for such problems.

For problems of *dynamic* ductile rupture, the situation is different since extremely large triaxialities may be encountered, and it is clear from Fig. 3 that no single value of q can match the values of $\dot{f}/\dot{f}^{\text{Gurson}}$ predicted by the second-order approximation over the full range of triaxialities. Of

[2]This of course assumes that the voids are initially spherical.

[3]Because of this influence of Lode's angle, the numerical porosity rate does not vanish for an exactly zero triaxiality, but for a small one, the sign of which depends upon the Lode angle; since for Gurson's model this rate vanishes for an exactly zero triaxiality, this implies that for the numerical results, the ratio $\dot{f}/\dot{f}^{\text{Gurson}}$ behaves oddly for very small triaxialities. This behavior is not represented in Figure 2 because it is of little interest, both $\dot{f}$ and $\dot{f}^{\text{Gurson}}$ being very small anyway under such conditions.

[4]GTN: Gurson-Tvergaard-Needleman.

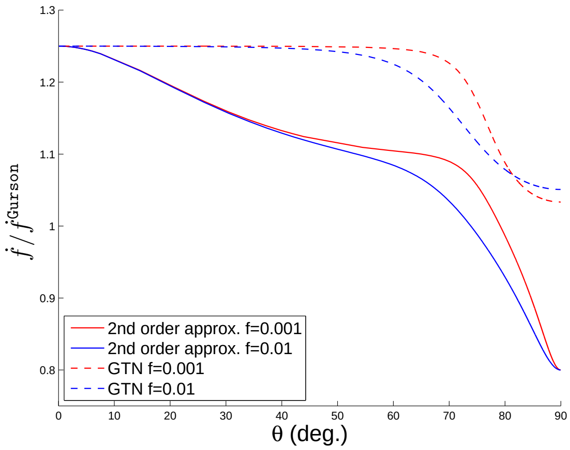

Fig. 3. Comparison of values of $\dot{f}/\dot{f}^{\text{Gurson}}$: second-order approximation and GTN model (with $q = 1.25$)

course, the problem could be solved by adopting the suggestion made by Sovik and Thaulow [18] and Pardoen and Hutchinson [19] of considering q as a function of the triaxiality; but doing so would be dangerous since the introduction of heuristic load-dependent parameters into the yield function may destroy the convexity of the reversibility domain. It therefore seems preferable, for such problems, to use the "second-order model" defined by the expression (21) of $\Pi^{(2)}$ rather than the GTN model.

Two objections to this proposal may be raised. First, Gurson's homogenization procedure did not include micro-inertia effects; therefore his model, and its improved variants such as the second-order model, become inadequate in the presence of such effects, and thus are inapplicable anyway to problems of dynamic ductile rupture. The answer to this objection lies in a paper of Molinari and Mercier [20], who proposed a convincing, though approximate, method of extension of overall yield criteria for plastic porous materials subjected to quasistatic loads to fully dynamic ones. This method may be applied without difficulty to the second-order model discussed above, resulting in a model incorporating both second-order corrections to Gurson's model *and* micro-inertia effects.

A second, natural objection to the possible use of the second-order model is that it is formally more complex than Gurson's model, since the expression (21) of the relevant plastic dissipation no longer permits to eliminate the parameter $\mathbf{D}$ in the expression (22) of the principal stresses. The answer to that objection is that the slightly greater complexity of the second-order model just makes it somewhat less elegant, but no less convenient for its implementation into some finite element programme; indeed yield criteria in parametrized form, such as (22), do not raise any special problems in this context.

A final remark must be made about the adequacy of the second-order model itself. Although this model brings a definite improvement to that of Gurson, it is still imperfect, since it does not predict any influence of the Lode angle upon the porosity rate, in clear contrast to the results of numerical unit cell calculations. In order to incorporate this effect into the GTN model, Gologanu [15] suggested to adopt a q-value depending on the Lode angle. The same proposal could be made to improve the second-order model; but again doing so would be dangerous since the convexity of the reversibility domain would no longer be guaranteed.

What is in question here is *not* the approximation resulting from the second-order Taylor expansion of the term $\sqrt{1+\eta(\mathbf{r})}$ in Gurson's expression of the plastic dissipation, since pursuing the expansion to the third order does not suffice to match the numerical results (although it does introduces a slight influence of the Lode angle). Clearly, the problem lies in the inaccuracy of Gurson's velocity fields defined by Eqn. (2) themselves. Matching the numerical values of the porosity rate would require using more realistic and complex fields. An interesting first step in this direction has been made by Monchiet *et al.* [7], who used Eshelby-like velocity fields. (Again, this work parallels, for a hollow sphere, Huang's [10] improvement of Rice and Tracey's [5] limit-analysis of an infinite medium containing a spherical hole).

7 Summary and conclusion

The aim of this paper was to revisit Gurson's [1, 2] classical limit-analysis of a hollow plastic sphere subjected to conditions of homogeneous boundary strain rate, with special emphasis on successive approximations arising from a Taylor expansion of one term arising in the expression of the overall plastic dissipation.

The second-order approximation has been shown to bring a small correction to Gurson's first-order one for the overall yield criterion, but a significant one for the predicted porosity rate. For problems of *quasistatic* ductile rupture for which the triaxiality never becomes very large, this correction may be considered as approximately constant, and incorporated within the variant of Gurson's model known as the GTN model by ascribing a suitable value to Tvergaard's q-parameter. For problems of *dynamic* ductile rupture for which the triaxiality may take arbitrary values, such a simple remedy becomes impossible, and the best solution seems to use the second-order model (suitably extended to incorporate micro-inertia effects) instead of that of Gurson.

The third-order approximation appears to be of little practical interest in that it has been found to bring only very small corrections to the second-order one, with respect to both the overall criterion and the predicted porosity rate.

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Appendix A: Proof of inequalities (16)

Consider the function $g(\eta) \equiv \sqrt{1+\eta}$. The n-th order Taylor expansion $T^{(n)}(\eta)$ of this function around the point $\eta = 0$ is defined by

$$T^{(n)}(\eta) \equiv 1 + \sum_{k=1}^{n} \frac{g^{(k)}(0)}{k!}\eta^k$$

where, obviously,

$$g^{(k)}(0) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\dots\left(\frac{1}{2}-(k-1)\right).$$

It follows that

$$\begin{aligned} T^{(2n+1)}(\eta) - T^{(2n-1)}(\eta) &= \frac{g^{(2n)}(0)}{(2n)!}\eta^{2n} + \frac{g^{(2n+1)}(0)}{(2n+1)!}\eta^{2n+1} \\ &= \frac{g^{(2n)}(0)}{(2n)!}\eta^{2n}\left(1 + \frac{\frac{1}{2}-2n}{2n+1}\eta\right). \end{aligned}$$

Now $g^{(2n)}(0) < 0$ since there are $2n-1$ negative terms in the product defining this derivative, $\eta^{2n} \geq 0$ and $1 + \frac{\frac{1}{2}-2n}{2n+1}\eta \geq 0$ since $-1 < \frac{\frac{1}{2}-2n}{2n+1} < 0$ and $-1 \leq \eta \leq 1$ (Eqn. (11)). Hence $T^{(2n+1)}(\eta) - T^{(2n-1)}(\eta)$ is non-positive, as stated by Eqn. $(16)_1$.

The proof of Eqn. $(16)_2$ is even simpler since

$$T^{(2n)}(\eta) - T^{(2n-1)}(\eta) = \frac{g^{(2n)}(0)}{(2n)!}\eta^{2n}$$

where $g^{(2n)}(0)$ has already been noted to be negative.

Appendix B: Convergence of the sequence of approximations $\Pi^{(n)}$

Step 1: study of the convergence of the Taylor expansions $T^{(n)}(\eta)$. The complex function $z \mapsto \sqrt{1+z}$ is analytic on the unit open disk $\{z \in \mathbb{C}, |z| < 1\}$. By a well-known theorem of complex analysis, this implies that for every z in this disk, $\sqrt{1+z}$ is the sum of its infinite Taylor series around the point $z = 0$; in particular, on the real line,

$$\lim_{n \to +\infty} T^{(n)}(\eta) = \sqrt{1+\eta} \qquad \text{if} \quad -1 < \eta < 1. \tag{30}$$

Step 2: study of the possibility that $\eta(\mathbf{r})$ *may take the values* ± 1. Assume that $\eta(\mathbf{r}) = \pm 1$. This is possible only if inequalities $(10)_1$ and $(10)_2$ are in fact equalities, that is if the tensors $\mathbf{d}^A(\mathbf{r})$ and $\mathbf{d}^B(\mathbf{r})$ are collinear, and $d^A_{eq}(\mathbf{r}) = d^B_{eq}(\mathbf{r})$; that is, if $\mathbf{d}^A(\mathbf{r}) = \pm \mathbf{d}^B(\mathbf{r})$. Then, since $\mathbf{d}^A(\mathbf{r}) = D_m \frac{b^3}{r^3}(\mathbf{1} - 3\mathbf{n} \otimes \mathbf{n})$ (see Eqn. $(18)_1$) has a double eigenvalue, the same must be true of $\mathbf{d}^B(\mathbf{r}) = \mathbf{D}'$ (see Eqn. $(18)_2$); that is, the tensor $\mathbf{D}$ must be axisymmetric. Let Ox_3 denote the axis passing through the center O of the sphere Ω and parallel to the principal direction of $\mathbf{D}$ corresponding to its other, simple eigenvalue. The equality $\mathbf{d}^A(\mathbf{r}) = \pm \mathbf{d}^B(\mathbf{r})$ implies that the principal directions of these tensors corresponding to their simple eigenvalues must coincide; that is, the vector $\mathbf{n}$ must be parallel to the axis Ox_3. But since $\mathbf{n} = \mathbf{r}/r$, this can occur only for points $\mathbf{r}$ lying on this axis. Furthermore, even on this axis, the equality $\mathbf{d}^A(\mathbf{r}) = \pm \mathbf{d}^B(\mathbf{r})$ may occur only for one specific value of r, that is at two points, since the norm $d^A_{eq}(\mathbf{r})$ of the first tensor varies proportionally to r^{-3} whereas the norm $d^B_{eq}(\mathbf{r})$ of the second is independent of r. The conclusion is that the equality $\eta(\mathbf{r}) = \pm 1$ may occur, depending on the values of a, b and $\mathbf{D}$, either nowhere in the domain $\Omega - \omega$, or at two points of this domain only.

Step 3: combination of Steps 1 and 2. The two possible points where $\eta(\mathbf{r})$ may take the values ± 1 may be excluded from the domain of integration $\Omega - \omega$ since they form a set of measure zero. Then, by Eqn. (30), for every $\mathbf{r}$ in this domain, $T^{(n)}(\eta(\mathbf{r}))$ goes to $\sqrt{1+\eta(\mathbf{r})}$ when n goes to infinity.

Step 4: study of the sign of $T^{(n)}(\eta)$ *for* $-1 < \eta < 1$.

1. If $\eta \le 0$, consider the difference

$$T^{(2n+1)}(\eta) - T^{(2n)}(\eta) = \frac{g^{(2n+1)}(0)}{(2n+1)!}\eta^{2n+1}$$

   where the notations of Appendix A are used again; the derivative $g^{(2n+1)}(0)$ is positive since there are $2n$ negative terms in the product defining it, and $\eta^{2n+1} \le 0$. Hence $T^{(2n+1)}(\eta) - T^{(2n)}(\eta) \le 0$. Since, by Eqn. $(16)_2$, $T^{(2n)}(\eta) - T^{(2n-1)}(\eta) \le 0$ also, the sequence of Taylor approximations $T^{(n)}(\eta)$ is decreasing. Since it converges toward the limit $\sqrt{1+\eta}$ which is positive, all the $T^{(n)}(\eta)$ are necessarily positive.
2. If $\eta > 0$, consider the difference

$$T^{(n)}(\eta) - T^{(n)}(-\eta) = \sum_{2k+1 \le n} 2\frac{g^{(2k+1)}(0)}{(2k+1)!}\eta^{2k+1}.$$

   Each term in this sum is positive, since $g^{(2k+1)}(0) > 0$ and $\eta^{2k+1} > 0$; hence $T^{(n)}(\eta) - T^{(n)}(-\eta) > 0$. Since $T^{(n)}(-\eta)$ is positive by what precedes, $T^{(n)}(\eta)$ is also necessarily positive.

The conclusion is that $T^{(n)}(\eta)$ is positive in all cases for $-1 < \eta < 1$.

Step 5: conclusion. Combining Eqn. (16) and the result of Step 4, one concludes that $0 < T^{(n)}(\eta(\mathbf{r})) \le T^{(1)}(\eta(\mathbf{r}))$ and therefore $|T^{(n)}(\eta(\mathbf{r}))| \le T^{(1)}(\eta(\mathbf{r}))$ within the domain of integration; and the integral $\Pi^{(1)}(\mathbf{D})$ involving $T^{(1)}(\eta(\mathbf{r}))$ converges, its value being given by Gurson's result (13). Combination of these properties and the result of Step 3 permits to apply Lebesgue's dominated convergence theorem, and conclude that the sequence of approximations $\Pi^{(n)}(\mathbf{D})$ converges toward $\Pi(\mathbf{D})$ for every $\mathbf{D}$.