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Gyromagnetic Emission and Absorption: Approximate Formulas of Wide Validity

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Abstract

Approximate formulas of wide validity are derived for gyromagnetic emission and absorption of gyromagnetic waves by mildly relativistic electrons. An averaged emissivity is defined by replacing the sum over harmonic number by an integral and averaging over the pitch angle distribution of the radiating particles. A method for performing the average over pitch angle without approximation to the Bessel functions is developed and the resulting expressions are then approximated using Wild–Hill formulas which interpolate between the non-relativistic and ultra-relativistic limits. The volume emissivity and the absorption coefficient are evaluated for Maxwellian and power-law (in energy) distributions by evaluating the integral over energy using the method of steepest descents. The resulting formulas reproduce and generalize known special cases including the synchrotron formulas, the exact results derived by Trubnikov for perpendicular propagation, and formulas derived more recently by Petrosian for the mildly relativistic case. The accuracy of the approximations is checked by comparison with numerical results based on exact formulas. The polarization is discussed in the limits of both strong and weak Faraday rotation, and a deficiency in Petrosian's approximation for near-perpendicular propagation is rectified. The line frequencies and line widths are estimated analytically and an interpolation formula is found between the Doppler width and the transverse Doppler width, known previously from the work of Trubnikov for perpendicular propagation.

1. Introduction

Gyromagnetic emission from mildly relativistic electrons is of interest in astrophysical applications, notably solar microwave bursts (see e.g. the review by Marsh and Hurford 1982), where it is usually called gyrosynchrotron radiation, and also in laboratory applications where it is both an energy loss mechanism for hot plasmas (see e.g. the review by Bornatici *et al.* 1983) and a useful diagnostic tool (Engelmann and Curatolo 1973; Costley *et al.* 1974). The mildly relativistic regime extends roughly from a few tens of keV to a few MeV; more formally it may be defined as the regime between the cyclotron and synchrotron limits. In the cyclotron or non-relativistic limit, gyromagnetic emission from an optically thin plasma consists of line emission at harmonics of the electron cyclotron frequency Ω_e with the intensity decreasing rapidly with increasing harmonic number s (see e.g. Bekefi 1966, p. 203). In the synchrotron or ultra-relativistic limit gyromagnetic emission is dominated by very high harmonics $s \approx (\gamma \sin\alpha)^3$, where γ is the Lorentz factor and α is the pitch angle of the electron, and the emission from neighbouring harmonics overlaps to form

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a smooth continuum with a peak at $\omega \approx \gamma^2 \Omega_e \sin\alpha$. Relatively simple analytic approximations are available in these two limits. General formulas for gyromagnetic emission involve Bessel functions and in the cyclotron and synchrotron limits these are approximated by the leading terms in their power series expansions and by Airy functions respectively. Trubnikov (1958) introduced the Carlini approximation to the Bessel functions (Watson 1944, p. 226) for the mildly relativistic regime, and derived relatively simple analytic formulas for emission perpendicular to the magnetic field lines ($\theta = \frac{1}{2}\pi$) in vacuo. However, most of the detailed results for gyrosynchrotron radiation at $\theta \neq \frac{1}{2}\pi$ in a plasma have been based on numerical treatments (Ramaty 1969; Takakura 1972) in which the Bessel functions are treated exactly. Recently, Petrosian (1981) has made considerable progress in deriving relatively simple expressions using the Carlini approximation. His technique involves evaluating the integrals over pitch angle and energy for a distribution of electrons using the method of steepest descents, and the resulting formulas are found to be quite accurate when compared with numerical results (Petrosian 1981; Dulk and Marsh 1982; Petrosian and McTiernan 1983).

Our main purpose in this paper is to derive analytic approximations for the gyromagnetic emission and absorption coefficients with as wide a range of validity as practicable. Our approach involves three steps:

(i) We modify an approach used by Melrose (1971) to derive and extend formulas for synchrotron radiation: the method involves carrying out the integral over pitch angle without making any approximations to the Bessel functions. The essential idea which allows the integral to be performed is that $\cos\alpha - \cos\theta$ is a parameter of order γ^{-1} and the synchrotron formulas are derived by expanding in γ^{-1}. In the ultra-relativistic limit the fact that all the emission is confined to a forward cone with half angle $\approx \gamma^{-1}$ implies that $\alpha - \theta$ is of order γ^{-1}.

(ii) Petrosian's (1981) evaluation of the integral over $\cos\alpha$ using steepest descents suggests that the integrand is strongly peaked about the particular value $\cos\alpha = n\beta\cos\theta$ (n is the refractive index and βc the electron speed). Consequently, it is reasonable to suppose that $\cos\alpha - n\beta\cos\theta$ is small and to expand in this parameter. Such an expansion enables one to perform the $\cos\alpha$ integral, as done by Melrose (1971), without making any approximation to the Bessel functions. That is, on replacing the small parameter $\cos\alpha - \cos\theta$ in the ultra-relativistic case by $\cos\alpha - n\beta\cos\theta$ in the more general case, one can extend the formulas derived by Melrose (1971) to the mildly relativistic case.

(iii) We replace the Bessel functions by approximate forms of the general type first considered by Wild and Hill (1971). The Wild–Hill approximation is basically of the Carlini type with interpolations, so that it reproduces the Airy functions accurately in their appropriate limit; it also reproduces the power series expansions within the accuracy of Stirling's formula in their appropriate limit.

These three steps, together with the fact that we allow the radiation to be into the magnetoionic modes, lead to relatively simple formulas of wide validity. These formulas reproduce both Petrosian's (1981) formulas for the mildly relativistic case and the well-known synchrotron formulas in the ultra-relativistic case. The cyclotron limit is not reproduced directly because the harmonic number s is continuous rather than discrete, but our results should still be useful in this limit (cf. Dulk and Marsh 1982).

Our derivation of these general formulas is given in Section 2, and they are applied to Maxwellian and to power-law distributions of electrons in Section 3. The polarization of the emission is treated in Section 4 in two complementary ways: by separating into magnetoionic components and in terms of polarization tensors or Stokes parameters. In Section 5 a variant of our method is used to estimate the emission frequency and the bandwidth of emission at individual harmonics; we derive an interpolation formula linking the usual Doppler bandwidth and the bandwidth derived by Trubnikov (1958) for perpendicular propagation (cf. Bekefi 1966, p. 202) due to the (relativistic) transverse Doppler effect.

2. Averaged Emissivity

In this section we sum the single particle emissivity over harmonics and average over pitch angle to obtain an averaged emissivity. Our method of performing the pitch angle integration is analogous to that used by Melrose (1971) in the synchrotron case: the integrands are expanded in the small parameter $\cos\alpha - n\beta\cos\theta$ and the leading terms are integrated exactly with no approximation to the Bessel functions. The resulting averaged emissivity still involves Bessel functions which we treat using approximations of the type considered by Wild and Hill (1971); these approximations lead to relatively simple expressions valid in both the mildly relativistic and synchrotron regimes. Finally, we consider the accuracy of the Wild–Hill approximations in detail, including comparison with numerical results.

(a) General Analysis

The radiating electrons are assumed to be distributed in energy (γmc^2) and pitch angle cosine ($\cos\alpha$) according to

$$F(\gamma, \cos\alpha) = N(\gamma)\,\phi(\cos\alpha), \tag{1}$$

with

$$\int_{-1}^{1} \mathrm{d}\cos\alpha\;\phi(\cos\alpha) = 2, \qquad 4\pi\int_{1}^{\infty} \mathrm{d}\gamma\; N(\gamma) = N, \tag{2a, b}$$

where $\gamma = (1-\beta^2)^{-\frac{1}{2}}$ is the Lorentz factor and N is the number density of particles. The waves are described by their frequency ω, angle of propagation θ relative to the magnetic field, and refractive index n. The waves are assumed to be in (magnetoionic) modes with polarization ellipse characterized by the axial ratio T and longitudinal part K (Melrose 1980, p. 43); n, K and T are known functions of ω, Ω_e, ω_p and θ for each mode.

We are chiefly interested in emission into the ordinary (o) and extraordinary (x) modes with $\omega \gg \omega_p, \Omega_e$, implying $K \ll 1$ and

$$T_o = -\cos\theta\{a + (1+a^2)^{\frac{1}{2}}\}/|\cos\theta| = -T_x^{-1}, \tag{3a}$$

with

$$a = \Omega_e \sin^2\theta / 2\omega\,|\cos\theta|. \tag{3b}$$

Thus the modes are then nearly transverse and are approximately circularly polarized ($T_o \approx -1$, $T_x \approx 1$) for $|\theta - \frac{1}{2}\pi| \gg \Omega_e/2\omega$ and linearly polarized ($T_o \approx -\infty$, $T_x \approx 0$) for $|\theta - \frac{1}{2}\pi| \ll \Omega_e/2\omega$.

The gyromagnetic emissivity for waves in a particular mode and harmonic s is given by (Melrose 1980, Ch. 4)

$$\eta(s,\omega,\theta) = \frac{A}{1+T^2}[\{K\sin\theta + (\cos\theta - n\beta\cos\alpha)T\}\mathrm{J}_s(sz)$$

$$+n\beta\sin\alpha\sin\theta\,\mathrm{J}_s'(sz)]^2\delta\big(\omega(1-n\beta\cos\alpha\cos\theta) - s\Omega_e/\gamma\big), \tag{4}$$

with

$$A = e^2\omega^2/4\pi\varepsilon_0\, 2\pi c\, n\sin^2\theta, \tag{5a}$$

$$z = n\beta\sin\alpha\sin\theta/(1-n\beta\cos\alpha\cos\theta). \tag{5b}$$

We define an *averaged emissivity* by

$$\bar{\eta}(\omega,\theta) \equiv \sum_{s=1}^{\infty} \frac{1}{2\phi(n\beta\cos\theta)}\int_{-1}^{1} \mathrm{d}\cos\alpha\; \phi(\cos\alpha)\,\eta(s,\omega,\theta). \tag{6}$$

Thus we sum the single particle emissivity over harmonics and average over the pitch angle distribution.

Since we are interested in high harmonics, we replace the sum over s in (6) by an integral. Calculation of the averaged emissivity then proceeds as follows:

(i) Change variables from β and $\cos\alpha$ to β' and $\cos\alpha'$ given by

$$\beta' = \frac{n\beta\sin\theta}{(1-n^2\beta^2\cos^2\theta)^{\frac{1}{2}}}, \qquad \cos\alpha' = \frac{\cos\alpha - n\beta\cos\theta}{1-n\beta\cos\alpha\cos\theta}. \tag{7a, b}$$

We find that

$$\mathrm{d}\cos\alpha = \frac{1-n^2\beta^2\cos^2\theta}{(1+n\beta\cos\alpha'\cos\theta)^2}\mathrm{d}\cos\alpha', \tag{8a}$$

$$z = \beta'\sin\alpha'. \tag{8b}$$

The limits $\cos\alpha' = \pm 1$ correspond with $\cos\alpha = \pm 1$. This change of variables corresponds to a Lorentz transformation into a frame in which the otherwise helical motion of the particles with $\cos\alpha = n\beta\cos\theta$ becomes purely circular; these particles have $\cos\alpha' = 0$ and our expansion presupposes that particles with small $|\cos\alpha'|$ dominate in the emission.

(ii) Expand the pitch angle distribution up to first order in $\cos\alpha'$:

$$\phi(\cos\alpha) = \phi(n\beta\cos\theta)\{1+\cos\alpha'(1-n^2\beta^2\cos^2\theta)(\ln\phi)'\}, \tag{9}$$

with

$$(\ln\phi)' = \left.\frac{\mathrm{d}\phi(\cos\alpha)}{\mathrm{d}\cos\alpha}\right|_{\cos\alpha = n\beta\cos\theta}.$$

(iii) Assume that $\cos\alpha'$ is a small quantity, of order $(\Omega_e/2\omega)^{\frac{1}{2}}$ according to Petrosian (1981), and expand the coefficients of J_s and J_s' in equation (4) up to first order in $\cos\alpha'$.

(iv) Evaluate the integral over s using the approximate equality

$$\int_0^\infty \mathrm{d}s\, H(s)\,\delta\big(\omega(1-n\beta\cos\alpha\cos\theta)-s\Omega_e/\gamma\big) \approx (\gamma/\Omega_e)\{H(s)-sn\beta\cos\alpha'\cos\theta\,\mathrm{d}H(s)/\mathrm{d}s\}_{s=s_0}, \quad (10)$$

with

$$s_0 = (\gamma\omega/\Omega_e)(1-n^2\beta^2\cos^2\theta) \quad (11)$$

and where $H(s)$ is any function of s which can be approximated by

$$H(s) \approx H(s_0)+(s-s_0)\{\mathrm{d}H(s)/\mathrm{d}s\}_{s=s_0},$$

for small $s-s_0$.

We now substitute (4) into (6) and carry out these steps to obtain

$$\bar{\eta}(\omega,\theta) = \frac{A}{1+T^2}\frac{\gamma}{\Omega_e}(1-n^2\beta^2\cos^2\theta)\,\tfrac{1}{2}\int_{-1}^{1}\mathrm{d}\cos\alpha'\,(1+\hat{c}_1\cos\alpha') \times\{(c_2+c_3\cos\alpha')\,\mathrm{J}_s(s\beta'\sin\alpha') + c_4\sin\alpha'\,\mathrm{J}_s'(s\beta'\sin\alpha')\}^2|_{s=s_0}, \quad (12)$$

where $\hat{c}_1$ is an operator and c_2, c_3 and c_4 are constants given by

$$\hat{c}_1 = -n\beta\cos\theta(4+s\,\mathrm{d}/\mathrm{d}s)+(1-n^2\beta^2\cos^2\theta)(\ln\phi)', \quad (13a)$$

$$c_2 = K\sin\theta+T\cos\theta(1-n^2\beta^2), \quad (13b)$$

$$c_3 = n\beta\sin\theta(K\cos\theta-T\sin\theta), \quad (13c)$$

$$c_4 = n\beta\sin\theta(1-n^2\beta^2\cos^2\theta)^{\frac{1}{2}}. \quad (13d)$$

Hereafter, all quantities involving s are to be evaluated at $s = s_0$ (equation 11), except in Section 5.

The integrals over $\cos\alpha'$ which appear in (12) are performed exactly in Appendix 1. We find that

$$\bar{\eta}(\omega,\theta) = \frac{A}{1+T^2}\frac{\gamma}{\Omega_e}(1-n^2\beta^2\cos^2\theta)\{c_2^2 I_s^{(1)}(\beta')+(c_3^2+2\hat{c}_1 c_2 c_3)I_s^{(2)}(\beta') + c_4^2 I_s^{(3)}(\beta')+c_2 c_4 I_s^{(4)}(\beta')+\hat{c}_1 c_3 c_4 I_s^{(5)}(\beta')\}, \quad (14)$$

with

$$I_s^{(1)}(\beta') = \tfrac{1}{2}\int_{-1}^{1}\mathrm{d}\cos\alpha'\,(\mathrm{J}_s)^2, \quad (15a)$$

$$I_s^{(2)}(\beta') = \tfrac{1}{2}\int_{-1}^{1}\mathrm{d}\cos\alpha'\,\cos^2\alpha'(\mathrm{J}_s)^2, \quad (15b)$$

$$I_s^{(3)}(\beta') = \tfrac{1}{2}\int_{-1}^{1}\mathrm{d}\cos\alpha'\,\sin^2\alpha'(\mathrm{J}_s')^2, \quad (15c)$$

$$I_s^{(4)}(\beta') = \tfrac{1}{2}\int_{-1}^{1} \mathrm{d}\cos\alpha'\; 2\sin\alpha'\, \mathrm{J}_s'\mathrm{J}_s\,, \tag{15d}$$

$$I_s^{(5)}(\beta') = \tfrac{1}{2}\int_{-1}^{1} \mathrm{d}\cos\alpha'\; 2\sin\alpha'\cos^2\alpha'\, \mathrm{J}_s\mathrm{J}_s'\,, \tag{15e}$$

where the argument of the Bessel functions is $s\beta'\sin\alpha'$.

(*b*) *Approximations to the Bessel Functions*

In Appendix 1 the integrals $I_s^{(j)}(\beta')$ are evaluated exactly in terms of Bessel functions of order $2s$. For small β' these functions may be adequately approximated by their power series expansions. In the synchrotron limit ($s \gg 1$, $\beta' \approx 1$) the $I_s^{(j)}(\beta')$ may be approximated by Airy functions, as shown explicitly by Melrose (1971). Petrosian (1981) made the Carlini approximation (Watson 1944, p. 226) to the Bessel functions; however, he did so before integrating over pitch angle by steepest descents. It is straightforward to apply the Carlini approximation to the integrals $I_s^{(j)}(\beta')$ and this is valid for $s(1-\beta'^2)^{3/2} \gg 1$. The Carlini and Airy function approximations to the $I_s^{(j)}(\beta')$ are derived in Appendix 1 and appear in Table 1 with

$$s_c \equiv \tfrac{3}{2}(1-\beta'^2)^{-3/2}. \tag{16}$$

Wild–Hill Approximations. Wild and Hill (1971) introduced approximations to the Bessel functions which are valid for virtually all orders and arguments of interest in gyroemission. The Wild–Hill approximations are of similar form to the Carlini approximations and interpolate between them and the Airy function approximations. In Appendix 1 approximations of the type considered by Wild and Hill (1971) are derived for the integrals $I_s^{(j)}(\beta')$. These approximations reproduce both the Carlini and Airy function approximations to the $I_s^{(j)}(\beta')$ in the appropriate limits. The Wild–Hill approximations and the limiting forms between which they interpolate are listed in Table 2 where the function Z is defined as

$$Z \equiv \beta' \exp\{(1-\beta'^2)^{\frac{1}{2}}\}/\{1+(1-\beta'^2)^{\frac{1}{2}}\}. \tag{17}$$

The accuracy of these approximations is discussed in Section 2*d*.

Averaged Emissivity. The averaged emissivity in terms of the Wild–Hill approximations is given by

$$\bar{\eta}(\omega,\theta) = \frac{A}{1+T^2}\frac{\gamma}{\Omega_e}(1-n^2\beta^2\cos^2\theta)\frac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}$$

$$\times\left(\frac{c_2^2}{2s}Q_1^{-\frac{1}{2}} + \frac{c_3^2+2\hat{c}_1 c_2 c_3}{4s^2}Q_2^{-5/6} + \frac{c_4^2}{2s\beta'^2}Q_3^{1/6}\right.$$

$$\left. + \frac{c_2 c_4}{s\beta'}Q_4^{-1/6} + \frac{\hat{c}_1 c_3 c_4}{2s^2\beta'}Q_5^{-\frac{1}{2}}\right), \tag{18}$$

with

$$Q_j = 3/2s_c + a_j/2s. \tag{19}$$

The a_j are given in the note to Table 2.

Table 1. Carlini and Airy function approximations to the five integrals introduced in (15)
In the Carlini approximation, $J_{2s}(2s\beta')$ is to be interpreted as $Z^{2s}/(4\pi s)^{\frac{1}{2}}(1-\beta'^2)^{\frac{1}{4}}$ with Z given by (17)

Integral	Approximation: Carlini	Approximation: Airy function
$I_s^{(1)}(\beta')$	$\dfrac{J_{2s}(2s\beta')}{2s(1-\beta'^2)^{\frac{1}{2}}}$	$\dfrac{1}{2s\pi\sqrt{3}}\int_{s/s_c}^{\infty} dt\, K_{1/3}(t)$
$I_s^{(2)}(\beta')$	$\dfrac{J_{2s}(2s\beta')}{4s^2(1-\beta'^2)}$	$\dfrac{1-\beta'^2}{4s\pi\sqrt{3}}\left(\int_{s/s_c}^{\infty} dt\, K_{5/3}(t) - K_{2/3}(s/s_c)\right)$
$I_s^{(3)}(\beta')$	$\dfrac{J_{2s}(2s\beta')(1-\beta'^2)^{\frac{1}{2}}}{2s\beta'^2}$	$\dfrac{1-\beta'^2}{4s\pi\sqrt{3}}\left(\int_{s/s_c}^{\infty} dt\, K_{5/3}(t) + K_{2/3}(s/s_c)\right)$
$I_s^{(4)}(\beta')$	$\dfrac{J_{2s}(2s\beta')}{s\beta'}$	$\dfrac{1-\beta'^2}{s\pi\sqrt{3}} K_{1/3}(s/s_c)$
$I_s^{(5)}(\beta')$	$\dfrac{J_{2s}(2s\beta')}{2s^2\beta'(1-\beta'^2)^{\frac{1}{2}}}$	$\dfrac{1}{2\pi s^2\sqrt{3}}\int_{s/s_c}^{\infty} dt\, K_{1/3}(t)$

Table 2. Limiting cases of approximations to the integrals $I_s^{(j)}(\beta')$
The columns labelled 'Carlini' and 'Airy' apply for $s \gg s_c$ and $s \ll s_c$ respectively, with s_c defined by (16). The 'Wild–Hill' entries are interpolations between the other two with

$$a_1 = a_5 = 9/2\pi = 1\cdot4324, \qquad a_2 = \{2^{1/3}3^{1/3}\Gamma(\tfrac{1}{3})/(2\pi)^{\frac{1}{2}}\}^{6/5} = 2\cdot2179,$$

$$a_3 = \{2^{1/6}3^{2/3}\pi^{\frac{1}{2}}/\Gamma(\tfrac{1}{3})\}^6 = 13\cdot5890, \qquad a_4 = \{2^{1/6}3^{2/3}\Gamma(\tfrac{2}{3})/(4\pi)^{\frac{1}{2}}\}^6 = 0\cdot5033$$

Integral	Approximation: Carlini	Approximation: Airy	Approximation: Wild–Hill
$I_s^{(1)}(\beta')$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s}\left(\dfrac{2s_c}{3}\right)^{\frac{1}{2}}$	$\dfrac{1}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s}\left(\dfrac{2s}{a_1}\right)^{\frac{1}{2}}$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s}\left(\dfrac{3}{2s_c}+\dfrac{a_1}{2s}\right)^{-\frac{1}{2}}$
$I_s^{(2)}(\beta')$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{4s^2}\left(\dfrac{2s_c}{3}\right)^{5/6}$	$\dfrac{1}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{4s^2}\left(\dfrac{2s}{a_2}\right)^{5/6}$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{4s^2}\left(\dfrac{3}{2s_c}+\dfrac{a_2}{2s}\right)^{-5/6}$
$I_s^{(3)}(\beta')$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s\beta'^2}\left(\dfrac{2s_c}{3}\right)^{-1/6}$	$\dfrac{1}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s}\left(\dfrac{2s}{a_3}\right)^{-1/6}$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s\beta'^2}\left(\dfrac{3}{2s_c}+\dfrac{a_3}{2s}\right)^{1/6}$
$I_s^{(4)}(\beta')$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{s\beta'}\left(\dfrac{2s_c}{3}\right)^{1/6}$	$\dfrac{1}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{s}\left(\dfrac{2s}{a_4}\right)^{1/6}$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{s\beta'}\left(\dfrac{3}{2s_c}+\dfrac{a_4}{2s}\right)^{-1/6}$
$I_s^{(5)}(\beta')$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s^2\beta'}\left(\dfrac{2s_c}{3}\right)^{\frac{1}{2}}$	$\dfrac{1}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s^2}\left(\dfrac{2s}{a_5}\right)^{\frac{1}{2}}$	$\dfrac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\dfrac{1}{2s^2\beta'}\left(\dfrac{3}{2s_c}+\dfrac{a_5}{2s}\right)^{-\frac{1}{2}}$

(c) *Isotropic Distributions*

In the case of anisotropic distributions, equation (18) cannot be simplified significantly since we must retain the term $(\ln\phi)'$ in $\hat{c}_1$. If the distribution is isotropic, however, the error involved in neglecting terms involving $\hat{c}_1$ is at most a few per cent, except in the degree of circular polarization (cf. Sections 2*d* and 4). Neglecting terms involving $\hat{c}_1$, we obtain

$$\bar{\eta}(\omega,\theta) \approx \frac{A}{1+T^2}\frac{\gamma}{\Omega_e}(1-n^2\beta^2\cos^2\theta)\frac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\frac{Q_3^{1/6}}{2s}$$

$$\times\left(c_2^2 Q_1^{-\frac{1}{2}} Q_3^{-1/6} + \frac{2c_2 c_4}{\beta'} Q_4^{-1/6} Q_3^{-1/6} + \frac{c_4^2}{\beta'^2} + \frac{c_3^2}{2s} Q_2^{-5/6} Q_3^{-1/6}\right). \tag{20}$$

Using approximations to the products $Q_i Q_j$ developed in Appendix 1 and setting $K = 0$ in c_3, we obtain our final expression for $\bar{\eta}(\omega,\theta)$:

$$\bar{\eta}(\omega,\theta) = \frac{A}{1+T^2}\frac{\gamma}{\Omega_e}(1-n^2\beta^2\cos^2\theta)\frac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\frac{\xi^{3/2}}{2s}\left(1+\frac{a_3 s_c}{3s}\right)^{1/6}$$

$$\times\,[\{c_2(1+0\cdot 85\, s_c/s)^{-1/3}+(1-n^2\beta^2)^{\frac{1}{2}}(1-n^2\beta^2\cos^2\theta)^{\frac{1}{2}}\}^2$$

$$+n^2\beta^2T^2\xi\sin^4\theta/2(s+s_c)], \tag{21}$$

with

$$\xi = (1-\beta'^2)^{-\frac{1}{2}} = (2s_c/3)^{1/3},$$

and where we recall that A is defined following (4), Z by (17), c_2 by (13b), s_c by (16) and a_3 is given in the note to Table 2.

(d) *Accuracy of the Wild–Hill Approximations*

When compared with direct numerical evaluations of the integrals $I_s^{(j)}(\beta')$, it is found that the Wild–Hill approximations systematically overestimate the $I_s^{(j)}(\beta')$ for all j. This is due in part to the dropping of some small (negative) correction terms before applying the Wild–Hill interpolation procedure, and in part to the interpolation procedure itself. In Fig. 1 we plot contours of the percentage error of our approximations. In each case our approximations are seen to reduce correctly in both the Airy and Carlini limits.

We make the following further points regarding the accuracy of our approximations.

(i) In Section 4 we find that our results reproduce Trubnikov's (1958) exact results for $\theta = \frac{1}{2}\pi$; in Sections 3 and 4 they are found to reproduce the approximate expressions of Petrosian (1981) and Petrosian and McTiernan (1983) in the appropriate limits. These authors have carried out extensive numerical checks of their results and found them to be remarkably accurate; this implies similar accuracy for our results in this limit.

(ii) As seen from Fig. 1, the largest errors occur for $s_c/s \approx 3$ in all cases. This general similarity, coupled with the systematic overestimate of the $I_s^{(j)}(\beta')$, indicates

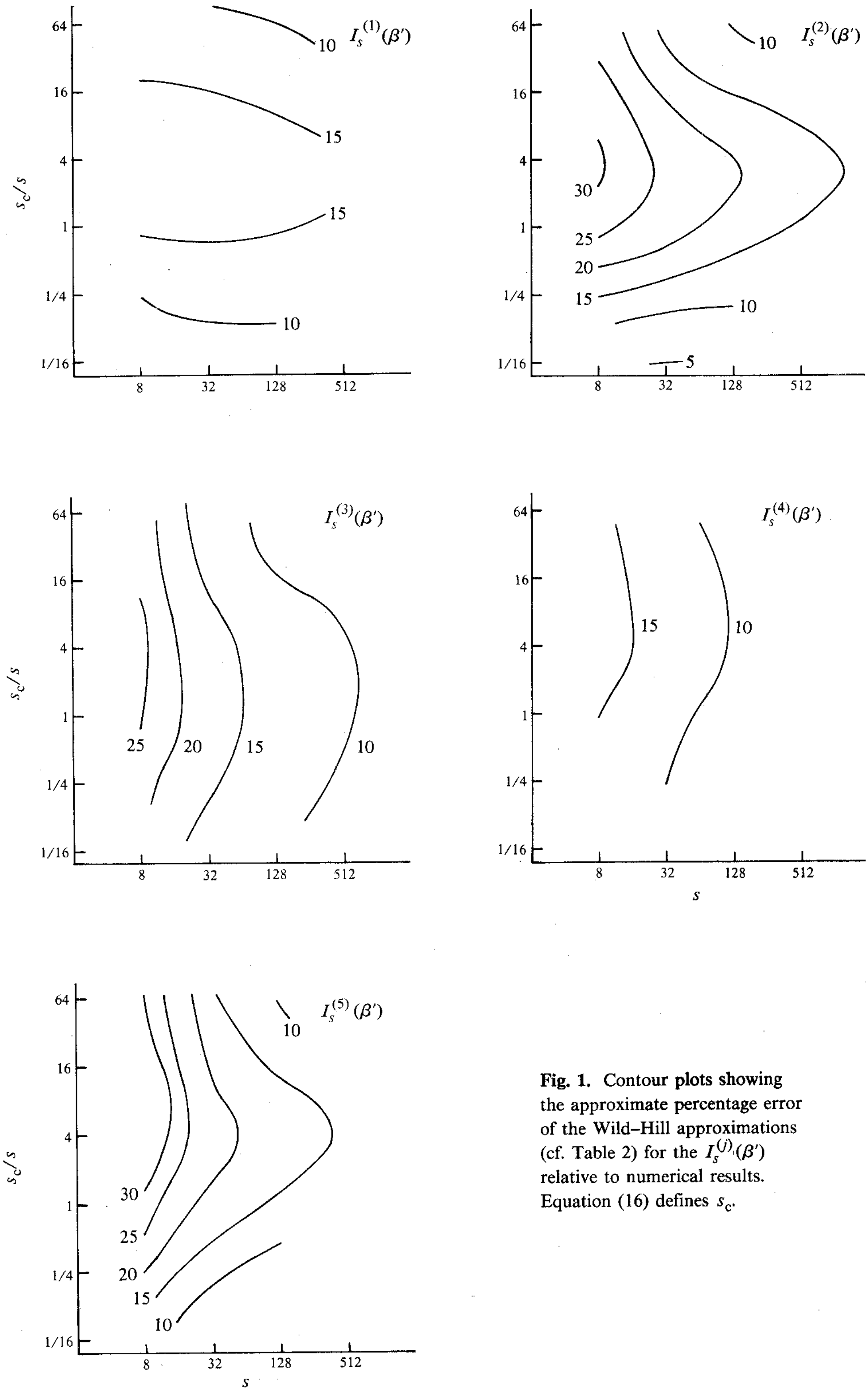

Fig. 1. Contour plots showing the approximate percentage error of the Wild–Hill approximations (cf. Table 2) for the $I_s^{(j)}(\beta')$ relative to numerical results. Equation (16) defines s_c.

that ratios of quantities obtained using our method are more accurate than the individual quantities themselves.

(iii) The largest errors are found in our approximations to $I_s^{(5)}$ and $I_s^{(2)}$. These are relatively unimportant because terms involving $I_s^{(5)}$ can be neglected in most circumstances for isotropic distributions, and in the region in which the error in approximating $I_s^{(2)}$ is large, terms involving $I_s^{(2)}$ comprise only small corrections to the emissivity.

(iv) The maximum error in our approximations could be reduced from $\sim 30\%$ to $\sim 15\%$ by relaxing the requirement that they reduce asymptotically to the Airy and Carlini approximations in the appropriate limits. We find that if we uniformly multiply our formulas by 0·85 we introduce an underestimate by 15% in the Airy and Carlini limits but the error between these limits nowhere exceeds $\sim 15\%$.

(v) Our procedure relies in part upon the smallness of terms involving the operator $s\,\mathrm{d}/\mathrm{d}s$ (in $\hat{c}_1$) with respect to the other terms in (14) or (18). These terms are known to be small in both the non-relativistic and ultra-relativistic limits (cf. e.g. Melrose 1980, Ch. 4). All non-vanishing integrals involving $s\,\mathrm{d}/\mathrm{d}s$ have integrands of second order in the small quantity $\cos\alpha'$ and hence are expected to be small relative to terms of zeroth order in this quantity; this is confirmed by direct analytic and numerical comparison of the magnitudes of the various terms in (18).

3. Volume Emissivity and Absorption Coefficient

The gyromagnetic volume emissivity $J(\omega,\theta)$ and absorption coefficient $\Gamma(\omega,\theta)$ are given by the following expressions (Melrose 1980, p. 214):

$$J(\omega,\theta) = 4\pi\int_1^\infty \mathrm{d}\gamma\, N(\gamma)\,\bar{\eta}(\omega,\theta)\,\phi(n\beta\cos\theta)\,, \tag{22}$$

$$\Gamma(\omega,\theta) = \Gamma_{\mathrm{E}}(\omega,\theta)+\Gamma_{\mathrm{A}}(\omega,\theta)\,, \tag{23}$$

with

$$\Gamma_{\mathrm{E}}(\omega,\theta) = -2\pi\sum_{s=1}^{\infty}\int_1^\infty \mathrm{d}\gamma\, N(\gamma)\int_{-1}^{1}\mathrm{d}\cos\alpha\;\phi(\cos\alpha)\,\eta(s,\omega,\theta) \times\frac{\partial\ln\{N(\gamma)/\beta\gamma^2\}}{\partial\gamma}\,\frac{\omega}{nmc^2}\left(\frac{2\pi c}{\omega}\right)^3, \tag{24}$$

$$\Gamma_{\mathrm{A}}(\omega,\theta) = -2\pi\sum_{s=1}^{\infty}\int_1^\infty \mathrm{d}\gamma\, N(\gamma)\int_{-1}^{1}\mathrm{d}\cos\alpha\;\phi(\cos\alpha)\,\eta(s,\omega,\theta) \times\frac{n\beta\cos\theta-\cos\alpha}{\beta^2\gamma}\,\frac{\partial\ln\phi(\cos\alpha)}{\partial\cos\alpha}\,\frac{\omega}{nmc^2}\left(\frac{2\pi c}{\omega}\right)^3. \tag{25}$$

The division of Γ into Γ_{E} and Γ_{A} is useful in that Γ_{E} is independent of any anisotropy [except through the normalization of $\phi(\cos\alpha)$], while Γ_{A} vanishes for an isotropic distribution. Using (6), Γ_{E} may be rewritten in terms of $\bar{\eta}(\omega,\theta)$:

$$\Gamma_{\mathrm{E}}(\omega,\theta) = -4\pi\int_1^\infty \mathrm{d}\gamma\, N(\gamma)\,\phi(n\beta\cos\theta) \times\frac{\mathrm{d}}{\mathrm{d}\gamma}\left\{\ln\left(\frac{N(\gamma)}{\beta\gamma^2}\right)\right\}\bar{\eta}(\omega,\theta)\left(\frac{2\pi c}{\omega}\right)^3\frac{\omega}{nmc^2}\,. \tag{26}$$

The quantity Γ_A must be evaluated separately, performing the integration over $\cos\alpha$ using the same procedure as in Section 2. This integration is carried out in Appendix 2. The result is

$$\Gamma_A(\omega,\theta) = 4\pi\int_1^\infty d\gamma\,\frac{N(\gamma)}{\beta^2\gamma}\,\frac{\omega}{nmc^2}\left(\frac{2\pi c}{\omega}\right)^3\frac{\gamma}{\Omega_e}\,\frac{A}{1+T^2}(1-n^2\beta^2\cos^2\theta)^2$$

$$\times\,\phi'(n\beta\cos\theta)\{(2c_2c_3+\hat{q}_1c_2^2)I_s^{(2)}(\beta')$$

$$+(c_3c_4+\hat{q}_1c_2c_4)I_s^{(5)}(\beta')+\hat{q}_1c_4^2I_s^{(6)}(\beta')\}, \quad (27)$$

with

$$I_s^{(6)}(\beta') = \tfrac{1}{2}\int_{-1}^{1} d\cos\alpha'\,\cos^2\alpha'\sin^2\alpha'\{J_s'(s\beta'\sin\alpha')\}^2, \quad (28a)$$

$$\hat{q}_1 = -n\beta\cos\theta(5+s\,d/ds). \quad (28b)$$

As in (12) all quantities are evaluated at $s = s_0$ and we neglect ϕ''.

We note that the method of Section 2 is restricted to distributions which may be approximated by (9) wherever the integrand in (6) is appreciable. This does not necessarily imply that Γ_A is negligible compared with Γ_E; however, our method breaks down if $\phi''(n\beta\cos\theta)$ is large.

(a) *Thermal Distributions*

Beginning with Trubnikov (1958), several authors have considered emission and absorption by an isotropic thermal plasma with distribution function of the form

$$N(\gamma) = C\gamma(\gamma^2-1)^{\frac{1}{2}}\exp\{-\mu(\gamma-1)\}, \quad (29)$$

where $\mu = mc^2/k_B T$, N is the number density of particles and

$$C = N(2\pi/\mu)^{-3/2}(1-15/8\mu+\ldots). \quad (30)$$

The integrals over γ involved in obtaining J and Γ_E ($\Gamma_A = 0$) are of the form

$$I = \int_1^\infty d\gamma\; G(\gamma)\exp\{-\mu Q(\gamma)\}, \quad (31)$$

where $G(\gamma)$ is slowly varying and $\exp\{-\mu Q(\gamma)\}$ is sharply peaked at $\gamma = \gamma_0$. Following Trubnikov (1958) and Petrosian (1981) we may perform these integrals by the method of steepest descents provided μ is large. We have (with $n = 1$)

$$Q(\gamma) = \gamma-1-(2s/\mu)\ln Z$$

$$= \gamma-1-\frac{x\xi^2}{\gamma\sin^2\theta}\left\{\ln\left(\frac{\xi-1}{\xi+1}\right)+\frac{2}{\xi}\right\}, \quad (32)$$

with

$$x = \omega\sin^2\theta/\Omega_e\mu, \qquad \xi = (1-\beta'^2)^{-\frac{1}{2}}. \quad (33a,b)$$

Differentiating (32) with respect to γ gives

$$Q'(\gamma) = 1 - \frac{x(\xi^2 - 2\cos^2\theta)}{\xi^2 - \cos^2\theta}\left\{\ln\left(\frac{\xi-1}{\xi+1}\right) + \frac{2}{\xi}\right\} - \frac{2x}{\xi(\xi^2-1)}. \tag{34}$$

Let $\gamma = \gamma_0$ be the solution of $Q'(\gamma_0) = 0$; then using (32) and (34) one finds

$$Q(\gamma_0) = -1 + \frac{2\gamma_0}{\xi_0^2 - 2\cos^2\theta}\left(-\cos^2\theta + \frac{x\xi_0}{\xi_0^2-1}\right), \tag{35}$$

where ξ_0 is found by setting $\gamma = \gamma_0$ in (33b). The second derivative of (32) gives

$$Q''(\gamma_0) = \frac{4x}{(\xi_0^2-1)^2}\frac{\gamma_0 \sin^2\theta}{\xi_0}, \tag{36}$$

and hence

$$I = \frac{\xi_0^2 - 1}{2\sin\theta}\left(\frac{2\pi\xi_0}{\mu x \gamma_0}\right)^{\frac{1}{2}} G(\gamma_0)\exp\{-\mu Q(\gamma_0)\}. \tag{37}$$

We solve (34) for γ_0 in two limits and express the result in terms of ξ_0:

$$\xi_0 = (\tfrac{4}{3}x)^{1/3}, \qquad x \gg 1; \tag{38a}$$

$$= 1 + x, \qquad x \ll 1. \tag{38b}$$

Interpolation between these limits may be achieved by setting (Petrosian 1981)

$$\gamma_0^2 - 1 = (2\omega/\mu\Omega_e)(1 + \tfrac{9}{2}x)^{-1/3}. \tag{39}$$

Volume Emissivity. Using (13), (21), (22) and (37) and substituting for the a_j from Table 2, our result for the volume emissivity is

$$\begin{aligned} J(\omega, \theta) = {} & \pi^{\frac{1}{2}} C \frac{\gamma^2}{\Omega_e}(\gamma^2 - 1)^{\frac{1}{2}} \frac{A}{1+T^2}(1 - \beta^2\cos^2\theta)(\xi/s)^{3/2}(1 + 4\cdot 53\, s_c/s)^{1/6} \\ & \times [\{T\cos\theta(1-\beta^2)(1 + 0\cdot 85\, s_c/s)^{-1/3} \\ & \quad + (1-\beta^2)^{\frac{1}{2}}(1-\beta^2\cos^2\theta)^{\frac{1}{2}}\}^2 + \beta^2 T^2 \xi \sin^4\theta/2(s+s_c)] \\ & \times \frac{\xi^2 - 1}{2\sin\theta}\left(\frac{2\pi\xi}{\mu x \gamma}\right)^{\frac{1}{2}} Z^{2s}\exp\{-\mu(\gamma - 1)\}, \end{aligned} \tag{40}$$

evaluated at $\gamma = \gamma_0$, $\beta = \beta_0$, $\xi = \xi_0$ with γ_0 given by (39) and $s = s_0$ given by (11). Appendix 3 summarizes the equations needed to evaluate (40).

Special Cases. We show how (40) reduces to known results in some special cases.

First, suppose we neglect the final term in the square brackets in (40); the resulting expression is equivalent to that of Petrosian (1981, equation 28) in the limit $s \gg s_c$. The neglect of the final term is justified except for the o mode for $|\theta - \frac{1}{2}\pi| \lesssim \Omega_e/2\omega$. The exceptional case of the o mode for $\theta \approx \frac{1}{2}\pi$, where our result does not reproduce that by Petrosian, corresponds to a weakness in Petrosian's analysis: his result is inconsistent with Trubnikov's exact result for this case, cf. (62b) below. Equation (40) does reproduce Trubnikov's result in the relevant limit $T_o \gg 1$, $|\cos\theta| \ll 1$.

Second, for $\xi_0 \gg 1$, $K = 0$, $n = 1$ and $s \gg s_c$ we find that

$$J(\omega,\theta) = \frac{\mu^{3/2}N\xi_0^3}{4\sqrt{\pi}}\frac{A}{1+T^2}\frac{\Omega_e}{\omega^2}\left\{\left(1+\frac{T\cos\theta}{\xi_0}\right)^2+\frac{\xi_0^3T^2}{2s}\right\}$$

$$\times\exp\{\mu-(\mu/\sin\theta)(P^{1/3}+\tfrac{9}{20}P^{-1/3}-\tfrac{3}{8}\cos^2\theta\,P^{-1/3})\}, \qquad (41)$$

with

$$P = \tfrac{9}{2}x.$$

Equation (41) reproduces the result due to Trubnikov (1958, equation 3.14) in this limit.

Third, for $\xi_0 - 1 \ll 1$ we have $\exp\{-\mu Q(\gamma_0)\} = (\tfrac{1}{2}ex)^{\omega/\Omega_e}$ [with e = exp(1)] and (40) becomes

$$J(\omega,\theta) = \frac{N}{(2\pi)^{\frac{1}{2}}}\frac{1}{(\omega\Omega_e)^{\frac{1}{2}}}\frac{A}{1+T^2}\left(\frac{e\,\omega\sin^2\theta}{2\mu\Omega_e}\right)^{\omega/\Omega_e}$$

$$\times\{(1+K\sin\theta+T\cos\theta)^2+T^2\sin^4\theta/\mu\}. \qquad (42)$$

Dulk *et al.* (1979) speculated that this result was the generalization of two previously known results; our derivation justifies this speculation.

Absorption Coefficient. For the thermal distribution assumed here and $n = 1$ we may obtain the absorption coefficient directly from the volume emissivity by using Kirchoff's law:

$$\Gamma(\omega,\theta) = \{(2\pi c)^3/\omega^2 k_B T\}J(\omega,\theta). \qquad (43)$$

(b) Power-law Distributions

Let us consider $J(\omega,\theta)$ and $\Gamma(\omega,\theta)$ for an isotropic power-law distribution of the form

$$N(\gamma) = 0, \qquad \gamma - 1 < \varepsilon_c; \qquad (44a)$$

$$= \frac{N(\delta-1)}{4\pi\varepsilon_c}\left(\frac{\gamma-1}{\varepsilon_c}\right)^{-\delta}, \qquad \varepsilon_c < \gamma - 1; \qquad (44b)$$

with

$$N = 4\pi\int_1^\infty N(\gamma)\,d\gamma \qquad (44c)$$

being the number density of electrons with $\gamma - 1 > \varepsilon_c$. We have

$$\left.\begin{matrix}J(\omega,\theta)\\ \Gamma(\omega,\theta)\end{matrix}\right\} = N(\delta-1)\,\varepsilon_c^{\delta-1}\int_{1+\varepsilon_c}^\infty d\gamma\,(\gamma-1)^{-\delta}\bar{\eta}(\omega,\theta)\begin{cases}1\\ \dfrac{(2\pi c)^3}{\omega^2 mc^2}\dfrac{\gamma(\delta+2)}{\gamma^2-1}.\end{cases} \qquad (45a, b)$$

The integrals in (45) are of the form

$$I = \int_{1+\varepsilon_c}^{\infty} \mathrm{d}\gamma\, G(\gamma) \exp\{-F(\gamma)\}, \tag{46}$$

with (cf. equations 32 and 33)

$$F(\gamma) = a\ln(\gamma-1) + b\ln\gamma - \frac{t\xi^2}{\gamma\sin^2\theta}\left\{\ln\left(\frac{\xi-1}{\xi+1}\right) + \frac{2}{\xi}\right\}, \tag{47a}$$

$$t = \omega\sin^2\theta/\Omega_e, \qquad \xi^2 = \gamma^2\sin^2\theta + \cos^2\theta, \tag{47b, c}$$

and where $G(\gamma)$ is slowly varying with γ, and a and b are constant. We perform the integrations by the method of steepest descents.

On differentiating $F(\gamma)$ we find that

$$F'(\gamma) = \frac{a}{\gamma-1} + \frac{b}{\gamma} - t\frac{\xi^2 - 2\cos^2\theta}{\xi^2 - \cos^2\theta}\left\{\ln\left(\frac{\xi-1}{\xi+1}\right) + \frac{2}{\xi}\right\} - \frac{2t}{\xi(\xi^2-1)}. \tag{48}$$

We obtain the following approximate solution $\gamma = \gamma_0$ of $F'(\gamma_0) = 0$ in the case $\omega/\Omega_e \gtrsim 1\cdot5(a+b)$:

$$(\gamma_0 + \tfrac{1}{2})^2 - 1 = 4\omega/\{3\Omega_e(a+b)\sin\theta\}, \tag{49}$$

with

$$a+b = \delta+1 \quad \text{for} \quad J(\omega,\theta); \tag{50a}$$

$$= \delta+2 \quad \text{for} \quad \Gamma(\omega,\theta). \tag{50b}$$

To a good approximation we have

$$F''(\gamma_0) = (a+b)^2 3\Omega_e \sin\theta/2\omega, \tag{51}$$

and hence [setting $n = 1$ in $\bar{\eta}(\omega,\theta)$]

$$\left.\begin{matrix} J(\omega,\theta) \\ \Gamma(\omega,\theta) \end{matrix}\right\} = \frac{N(\delta-1)\surd 3}{8}\,\frac{A}{1+T^2}\,\frac{\Omega_e}{\omega^2}\sin^3\theta\left(1+\frac{1}{\gamma_0}\right)$$

$$\times\left\{\left(1+\frac{T\cot\theta}{\gamma_0}\right)^2 + \frac{2T^2}{3(a+b+2)}\right\}$$

$$\times\left(\frac{\gamma_0-1}{\varepsilon_c}\right)^{1-\delta} Z^{2s}\left\{\begin{matrix} 1 \\ \dfrac{(2\pi c)^3}{\omega^2 mc^2}\,\dfrac{\delta+2}{\varepsilon_c}\left(\dfrac{\gamma_0-1}{\varepsilon_c}\right)^{-1}\dfrac{\gamma_0}{1+\gamma_0}, \end{matrix}\right. \tag{52a, b}$$

with

$$s = \gamma_0\,\omega\sin^2\theta/\Omega_e, \tag{53a}$$

$$Z = \{(\xi_0-1)^{\frac{1}{2}}/(\xi_0+1)^{\frac{1}{2}}\}\exp\xi_0^{-1}, \tag{53b}$$

$$\xi_0^2 = \gamma_0^2\sin^2\theta + \cos^2\theta. \tag{53c}$$

Equations (52) and their subsidiary equations are summarized in Appendix 3.

If $\gamma_0 \gtrsim 1\cdot5$ then $Z^{2s} \approx \exp\{-\frac{1}{2}(a+b)\}$, and if $\gamma_0 \gg 1$ we may replace $\gamma_0 - 1$ by γ_0. On making these replacements equations (52) become identical with the corresponding results obtained by Petrosian and McTiernan (1983), apart from the second term in the large braces, which is absent in their treatment; inclusion of this term is essential in treating the polarization of synchrotron emission.

Synchrotron Limit. For $\gamma_0 \gg 1$ the gyrosynchrotron formulas (52) differ in only small respects from the well-known formulas for synchrotron radiation from a power-law distribution (see e.g. Ginzburg and Syrovatskii 1965; Melrose 1971, 1980, pp. 123, 217). The numerical coefficients of (52) differ from those of synchrotron theory, but by no more than a factor of 2 for $\delta = 2$ and by a smaller factor for larger δ where integration over γ by steepest descents is better justified. The chief difference between equations (52) and their counterparts in synchrotron theory lies in the second term in the large braces: $3\delta+9$ and $3\delta+12$ in (52a) and (52b) are replaced by $3\delta+5$ and $3\delta+8$, respectively, in the synchrotron formulas. Nonetheless, the degree of linear polarization, for example, obtained using (52a) (in the limit of weak Faraday rotation) is within 10% of the synchrotron value for $\delta \gtrsim 2$.

4. Polarizations

We discuss the polarization of gyrosynchrotron radiation in an isotropic plasma in two complementary ways. First, in Section 4*a*, we consider emission into natural modes which are assumed to propagate independently; this corresponds to treating the polarization in the limit of strong Faraday rotation. The resulting degrees of total, linear and circular polarization reproduce and generalize results obtained by Petrosian and McTiernan (1983), and also reproduce the exact results obtained by Trubnikov (1958) for $\theta = \frac{1}{2}\pi$. In Section 4*b* we treat the polarization in terms of polarization tensors, which are equivalent to an algebra involving the Stokes parameters. This corresponds to the weak anisotropy limit (Sazonov and Tsytovich 1968; Melrose 1980, p. 193) or equivalently, the limit of weak Faraday rotation.

(*a*) *Strong Faraday Rotation*

Consider emission into the o and x modes. In the limit of strong Faraday rotation in the source, the degrees of total polarization r, linear polarization r_l and circular polarization r_c are determined by the degree of polarization in the sense of one mode (the o mode say) and the shape of the polarization ellipse for that mode. For an optically thin source we have (Petrosian and McTiernan 1983, equation 22)

$$r = \frac{\bar{\eta}_o - \bar{\eta}_x}{\bar{\eta}_o + \bar{\eta}_x}, \qquad r_l = \frac{T_o^2 - 1}{T_o^2 + 1} r, \qquad r_c = \frac{2T_o}{T_o^2 + 1} r, \tag{54}$$

where T_o is given by (3a).

Using (21) we find that (for $n = 1$)

$$\frac{\bar{\eta}_o(\omega,\theta)}{\bar{\eta}_x(\omega,\theta)} = \frac{(T_o\,\psi + \xi)^2 + \beta^2 T_o^2(\gamma\sin\theta)^4\xi/2(s+s_c)}{(\xi T_o - \psi)^2 + \beta^2(\gamma\sin\theta)^4\xi/2(s+s_c)}, \tag{55}$$

with

$$\psi = (1 + 0\cdot85\, s_c/s)^{-1/3}\cos\theta\,.$$

For the total polarization this yields

$$r = \frac{(T_o^2-1)\{\psi^2-\xi^2+\beta^2\xi(\gamma\sin\theta)^4/2(s+s_c)\}+4T_o\xi\psi}{(T_o^2+1)\{\psi^2+\xi^2+\beta^2\xi(\gamma\sin\theta)^4/2(s+s_c)\}}. \tag{56}$$

Special Cases. We consider a number of special cases of equations (55) and (56) and compare them with previously known results. Firstly, in the case of perpendicular propagation ($\theta = \frac{1}{2}\pi$, $\xi = \gamma$, $T_o = -\infty$) we find that

$$\bar{\eta}_o(\omega, \tfrac{1}{2}\pi)/\bar{\eta}_x(\omega, \tfrac{1}{2}\pi) = \beta^2\gamma^3/2(s+s_c), \tag{57}$$

which yields

$$r = -\frac{1+(3-\beta^2)s_c/3s}{1+(3+\beta^2)s_c/3s}. \tag{58}$$

In the ultra-relativistic limit the total polarization then becomes

$$r = -(3s+2s_c)/(3s+4s_c). \tag{59}$$

Trubnikov (1958) obtained the following exact expression for the ratio of the averaged emissivities for perpendicular propagation in two orthogonal modes, which correspond to the o and x modes in the limit of vanishing number density of particles ($n = 1$, $K = 0$, $T_o = -\infty$, $T_x = 0$):

$$\frac{\bar{\eta}_o(\omega, \frac{1}{2}\pi)}{\bar{\eta}_x(\omega, \frac{1}{2}\pi)} = \frac{\chi_s(\beta)-(2s/\beta^2\gamma^2)\int_0^\beta dy\, J_{2s}(2sy)}{3\chi_s(\beta)-(2s/\beta^2\gamma^2)\int_0^\beta dy\, J_{2s}(2sy)}, \tag{60}$$

with

$$\chi_s(\beta) = J'_{2s}(2s\beta) - \frac{1}{2s\beta}J_{2s}(2s\beta) + \frac{1}{2s\beta^2}\int_0^\beta dy\, J_{2s}(2sy). \tag{61}$$

Two important limits of this result are

$$\bar{\eta}_o(\omega, \tfrac{1}{2}\pi)/\bar{\eta}_x(\omega, \tfrac{1}{2}\pi) \approx \tfrac{1}{3}, \qquad s \ll s_c = \tfrac{3}{2}\gamma^3; \tag{62a}$$

$$\approx \beta^2\gamma^3/2s, \qquad s_c \ll s. \tag{62b}$$

Both of these limits are reproduced by (57). In Fig. 2 we compare the exact polarization for $\beta \approx 1$ with (59) and with the value obtained from (18) without approximating the Q_j. Our expressions are seen to be in excellent agreement with (60), the largest discrepancy being $\sim 4\%$ for $s_c \approx 8s$.

Another important special case of (56) occurs when $s \gg s_c$ and $|\theta - \frac{1}{2}\pi| \gg \Omega_e/2\omega$. In this case $T_o \approx -(1+a)$ and $a \ll 1$ (cf. equations 3), and (56) becomes

$$r = -\frac{2\xi\cos\theta/\gamma^2\sin^2\theta + a}{1+2\cot^2\theta/\gamma^2}. \tag{63}$$

Equation (63) is equivalent to equation (22) of Petrosian and McTiernan (1983); these equations are invalid for $|\theta - \frac{1}{2}\pi| \lesssim \Omega_e/2\omega$. It is clear from the discussions following equation (40) and equation (62b) that (56) provides the appropriate generalization to near-perpendicular propagation for all values of s/s_c.

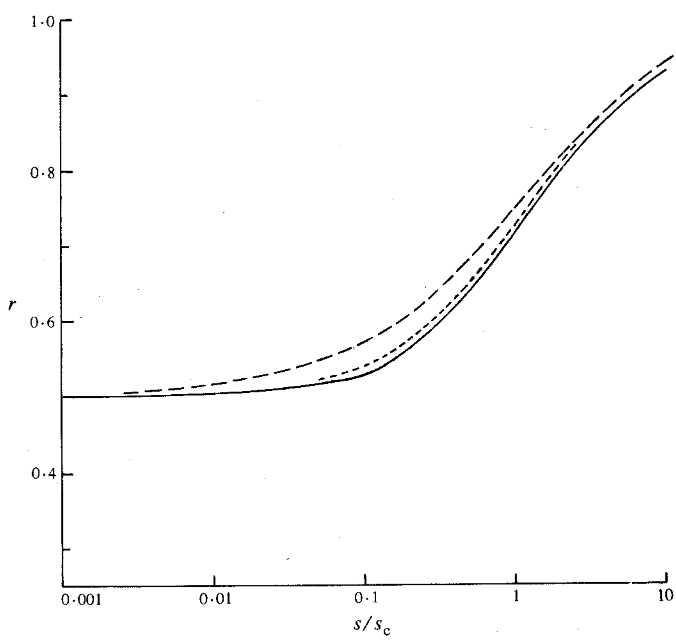

Fig. 2. Degree of polarization for emission by strongly relativistic (mono-energetic) electrons at $\theta = \frac{1}{2}\pi$. The results of synchrotron theory are given by the upper curve, while the degrees of polarization obtained from equations (18) and (59) correspond to the middle and lower curves respectively.

(*b*) *Weak Faraday Rotation*

In the weak anisotropy limit (Sazonov and Tsytovich 1968; Melrose 1980, p. 193) the two natural modes are assumed to be transverse and to have identical ray paths. In this limit the polarization can be described in terms of polarization tensors which are equivalent to an algebra involving the Stokes parameters. In this subsection we derive the emissivity in the form of a polarization tensor and use this quantity to obtain the Stokes parameters and the degrees of polarization in the limit of weak Faraday rotation in an isotropic plasma.

The polarization tensors $J^{\alpha\beta}$ and $\Gamma^{\alpha\beta}$ are defined by analogy with (22) and (24) for an isotropic plasma. We choose the real polarization basis vectors

$$e^1 = (\cos\theta,\, \theta,\, -\sin\theta), \qquad e^2 = (0, 1, 0), \tag{64}$$

to obtain

$$\bar{\eta}^{11}(\omega,\theta) = \sum_{s=1}^{\infty}\int_{-1}^{1} \mathrm{d}\cos\alpha\, A\{(\cos\theta - n\beta\cos\alpha)\,\mathrm{J}_s(sz)\}^2\delta(u),$$

$$\bar{\eta}^{22}(\omega,\theta) = \sum_{s=1}^{\infty}\int_{-1}^{1} \mathrm{d}\cos\alpha\, A\{n\beta\sin\alpha\sin\theta\,\mathrm{J}_s'(sz)\}^2\delta(u),$$

$$\begin{aligned}\bar{\eta}^{12}(\omega,\theta) &= -\bar{\eta}^{21}(\omega,\theta)\\ &= -\mathrm{i}\sum_{s=1}^{\infty}\int_{-1}^{1} \mathrm{d}\cos\alpha\, A(\cos\theta - n\beta\cos\alpha)\\ &\qquad\qquad \times n\beta\sin\alpha\sin\theta\,\mathrm{J}_s(sz)\,\mathrm{J}_s'(sz)\,\delta(u),\end{aligned} \tag{65}$$

with

$$u = \omega(1 - n\beta\cos\alpha\cos\theta) - s\Omega_e/\gamma .$$

Comparison with the analysis in Section 2 yields the Wild–Hill approximations to (65):

$$\begin{aligned}
\bar\eta^{11}(\omega,\theta) &= (A\gamma/\Omega_e)(1-n^2\beta^2\cos^2\theta)\{c_2^2 I_s^{(1)}(\beta') + c_3^2 I_s^{(2)}(\beta')\},\\
\bar\eta^{22}(\omega,\theta) &= (A\gamma/\Omega_e)(1-n^2\beta^2\cos^2\theta)c_4^2 I_s^{(3)}(\beta'),\\
\bar\eta^{12}(\omega,\theta) &= -\bar\eta^{21}(\omega,\theta)\\
&= -\mathrm{i}(A\gamma/\Omega_e)(1-n^2\beta^2\cos^2\theta)\\
&\quad\times\{c_2 c_4 I_s^{(4)}(\beta') + \hat c_1 c_3 c_4 I_s^{(5)}(\beta')\},
\end{aligned} \tag{66}$$

with the $I_s^{(j)}(\beta')$ as listed in Table 2 and the c_j given by (13) with $K = 0$ and $T = 1$. We have neglected a term involving $\hat c_1$ in $\bar\eta^{11}(\omega,\theta)$.

Polarization and the Stokes Parameters. If the volume emissivity polarization tensor $J^{\alpha\beta}$ is obtained from $\bar\eta^{\alpha\beta}$ using steepest descents to integrate over energy we find the following degrees of linear and circular polarization:

$$\begin{aligned}
r_l &= \frac{J^{11}-J^{22}}{J^{11}+J^{22}} \approx \left.\frac{\bar\eta^{11}-\bar\eta^{22}}{\bar\eta^{11}+\bar\eta^{22}}\right|_{\gamma=\gamma_0}\\
&\approx \left.\frac{c_2^2 I_s^{(1)}(\beta') + c_3^2 I_s^{(2)}(\beta') - c_4^2 I_s^{(3)}(\beta')}{c_2^2 I_s^{(1)}(\beta') + c_3^2 I_s^{(2)}(\beta') + c_4^2 I_s^{(3)}(\beta')}\right|_{\gamma=\gamma_0},
\end{aligned} \tag{67}$$

$$\begin{aligned}
r_c &= \mathrm{i}(J^{12}-J^{21})/(J^{11}+J^{22})\\
&\approx \left.\frac{2c_4\{c_2 I_s^{(4)}(\beta') + \hat c_1 c_3 I_s^{(5)}(\beta')\}}{c_2^2 I_s^{(1)}(\beta') + c_3^2 I_s^{(2)}(\beta') + c_4^2 I_s^{(3)}(\beta')}\right|_{\gamma=\gamma_0}.
\end{aligned} \tag{68}$$

In the ultra-relativistic limit, (67) and (68) reproduce the appropriate synchrotron expressions in the case of weak Faraday rotation (Melrose 1980, p. 121).

In terms of the $J^{\alpha\beta}$ the Stokes parameters are (Melrose 1980, p. 198)

$$I = J^{11}+J^{22}, \quad Q = J^{11}-J^{22}, \quad U = 0, \quad V = -2\mathrm{i}J^{12}. \tag{69}$$

For Maxwellian and power-law distributions, these may readily be obtained by comparison with the results of Section 3.

5. Line Frequencies and Line Widths for Thermal Distributions

The resonance condition for emission in the sth harmonic is

$$\omega - k_\parallel v_\parallel - s\Omega_e/\gamma = 0. \tag{70}$$

This implies that emission in the sth harmonic from a distribution of particles has finite bandwidth; if $k_\parallel \neq 0$ the spread in $v_\parallel$ across the distribution gives rise to the well-known Doppler line width, while even if $k_{\parallel} = 0$ the spread in γ over the distribution leads to a transverse Doppler line width of purely relativistic origin. In this section we calculate the mean frequency and the line width of emission in the sth harmonic from Maxwellian distributions. Firstly, we obtain results for $\theta = \frac{1}{2}\pi$ which generalize Trubnikov's (1958) expressions for the line frequency and line width to higher harmonics and plasma temperatures than previously considered; secondly, we consider emission at arbitrary θ by a non-relativistic Maxwellian plasma and show that the line width in this case has the form of a simple interpolation between the Doppler and transverse Doppler limits. These results are of use in connection with certain laboratory plasma diagnostics (see e.g. Engelmann and Curatolo 1973).

Analysis. We define the mth moment of the volume emissivity in the sth harmonic and a given mode by

$$\langle\omega^m(s,\theta)\rangle \equiv \int_0^\infty \mathrm{d}\omega\, \omega^{m-1} J(s,\omega,\theta) \Big/ \int_0^\infty \mathrm{d}\omega\, \omega^{-1} J(s,\omega,\theta). \tag{71}$$

We make this choice rather than say ω^m and unity in the numerator and denominator respectively, because the emission rate for photons is proportional to $J(s,\omega,\theta)/\omega$.

Equation (71) may be written as

$$\langle\omega^m(s,\theta)\rangle = H_m(s,\theta)/H_0(s,\theta), \tag{72}$$

with

$$H_m(s,\theta) = \int_1^\infty \mathrm{d}\gamma\, N(\gamma) \int_{-1}^1 \mathrm{d}\cos\alpha \int_0^\infty \mathrm{d}\omega\, \omega^{m-1} \phi(\cos\alpha)\, \eta(s,\omega,\theta), \tag{73}$$

where $\eta(s,\omega,\theta)$ is given by (4) and we assume that the plasma distribution satisfies (1) and (2). Setting $K = 0$, integrating over ω using the δ function in $\eta(s,\omega,\theta)$ and assuming an isotropic distribution $[\phi(\cos\alpha) = 1]$, equation (73) becomes

$$H_m(s,\theta) = \int_1^\infty \mathrm{d}\gamma\, N(\gamma) \int_{-1}^1 \mathrm{d}\cos\alpha\, \frac{A_1}{1+T^2} \frac{\omega_0^{m+1}}{1-n\beta\cos\alpha\cos\theta}$$
$$\times \{(\cos\theta - n\beta\cos\alpha) T\, \mathrm{J}_s(sz) + n\beta \sin\alpha \sin\theta\, \mathrm{J}_s'(sz)\}^2, \tag{74}$$

with

$$\omega_0 = \frac{s\Omega_\mathrm{e}}{\gamma} \frac{1}{1-n\beta\cos\alpha\cos\theta}, \qquad A_1 = A/\omega^2.$$

Perpendicular Propagation. In the case of perpendicular propagation the line frequency and line width depend on intrinsically relativistic effects as shown by Trubnikov (1958). On making the Carlini approximation (A18) to the Bessel functions (which is accurate to within ~10% for $s \geqslant 1$ provided β is not too close to unity), equation (74) leads to

$$\langle\omega^m(s,\tfrac{1}{2}\pi)\rangle = (s\Omega_\mathrm{e})^m I_m/I_0, \tag{75}$$

with

$$I_m = \int_1^\infty \mathrm{d}\gamma\, N(\gamma) \gamma^{-(m+3/2)} (\beta^2\gamma^3/2s,\, 1) Z^{2s}, \tag{76}$$

where we have divided both numerator and denominator in (74) by some common factors and set $n = 1$. The first and second terms in the parentheses in (76) refer to the o and x modes respectively. The integrals in (76) are of the form

$$I_m = \int_1^\infty \mathrm{d}\gamma \exp\{-Q(\gamma)\}\, P(\gamma)\gamma^{-m}, \tag{77}$$

where $P(\gamma)$ is a slowly varying function of γ, independent of m, and $\exp\{-Q(\gamma)\}$ is sharply peaked at a particular value of γ, labelled γ_{P}.

We consider a relativistic Maxwellian as in (29). In this case we have

$$Q(\gamma) = \mu(\gamma-1) - 2s\ln Z \quad (s \text{ fixed}), \tag{78}$$

$$Q'(\gamma) = \mu - 2s/\gamma^2(\gamma^2-1), \tag{79}$$

$$Q''(\gamma) = 4s(2\gamma^2-1)/\gamma^3(\gamma^2-1)^2, \tag{80}$$

where the primes denote differentiation with respect to γ. Now γ_{P} is determined by the condition $Q'(\gamma_{\mathrm{P}}) = 0$, which gives

$$\gamma_{\mathrm{P}} = [\tfrac{1}{2}\{1+(1+8s/\mu)^{\frac{1}{2}}\}]^{\frac{1}{2}}. \tag{81}$$

To obtain nonzero line widths we must consider the dependence of (77) on m. We suppose that the peak of $\exp\{-Q(\gamma)\}\gamma^{-m}$ occurs at

$$\gamma_m = \gamma_{\mathrm{P}} + \varepsilon_m, \tag{82}$$

with $\varepsilon_m \ll \gamma_{\mathrm{P}} - 1$ for small m. We determine ε_m from the condition

$$(\mathrm{d}/\mathrm{d}\gamma)\ln(\gamma^{-m}) + Q'(\gamma) = 0 \tag{83}$$

at $\gamma = \gamma_m$. Using $Q'(\gamma_{\mathrm{P}}) = 0$, equation (83) gives

$$\varepsilon_m \approx -m/\gamma_{\mathrm{P}}\, Q''(\gamma_{\mathrm{P}}) = -m(\gamma_{\mathrm{P}}^2-1)/2\mu(2\gamma_{\mathrm{P}}^2-1). \tag{84}$$

For the mean frequency of emission we find

$$\langle\omega(s,\theta)\rangle = s\Omega_{\mathrm{e}}/\gamma_{\mathrm{P}}. \tag{85}$$

For $s/\mu \ll 1$ this reproduces Trubnikov's (1958) result for the mean frequency of emission in the sth harmonic:

$$\langle\omega(s,\theta)\rangle = s\Omega_{\mathrm{e}}(1-s/\mu). \tag{86}$$

The line width $\Delta\omega$ in the sth harmonic is obtained from (75) using (82) and (84), giving

$$(\Delta\omega)^2 \equiv \langle\omega^2\rangle - \langle\omega\rangle^2 = \frac{(s\Omega_{\mathrm{e}})^2}{\mu^2}\frac{2s}{\gamma_{\mathrm{P}}^5(2\gamma_{\mathrm{P}}^2-1)}. \tag{87}$$

For $\mu \gg 1$, Trubnikov's (1958) corresponding result is

$$(\Delta\omega)^2 = (s\Omega_{\mathrm{e}})^2(s+r)/\mu^2, \tag{88}$$

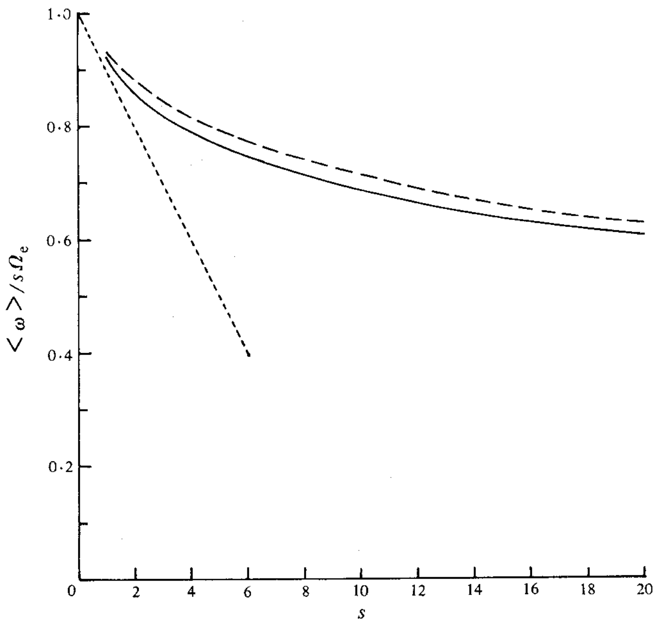

Fig. 3. Mean frequency of emission at the sth harmonic from a 50 keV Maxwellian distribution. Numerical results are represented by the solid curve. Our expression (85) and Trubnikov's estimate (86) correspond to the long and short dashed curves respectively.

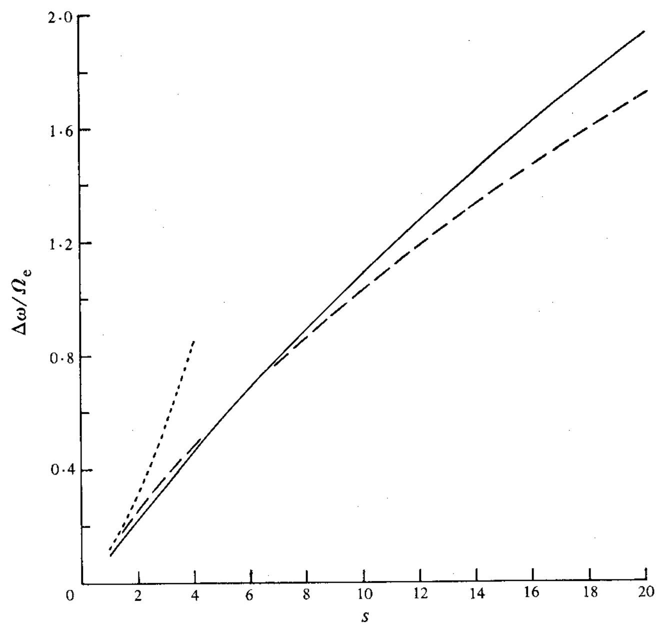

Fig. 4. Line width of emission at the sth harmonic from a 50 keV Maxwellian distribution. Numerical results are represented by the solid curve. Our expression (87) and Trubnikov's estimate (88) correspond to the long and short dashed curves respectively.

Appendix 1. Integrals over Pitch Angle

In Section 2 the averaged emissivity is expressed in terms of five integrals over pitch angle (15). In this appendix these integrals are simplified and various approximations, including the Wild–Hill approximations, are applied to them.

Reduction Procedure

We consider the integral

$$A_t^{(n)}(s\beta) = \int_{-1}^{1} \mathrm{d}\cos\alpha\,(\sin\alpha)^{2n} \mathrm{J}_t^2(s\beta\sin\alpha)\,. \tag{A1}$$

Using the equations

$$\{\mathrm{J}_t(z)\}^2 = \sum_{m=0}^{\infty} \frac{(-1)^m(2t+2m)!}{m!\,(2t+m)!\,\{(t+m)!\}^2}(\tfrac{1}{2}z)^{2t+2m}\,, \tag{A2}$$

$$\mathrm{J}_{2t}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(2t+m)!}(\tfrac{1}{2}z)^{2t+2m}\,, \tag{A3}$$

$$\int_{-1}^{1} \mathrm{d}\cos\alpha\,(\sin\alpha)^{2r} = \frac{2^{2r+1}(r!)^2}{(2r+1)!}\,, \tag{A4}$$

we obtain

$$A_t^{(0)}(s\beta) = \frac{2}{\beta}\int_0^{\beta} \mathrm{d}y\,\mathrm{J}_{2t}(2sy)\,, \tag{A5}$$

$$A_t^{(-1)}(s\beta) = 2\int_0^{\beta} \frac{\mathrm{d}y}{y}\,\mathrm{J}_{2t}(2sy)\,, \tag{A6}$$

$$A_t^{(1)}(s\beta) = \frac{1}{\beta}\int_0^{\beta} \mathrm{d}y\,\mathrm{J}_{2t}(2sy) + \frac{1}{\beta^3}\int_0^{\beta} \mathrm{d}y\,y^2\mathrm{J}_{2t}(2sy)\,, \tag{A7}$$

$$A_t^{(2)}(s\beta) = \frac{3}{4\beta}\int_0^{\beta} \mathrm{d}y\,\mathrm{J}_{2t}(2sy) + \frac{1}{2\beta^3}\int_0^{\beta} \mathrm{d}y\,y^2\mathrm{J}_{2t}(2sy) + \frac{3}{4\beta^5}\int_0^{\beta} \mathrm{d}y\,y^4\mathrm{J}_{2t}(2sy)\,. \tag{A8}$$

On multiplying Bessel's equation

$$\mathrm{J}_{2s}''(2sy) + (1/2sy)\mathrm{J}_{2s}'(2sy) + (1-1/y^2)\mathrm{J}_{2s}(2sy) = 0 \tag{A9}$$

by y^2 and integrating, one obtains the identities

$$\int_0^{\beta} \mathrm{d}y\,y^2\mathrm{J}_{2s}(2sy) = \left(1-\frac{1}{4s^2}\right)\int_0^{\beta} \mathrm{d}y\,\mathrm{J}_{2s}(2sy) + \frac{\beta}{4s^2}\mathrm{J}_{2s}(2s\beta) - \frac{\beta^2}{2s}\mathrm{J}_{2s}'(2s\beta)\,, \tag{A10}$$

$$\int_0^\beta \mathrm{d}y\, y^4 J_{2s}(2sy) = \left(1-\frac{1}{4s^2}\right)\left(1-\frac{9}{4s^2}\right)\int_0^\beta \mathrm{d}y\, J_{2s}(2sy)$$
$$+\frac{\beta}{4s^2}\left(3\beta^2+1-\frac{9}{4s^2}\right)J_{2s}(2s\beta)$$
$$-\frac{\beta^2}{2s}\left(\beta^2+1-\frac{9}{4s^2}\right)J'_{2s}(2s\beta). \tag{A11}$$

We can now evaluate the integrals $I_s^{(1)}(\beta), \ldots, I_s^{(5)}(\beta)$ which appear in Section 2 (in this appendix we omit the prime on β' for convenience):

$$I_s^{(1)}(\beta) = \frac{1}{\beta}\int_0^\beta \mathrm{d}y\, J_{2s}(2sy), \tag{A12}$$

$$I_s^{(2)}(\beta) = \tfrac{1}{2}\{A_s^{(0)}(s\beta) - A_s^{(1)}(s\beta)\}$$
$$= \frac{1}{4s\beta}\left(\chi_s(\beta) - \frac{2s(1-\beta^2)}{\beta^2}\int_0^\beta \mathrm{d}y\, J_{2s}(2sy)\right), \tag{A13}$$

with

$$\chi_s(\beta) = J'_{2s}(2s\beta) - \frac{1}{2s\beta}J_{2s}(2s\beta) + \frac{1}{2s\beta^2}\int_0^\beta \mathrm{d}y\, J_{2s}(2sy), \tag{A14}$$

$$I_s^{(3)}(\beta) = \tfrac{1}{4}\{A_{s-1}^{(1)}(s.\,) + A_{s+1}^{(1)}(s\beta)\} - (1/2\beta^2)A_s^{(0)}(s\beta)$$
$$= \frac{1}{4s\beta}\left(3\chi_s(\beta) - \frac{2s(1-\beta^2)}{\beta^2}\int_0^\beta \mathrm{d}y\, J_{2s}(2sy)\right), \tag{A15}$$

$$I_s^{(4)}(\beta) = \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}\beta}I_s^{(1)}(\beta)$$
$$= \frac{1}{s\beta}\left(-\frac{1}{\beta}\int_0^\beta \mathrm{d}y\, J_{2s}(2sy) + J_{2s}(2s\beta)\right), \tag{A16}$$

$$I_s^{(5)}(\beta) = \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}\beta}I_s^{(2)}(\beta)$$
$$= -\frac{3}{4\beta^2 s^2}\chi_s(\beta) + \frac{3-\beta^2}{2s\beta^4}\int_0^\beta \mathrm{d}y\, J_{2s}(2sy). \tag{A17}$$

These integrals are now in a form in which they can be usefully approximated.

Approximations to the Bessel Functions

Approximations to $J'_{2s}(2s\beta)$, $J_{2s}(2s\beta)$ and $\int_0^\beta J_{2s}(2sy)\,\mathrm{d}y$ are required to obtain simple expressions for the $I_s^{(j)}(\beta)$.

Carlini Approximation. This approximation (Abramowitz and Stegun 1970, 9.3.7; Watson 1944, p. 226) corresponds to

$$\mathrm{J}_{2s}(2s\beta) = \frac{1}{(4\pi s)^{\frac{1}{2}}}\frac{Z^{2s}}{(1-\beta^2)^{\frac{1}{4}}}\left(1 - \frac{2+3\beta^2}{48s(1-\beta^2)^{3/2}} + \dots\right), \tag{A18}$$

with

$$Z = \beta\exp\{(1-\beta^2)^{\frac{1}{2}}\}/\{1+(1-\beta^2)^{\frac{1}{2}}\}. \tag{A19}$$

Differentiating (A18) gives

$$\mathrm{J}'_{2s}(2s\beta) = \frac{(1-\beta^2)^{\frac{1}{2}}}{\beta}\mathrm{J}_{2s}(2s\beta)\left(1 + \frac{\beta^2}{4s(1-\beta^2)^{3/2}} + \dots\right), \tag{A20}$$

and integrating (A18) gives

$$\int_0^\beta \mathrm{d}y\, \mathrm{J}_{2s}(2sy) = \frac{\beta}{2s(1-\beta^2)^{\frac{1}{2}}}\mathrm{J}_{2s}(2s\beta)\left(1 - \frac{2+\beta^2}{4s(1-\beta^2)^{3/2}} + \dots\right). \tag{A21}$$

These expressions are valid provided that $s(1-\beta^2)^{3/2} \gg 1$.

Airy Function Approximation. This approximation applies when $\beta \approx 1$. Setting

$$s_c = \tfrac{3}{2}(1-\beta^2)^{-3/2} \tag{A22}$$

(again we omit the prime on β' as used in the text), we find that (e.g. Melrose 1980, p. 117)

$$\mathrm{J}'_{2s}(2s\beta) = \frac{1-\beta^2}{\pi\sqrt{3}}K_{2/3}(s/s_c), \tag{A23}$$

$$\mathrm{J}_{2s}(2s\beta) = \frac{(1-\beta^2)^{\frac{1}{2}}}{\pi\sqrt{3}}K_{1/3}(s/s_c), \tag{A24}$$

$$\int_0^\beta \mathrm{d}y\, \mathrm{J}_{2s}(2sy) = \frac{1}{2s\pi\sqrt{3}}\int_{s/s_c}^\infty \mathrm{d}t\, K_{1/3}(t). \tag{A25}$$

For $s \ll s_c$ one has

$$K_\nu(s/s_c) \approx \tfrac{1}{2}\Gamma(\nu)(s/2s_c)^{-\nu}. \tag{A26}$$

Using these approximations and the identity

$$2K_{2/3}(s/s_c) - \int_{s/s_c}^\infty \mathrm{d}t\, K_{1/3}(t) = \int_{s/s_c}^\infty \mathrm{d}t\, K_{5/3}(t), \tag{A27}$$

we find the forms for $I_s^{(j)}(\beta)$ given in Table 1.

Wild–Hill Approximations. Wild and Hill (1971) interpolated between the Airy function approximation for $\mathrm{J}_s(s\beta)$ and $\mathrm{J}'_s(s\beta)$ for $s \ll s_c$ and the Carlini approximation. The corresponding limiting cases of the $I_s^{(j)}(\beta)$ appear in Table 2. Note that for a given $I_s^{(j)}(\beta)$ a factor $(2s_c/3)^b$ (b constant) occurs in the Carlini approximation while a similar factor $(2s/a_j)^b$ occurs in the Airy function expression. The Wild–Hill approximation to the $I_s^{(j)}(\beta)$ consists of replacing these factors by

$$Q_j \equiv \left(\frac{3}{2s_c}+\frac{a_j}{2s}\right)^{-b}, \tag{A28}$$

and noting that $Z^{2s} \to 1$ as $\beta \to 1$. The Wild–Hill approximations to the $I_s^{(j)}(\beta)$ are given in the last column of Table 2.

Approximations to Products of the Q_j

In Section 2 we exploit several approximations to products of the Q_j of (A28) in order to obtain compact expressions for $\bar{\eta}(\omega,\theta)$. We briefly outline these approximations here.

Firstly, we note that

$$Q_1^{-1/4}Q_3^{-1/12} = Q_4^{-1/6}Q_3^{-1/6} \tag{A29}$$

to within a few per cent for all values of s/s_c. This approximate equality is easily verified by direct evaluation and is used in obtaining (21).

Secondly, by interpolating between their asymptotic forms for $s \ll s_c$ and $s \gg s_c$, we find the following approximations for $Q_4^{-1/6}Q_3^{-1/6}$ and $Q_2^{-5/6}Q_3^{-1/6}$, also used in (21):

$$Q_4^{-1/6}Q_3^{-1/6} \approx \xi(1+0{\cdot}85\,s_c/s)^{-1/3} \approx Q_1^{-1/4}Q_3^{-1/12}, \tag{A30}$$

$$Q_2^{-5/6}Q_3^{-1/6} \approx \xi^3(1+s_c/s)^{-1}, \tag{A31}$$

with ξ given by

$$\xi = (2s_c/3)^{1/3}. \tag{A32}$$

Equations (A30) and (A31) are also found to be accurate to within a few per cent for all s/s_c.

Appendix 2. Calculation of Γ_A

In this appendix we re-express Γ_A in terms of the $I_s^{(j)}(\beta')$, with the addition of a further integral $I_s^{(6)}(\beta')$ for which we obtain a Wild–Hill approximation.

Analysis

The expression (25) for Γ_A is

$$\Gamma_A(\omega,\theta) = 4\pi\int_1^{\infty} d\gamma\,\frac{N(\gamma)}{\beta^2\gamma}\,\frac{\omega}{nmc^2}\left(\frac{2\pi c}{\omega}\right)^3 I, \tag{A33}$$

with

$$I = \tfrac{1}{2}\int_{-1}^{1} d\cos\alpha \sum_{s=1}^{\infty}\eta(s,\omega,\theta)(\cos\alpha - n\beta\cos\theta)\,\frac{d\,\phi(\cos\alpha)}{d\cos\alpha}. \tag{A34}$$

As in Section 2 we convert the sum over s into an integral and transform to the new variables β' and $\cos\alpha'$. We expand ϕ' up to first order in $\cos\alpha'$ and obtain

$$I = \tfrac{1}{2}\frac{\gamma}{\Omega_e}\int_{-1}^{1} \mathrm{d}\cos\alpha' \frac{(1-n^2\beta^2\cos^2\theta)^2}{(1-n\beta\cos\alpha'\cos\theta)^3} \times \{(1-sn\beta\cos\alpha'\cos\theta\,\mathrm{d}/\mathrm{d}s)\eta_1(s,\omega,\theta)\} \times \cos\alpha'\{\phi' + (1-n^2\beta^2\cos^2\theta)\cos\alpha'\,\phi''\}, \tag{A35}$$

evaluated at $s = s_0$, with $\eta_1(s,\omega,\theta)$ defined by

$$\eta(s,\omega,\theta) = \eta_1(s,\omega,\theta)\,\delta\big(\omega(1-n\beta\cos\alpha\cos\theta) - s\Omega_e/\gamma\big).$$

Then, as with (14), we have

$$I = \frac{\gamma}{\Omega_e}\frac{A}{1+T^2}(1-n^2\beta^2\cos^2\theta)^2\{(2c_2c_3\phi' + \hat{q}c_2^2)I_s^{(2)}(\beta') + (c_3c_4\phi' + \hat{q}c_2c_4)I_s^{(5)}(\beta') + \hat{q}c_4^2 I_s^{(6)}(\beta')\}, \tag{A36}$$

with

$$I_s^{(6)}(\beta') = \tfrac{1}{2}\int_{-1}^{1} \mathrm{d}\cos\alpha' \cos^2\alpha' \sin^2\alpha' \{\mathrm{J}_s'(s\beta'\sin\alpha')\}^2, \tag{A37}$$

$$\hat{q} = -n\beta\cos\theta(5 + s\,\mathrm{d}/\mathrm{d}s)\phi' + (1-n^2\beta^2\cos^2\theta)\phi''. \tag{A38}$$

Wild–Hill Approximation of $I_s^{(6)}(\beta')$

The re-expression of $I_s^{(6)}(\beta')$ in terms of J_{2s}', J_{2s} and $\int \mathrm{d}y\,\mathrm{J}_{2s}$ involves some lengthy algebra using the methods outlined in Appendix 1. Our method in this case is to obtain the Carlini approximation and use this to infer the form of the Wild–Hill approximation. The constant a_6 (cf. Table 2) is then obtained numerically.

In the Carlini approximation we have

$$I_s^{(6)}(\beta') = \tfrac{1}{2}\int_{-1}^{1} \mathrm{d}\cos\alpha \cos^2\alpha \sin^2\alpha \{\mathrm{J}_s'(s\beta'\sin\alpha)\}^2 \approx \frac{1}{2\beta'^2}\int_{-1}^{1} \mathrm{d}\cos\alpha \cos^2\alpha (1-\beta'^2\sin^2\alpha)\{\mathrm{J}_s(s\beta'\sin\alpha)\}^2. \tag{A39}$$

For large s the integrand in (A39) is small except for $|\alpha - \tfrac{1}{2}\pi| \ll 1$, implying that

$$I_s^{(6)}(\beta') \approx \frac{1-\beta'^2}{\beta'^2} I_s^{(2)}(\beta') \approx \frac{1}{4s^2\beta'^2}\frac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}(2s_c/3)^{1/6}. \tag{A40}$$

Comparison of (A40) with the approximations in Table 2 suggests a Wild–Hill approximation of the form

$$I_s^{(6)}(\beta') \approx \frac{1}{4s^2\beta'^2}\frac{Z^{2s}}{(4\pi s)^{\frac{1}{2}}}\left(\frac{3}{2s_c} + \frac{a_6}{2s}\right)^{-1/6}. \tag{A41}$$

Numerical evaluation of $I_s^{(6)}(\beta')$ for $8 \leqslant s \leqslant 512$ confirms this suggestion and yields $a_6 \approx 0 \cdot 15$.

Appendix 3. Emissivity and Absorption Coefficients for Thermal and Power–law Distributions

Among the most important results of this work are the expressions (40) with (43) and (52a) and (52b) for the emissivity and the absorption coefficient of thermal and power-law electron distributions. In this appendix we summarize these results and their subsidiary equations. The equation numbers from the text are included here for reference (in some cases the indicated definition occurs just after the equation cited).

Thermal Distributions

Defining the quantity

$$\mu_0 = \beta\cos\theta, \tag{A42}$$

we may write (40) in the form (with $n \approx 1$)

$$J(\omega,\theta) = \pi^{\frac{1}{2}} C\frac{\gamma^2}{\Omega_e}(\gamma^2-1)^{\frac{1}{2}}\frac{A}{1+T^2}(1-\mu_0^2)(\xi/s)^{3/2}(1+4\cdot 53\,s_c/s)^{1/6}$$
$$\times\left(\{T(1-\beta^2)\cos\theta(1+0\cdot 85\,s_c/s)^{-1/3}+(1-\beta^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\}^2\right.$$
$$\left.+\frac{\beta^2T^2\xi\sin^4\theta}{2(s+s_c)}\right)\frac{\xi^2-1}{2\sin\theta}\left(\frac{2\pi\xi}{\mu x\gamma}\right)^{\frac{1}{2}}Z^{2s}\exp\{-\mu(\gamma-1)\}, \tag{40}$$

with

$$\mu = mc^2/k_B T, \tag{29}$$

$$C = N(\mu/2\pi)^{3/2}(1-15/8\mu\ldots) \tag{30}$$

(N is the number density of electrons),

$$A = e^2\omega^2/4\pi\varepsilon_0\, 2\pi c\sin^2\theta, \tag{5a}$$

$$T_o = -T_x^{-1} = -\cos\theta\{a+(1+a^2)^{\frac{1}{2}}\}/|\cos\theta|, \tag{3a}$$

$$a = \Omega_e\sin^2\theta/2\omega|\cos\theta|, \tag{3b}$$

$$\gamma = \gamma_0 = \{1+(2\omega/\mu\Omega_e)(1+\tfrac{9}{2}x)^{-1/3}\}^{\frac{1}{2}}, \tag{39}$$

$$x = \omega\sin^2\theta/\Omega_e\mu, \tag{33a}$$

$$\beta = \beta_0 = (1-\gamma_0^{-2})^{\frac{1}{2}}, \tag{A43}$$

$$\xi = \xi_0 = (1-\beta'^2)^{-\frac{1}{2}} = (1-\mu_0^2)^{\frac{1}{2}}(1-\beta^2)^{-\frac{1}{2}}, \tag{33b}$$

$$s = s_0 = \gamma_0\omega(1-\mu_0^2)/\Omega_e, \qquad s_c = \tfrac{3}{2}\xi_0^3, \tag{11}$$

$$Z = \beta'\exp\{(1-\beta'^2)^{\frac{1}{2}}\}/\{1+(1-\beta'^2)^{\frac{1}{2}}\}, \tag{17}$$

$$\beta' = \beta\sin\theta/(1-\mu_0^2)^{\frac{1}{2}}. \tag{7a}$$

The absorption coefficient is obtained from Kirchoff's law:

$$\Gamma(\omega,\theta) = \{(2\pi c)^3/\omega^2 k_B T\} J(\omega,\theta). \tag{43}$$

Power-law Distributions

In the case of a power-law distribution of the form

$$N(\gamma) = 0, \qquad \gamma-1 < \varepsilon_c ; \tag{44a}$$

$$= N(\delta-1)\,\varepsilon_c^{\delta-1}(\gamma-1)^{-\delta}/4\pi, \qquad \varepsilon_c < \gamma-1 ; \tag{44b}$$

with

$$N = 4\pi \int_1^\infty d\gamma\, N(\gamma), \tag{44c}$$

we have

$$\left.\begin{matrix} J(\omega,\theta) \\ \Gamma(\omega,\theta)\end{matrix}\right\} = \frac{N(\delta-1)\surd 3}{8}\,\frac{A}{1+T^2}\,\frac{\Omega_e}{\omega^2}\sin^3\theta\left(1+\frac{1}{\gamma_0}\right) \times\left\{\left(1+\frac{T\cot\theta}{\gamma_0}\right)^2 + \frac{2T^2}{3(a+b+2)}\right\} \times\left(\frac{\gamma_0-1}{\varepsilon_c}\right)^{1-\delta} Z^{2s}\left\{\begin{matrix} 1 \\ \dfrac{(2\pi c)^3}{\omega^2 mc^2}\,\dfrac{\delta+2}{\varepsilon_c}\left(\dfrac{\gamma_0-1}{\varepsilon_c}\right)^{-1}\dfrac{\gamma_0}{1+\gamma_0}\end{matrix}\right., \tag{52a, b}$$

with

$$a+b = \delta+1 \quad \text{for} \quad J(\omega,\theta); \tag{50a}$$

$$= \delta+2 \quad \text{for} \quad \Gamma(\omega,\theta); \tag{50b}$$

$$(\gamma_0+\tfrac{1}{2})^2 - 1 = 4\omega/\{3\Omega_e(a+b)\sin\theta\}, \tag{49}$$

$$\xi_0 = (\gamma_0^2\sin^2\theta + \cos^2\theta)^{\frac{1}{2}} \approx \gamma_0\sin\theta, \tag{53c}$$

$$\beta_0 = (1-\gamma_0^{-2})^{\frac{1}{2}}, \tag{A43}$$

$$s = s_0 = (\gamma_0\,\omega/\Omega_e)(1-\beta_0^2\cos^2\theta) \approx (\gamma_0\,\omega/\Omega_e)\sin^2\theta, \tag{11}$$

$$Z = \{(\xi_0-1)/(\xi_0+1)\}^{\frac{1}{2}}\exp\xi_0^{-1}. \tag{53b}$$

The approximate equalities for ξ_0 and s_0 hold for γ_0 large or $|\theta-\frac{1}{2}\pi|$ small. Equation (49) is valid for $\omega/\Omega_e \gtrsim 1\cdot5(a+b)$. If $\gamma_0 \gtrsim 2$ we have

$$Z^{2s} \approx \exp\{-\tfrac{1}{2}(a+b)\}. \tag{53}$$

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