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GYROMETRIC PRESERVING MAPS ON EINSTEIN GYROGROUPS, MÖBIUS GYROGROUPS AND PROPER VELOCITY GYROGROUPS

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Abstract. In this paper we describe all surjective gyrometric preserving maps on the three models of gyrovector spaces, the Einstein gyrogroup, the Möbius gyrogroup and the Proper Velocity gyrogroup.

1. Introduction

In the book [4] Ungar studied the gyrocommutative gyrogroup. The (gyrocommutative) gyrogroup is a generalization of the (commutative) group, which is not necessarily (commutative and) associative. In his book Ungar studied the three models of gyrocommutative gyrogroup, the Einstein gyrogroup, the Möbius gyrogroup and the Proper Velocity gyrogroup. These models also have the structure of the gyrovector space and the gyrometric. The gyrometric on the Einstein gyrogroup is related with the Bergman metric, and the gyrometric on the Möbius gyrogroup is related with the Poincaré metric. In the paper [2], Kim established an isomorphism between the Einstein gyrogroup and the set of all qubit density matrices representing mixed states endowed with an appropriate addition. Moreover, he established a relation between the trace metric for the qubit density matrices and the Einstein metric on the Einstein gyrogroup.

Let (X, d_X) and (Y, d_Y) be metric spaces. $T : (X, d_X) \to (Y, d_Y)$ is said to be an isometry if it satisfies $d_Y(Ta, Tb) = d_X(a, b)$ for any pair $a, b \in X$.

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The study of isometry maps is important and many results are known. The celebrated Mazur-Ulam Theorem states that a surjective isometry T from a normed vector space X onto a normed vector space Y is of the form $T(\cdot) = T(\mathbf{0}) + T_0(\cdot)$, where T_0 is linear. This theorem referred to about algebraic structure of an isometry.

In this paper, we study surjective gyrometric preserving self-maps on Einstein gyrogroups, Möbius gyrogroups and PV gyrogroups. We will show the algebraic structures of these maps and representation theorems.

In the following of the paper, $\mathbb{V}$ denotes a real inner product space with the vector addition $+$ and a positive definite inner product $\langle \cdot, \cdot \rangle$. We say that an inner product $\langle \cdot, \cdot \rangle$ is positive definite if the following holds; $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{V}$ implies $\boldsymbol{v} = \mathbf{0}$. We denote by $\| \cdot \|$ the norm on $\mathbb{V}$ induced by $\langle \cdot, \cdot \rangle$ and $\boldsymbol{B}$ denotes the open unit ball of $\mathbb{V}$; $\boldsymbol{B} = \{\boldsymbol{u} \in \mathbb{V} : \|\boldsymbol{u}\| < 1\}$.

2. Gyrogroups

In this section, we recall necessary definitions and briefly sammarize fundamental results of gyrogroups based on the Ungar's book [4].

Definition 2.1. ([4, Definition 2.1]) A groupoid $(S, +)$ is a nonempty set, S, with a binary operation, $+ : S \times S \to S$. An automorphism ϕ of a groupoid $(S, +)$ is a bijective self-map of S, $\phi : S \to S$, which preserves its groupoid operation, that is, $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in S$. $\mathrm{Aut}(S, +)$ is the set of all automorphism of a groupoid $(S, +)$.

Definition 2.2. ([4, Definition 2.7]) A groupoid $(G, \oplus)$ is a gyrogroup if it satisfies the following axioms.

(G1) There is a element, $\mathbf{0} \in G$, called a left identity, satisfying

$$\mathbf{0} \oplus \boldsymbol{a} = \mathbf{0},$$

for all $\boldsymbol{a} \in G$;

(G2) For each $\boldsymbol{a} \in G$ there is an element $\ominus \boldsymbol{a}$, called a left inverse of $\boldsymbol{a}$, satisfying

$$\ominus \boldsymbol{a} \oplus \boldsymbol{a} = \mathbf{0};$$

(G3) For any triple $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G$ there exists a unique element $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in G$ such that the binary operation obeys the left gyroassociative law

$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c};$$

(G4) The map $\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}] : G \to G$ given by $\boldsymbol{c} \mapsto \mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}$ is an automorphism of the groupoid $(G, \oplus)$,

$$\mathrm{gyr}[\boldsymbol{a}, \boldsymbol{b}] \in \mathrm{Aut}(G, \oplus).$$

The automorphism $\mathrm{gyr}[\boldsymbol{a},\boldsymbol{b}]$ of G is called gyroautomorphism of G generated by $\boldsymbol{a},\boldsymbol{b}\in G$. The operator $\mathrm{gyr} : G\times G\to \mathrm{Aut}(G,\oplus)$ is called gyrator of G;

(G5) The gyroautomorphism $\mathrm{gyr}[a,b]$ generated by any $\boldsymbol{a},\boldsymbol{b}\in G$ possesses the left loop property

$$\mathrm{gyr}[\boldsymbol{a}\oplus\boldsymbol{b},\boldsymbol{b}]=\mathrm{gyr}[\boldsymbol{a},\boldsymbol{b}]$$

for any $\boldsymbol{a},\boldsymbol{b}\in G$.

Definition 2.3. ([4, Definition 2.8]) A gyrogroup $(G,\oplus)$ is gyrocommutative if its binary operation obey the gyrocommutative law

(G6) $$\boldsymbol{a}\oplus\boldsymbol{b}=\mathrm{gyr}[\boldsymbol{a},\boldsymbol{b}](\boldsymbol{b}\oplus\boldsymbol{a}) \text{ for all } \boldsymbol{a},\boldsymbol{b}\in G.$$

Example 2.4. A (commutative) group is a (gyrocommutative) gyrogroup which all the gyroautomorphism is the identity map on G.

Example 2.5. ([4, Definition 3.45]) Let $\boldsymbol{B}$ be the open unit ball of $\mathbb{V}$. Einstein addition $\oplus_E$ is a binary operation in $\boldsymbol{B}$ given by the equation

$$\begin{aligned}\boldsymbol{u}\oplus_E\boldsymbol{v}&=\frac{1}{1+\langle\boldsymbol{u},\boldsymbol{v}\rangle}\left\{\left(1+\frac{\gamma_u\langle\boldsymbol{u},\boldsymbol{v}\rangle}{1+\gamma_u}\right)\boldsymbol{u}+\frac{1}{\gamma_u}\boldsymbol{v}\right\}\\&=\frac{1}{1+\langle\boldsymbol{u},\boldsymbol{v}\rangle}\left\{\left(1+\frac{\langle\boldsymbol{u},\boldsymbol{v}\rangle}{1+\alpha_u}\right)\boldsymbol{u}+\alpha_u\boldsymbol{v}\right\},\end{aligned}\tag{2.1}$$

where $\alpha_u=\sqrt{1-\|\boldsymbol{u}\|^2}$, $\gamma_u={\alpha_u}^{-1}$. $(\boldsymbol{B},\oplus_E)$ is a gyrocommutative gyrogroup and called the Einstein gyrogroup. By a simple calculation we see that the identity of $(\boldsymbol{B},\oplus_E)$ is the zero vector of $\mathbb{V}$ and the inverse element of $\boldsymbol{u}\in(\boldsymbol{B},\oplus_E)$ is $-\boldsymbol{u}$.

Example 2.6. ([4, p75]) Möbius addition $\oplus_M$ is a binary operation in the open unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ given by the equation

$$a\oplus_M b=\frac{a+b}{1+\overline{a}b}.\tag{2.2}$$

$(\mathbb{D},\oplus_M)$ is a gyrocommutative gyrogroup. The identity of $(\mathbb{D},\oplus_M)$ is 0 and the inverse element of $a\in(\mathbb{D},\oplus_M)$ is $-a$.

Let us identify the complex plane $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$ in the usual sense, we have a natural extension as the following.

Example 2.7. ([4, Definition 3.40]) Let $\boldsymbol{B}$ be the open unit ball of $\mathbb{V}$. Möbius addition $\oplus_M$ is a binary operation in $\boldsymbol{B}$ given by the equation

$$\boldsymbol{u} \oplus_M \boldsymbol{v} = \frac{(1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2)\boldsymbol{u} + (1 - \|\boldsymbol{u}\|^2)\boldsymbol{v}}{1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2}. \tag{2.3}$$

$(\boldsymbol{B}, \oplus_M)$ is a gyrocommutative gyrogroup and called the Möbius gyrogroup. The identity of $(\boldsymbol{B}, \oplus_M)$ is the zero vector of $\mathbb{V}$ and the inverse element of $\boldsymbol{u} \in (\boldsymbol{B}, \oplus_M)$ is $-\boldsymbol{u}$.

Example 2.8. ([4, Definition 3.47]) Let $\mathbb{V}$ be a real inner product space. The PV (Proper Velocity) addition $\oplus_P$ is a binary operation in $\mathbb{V}$ given by the equation

$$\begin{aligned} \boldsymbol{u} \oplus_P \boldsymbol{v} &= \left\{ \frac{\beta_u}{1 + \beta_u} \langle \boldsymbol{u}, \boldsymbol{v}\rangle + \frac{1}{\beta_v} \right\} \boldsymbol{u} + \boldsymbol{v} \\ &= \left\{ \frac{1}{1 + \delta_u} \langle \boldsymbol{u}, \boldsymbol{v}\rangle \right\} \boldsymbol{u} + \boldsymbol{v}, \end{aligned} \tag{2.4}$$

where $\delta_u = \sqrt{1 + \|\boldsymbol{u}\|^2}$, $\beta_u = \delta_u^{-1}$. $(\mathbb{V}, \oplus_P)$ is a gyrocommutative gyrogroup and called the PV (Proper Velocity) gyrogroup. The identity of $(\mathbb{V}, \oplus_P)$ is the zero vector of $\mathbb{V}$ and the inverse element of $\boldsymbol{u} \in (\mathbb{V}, \oplus_P)$ is $-\boldsymbol{u}$.

Definition 2.9. ([4, Definition 6.2]) Let G be a subset of a real inner product space $\mathbb{V}$. A real inner product gyrovector space (gyrovector space, in short) $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ with a scalar multiplication $\otimes : \mathbb{R} \times G \to G$ that obey the following axioms:

(V0) $\langle \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}, \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G$;
(V1) $1 \otimes \boldsymbol{a} = \boldsymbol{a}$ for all $\boldsymbol{a} \in G$;
(V2) $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$ for all $\boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;
(V3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$ for all $\boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;
(V4) $\dfrac{|r| \otimes \boldsymbol{a}}{\|r \otimes \boldsymbol{a}\|} = \dfrac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$ for all $\boldsymbol{a} \in G,\ r \in \mathbb{R}$;
(V5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G,\ r \in \mathbb{R}$;
(V6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G$ for all $\boldsymbol{v} \in G,\ r_1, r_2 \in \mathbb{R}$;
(VV) $\|G\| = \{\pm\|\boldsymbol{a}\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is an one-dimensional real vector space with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in G$,
(V7) $\|r \otimes \boldsymbol{a}\| = |r| \otimes \|\boldsymbol{a}\|$;
(V8) $\|\boldsymbol{a} \oplus \boldsymbol{b}\| \leq \|\boldsymbol{a}\| \oplus \|\boldsymbol{b}\|$.

Gyrovector spaces are defined only on a subset G of a real inner product space $\mathbb{V}$, and $\mathbb{V}$ is called a carrier of G. The gyrometric ϱ on a gyrovector space $(G, \oplus, \otimes)$ is given by $\varrho(\boldsymbol{a}, \boldsymbol{b}) = \| \ominus \boldsymbol{a} \oplus \boldsymbol{b}\|$ for any pair $\boldsymbol{a}, \boldsymbol{b} \in G$, where $\| \cdot \|$ be the norm on the carrier of G. Einstein gyrogroups, Möbius gyrogroups and PV gyrogroups turn themselves into gyrovector spaces and we can cosider their gyrometrics.

Example 2.10. ([4, Definition 6.86]) An Einstein gyrogroup $(\boldsymbol{B}, \oplus_E)$ forms a gyrovector space $(\boldsymbol{B}, \oplus_E, \otimes_E)$ with the scalar multiplication $\otimes_E$ on $(\boldsymbol{B}, \oplus_E)$ defined by

$$r \otimes_E \boldsymbol{v} = \tanh(r \tanh^{-1} \|\boldsymbol{v}\|) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \tag{2.5}$$

where $r \in \mathbb{R}$, $\boldsymbol{v} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$; and $r \otimes_E \boldsymbol{0} = \boldsymbol{0}$. The gyrometric on the Einstein gyrogroup $\varrho_E(\boldsymbol{u}, \boldsymbol{v}) = \| - \boldsymbol{u} \oplus_E \boldsymbol{v}\|$ called Einstein gyrometric and $(\boldsymbol{B}, \varrho_E)$ is a metric space. Let $d_E(\boldsymbol{u}, \boldsymbol{v}) = \tanh^{-1} \varrho_E(\boldsymbol{u}, \boldsymbol{v})$ then d_E is also the metric on $\boldsymbol{B}$. In fact, the metric d_E on $\boldsymbol{B}$ corresponds to the Bergman metric on the ball in $\mathbb{C}^n$ if $\mathbb{V} = \mathbb{R}^n$.

Example 2.11. ([4, Definition 6.83, Theorem 6.84]) A Möbius gyrogroup $(\boldsymbol{B}, \oplus_M)$ forms a gyrovector space $(\boldsymbol{B}, \oplus_M, \otimes_M)$ with the scalar multiplication $\otimes_M$ on $(\boldsymbol{B}, \oplus_M)$ defined by

$$r \otimes_M \boldsymbol{v} = \tanh(r \tanh^{-1} \|\boldsymbol{v}\|) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \tag{2.6}$$

where $r \in \mathbb{R}$, $\boldsymbol{v} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$; and $r \otimes_E \boldsymbol{0} = \boldsymbol{0}$. The gyrometric on the Möbius gyrogroup $\varrho_M(\boldsymbol{u}, \boldsymbol{v}) = \| - \boldsymbol{u} \oplus_M \boldsymbol{v}\|$ called the Möbius gyrometric and $(\boldsymbol{B}, \varrho_M)$ is a metric space. Let $d_M(\boldsymbol{u}, \boldsymbol{v}) = \tanh^{-1} \varrho_M(\boldsymbol{u}, \boldsymbol{v})$ then $(\boldsymbol{B}, d_M)$ is also the metric space and we call d_M the Möbius metric. In the special case when we consider the Möbius gyrogroup on the complex open unit disc $(\mathbb{D}, \oplus_M)$, Möbius gyrometric reduces to

$$\varrho_M(a, b) = | - a \oplus_M b| = \left| \frac{a - b}{1 - \overline{a} b} \right|.$$

The Möbius gyrometric on $\mathbb{D}$ is known as the pesudo-hyperbolic metric and Möbius metric d_M on $\mathbb{D}$ is also known as the Poincaré metric.

Example 2.12. ([4, Definition 6.87]) The PV gyrogroup $(\mathbb{V}, \oplus_P)$ forms a gyrovector space $(\mathbb{V}, \oplus_P, \otimes_P)$ with the scalar multiplication $\otimes_P$ on $(\mathbb{V}, \oplus_P)$ defined by

$$r \otimes_P \boldsymbol{v} = \sinh\left(r \sinh^{-1} \|\boldsymbol{v}\|\right) \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}, \tag{2.7}$$

where $r \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$; and $r \otimes_P \boldsymbol{0} = \boldsymbol{0}$. ϱ_P denotes the gyrometric on the PV gyrogroup; $\varrho_P(\boldsymbol{u}, \boldsymbol{v}) = \| - \boldsymbol{u} \oplus_P \boldsymbol{v}\|$.

Definition 2.13. ([4, Definition 6.15]) An automorphism τ of a gyrovector space $(G, \oplus, \otimes)$ is a bijective self-map of G, $\tau : G \to G$, which satisfies the following conditions (a), (b) and (c):

(a) $\tau(\boldsymbol{a} \oplus \boldsymbol{b}) = \tau\boldsymbol{a} \oplus \tau\boldsymbol{b}$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$;
(b) $\tau(r \otimes \boldsymbol{a}) = r \otimes \tau\boldsymbol{a}$ for any $r \in \mathbb{R}$, $\boldsymbol{a} \in G$;
(c) $\langle \tau\boldsymbol{a}, \tau\boldsymbol{b}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$ where $\langle \cdot, \cdot\rangle$ be the inner product on carrier of G.

Denote $\mathrm{Aut}(G, \oplus, \otimes)$ the set of all automorphism of a gyrovector space $(G, \oplus, \otimes)$.

Ungar has shown that the gyrometric is invariant under automorphisms and left gyrotranslations.

Theorem 2.14. ([4, Theorem 6.12]) *Suppose that ϱ be the gyrometric on a gyrovector space $(G, \oplus, \otimes)$. We have $\varrho(\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{a} \oplus \boldsymbol{c}) = \varrho(\boldsymbol{b}, \boldsymbol{c})$ and $\varrho(\tau\boldsymbol{b}, \tau\boldsymbol{c}) = \varrho(\boldsymbol{b}, \boldsymbol{c})$ for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G$, $\tau \in \mathrm{Aut}(G, \oplus, \otimes)$.*

Gyrometrics ϱ_E, ϱ_M and ϱ_P can be treated into the equations in the following proposition.

Proposition 2.15. *For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$,*

$$\varrho_E(\boldsymbol{u}, \boldsymbol{v}) = \left\{1 - \frac{(1 - \|\boldsymbol{u}\|^2)(1 - \|\boldsymbol{v}\|^2)}{(1 - \langle \boldsymbol{u}, \boldsymbol{v}\rangle)^2}\right\}^{\frac{1}{2}}, \tag{2.8}$$

$$\varrho_M(\boldsymbol{u}, \boldsymbol{v}) = \left\{1 - \frac{(1 - \|\boldsymbol{u}\|^2)(1 - \|\boldsymbol{v}\|^2)}{1 + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle}\right\}^{\frac{1}{2}}. \tag{2.9}$$

For any $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$,

$$\varrho_P(\boldsymbol{u}, \boldsymbol{v}) = (\langle \boldsymbol{u}, \boldsymbol{v}\rangle^2 - 2\delta_u\delta_v\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2)^{\frac{1}{2}}. \tag{2.10}$$

Proof. Put $a = \|\boldsymbol{u}\|$, $b = \|\boldsymbol{v}\|$ and $x = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$. Since $\|\alpha\boldsymbol{u} + \beta\boldsymbol{v}\| = \alpha^2\|\boldsymbol{u}\|^2 + \beta^2\|\boldsymbol{v}\|^2 + 2\alpha\beta\langle \boldsymbol{u}, \boldsymbol{v}\rangle$ for any $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}
\| -\boldsymbol{u} \oplus_E \boldsymbol{v}\|^2 &= \left\|\frac{1}{1 + \langle -\boldsymbol{u}, \boldsymbol{v}\rangle}\left\{\left(1 + \frac{\langle -\boldsymbol{u}, \boldsymbol{v}\rangle}{1 + \alpha_{-u}}\right)(-\boldsymbol{u}) + \alpha_{-u}\boldsymbol{v}\right\}\right\|^2 \\
&= \left\|\frac{1}{1 - x}\left\{\left(-1 + \frac{x}{1 + \alpha_u}\right)\boldsymbol{u} + \alpha_u\boldsymbol{v}\right\}\right\|^2 \\
&= \frac{\left(1 - \frac{x}{1+\alpha_u}\right)^2 a^2 + (1 - a^2)b^2 - 2\alpha_u\left(1 - \frac{x}{1+\alpha_u}\right)x}{(1 - x)^2} \\
&= \frac{x^2 - 2x + a^2 + (1 - a^2)b^2}{(1 - x)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-x)^2-(1-a^2)(1-b^2)}{(1-x)^2}\\
&= 1-\frac{(1-a^2)(1-b^2)}{(1-x)^2},
\end{aligned}
$$

$$
\begin{aligned}
\|-\boldsymbol{u}\oplus_M \boldsymbol{v}\|^2 &= \left\|\frac{(1+2\langle -\boldsymbol{u},\boldsymbol{v}\rangle+\|\boldsymbol{v}\|^2)(-\boldsymbol{u})+(1-\|\boldsymbol{u}\|^2)\boldsymbol{v}}{1+2\langle -\boldsymbol{u},\boldsymbol{v}\rangle+\|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2}\right\|^2\\
&= \left\|\frac{-(1+b^2-2x)\boldsymbol{u}+(1-a^2)\boldsymbol{v}}{1+a^2b^2-2x}\right\|^2\\
&= \frac{(1+b^2-2x)^2a^2+(1-a^2)^2b^2-2(1+b^2-2x)(1-a^2)x}{(1+a^2b^2-2x)^2}\\
&= \frac{4x^2-2(1+a^2)(1+b^2)x+(a^2+b^2)(1+a^2b^2)}{(1+a^2b^2-2x)^2}\\
&= \frac{(1+a^2b^2-2x)^2-(1-a^2)(1-b^2)(1+a^2b^2-2x)}{(1+a^2b^2-2x)^2}\\
&= 1-\frac{(1-a^2)(1-b^2)}{1+a^2b^2-2x},
\end{aligned}
$$

$$
\begin{aligned}
\|-\boldsymbol{u}\oplus_P \boldsymbol{v}\|^2 &= \left\|\left\{\frac{1}{1+\delta_u}\langle -\boldsymbol{u},\boldsymbol{v}\rangle+\delta_v\right\}(-\boldsymbol{u})+\boldsymbol{v}\right\|\\
&= \left\{\frac{-x}{1+\delta_u}+\delta_v\right\}^2 a^2+2\left\{\frac{-x}{1+\delta_u}+\delta_v\right\}(-x)+b^2\\
&= \frac{-(1-\delta_u)+2}{1+\delta_u}x^2+\{2\delta_v(1-\delta_u)-2\delta_v\}x+a^2\delta_v^2+b^2\\
&= x^2-2\delta_u\delta_v x+a^2+b^2+a^2b^2.
\end{aligned}
$$

□

3. Main Results

Let X and Y be subsets of two inner product spaces respectively. $S : X \to Y$ is an inner product preserving map if it satisfies $\langle S\boldsymbol{u}, S\boldsymbol{v}\rangle = \langle \boldsymbol{u},\boldsymbol{v}\rangle$ for any pair $\boldsymbol{u},\boldsymbol{v} \in X$. Let $(G,\oplus)$ be a gyrovector space and ϱ denotes the gyrometric on $(G,\oplus)$. We say $T : (G,\oplus) \to (G,\oplus)$ is a gyrometric preserving map if it satisfies $\varrho(T\boldsymbol{a}, T\boldsymbol{b}) = \varrho(\boldsymbol{a},\boldsymbol{b})$ for any pair $\boldsymbol{a},\boldsymbol{b} \in G$.

The following three theorems are main results of the paper.

Theorem 3.1. *Let $\boldsymbol{B}$ be the open unit ball of a real inner product space $\mathbb{V}$ and $T : (\boldsymbol{B},\oplus_E) \to (\boldsymbol{B},\oplus_E)$ be a map from the Einstein gyrogroup $(\boldsymbol{B},\oplus_E)$ into itself. Then the following conditions* (E1)*,* (E2) *and* (E3) *are equivalent.*

(E1) *The map T satisfies the following conditions* (a), (b) *and* (c)*:*
  (a) $T(\mathbf{0}) = \mathbf{0}$,
  (b) *T is a surjection,*
  (c) *T is an Einstein gyrometric preserving map.*
(E2) $T = O|_{\boldsymbol{B}}$ *for some surjective inner product preserving linear operator* $O : \mathbb{V} \to \mathbb{V}$.
(E3) $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_E, \otimes_E)$.

In particular, if $\dim \mathbb{V} < \infty$*, then the condition* (a) *and* (c) *of* (E1) *together imply the condition* (b) *of* (E1)*.*

Theorem 3.2. *Let $\boldsymbol{B}$ be the open unit ball of a real inner product space $\mathbb{V}$ and $T : (\boldsymbol{B}, \oplus_M) \to (\boldsymbol{B}, \oplus_M)$ be a map from the Möbius gyrogroup $(\boldsymbol{B}, \oplus_M)$ into itself. Then the following conditions* (M1), (M2) *and* (M3) *are equivalent.*

(M1) *The map T satisfies the following conditions* (a), (b) *and* (c)*:*
  (a) $T(\mathbf{0}) = \mathbf{0}$,
  (b) *T is a surjection,*
  (c) *T is a Möbius gyrometric preserving map.*
(M2) $T = O|_{\boldsymbol{B}}$ *for some surjective inner product preserving linear operator* $O : \mathbb{V} \to \mathbb{V}$.
(M3) $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_M, \otimes_M)$.

In particular, if $\dim \mathbb{V} < \infty$*, then the condition* (a) *and* (c) *of* (M1) *together imply the condition* (b) *of* (M1)*.*

Theorem 3.3. *Let $\mathbb{V}$ be a real inner product space. $T : (\mathbb{V}, \oplus_P) \to (\mathbb{V}, \oplus_P)$ be a map from the PV gyrogroup $(\mathbb{V}, \oplus_P)$ into itself. Then the following conditions* (P1), (P2) *and* (P3) *are equivalent.*

(P1) *The map T satisfies the following conditions* (a), (b) *and* (c)*:*
  (a) $T(\mathbf{0}) = \mathbf{0}$,
  (b) *T is a surjection,*
  (c) *T is a gyrometric preserving map on the PV gyrogroup.*
(P2) *T is a surjective inner product preserving linear operator on $\mathbb{V}$.*
(P3) $T \in \mathrm{Aut}(\mathbb{V}, \oplus_P, \otimes_P)$.

In particular, if $\dim \mathbb{V} < \infty$*, then the condition* (a) *and* (c) *of* (P1) *together imply the condition* (b) *of* (P1)*.*

4. Lemmas

In this section we give some lemmas with which Theorem 3.1, 3.2 and 3.3 will be proved.

For any nonnegative real number a, define $\pi_a = \{\boldsymbol{u} \in \mathbb{V} : \|\boldsymbol{u}\| = a\}$.

Lemma 4.1. *Let* $T : (\boldsymbol{B}, \oplus_E) \to (\boldsymbol{B}, \oplus_E)$ *be an Einstein gyrometric preserving map from an Einstein gyrogroup into itself. Suppose that* $T(\mathbf{0}) = \mathbf{0}$*. Then* $\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$ *for any* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$*.*

Proof. We first note that $T(\pi_a) \subseteq \pi_a$ for any $0 \leq a < 1$ since $\|T\boldsymbol{u}\| = \varrho_E(\mathbf{0}, T\boldsymbol{u}) = \varrho_E(T\mathbf{0}, T\boldsymbol{u}) = \varrho_E(\mathbf{0}, \boldsymbol{u}) = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. Let $0 \leq a, b < 1$ be arbitrary and put

$$f(x) = \left\{1 - \frac{(1-a^2)(1-b^2)}{(1-x)^2}\right\}^{\frac{1}{2}}. \tag{4.1}$$

Then the function $f : [-ab, ab] \to \mathbb{R}$ is a monotone function because $x \leq ab < 1$ for any $x \in [-ab, ab]$. We infer that f is injective. Let $\boldsymbol{u} \in \pi_a, \boldsymbol{v} \in \pi_b$. We have $-ab \leq \langle T\boldsymbol{u}, T\boldsymbol{v}\rangle \leq ab$ and $\varrho_E(\boldsymbol{u}, \boldsymbol{v}) = f(\langle \boldsymbol{u}, \boldsymbol{v}\rangle)$. We also have $\varrho_E(T\boldsymbol{u}, T\boldsymbol{v}) = f(\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle)$ because $T\boldsymbol{u} \in \pi_a$ and $T\boldsymbol{v} \in \pi_b$. Hence

$$f(\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle) = \varrho_E(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_E(\boldsymbol{u}, \boldsymbol{v}) = f(\langle \boldsymbol{u}, \boldsymbol{v}\rangle).$$

We conclude that $\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$. □

Lemma 4.2. *Let* $T : (\boldsymbol{B}, \oplus_M) \to (\boldsymbol{B}, \oplus_M)$ *be a Möbius gyrometric preserving map from a Möbius gyrogroup into itself. Suppose that* $T(\mathbf{0}) = \mathbf{0}$*. Then* $\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$ *for any* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$*.*

Proof. Note $T(\pi_a) \subseteq \pi_a$ for any $0 \leq a < 1$ since $\|T\boldsymbol{u}\| = \varrho_M(\mathbf{0}, T\boldsymbol{u}) = \varrho_M(T\mathbf{0}, T\boldsymbol{u}) = \varrho_M(\mathbf{0}, \boldsymbol{u}) = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. Choose $0 \leq a, b < 1$ be arbitrary and put

$$g(x) = \left\{1 - \frac{(1-a^2)(1-b^2)}{1 + a^2b^2 - 2x}\right\}^{\frac{1}{2}}. \tag{4.2}$$

Then the function $g : [-ab, ab] \to \mathbb{R}$ is a monotone function because $1 + a^2b^2 > 2ab \geq 2x$ for any $x \in [-ab, ab]$. We infer that f is injective. Let $\boldsymbol{u} \in \pi_a, \boldsymbol{v} \in \pi_b$. We have $-ab \leq \langle \boldsymbol{u}, \boldsymbol{v}\rangle \leq ab$ and $\varrho_M(\boldsymbol{u}, \boldsymbol{v}) = f(\langle \boldsymbol{u}, \boldsymbol{v}\rangle)$. We also have $\varrho_M(T\boldsymbol{u}, T\boldsymbol{v}) = g(\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle)$ because $T\boldsymbol{u} \in \pi_a$ and $T\boldsymbol{v} \in \pi_b$. Hence

$$g(\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle) = \varrho_M(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_M(\boldsymbol{u}, \boldsymbol{v}) = g(\langle \boldsymbol{u}, \boldsymbol{v}\rangle).$$

We conclude that $\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$. □

Lemma 4.3. *Let* $T : (\mathbb{V}, \oplus_P) \to (\mathbb{V}, \oplus_P)$ *be a PV gyrometric preserving map from a PV gyrogroup into itself. Suppose that* $T(\mathbf{0}) = \mathbf{0}$*. Then* $\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{v}\rangle$ *for any* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$*.*

Proof. Note $T(\pi_a) \subseteq \pi_a$ for any $0 \leq a$ since $\|T\boldsymbol{u}\| = \varrho_P(\mathbf{0}, T\boldsymbol{u}) = \varrho_P(T\mathbf{0}, T\boldsymbol{u}) = \varrho_P(\mathbf{0}, \boldsymbol{u}) = \|\boldsymbol{u}\|$ for any $\boldsymbol{u} \in \mathbb{V}$. Let a, b be arbitrary non-negative real numbers. Put

$$h(x) = (x^2 - 2\delta_u\delta_v x + a^2 + b^2 + a^2b^2)^{\frac{1}{2}}. \tag{4.3}$$

We show that $h : [-ab, ab] \to \mathbb{R}$ is a monotone function. We have

$$(h^2)'(x) = \frac{dh^2(x)}{dx} = 2x - 2\delta_u\delta_v. \tag{4.4}$$

$(h^2)'(x) < 0$ for any $x \in [-ab, ab]$ because $x \le ab < \delta_u\delta_v$. Hence we have that h^2 is a monotone function and hence h is also monotone. Therefore h is injective. Let $\boldsymbol{u} \in \pi_a, \boldsymbol{v} \in \pi_b$. We have $-ab \le \langle \boldsymbol{u}, \boldsymbol{v} \rangle \le ab$ and $\varrho_P(\boldsymbol{u}, \boldsymbol{v}) = h(\langle \boldsymbol{u}, \boldsymbol{v} \rangle)$. We also have $\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = h(\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle)$ because $T\boldsymbol{u} \in \pi_a$ and $T\boldsymbol{v} \in \pi_b$. Hence

$$h(\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle) = \varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_P(\boldsymbol{u}, \boldsymbol{v}) = h(\langle \boldsymbol{u}, \boldsymbol{v} \rangle).$$

It implies that $\langle T\boldsymbol{u}, T\boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. □

The following lemma shows that an inner product preserving map on $\boldsymbol{B}$ is extensible to the whole space.

Lemma 4.4. *Let $\boldsymbol{B}$ be the open unit ball of a real inner product space $\mathbb{V}$. Suppose that $T : \boldsymbol{B} \to \boldsymbol{B}$ be an inner product preserving map. Then T can be extended to an inner product preserving map $S : \mathbb{V} \to \mathbb{V}$ defined by*

$$S(\boldsymbol{w}) = 2\|\boldsymbol{w}\| T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right) \tag{4.5}$$

for any $\boldsymbol{w} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$; and $S(\boldsymbol{0}) = \boldsymbol{0}$. Moreover, S is a linear operator if T is surjective.

Proof. Put

$$S(\boldsymbol{w}) = 2\|\boldsymbol{w}\| T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right) \tag{4.6}$$

for any $\boldsymbol{w} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$; and $S(\boldsymbol{0}) = \boldsymbol{0}$.

First we show that $T = S|_{\boldsymbol{B}}$. Let $\boldsymbol{u} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$, $r > 0$ which satisfy $r\boldsymbol{u} \in \boldsymbol{B}$. We have

$$\langle T(r\boldsymbol{u}), T\boldsymbol{u} \rangle = \langle r\boldsymbol{u}, \boldsymbol{u} \rangle = r\|\boldsymbol{u}\|^2 = \|r\boldsymbol{u}\|\|\boldsymbol{u}\| = \|T(r\boldsymbol{u})\|\|T\boldsymbol{u}\|.$$

On the other hand, by the Cauchy-Schwarz inequality, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\|\|\boldsymbol{y}\|$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are parallel. This means that $T(r\boldsymbol{u})$ and $T\boldsymbol{u}$ are parallel, that is $T(r\boldsymbol{u}) = sT\boldsymbol{u}$ for some $s \in \mathbb{R}$. We have $s = t$ by the equation

$$r\|\boldsymbol{u}\|^2 = \langle r\boldsymbol{u}, \boldsymbol{u} \rangle = \langle T(r\boldsymbol{u}), T\boldsymbol{u} \rangle = \langle sT\boldsymbol{u}, T\boldsymbol{u} \rangle = s\|T\boldsymbol{u}\|^2 = s\|\boldsymbol{u}\|^2.$$

Hence we have $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$ for any $\boldsymbol{u} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$ and $r > 0$ which satisfy $r\boldsymbol{u} \in \boldsymbol{B}$. In particular,

$$S(\boldsymbol{u}) = 2\|\boldsymbol{u}\| T\left(\frac{\boldsymbol{u}}{2\|\boldsymbol{u}\|}\right) = T(\boldsymbol{u})$$

for any $\boldsymbol{u} \in \boldsymbol{B} \setminus \{\boldsymbol{0}\}$.

Next we show that S is an inner product preserving map. It is clear that $\langle S\boldsymbol{w}, S\boldsymbol{z}\rangle = 0 = \langle \boldsymbol{w}, \boldsymbol{z}\rangle$ if $\boldsymbol{w} = 0$ or $\boldsymbol{z} = 0$. For any pair $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{V} \setminus \{\boldsymbol{0}\}$, we have

$$\begin{aligned}\langle S\boldsymbol{w}, S\boldsymbol{z}\rangle &= \left\langle 2\|\boldsymbol{w}\| T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right), 2\|\boldsymbol{z}\| T\left(\frac{\boldsymbol{z}}{2\|\boldsymbol{z}\|}\right)\right\rangle \\ &= 2\|\boldsymbol{w}\|2\|\boldsymbol{z}\| \left\langle T\left(\frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}\right), T\left(\frac{\boldsymbol{z}}{2\|\boldsymbol{z}\|}\right)\right\rangle \\ &= 2\|\boldsymbol{w}\|2\|\boldsymbol{z}\| \left\langle \frac{\boldsymbol{w}}{2\|\boldsymbol{w}\|}, \frac{\boldsymbol{z}}{2\|\boldsymbol{z}\|}\right\rangle \\ &= \langle \boldsymbol{w}, \boldsymbol{z}\rangle.\end{aligned}$$

Finally, we show that S is a linear operator if T is surjective. Assume that T is surjective. We can prove that $S(s\boldsymbol{v}) = sS(\boldsymbol{v})$ for any $s > 0$ and $\boldsymbol{v} \in \mathbb{V}$ in way similar as a $T(r\boldsymbol{u}) = rT(\boldsymbol{u})$ for any $r > 0$ and $\boldsymbol{u} \in \mathbb{V}$ such that $r\boldsymbol{u} \in \boldsymbol{B}$. Therefore, we have

$$\begin{aligned}S\left(2\|\boldsymbol{y}\| T^{-1}\left(\frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|}\right)\right) &= 2\|\boldsymbol{y}\| S\left(T^{-1}\left(\frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|}\right)\right) \\ &= 2\|\boldsymbol{y}\| T\left(T^{-1}\left(\frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|}\right)\right) \\ &= 2\|\boldsymbol{y}\| \frac{\boldsymbol{y}}{2\|\boldsymbol{y}\|} = \boldsymbol{y}\end{aligned}$$

for any $\boldsymbol{y} \in \mathbb{V} \setminus \boldsymbol{0}$. Clearly, $S(\boldsymbol{0}) = \boldsymbol{0}$. Thus S is a surjective isometry from a normed space onto itself. The Mazur-Ulam Theorem asserts that S is a linear operator. □

The following lemma is appeared in [1, Excercise 2.4.1]. A proof is elementary and is omitted.

Lemma 4.5. *Let (X, d) be a compact metric space. Suppose that $T : (X, d) \to (X, d)$ is an isometry. Then T is surjective.*

5. Proof of Main Results

Proof of Theorem 3.1. (E2)⇒(E3): Suppose that $T = O|_{\boldsymbol{B}}$ for a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. First, T is an inner product preserving map because O is so. Next,

$$\begin{aligned}
r \otimes_E T(\boldsymbol{u}) &= \tanh(r \tanh^{-1} \|T\boldsymbol{u}\|) \frac{T\boldsymbol{u}}{\|T\boldsymbol{u}\|} \\
&= \tanh(r \tanh^{-1} \|O\boldsymbol{u}\|) \frac{O\boldsymbol{u}}{\|O\boldsymbol{u}\|} \\
&= O\left(\tanh(r \tanh^{-1} \|\boldsymbol{u}\|) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right) \\
&= T\left(\tanh(r \tanh^{-1} \|\boldsymbol{u}\|) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} \right) = T(r \otimes_E \boldsymbol{u})
\end{aligned}$$

for every $r \in \mathbb{R}$, $\boldsymbol{u} \in \boldsymbol{B}$. Moreover,

$$\begin{aligned}
T(\boldsymbol{u}) \oplus_E T(\boldsymbol{v}) &= \frac{1}{1 + \langle T\boldsymbol{u}, T\boldsymbol{v}\rangle} \left\{ \left(1 + \frac{\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle}{1 + \alpha_{Tu}} \right) T\boldsymbol{u} + \alpha_{Tu} T\boldsymbol{v} \right\} \\
&= \frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v}\rangle} \left\{ \left(1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{1 + \alpha_u} \right) O\boldsymbol{u} + \alpha_u O\boldsymbol{v} \right\} \\
&= O\left(\frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v}\rangle} \left\{ \left(1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{1 + \alpha_u} \right) \boldsymbol{u} + \alpha_u \boldsymbol{v} \right\} \right) \\
&= T(\boldsymbol{u} \oplus_E \boldsymbol{v})
\end{aligned}$$

for any pair $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$. Finally, $O(\boldsymbol{B}) = \boldsymbol{B}$ since O is a surjective isometry. It implies that T is surjctive.
(E3)⇒(E1): Suppose that $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_E, \otimes_E)$. By Theorem2.14 T is a gyrometric preserving map. By the definition of the automorphism of a gyrovector space, T is surjective. Moreover, $T(\mathbf{0}) = \mathbf{0}$ because T is an inner product preserving map.
(E1)⇒(E2): Suppose that T satisfies the condition (E1). Then Lemma 4.1 asserts that T is a surjective inner product preserving map. Moreover, Lemma 4.4 asserts that T is extended to be a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$.

Finally, we show that if $\dim \mathbb{V} < \infty$, then the condition (a) and (c) imply the condition (b). The condition (a) and (c) imply that $T(\pi_a) \subset \pi_a$ for any $0 \leq a < 1$ since $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. If $\dim \mathbb{V} < \infty$, then π_a is compact for all $0 \leq a < 1$. Lemma 4.5 asserts that $T(\pi_a) = \pi_a$ for all $0 \leq a < 1$ and hence $T(\boldsymbol{B}) = \boldsymbol{B}$. □

Proof of Theorem 3.2. (M2)⇒(M3): Suppose that $T = O|_{\boldsymbol{B}}$ for a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. Then T is an inner product preserving operator. Moreover,

$$\begin{aligned}
r \otimes_M T(\boldsymbol{u}) &= \tanh(r \tanh^{-1} \|T\boldsymbol{u}\|) \frac{T\boldsymbol{u}}{\|T\boldsymbol{u}\|} \\
&= \tanh(r \tanh^{-1} \|O\boldsymbol{u}\|) \frac{O\boldsymbol{u}}{\|O\boldsymbol{u}\|} \\
&= O\left(\tanh(r \tanh^{-1} \|\boldsymbol{u}\|) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right) \\
&= T\left(\tanh(r \tanh^{-1} \|\boldsymbol{u}\|) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right) = T(r \otimes_M \boldsymbol{u})
\end{aligned}$$

for every $r \in \mathbb{R}$, $\boldsymbol{u} \in \boldsymbol{B}$ and

$$\begin{aligned}
T(\boldsymbol{u}) \oplus_M T(\boldsymbol{v}) &= \frac{(1 + 2\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle + \|T\boldsymbol{v}\|^2)T\boldsymbol{u} + (1 - \|T\boldsymbol{u}\|^2)T\boldsymbol{v}}{1 + 2\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle + \|T\boldsymbol{u}\|^2\|T\boldsymbol{v}\|^2} \\
&= \frac{(1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2)O\boldsymbol{u} + (1 - \|\boldsymbol{u}\|^2)O\boldsymbol{v}}{1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2} \\
&= O\left(\frac{(1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{v}\|^2)\boldsymbol{u} + (1 - \|\boldsymbol{u}\|^2)\boldsymbol{v}}{1 + 2\langle \boldsymbol{u}, \boldsymbol{v}\rangle + \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2}\right) \\
&= T(\boldsymbol{u} \oplus_M \boldsymbol{v})
\end{aligned}$$

for any pair $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$. Finally, $O(\boldsymbol{B}) = \boldsymbol{B}$ since O is a surjective isometry.
(M3)⇒(M1): Suppose that $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_M, \otimes_M)$. By Theorem 2.14, T is a gyrometric preserving map. By the definition of the automorphism of a gyrovector space, T is a surjection. Moreover, $T(\boldsymbol{0}) = \boldsymbol{0}$ because T is an inner product preserving map.
(M1)⇒(M2): Suppose that T satisfies the condition (M1). Then Lemma 4.2 asserts that T is a surjective inner product preserving map. Moreover, Lemma 4.4 asserts that T is extended to be a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$.

Finally, we show that if $\dim \mathbb{V} < \infty$, then the condition (a) and (c) imply the condition (b). The condition (a) and (c) imply that $T(\pi_a) \subset \pi_a$ for any $0 \le a < 1$ since $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \boldsymbol{B}$. If $\dim \mathbb{V} < \infty$, then π_a is compact for all $0 \le a < 1$. Lemma 4.5 asserts that $T(\pi_a) = \pi_a$ for all $0 \le a < 1$, hence $T(\boldsymbol{B}) = \boldsymbol{B}$. □

Proof of Theorem 3.3. (P2)⇒(P3): Suppose that T is a surjective inner product preserving linear operator $T : \mathbb{V} \to \mathbb{V}$. Needless to say, T is surjective and an inner product preserving map. Moreover,

$$\begin{aligned} r \otimes_P T(\boldsymbol{u}) &= \sinh\left(r \sinh^{-1}\|T\boldsymbol{u}\|\right) \frac{T\boldsymbol{u}}{\|T\boldsymbol{u}\|} \\ &= \sinh\left(r \sinh^{-1}\|\boldsymbol{u}\|\right) \frac{T\boldsymbol{u}}{\|\boldsymbol{u}\|} \\ &= T\left(\sinh\left(r \sinh^{-1}\|\boldsymbol{u}\|\right) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}\right) \\ &= T(r \otimes_P \boldsymbol{u}) \end{aligned}$$

for every $r \in \mathbb{R}$, $\boldsymbol{u} \in \boldsymbol{B}$ and

$$\begin{aligned} T(\boldsymbol{u}) \oplus_P T(\boldsymbol{v}) &= \left(\frac{1}{1+\delta_{Tu}}\langle T\boldsymbol{u}, T\boldsymbol{v}\rangle\right) T\boldsymbol{u} + T\boldsymbol{v} \\ &= \left(\frac{1}{1+\delta_u}\langle \boldsymbol{u}, \boldsymbol{v}\rangle\right) T\boldsymbol{u} + T\boldsymbol{v} \\ &= T\left(\left(\frac{1}{1+\delta_u}\langle \boldsymbol{u}, \boldsymbol{v}\rangle\right) \boldsymbol{u} + \boldsymbol{v}\right) \\ &= T(\boldsymbol{u} \oplus_P \boldsymbol{v}) \end{aligned}$$

for any pair $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$.
(P3)⇒(P1): Suppose that $T \in \mathrm{Aut}(\boldsymbol{B}, \oplus_P, \otimes_P)$. By Theorem 2.14, T is a gyrometric preserving map. By the definition of the automorphism of a gyrovector space, T is a surjection. Moreover, $T(\mathbf{0}) = \mathbf{0}$ because T is an inner product preserving map.
(P1)⇒(P2): Suppose that T satisfies the condition (P1). Since T satisfies the conditions (a) and (c), Lemma 4.3 asserts that T is an inner product preserving map. Moreover, T is surjective and hence the Mazur-Ulam theorem asserts that T is a linear operator. Indeed, $T : \mathbb{V} \to \mathbb{V}$ is a surjective inner product preserving linear operator.

Finally, we show that if $\dim \mathbb{V} < \infty$, then the condition (a) and (c) imply the condition (b). The condition (a) and (c) imply that $T(\pi_a) \subset \pi_a$ for any $0 \leq a$ since $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$ for all $\boldsymbol{u} \in \mathbb{V}$. If $\dim \mathbb{V} < \infty$, then π_a is compact for all $0 \leq a$. Lemma 4.5 asserts that $T(\pi_a) = \pi_a$ for all $0 \leq a$ and hence $T(\mathbb{V}) = \mathbb{V}$. □

6. Gyrometric preserving maps on the Einstein gyrogroup, the Möbius gyorogroup and the PV gyrogroup

We have exhibited in Theorems 3.1, 3.2 and 3.3 the form of gyrometric preserving maps with the assumption $T(\mathbf{0}) = \mathbf{0}$. Generally, a gyrometric preserving map does not necessarily preserve the identity $\mathbf{0}$.

We note that any gyrogroup $(G, \oplus)$ satisfies $\boldsymbol{a} \oplus (\ominus \boldsymbol{a} \oplus \boldsymbol{b}) = \boldsymbol{b}$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$ (the left cancellation law) (See [4, p30])). Let $T : (G, \oplus, \otimes) \to (G, \oplus, \otimes)$ be a map from a gyrovector space into itself. Put $T_0(\cdot) = \ominus T(\mathbf{0}) \oplus T(\cdot)$. Then Theorem 2.14 shows that $\varrho(T_0(\boldsymbol{a}), T_0(\boldsymbol{b})) = \varrho(T(\boldsymbol{a}), T(\boldsymbol{b}))$ for any pair $\boldsymbol{a}, \boldsymbol{b} \in G$. Hence T_0 is a gyrometric preserving map if and only if T is so. Clearly, $T_0(\mathbf{0}) = \mathbf{0}$. Moreover, T_0 is surjective if and only if T is so. Applying Theorem 3.1, 3.2, 3.3 to T_0 we obtain Corollary 6.1, 6.2, 6.3. These corollaries gives us complete descriptions of all surjective gyrometric preserving self-maps of our three models without the assumption $T(\mathbf{0}) = \mathbf{0}$.

Corollary 6.1. *Let $\boldsymbol{B}$ be the open unit ball of a real inner product space $\mathbb{V}$. Suppose that $T : (\boldsymbol{B}, \oplus_E) \to (\boldsymbol{B}, \oplus_E)$ is a map from the Einstein gyrogroup into itself. Then the following conditions are equivalent.*

(E-A) *T is a surjective Einstein gyrometric preserving map.*

(E-B) *There exists a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.*

(E-C) *T is a surjective isometry with respect to the metric d_E.*

Proof. (E-A)⇔(E-C): It is easy to be verified since $d_E(\boldsymbol{u}, \boldsymbol{v}) = \tanh^{-1} \varrho_E(\boldsymbol{u}, \boldsymbol{v})$ for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$.

(E-A)⇔(E-B): Let $T : (\boldsymbol{B}, \oplus_E) \to (\boldsymbol{B}, \oplus_E)$. Put $T_0(\cdot) = -T(\mathbf{0}) \oplus_E T(\cdot)$. First, we assume that T satisfies the condition (E-A). Then we have $T_0(\mathbf{0}) = \mathbf{0}$ and T_0 is a surjective Einstein gyrometric preserving map. Theorem 3.1 shows that T_0 is the restriction of some surjective inner product preserving operator $O : \mathbb{V} \to \mathbb{V}$. It follows that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$ because $T(\mathbf{0}) \oplus_E T_0(\cdot) = T(\mathbf{0}) \oplus_E (\ominus_E T(\mathbf{0}) \oplus_E T(\cdot)) = T(\cdot)$.

Conversely, let $O : \mathbb{V} \to \mathbb{V}$ be a surjective inner product preserving linear operator and $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Then we have $T_0(\boldsymbol{u}) = O(\boldsymbol{u})$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Theorem 3.1 asserts that T_0 is a surjective Einstein gyrometric preserving map, hence T is so. □

Corollary 6.2. *Let $\boldsymbol{B}$ be the open unit ball of a real inner product space $\mathbb{V}$. Suppose that $T : (\boldsymbol{B}, \oplus_M) \to (\boldsymbol{B}, \oplus_M)$ is a map from the Möbius gyrogroup into itself. Then the following conditions are equivalent.*

(M-A) *T is a surjective Möbius gyrometric preserving map.*

(M-B) *There exists a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_M O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.*

(M-C) *T is a surjective isometry with respect to the Möbius metric d_M.*

Proof. (M-A)⇔(M-C): It is obvious since $d_M(\boldsymbol{u}, \boldsymbol{v}) = \tanh^{-1} \varrho_M(\boldsymbol{u}, \boldsymbol{v})$ for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{B}$.

(M-A)⇔(M-B): Let $T : (\boldsymbol{B}, \oplus_M) \to (\boldsymbol{B}, \oplus_M)$. Put $T_0(\cdot) = -T(\mathbf{0}) \oplus_M T(\cdot)$. First, we assume that T satisfies the condition (M-A). Then we have $T_0(\mathbf{0}) = \mathbf{0}$

and T_0 is a surjective Möbius gyrometric preserving map. Theorem 3.2 asserts that T_0 is the restriction of some surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$. It follows that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.

Conversely, let $O : \mathbb{V} \to \mathbb{V}$ be a surjective inner product preserving linear operator and $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_M O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Then we have $T_0(\boldsymbol{u}) = O(\boldsymbol{u})$ for any $\boldsymbol{u} \in \boldsymbol{B}$. Theorem 3.2 show that T_0 is a surjective Möbius gyrometric preserving map, hence T is so. □

Corollary 6.3. *Let $\mathbb{V}$ be a real inner product space. Suppose that $T : (\mathbb{V}, \oplus_P) \to (\mathbb{V}, \oplus_P)$ is a map from the PV gyrogroup into itself. Then the following conditions are equivalent.*

- (P-A) *T is a surjection such that $\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_P(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$.*
- (P-B) *There exists a surjective inner product preserving linear operator $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_P O\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}$.*

Proof. Let $T : (\boldsymbol{B}, \oplus_P) \to (\boldsymbol{B}, \oplus_P)$. Put $T_0(\cdot) = -T(\mathbf{0}) \oplus_P T(\cdot)$.

First, we assume that T satisfies the condition (P-A). Then we have $T_0(\mathbf{0}) = \mathbf{0}$ and T_0 is a surjective PV gyrometric preserving map. Theorem 3.3 shows that T_0 is a surjective inner product preserving linear operator on $\mathbb{V}$. Since $T(\cdot) = T(\mathbf{0}) \oplus_P T_0(\cdot)$, T satisfies the condition (P-B).

Conversely, let $O : \mathbb{V} \to \mathbb{V}$ be a surjective inner product preserving linear operator and $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_P O\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}$. Then we have $T_0(\boldsymbol{u}) = O(\boldsymbol{u})$ for any $\boldsymbol{u} \in \mathbb{V}$. Theorem 3.3 asserts that T_0 is surjective and preserves the gyrometric on the PV gyrogroup, hence T is so. □

In particular, if $\mathbb{V} = \mathbb{R}^n$ then gyrometric preserving maps on our three models are automatically surjective. We have the following three corollaries.

Corollary 6.4. *Suppose that $\boldsymbol{B}$ be the open unit ball of the Euclidean space $\mathbb{R}^n$ and $T : (\boldsymbol{B}, \oplus_E) \to (\boldsymbol{B}, \oplus_E)$ a map from the Einstein gyrogroup into itself. Then the following conditions are equivalent.*

- (E-A') *T is an Einstein gyrometric preserving map.*
- (E-B') *There exists a $n \times n$ orthogonal matrix $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_E O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.*
- (E-C') *T is an isometry with respect to the metric d_E.*

Corollary 6.5. *Suppose that $\boldsymbol{B}$ be the open unit ball of the Euclidean space $\mathbb{R}^n$ and $T : (\boldsymbol{B}, \oplus_M) \to (\boldsymbol{B}, \oplus_M)$ a map from the Möbius gyrogroup into itself. Then the following conditions are equivalent.*

- (M-A') *T is a Möbius gyrometric preserving map.*
- (M-B') *There exists a $n \times n$ orthogonal matrix $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\mathbf{0}) \oplus_M O\boldsymbol{u}$ for any $\boldsymbol{u} \in \boldsymbol{B}$.*

(M-C') *T is an isometry with respect to the Möbius metric d_M.*

Corollary 6.6. *Suppose that $T : (\mathbb{R}^n, \oplus_P) \to (\mathbb{R}^n, \oplus_P)$ is a map from the PV gyrogroup into itself. The following conditions are equivalent.*

(P-A') *$\varrho_P(T\boldsymbol{u}, T\boldsymbol{v}) = \varrho_P(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}$.*

(P-B') *There exists a $n \times n$ orthogonal matrix $O : \mathbb{V} \to \mathbb{V}$ such that $T(\boldsymbol{u}) = T(\boldsymbol{0}) \oplus_P O\boldsymbol{u}$ for any $\boldsymbol{u} \in \mathbb{V}$.*

Remark 6.7. If $\dim \mathbb{V} = \infty$, then some Einstein gyrometric preserving map is not necessarily surjective. Let $\mathbb{V} = \ell^2(\mathbb{N})$ and S be the shift operator on ℓ^2, that is $S(x_1, x_2, x_3, ...) = (0, x_1, x_2, ...)$ for any $(x_1, x_2, x_3, ...) \in \ell^2$. Then S is an inner product preserving linear operator and hence S is an Einstein gyrometric preserving map. Nevertheless, $S(\boldsymbol{B})$ is a proper subset of $\boldsymbol{B}$. Moreover S preserves the Möbius gyrometric and the gyrometric on the PV gyrogroup.

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