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$H^2(\mu)$ SPACES AND BOUNDED POINT EVALUATIONS

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Let $H^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$, where μ is a positive finite compactly supported Borel measure carried by the closed unit disc $\bar{D}$. For $\lambda \in \bar{D}$, define $E(\lambda)=\sup\{|p(\lambda)|/\|p\|_\mu\}$, where the suprenum is taken over all polynomials whose $L^2(\mu)$ norm is not zero. If $E(\lambda)<\infty$ we say that μ has a bounded point evaluation at λ, abbreviated b.p.e. at λ. Whenever $E(\lambda)<\infty$ we may fix the value of $f \in H^2(\mu)$ at λ. We determine the set on which all functions in $H^2(\mu)$ have (fixed) analytic values in terms of the parts of the spectrum of a certain operator.

In the case that the support of μ has a hole H bounded by an exposed arc Γ contained in ∂D and $E(z)$ is finite in H, we show how to recover the absolutely continuous part (with respect to Lebesgue measure on ∂D) of $d\mu|_\Gamma$ from a knowledge of the $E(z)$'s in H. A corollary of this is that for such measures μ the functions in $H^2(\mu)$ behave locally near Γ like those of classical Hardy space. That is, they have boundary values and their zero sets near Γ satisfy a Blaschke type growth condition. We apply this corollary to measures of the form $d\nu = GdA + wd\sigma$ to study the local behavior of functions in $H^2(\nu)$ near Γ (A denotes planar measure on $\bar{D}$, $d\sigma$ denotes linear Lebesgue measure on ∂D, and G and w are in an appropriate sense not too small on D and Γ respectively).

1. Bounded evaluations and analytic extensions of functions in $H^2(\mu)$.

Let μ be a finite positive compactly supported Borel measure carried by the closed unit disc $\bar{D}$. We note that for λ a complex number, the point evaluation functional defined on polynomials by

$$p \longrightarrow p(\lambda)$$

is bounded with respect to the $L^2(\mu)$ norm if and only if $E(\lambda)<\infty$. In this latter case, by the Riesz representation theorem there is a unique element of $H^2(\mu)$, denoted by k_λ, satisfying

$$p(\lambda) = \langle p, k_\lambda \rangle$$

for all polynomials p and $\|k_\lambda\| = E(\lambda)$. We call k_λ the bounded evaluation functional for μ at λ, abbreviated b.e.f. for μ at λ.

If μ has a b.p.e. at λ with b.e.f. k_λ and $f \in H^2(\mu)$, then we fix the value of f at λ by

$$(1)\qquad \tilde{f}(\lambda) = \langle f, k_\lambda \rangle .$$

We remark that if μ has b.p.e's on a set of positive μ measure then the values $\tilde{f}$ of f fixed by (1) agree μ-a.e. with any representative of f. Also the "filling in holes" theorem due to Bram [1], interpreted in this context, says that if H is a hole of the support of μ then either

(2) μ has b.p.e.'s at every $\lambda \in H$

or else

(3) μ has no b.p.e.'s in H .

Whenever (2) occurs the functions in $H^2(\mu)$ can be extended into the hole H.

It is well known that if $f \in H^2(\mu)$ then $\tilde{f}$ is analytic in any holes satisfying (2). We specify the largest open set on which all extensions of functions in $H^2(\mu)$ are analytic.

Let M_μ denote the bounded linear operator multiplication by z on $H^2(\mu)$. $\Lambda(M_\mu)$, $\Gamma(M_\mu)$, and $\Pi(M_\mu)$ will designate the spectrum, the compression spectrum, and the approximate point spectrum of M_μ, respectively [see 12]. If O is an open set on which all extensions of functions in $H^2(\mu)$ are analytic, then we call O an *analytic set for* μ. If $G \subset C$ then we denote the interior of G by int G.

THEOREM 1.1. *The largest analytic set for* μ *is* $\text{int}(\Gamma(M_\mu) - \Pi(M_\mu))$.

Proof. If O is any analytic set for μ and $F \subset O$ is compact, then using the Banach Steinhaus theorem [16] we see that

$$\sup\{\|k_\lambda\| \colon \lambda \in F\} < \infty .$$

Also if O is an open set and $\lambda \to \|k_\lambda\|$ is bounded on compact subsets of O, then using (1) and the Cauchy-Schwartz inequality it follows that O is an analytic set for μ.

Assume that O is an analytic set for μ. It is well known that $O \subset \Gamma(M_\mu)$. (This is just the statement that $M_\mu^* k_\lambda = \bar{\lambda} k_\lambda$ for $\lambda \in O$.) We show that

$$(4)\qquad O \cap \Pi(M_\mu) = \varnothing .$$

If (4) fails then there exists a λ in O and a sequence of polynomials p_n satisfying

$$(5)\qquad \|(z-\lambda)p_n(z)\|^2 < \frac{1}{n}$$

and

$$\|p_n\|^2 \geqq \frac{1}{2} . \tag{6}$$

Let B be the closed disc of radius r centered at λ and contained in O. Since O is an analytic set for μ,

$$\sup\{\|k_z\|: z \in B\} = C < \infty .$$

For w with $|w - \lambda| = r$,

$$\frac{1}{n} > \|(z - \lambda)p_n(z)\|^2 \geqq \frac{|w - \lambda|^2 |p_n(w)|^2}{\|k_w\|^2} \geqq \frac{r^2}{C^2} |p_n(w)|^2 .$$

So by the maximum modulus principle,

$$|p_n(u)|^2 \leqq \frac{C^2}{nr^2} \tag{7}$$

for all $u \in B$. But using (5) and (7),

$$\|p_n\|^2 = \int_{\bar{D}-B} |p_n|^2 d\mu + \int_B |p_n|^2 d\mu$$
$$\leqq \frac{1}{r^2 n} + \frac{C^2}{nr^2} \mu(B) .$$

Letting $n \to \infty$, we see that (6) is contradicted so (4) holds.

Conversely, assume that O is an open set satisfying $O \cap \Pi(M_\mu) = \emptyset$ and $O \subset \Gamma(M_\mu)$. By our opening remark in the proof, it will be sufficient to show that $\lambda \to \|k_\lambda\|$ is bounded in a neighborhood of λ. Fix $a \in O$. Since $a \notin \Pi(M_\mu)$ there is a $C < \infty$ so that

$$\|f\| \leqq C \|(z - a)f(z)\|$$

for all $f \in H^2(\mu)$. A computation shows that

$$\|f\| \leqq 2C \|(z - w)f(z)\| \tag{8}$$

whenever $|w - a| \leqq 1/2C$.

Let $q(z) = (p(z) - p(\lambda))/(z - \lambda)$ for p a polynomial and let $C_1 = \min\{1/2C, 1/(4C \|k_a\|)\}$. By (8), for $|\lambda - a| < C_1 \leqq 1/2C$,

$$|q(a)| \leqq \|k_a\| \|q\| \leqq \|k_a\| 2C \|(z - \lambda)q(z)\|$$
$$\leqq 2C \|k_a\|[\|p\| + |p(\lambda)|] .$$

Hence

$$|p(\lambda)| \leqq |p(a)| + |\lambda - a| 2C \|k_a\|[\|p\| + |p(\lambda)|] .$$

So for $|\lambda - a| < C_1$,

$$|p(\lambda)| \leqq ||k_a|| \, ||p|| + \frac{1}{2}[||p|| + |p(\lambda)|] .$$

Thus

$$||k_\lambda|| \leqq 2\,||k_a|| + 1$$

so we are done.

COROLLARY 1.1. *If H is a hole of the support of μ and $H \subset \Lambda(M_\mu)$ then H is an analytic set for μ.*

Proof. $\Lambda(M_\mu) = \Pi(M_\mu) \cup \Gamma(M_\mu)$. If $\lambda \in H$ then $1/(z-\lambda) \in L^\infty(\mu)$ and hence $\lambda \notin \Pi(M_\mu)$.

Denote the essential spectrum of M_μ by $\Lambda_e(M_\mu)$ [9].

COROLLARY 1.2. *If M_μ has no point spectrum, then the maximal analytic set for μ is $\Lambda(M_\mu) - \Lambda_e(M_\mu)$.*

Proof. If M_μ has no point spectrum then [9] says that $\mathrm{int}(\Gamma(M_\mu) - \Pi(M_\mu)) = \Lambda(M_\mu) - \Lambda_e(M_\mu)$. Now apply Theorem 1.1.

Let M'_μ denote the pure subnormal part of M_μ [7].

COROLLARY 1.3. *The maximal analytic set for μ is $\Lambda(M'_\mu) - \Lambda_e(M'_\mu)$.*

Proof. It is easy to see that the maximal analytic sets of M_μ and M'_μ are equal. If M'_μ is a pure subnormal operator, then M'_μ has empty point spectrum so Corollary 1.2 applies.

If $\mathscr{B}$ denotes the set of b.p.e.'s for μ, the obvious question is whether int $\mathscr{B}$ is the largest analytic set for μ. While we cannot answer this, we have the following partial result.

THEOREM 1.2. *There exists a dense open subset $\mathscr{S}$ of $\mathscr{B}$ so that $\mathscr{S}$ is an analytic set for μ.*

Proof. We show that if $\mathscr{S} = \{z \in \mathscr{B}$: there is some neighborhood U of z with $\bar{U} \subset \mathscr{B}$ and $\sup\{||k_\lambda||: \lambda \in U\} < \infty\}\}$ then $\mathscr{S}$ is a dense subset of $\mathscr{B}$. Let V be any open subset of $\mathscr{B}$ with $\bar{V} \subset \mathscr{B}$. We are done if we show that $\bar{V} \cap \mathscr{S} \neq \varnothing$. Define

$$E_N = \{z \in \bar{V}: ||k_z|| \leqq N\} .$$

Clearly,

$$\bigcup_{N=1}^{\infty} E_N = \bar{V} .$$

Now

$$||k_z|| = E(z) = \sup\{|p(z)|/||p||\}$$

where the suprenum is taken over polynomials p with rational complex coefficients and $||p|| \neq 0$. Thus $z \to ||k_z||$ is a lower semi-continuous function on $\mathscr{B}$, so E_N is a closed set. An application of the Baire category theorem completes the proof.

It may be useful to note that by Corollary 1.3 $\mathscr{S} = \{z \in D$: $z - M_\mu$ is a Fredholm operator and $\operatorname{ind}(z - M_\mu) = -1\}$.

2. Recovering a part of the measure μ from $E(z)$. It is a well known result of Bram [1] that the operator M_μ, multiplication by z on $H^2(\mu)$, is a model for a general contractive cyclic subnormal operator. Some subnormal operators have been shown to have (nontrivial, closed) invariant subspaces by establishing that if $H^2(\mu) \neq L^2(\mu)$ then μ has a bounded point evaluation [2], [3], [4]. This provides a basic motivation for the study of the relationship of the measure μ to the possible existence of b.p.e.'s.

Let $d\sigma$ denote normalized Lebesgue measure on ∂D. For a measure ν carried by ∂D, it is a classical result of Szegö and Kolomogorov [see 13] that $H^2(\nu) \neq L^2(\nu)$ if and only if $\log h \in L^1(d\sigma)$, where h denotes the absolutely continuous part of ν with respect to σ. Whenever $H^2(\nu) \neq L^2(\nu)$, then ν has b.p.e.'s in D with b.e.f.'s k_λ for $\lambda \in D$. It was observed in [14] that h can be recovered from $||k_\lambda||$ as follows:

$$\lim_{\lambda \to e^{i\theta}} (1 - |\lambda|^2)||k_\lambda||^2 = \frac{1}{h(e^{i\theta})} \quad \text{for} \quad \sigma - \text{a.e. } e^{i\theta} \tag{9}$$

where $\lambda \to e^{i\theta}$ nontangentially. Suppose that μ is a measure carried by $\bar{D}$. Let

$$d\mu = d\mu|_D + \left(\frac{d\mu}{d\sigma}\right)d\sigma + d\mu_s$$

where $d\mu_s$ is carried by ∂D and is singular to $d\sigma$. Just as in the classical case (ν as above) a result of Clary [6] says that μ has a b.p.e. at $\lambda \in D$ if and only if $d\mu - d\mu_s$ does. Since $d\mu_s$ is not involved in the existence of b.p.e.'s, it is clear that there is no hope of recovering $d\mu_s$ from a knowledge of the norms of b.e.f.'s for

$d\mu$ (in fact, $E^{\mu}(\lambda) \equiv E^{\mu-\mu_s}(\lambda)$ for all λ).

We are interested in the interplay between $\mu|_D$ and $\mu|_{\partial D}$ and the existence of b.p.e.'s in D. By the previous discussion μ_s has no bearing on this problem. We investigate a class of measures μ for which the absolutely continuous part of μ with respect to σ can be recovered on an arc of ∂D in an analogous fashion to (9).

DEFINITION. Let K be a compact set. Then K contains *an exposed arc* J if there exists a simply connected open set $\mathscr{D}$ such that $\mathscr{D} \cap K = J$ and J is the range of a smooth Jordan curve.

Let μ be a measure carried by $\bar{D}$ satisfying:

(A) there is a hole H of the support of μ so that $\bar{H}$ has an exposed arc Γ with $\Gamma \subset \partial D$.

(B) μ has b.p.e.'s in the hole H.

We remark that by a result of Brown, Shields, and Zellar [5], it is possible to construct a measure μ carried by D whose support has a hole H for which (B) holds, $\mu(\partial D) = 0$, and $\partial H \supset \partial D$. For such a measure, it is clear that $\mu|_{\partial D}$ is not involved in the existence of b.p.e.'s in H. Thus condition (A) is a guarantee that if (B) is to hold, then $\mu|_\Gamma$ and $\mu|_D$ must interrelate in some way. Hence if μ satisfies (A) and (B), it is plausible that a knowledge of the norms of b.e.f.'s in H would lead to a recovery of the absolutely continuous part of μ with respect to σ restricted to Γ. This is indeed the case. Before proving this result, we will need a few lemmas.

Suppose that α is any measure whose support contains a hole H. Assume, furthermore, that α has b.p.e.'s in H. For $\lambda \in H$, k_λ is the b.e.f. of α at λ. Denote the orthogonal projections of $L^2(\alpha)$ onto $H^2(\alpha)$ and $H^2(\alpha)^\perp$ by P_1 and P_2, respectively. We have the following lemma.

LEMMA 2.1. (i) *Let* $a \in H$. *If* $g \in H^2(\alpha)^\perp$ *and* $\langle 1/(z-a), g\rangle \neq 0$ *then*

$$k_a = P_1\Big(\frac{g(z)}{\bar{z}-\bar{a}} + f\Big)\Big/\Big\langle g, \frac{1}{z-a}\Big\rangle \tag{10}$$

where f *is any element of* $H^2(\alpha)^\perp$.

(ii) *If* $g = P_2(1/(z-a))$ *then* $\langle 1/(z-\lambda), g(z)\rangle = 0$ *for at most a countable number of* λ*'s in* H.

Proof. Let $\hat{g}(a)$ denote $\langle 1/(z-a), g(z)\rangle$. If p is a polynomial then $(p(z)-p(a))/(z-a)$ is a polynomial so

$$0 = \Big\langle \frac{p(z)-p(a)}{z-a}, g\Big\rangle = \Big\langle p, \frac{g(z)}{\bar{z}-\bar{a}}\Big\rangle - p(a)\hat{g}(a) .$$

Hence

$$p(a) = \left\langle p, \left(\frac{g}{\bar{z} - \bar{a}}\right) \Big/ \overline{\hat{g}(a)} \right\rangle$$

for all polynomials p. Now $1/(z - a)$ is in $L^\infty(\alpha)$ since $a \in H$, so $g/(\bar{z} - \bar{a}) \in L^2(\alpha)$. Thus (10) follows by the uniqueness of the b.e.f. at a.

Let $g = P_2(1/(z-a))$. Since α has a b.p.e. at a, $1/(z-a) \notin H^2(\alpha)$. (Else we would have $1 = \langle 1, k_a \rangle = \langle (z-a)(1/(z-a)), k_a \rangle = (a-a) \langle 1/(z-a), k_a \rangle = 0$.) Thus

$$\left\langle \frac{1}{z-a}, g \right\rangle = \left\| P_2\left(\frac{1}{z-a}\right) \right\|^2 > 0 .$$

Now we need only notice that $\lambda \to \langle 1/(z-\lambda), g(z) \rangle$ is analytic and not identically zero in H to complete the proof of (ii).

Suppose that μ is a measure supported on $\bar{D}$ satisfying (A) and (B) for a hole H of the support of μ with exposed arc Γ. Let $a \in H$ and denote $P_2(1/(z-a))$ by g and $\langle 1/(z-a), g \rangle$ by $\hat{g}(a)$.

LEMMA 2.2. *g vanishes on no subset of Γ with positive Lebesgue measure.*

Proof. Define

$$d\beta = \frac{\overline{g(z)}}{(z-a)\hat{g}(a)} \, d\mu .$$

Then $d\beta$ is a complex representing measure for evaluation at a on the space of the polynomials with respect to sup norm on the support of μ [see 10]. It follows from Theorem 2.2 of [10] that there exists a positive representing measure $d\nu$ for evaluation at a which is absolutely continuous with respect to $|d\beta|$. It is easy to see that ν has a b.p.e. at a. Applying Lemma 2 of [17] shows that

$$\int_{\Gamma_1} \log \frac{d\nu}{d\sigma} d\sigma > -\infty$$

for every closed subarc Γ_1 of Γ. This completes the proof.

We are now ready for the main result of this section. Assume that μ is a measure supported on $\bar{D}$ satisfying (A) and (B) for a hole H of the support of μ with exposed arc Γ. Let w denote the Radon-Nikodym derivative of the absolutely continuous part of $\mu|_{\partial D}$ with respect to σ. Fix a point $a \in H$ and again denote

$P_2(1/(z-a))$ by g and $\langle 1/(z-a), g\rangle$ by $\hat{g}(a)$.

THEOREM 2.1.

$$\text{(11)} \qquad \lim_{\lambda\to e^{i\theta}} (1-|\lambda|^2)\|k_\lambda\|^2 = \frac{1}{w(e^{i\theta})} \textit{ for } \sigma\text{-a.e. } e^{i\theta}\in\Gamma$$

as $\lambda\to e^{i\theta}$ *nontangentially.*

Proof. By a theorem of [14] it is shown that for any measure β on $\bar{D}$,

$$\text{(12)} \qquad \varliminf_{\lambda\to e^{i\theta}} (1-|\lambda|^2)(E^\beta(\lambda))^2 \geqq 1\Big/\frac{d\beta}{d\sigma}(e^{i\theta}) \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\partial D$$

where $\lambda\to e^{i\theta}$ nontangentially. Thus we need only show that

$$\text{(13)} \qquad \varlimsup_{\lambda\to e^{i\theta}} (1-|\lambda|^2)\|k_\lambda\|^2 \leqq \frac{1}{w(e^{i\theta})} \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\Gamma$$

where $\lambda\to e^{i\theta}$ nontangentially. From Lemma 2.1 we see that

$$\text{(14)} \qquad \|k_\lambda\|^2 \leqq \left\|\frac{g}{z-\lambda}\right\|^2 \Big/ |\hat{g}(\lambda)|^2 .$$

(Note that from Lemma 2.1, $\hat{g}(\lambda)$ can vanish on at most a countable set of H. If for some $\lambda\in H$, $\hat{g}(\lambda)=0$, then the right hand side of (14) is to be interpreted as ∞.) Denote $(1-|\lambda|^2)/|1-\lambda e^{-i\theta}|^2$ by $P(\lambda, e^{i\theta})$. Define Ω to be the support of μ minus Γ. Then

$$\text{(15)} \qquad (1-|\lambda|^2)\left\|\frac{g(z)}{z-\lambda}\right\|^2 = \int_\Gamma P(\lambda, e^{it})|g(e^{it})|^2 w(e^{it})d\sigma(t) + \int_\Omega \frac{1-|\lambda|^2}{|\lambda-z|^2}|g(z)|^2 d\mu(z) .$$

Now

$$\hat{g}(\lambda) = \left\langle \frac{1}{z-\lambda}, g\right\rangle = \left\langle \frac{1}{z-\lambda} + \frac{\bar{\lambda}}{1-\bar{\lambda}z}, g\right\rangle$$

since $z\to\bar{\lambda}/(1-\bar{\lambda}z)$ is analytic in $\bar{D}$ and $g=P_2(1/(z-a))$ is in $H^2(\mu)^\perp$. Writing this out, we see that

$$\text{(16)} \qquad \hat{g}(\lambda) = \int_\Gamma P(\lambda, e^{it})e^{-it}\overline{g(e^{it})}w(e^{it})d\sigma(t) + \int_\Omega \frac{1-|\lambda|^2}{(z-\lambda)(1-\bar{\lambda}z)} g(z)d\mu(z) .$$

Since $e^{i\theta}\notin\bar{\Omega}$ it is easy to see that the second integrals of (15) and (16) converge to 0 as $\lambda \to e^{i\theta}$. Hence by a theorem of Fatou [see 13], we get

$$(17) \qquad \lim_{\lambda \to e^{i\theta}} (1 - |\lambda|^2) \left\| \frac{g(z)}{z - \lambda} \right\|^2 = |g(e^{i\theta})|^2 w(e^{i\theta}) ,$$

$$(18) \qquad \lim_{\lambda \to e^{i\theta}} \hat{g}(\lambda) = e^{-i\theta} \overline{g(e^{i\theta})} w(e^{i\theta}) \quad \text{for } \sigma\text{-a.e. } e^{i\theta} \in \Gamma$$

where $\lambda \to e^{i\theta}$ nontangentially. Recall that by Lemma 2.2, g cannot vanish on a subset of Γ with positive Lebesgue measure. Thus, combining (14), (17), and (18), we establish (13) to complete the proof.

Suppose that μ is a measure on $\bar{D}$ satisfying (A) and (B) for a hole H of the support of μ with exposed arc Γ. Assume that $d\mu|_\Gamma$ is absolutely continuous with respect to $d\sigma$. In [17] it was shown that if $f \in H^2(\mu)$ and f does not vanish identically on Γ then

$$\int_{\Gamma_1} \log |f| d\sigma > -\infty$$

for Γ_1 any closed subarc of Γ. Thus the functions of $H^2(\mu)$ exhibit one of the properties of Hardy space functions locally on Γ. Thus if $f \in H^2(\mu)$ the question is raised as to whether f can be recovered as the boundary values of $\tilde{f}$ on Γ. J. Thompson and R. Olin have informed us that the answer to this question is yes. Subsequently, we have established this result together with a Blaschke type growth condition based on Theorem 2.1 and a result of Kriete and Trutt [15].

Let μ satisfy the hypothesis of Theorem 2.1. Also assume that $d\mu|_\Gamma$ is absolutely continuous with respect to Lebesgue measure. We have the following regularity theorem for extensions of functions in $H^2(\mu)$.

THEOREM 2.2. *Let* $f \in H^2(\mu)$.

(i) $\lim_{\lambda \to e^{i\theta}} \tilde{f}(\lambda) = f(e^{i\theta})$ *for* σ-a.e. $e^{i\theta} \in \Gamma$ *where* $\lambda \to e^{i\theta}$ *nontangentially.*

(ii) *Assume that* f *is not equal to* 0 σ-a.e. *on* Γ. *If* Γ_1 *is any proper closed subarc of* Γ *and* $\tilde{f}$ *vanishes on the set* $\{z_n\}_1^\infty$ *which has no limit points outside of* Γ_1 *then*

$$\sum_1^\infty (1 - |z_n|) p_n < \infty$$

where p_n *is the multiplicity of* z_n *as a zero of* $\tilde{f}$.

Proof. The proof will be established by showing that any f in $H^2(\mu)$ may be viewed as an element of a space $H^2(\beta)$. The corresponding extensions of f as an element of $H^2(\mu)$ and $H^2(\beta)$ have the same values at the points which are bounded point evaluations of both μ and β. Once this is done it will be sufficient to show that extensions of functions in $H^2(\beta)$ satisfy (i) and (ii). This will follow from a conformal mapping argument.

Let Γ_1 be any closed subarc of Γ. Let a and b be elements of $\Gamma - \Gamma_1$, one on each side of Γ_1, for which equality holds in (11). Let M denote the arc connecting a with b and containing Γ_1. By hypothesis (B) on the support of μ, we can find a polar rectangle R with int $R \subset H$, and $\partial R \cap \partial D = M$. Let L denote $\partial R \cap D$.

Define a finite Borel measure, $d\beta$, with support ∂R by

$$d\beta = (1 - |z|^2)|dz|\Big|_L + \frac{w(z)}{2\pi}|dz|\Big|_M$$

where $|dz|$ denotes arc length measure.

Let p be a polynomial. Then

$$\begin{aligned}\|p\|_\beta^2 &= \int_L |p|^2(1 - |z|^2)|dz| + \int_M |p|^2 w\,d\sigma \\ &\leqq \|p\|_\mu^2 \int_L \|k_z^\mu\|^2(1 - |z|^2)|dz| + \|p\|_\mu^2 .\end{aligned}$$

Now the hypothesis that a and b satisfy the equality in (11) enables us to find a constant $K < \infty$ so that

$$\|p\|_\beta^2 \leqq K\|p\|_\mu^2 . \tag{19}$$

Hence by (19), the mapping defined on polynomials by $p \to p$ extends to a bounded linear map T of $H^2(\mu)$ into $H^2(\beta)$.

Notice that

$$\int_M |\log w|\,|dz| + \int_L |\log(1 - |z|^2)|\,|dz| < \infty .$$

The first integral is finite by Lemma 2 of [4] since μ has b.p.e.'s in H and the second integral is finite by a routine computation. Thus if

$$W(z) = \begin{cases} w(z) & z \in M \\ (1 - |z|^2) & z \in L \end{cases}$$

then

$$d\beta = W(z)|dz| \text{ where } \int_{\partial R} |\log W(z)| < \infty . \tag{20}$$

If ψ is a simple conformal mapping of D onto R extended to a mapping of $\bar{D}$ onto $\bar{R}$ then $\psi^{-1\prime}$ is bounded above by a modification of Theorem 9.8 of [18]. Using a theorem of Szegö [see 13] and a conformal mapping argument, it is not hard to show that β has b.p.e.'s in R if and only if $\log[(W\circ\psi)|\psi'|]\in L^1(d\sigma)$. By Theorem 3.12 of [8] (since ψ is rectifiable), $\psi'\in H^1(d\sigma)$ so $\log|\psi'|\in L^1(d\sigma)$. Combining (20) and the boundedness of $\psi^{-1\prime}$ we see that

$$\int_{\partial D}|\log W\circ\psi|d\sigma = \frac{1}{2\pi}\int_{\partial R}|\log W|\,|\psi^{-1\prime}|\,|dz| < \infty\,.$$

Fix $f\in H^2(\mu)$. By the definition of T, a sequence of polynomials converging to f in $H^2(\mu)$ will converge to Tf in $H^2(\beta)$. Also the existence of b.p.e.'s in the hole R implies by Theorem 1.1 that the convergence of polynomials is uniform on compact subsets of int R. Hence $\tilde{f} = T\tilde{f}$ in R.

To show that extensions of functions in $H^2(\beta)$ satisfy (i) and (ii), we refer to the proof of Theorem 8 in [15]. This completes the proof.

3. An application.

Let dA denote planar Lebesgue measure on D and let Γ be an open subarc of ∂D. We shall apply the results of §2 to finite positive measures of the form

$$d\nu = GdA + wd\sigma$$

satisfying

$$\log G \text{ is in } L^1(dA) \text{ and } \int_\Gamma \log wd\sigma > -\infty\,. \tag{21}$$

THEOREM 3.1. *Suppose that* $d\nu = GdA + wd\sigma$ *satisfies* (21). *Then*

$$\operatorname*{Lim}_{\lambda\to e^{i\theta}}(1-|\lambda|^2)\|k_\lambda\|^2 = \frac{1}{w(e^{i\theta})}\ \textit{for}\ \sigma\text{-a.e. } e^{i\theta}\in\Gamma$$

where $\lambda\to e^{i\theta}$ *nontangentially.*

Proof. Remove the open region S from D which is bounded by a proper closed subarc Γ_1 of Γ and the chord connecting the endpoints of Γ_1. Define $\tau = \nu|_{\bar{D}-S}$. Clearly, $\|p\|_\tau \leqq \|p\|_\nu$ so by definition

$$E^\nu(z) = \|k_z^\nu\| \leqq E^\tau(z)\,.$$

Appealing to (12), it is enough to show that

(22) $$\overline{\lim_{\lambda\to e^{i\theta}}}(1-|\lambda|^2)(E^{\tau}(\lambda))^2 \leqq \frac{1}{w(e^{i\theta})} \text{ for } \sigma\text{-a.e. } e^{i\theta}\in\Gamma_1$$

where $\lambda \to e^{i\theta}$ nontangentially.

The support of the measure τ satisfies condition (A) with respect to S and Γ_1 by definition. If we show that τ satisfies (B), then we may apply Theorem 2.1 to establish (22). The remainder of the proof is a lengthy calculation to show that (B) holds.

First we need some notation. Without loss of generality let us assume that for some α with $-1 < \alpha < 1$, $S = \{z \in D: \alpha < \operatorname{Re} z < 1\}$. For $-1 < x < \alpha$, let L_x denote the chord $\{z \in \bar{D}: \operatorname{Re} z = x\}$. Choose $-1 < \beta < \alpha$ so that for every x with $\beta \leqq x \leqq \alpha$ L_x intersects $\Gamma - \Gamma_1$ in two points. (Since Γ is an open arc and Γ_1 is a proper closed subarc of Γ this can be done.) For $-1 < x < \alpha$, let S_x denote the open segment of D with chord L_x and containing S. Denote $\partial S_x \cap \partial D$ by Γ_x.

Let $E_n = \{t \in [\beta, \alpha]: \int_{L_t} G(t+iy)|dy| < \infty$ and $\int_{\Gamma_t} |\log w/2\pi||dz| + \int_{L_t} |\log G(t+iy)||dy| < n\}$. It is clear from the hypotheses on G and w that for some $n < \infty$, $m(E_n) > 0$, where m is linear Lebesgue measure. Let E be any set E_n with $m(E_n) > 0$. If $t \in E$, define the measures ν_t with support ∂S_t by

$$d\nu_t = \frac{w}{2\pi}|dz|\,|_{\Gamma_t} + m(E)G(t+iy)|dy|\,|_{L_t}\,.$$

Let

$$h_t = \begin{cases} \dfrac{w}{2\pi} & \text{on } \Gamma_t \\ m(E)G(t+iy) & \text{on } L_t\,. \end{cases}$$

Then

$$d\nu_t = h_t|dz|\,|_{\partial S_t}$$

and

$$\int_{\partial S_t} |\log h_t|\,|dz| \leqq n < \infty\,.$$

Notice that ν_t has b.p.e.'s in S_t (and hence in S) by an argument similar to that employed in the proof of Theorem 2.2.

Fix any $a \in S$. For any polynomial p

(23) $$|p(a)|^2 \leqq \|k_a^{\nu_t}\|^2\,\|p\|_{\nu_t}^2$$

where $t \in E$. Integrating (23) on E with respect to dm, we obtain

$$m(E)\,|p(a)|^2 \leqq \sup_{t\in E} ||k_a^{\nu_t}||^2\Big[\int_E\int_{L_t} |p|^2 Gm(E)|\,dy\,|dm + m(E)\int_{\Gamma_t} |p|^2 w d\sigma\Big]$$
$$\leqq \sup_{t\in E} ||k_a^{\nu_t}||^2 m(E)\,||p||_\tau^2 .$$

We need only show that $\sup_{t\in E} ||k_a^{\nu_t}||^2$ is finite to establish (B). Let ψ_t denote the simple conformal map of D onto S_t with $\psi_t(a) = a$ and $\psi_t'(a) > 0$. Denote $\sup\{|\psi_t^{-1\prime}(z)|: \beta \leqq t \leqq \alpha, z \in \bar{S}_t\}$ by C. Let A stand for the set of angles measured in radians of the corners of S_t with $t \in [\beta, \alpha]$. Referring to the proof of Theorem 9.8 of [18], we see that $C < \infty$, since $0 < \inf A \leqq \sup A < \pi$. (Because these conformal maps can be given explicitly, this also follows by a direct computation.) It follows from a conformal mapping and a theorem of Szegö [see 13] that

$$||k_a^{\nu_t}||^2 = \frac{\exp - \int_{\partial S_t} P(a, \psi_t^{-1}(z))\log h(z)\,|\psi_t^{-1\prime}(z)|\,\frac{|dz|}{2\pi}}{2\pi(1 - |a|^2)|\psi_t'(a)|}$$

so

$$\sup_{t\in E} ||k_a^{\nu_t}||^2 \leqq \frac{\exp\left(\frac{1+|a|}{1-|a|}\right)Cn}{2\pi(1-|a|)^2}\,C .$$

This completes the proof.

We remark that functions in $H^2(dA)$ do not in general have Hardy space properties. However, if $d\nu = GdA + wd\sigma$ satisfies (21) then we have the following theorem.

THEOREM 3.2. *Suppose that $d\nu = GdA + wd\sigma$ satisfies* (21). *Let $f \in H^2(\nu)$.*

(i) $\lim_{\lambda\to e^{i\theta}} \tilde{f}(z) = f(e^{i\theta})$ *for σ-a.e. $e^{i\theta} \in \Gamma$.*

(ii) *Suppose that f is not the zero function. If Γ_1 is any proper closed subarc of Γ and $\tilde{f}$ vanishes on the set $\{z_n\}_1^\infty$ which has no limit points not in Γ_1, then*

$$\sum_1^\infty (1 - |z_n|)p_n < \infty$$

where the p_n is the multiplicity of z_n as a zero of $\tilde{f}$.

(iii) *Suppose f is not the zero function. Let Γ_1 be any proper closed subarc of Γ, then*

$$\int_{\Gamma_1} \log|f|\,d\sigma > -\infty .$$

Proof. The proof is similar to that given for Theorem 3.1 and will be omitted.

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