

H-ANTI-INVARIANT SUBMERSIONS FROM ALMOST QUATERNIONIC HERMITIAN MANIFOLDS

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Abstract. As a generalization of anti-invariant Riemannian submersions and Lagrangian Riemannian submersions, we introduce the notions of h-anti-invariant submersions and h-Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We obtain characterizations and investigate some properties: the integrability of distributions, the geometry of foliations, and the harmonicity of such maps. We also find a condition for such maps to be totally geodesic and give some examples of such maps. Finally, we obtain some types of decomposition theorems.

Keywords: Riemannian submersion; Lagrangian Riemannian submersion; decomposition theorem; totally geodesic

MSC 2010: 53C15, 53C26

1. INTRODUCTION

In 1960s, O’Neill in [17] and Gray in [10] introduced independently the notion of a Riemannian submersion, which is useful in many areas: physics ([6], [25], [5], [12], [13], [16]), medical imaging [15], robotic theory [1] (see [23]).

In 1976, Watson in [24] defined almost Hermitian submersions, which are Riemannian submersions from almost Hermitian manifolds onto almost Hermitian manifolds. Using this notion, he investigates a kind of structural problems among base manifold, fibers, total manifold. This notion was extended to almost contact manifolds in [7], locally conformal Kähler manifolds in [14], and quaternion Kähler manifolds in [11].

In 2010, Sahin in [22] introduced the notions of anti-invariant Riemannian submersions and Lagrangian Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. Using this notions, he studies total manifolds. In particular, he investigates some kinds of decomposition theorems.

We know that Riemannian submersions are related with physics and have applications in Yang-Mills theory ([6], [25]), Kaluza-Klein theory ([5], [12]), supergravity and superstring theories ([13], [16]). And quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear σ -models with supersymmetry, see [8].

The paper is organized as follows. In Section 2 we recall some notions, which are needed in the later sections. In Section 3 we introduce the notions of h-anti-invariant submersions and h-Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds, give examples, and investigate some properties: the integrability of distributions, the geometry of foliations, the condition for such maps to be totally geodesic, and the condition for such maps to be harmonic. In Section 4 under h-anti-invariant submersions and h-Lagrangian submersions, we consider some decomposition theorems.

2. PRELIMINARIES

Let (M, g, J) be an almost Hermitian manifold, where M is a C^∞ -manifold, g is a Riemannian metric on M , and J is a compatible almost complex structure on (M, g) (i.e., $J \in \text{End}(TM)$, $J^2 = -\text{id}$, $g(JX, JY) = g(X, Y)$ for $X, Y \in \Gamma(TM)$).

We call (M, g, J) a *Kähler manifold* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g .

Let (M, g_M) and (N, g_N) be Riemannian manifolds.

Let $F: (M, g_M) \rightarrow (N, g_N)$ be a C^∞ -map.

The *second fundamental form* of F is given by

$$(\nabla F_*)(U, V) := \nabla_U^F F_* V - F_*(\nabla_U V) \quad \text{for } U, V \in \Gamma(TM),$$

where ∇^F is the pullback connection along F and ∇ is the Levi-Civita connection of g_M , see [3].

Then the map F is *harmonic* if and only if $\text{trace}(\nabla F_*) = 0$, see [3].

We call F a *totally geodesic map* if $(\nabla F_*)(U, V) = 0$ for $U, V \in \Gamma(TM)$, see [3].

The map F is said to be a *C^∞ -submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

We call F a *Riemannian submersion* ([17], [9]) if F is a C^∞ -submersion and

$$(2.1) \quad (F_*)_p: ((\ker(F_*)_p)^\perp, (g_M)_p) \rightarrow (T_{F(p)}N, (g_N)_{F(p)})$$

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM to M at p .

Let $F: (M, g_M) \rightarrow (N, g_N)$ be a Riemannian submersion.

For any vector field $U \in \Gamma(TM)$ we write

$$(2.2) \quad U = \mathcal{V}U + \mathcal{H}U,$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Define the O'Neill tensors \mathcal{T} and \mathcal{A} by

$$(2.3) \quad \mathcal{A}_U V = \mathcal{H}\nabla_{\mathcal{H}U}\mathcal{V}V + \mathcal{V}\nabla_{\mathcal{H}U}\mathcal{H}V,$$

$$(2.4) \quad \mathcal{T}_U V = \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V + \mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}V$$

for $U, V \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M ([17], [9]).

Let

$$(2.5) \quad \widehat{\nabla}_V W := \mathcal{V}\nabla_V W \quad \text{for } V, W \in \Gamma(\ker F_*).$$

Then we have

$$(2.6) \quad \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y],$$

$$(2.7) \quad \mathcal{T}_U V = \mathcal{T}_V U$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$.

Proposition 2.1 ([17], [9]). *Let F be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) . Then we obtain*

$$(2.8) \quad g_M(\mathcal{T}_U V, W) = -g_M(V, \mathcal{T}_U W),$$

$$(2.9) \quad g_M(\mathcal{A}_U V, W) = -g_M(V, \mathcal{A}_U W),$$

$$(2.10) \quad (\nabla F_*)(U, V) = (\nabla F_*)(V, U),$$

$$(2.11) \quad (\nabla F_*)(X, Y) = 0$$

for $U, V, W \in \Gamma(TM)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

We recall the notions of an anti-invariant Riemannian submersion and a Lagrangian Riemannian submersion.

Let F be a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) . The map F is said to be an *anti-invariant Riemannian submersion*, see [22], if $J(\ker F_*) \subset (\ker F_*)^\perp$.

We call F a *Lagrangian Riemannian submersion*, see [22], if $J(\ker F_*) = (\ker F_*)^\perp$.

Let M be a $4m$ -dimensional C^∞ -manifold and let E be a rank 3 subbundle of $\text{End}(TM)$ such that for any point $p \in M$ with a neighborhood U there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on M and (M, E) an *almost quaternionic manifold*, see [2].

Moreover, let g be a Riemannian metric on M such that for any point $p \in M$ with a neighborhood U there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$(2.12) \quad J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

$$(2.13) \quad g(J_\alpha X, J_\alpha Y) = g(X, Y)$$

for $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call (M, E, g) an *almost quaternionic Hermitian manifold*, see [11].

For convenience, the above basis $\{J_1, J_2, J_3\}$ satisfying (2.12) and (2.13) is said to be a *quaternionic Hermitian basis*.

Let (M, E, g) be an almost quaternionic Hermitian manifold.

We call (M, E, g) a *quaternionic Kähler manifold* if given a point $p \in M$ with a neighborhood U , there exist 1-forms $\omega_1, \omega_2, \omega_3$ on U such that for any $\alpha \in \{1, 2, 3\}$,

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}$$

for $X \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3, see [11].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M (i.e., $\nabla J_\alpha = 0$ for $\alpha \in \{1, 2, 3\}$, where ∇ is the Levi-Civita connection of g), then (M, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric*, see [4].

Now, we recall the notions of almost h-slant submersions, almost h-semi-invariant submersions, and almost h-semi-slant submersions.

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold.

A Riemannian submersion $F: (M, E, g_M) \rightarrow (N, g_N)$ is said to be an *almost h-slant submersion* if given a point $p \in M$ with a neighborhood U , there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, see [19].

A Riemannian submersion $F: (M, E, g_M) \rightarrow (N, g_N)$ is called an *almost h-semi-invariant submersion* if given a point $p \in M$ with a neighborhood U , there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$ there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \quad R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \quad R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,$$

where \mathcal{D}_2^R is the orthogonal complement of \mathcal{D}_1^R in $\ker F_*$, see [18].

A Riemannian submersion $F: (M, E, g_M) \rightarrow (N, g_N)$ is called an *almost h-semi-slant submersions* if given a point $p \in M$ with a neighborhood U , there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for each $R \in \{I, J, K\}$ there is a distribution $\mathcal{D}_1^R \subset \ker F_*$ on U such that

$$\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \quad R(\mathcal{D}_1^R) = \mathcal{D}_1^R,$$

and the angle $\theta_R = \theta_R(X)$ between RX and the space $(\mathcal{D}_2^R)_q$ is constant for nonzero $X \in (\mathcal{D}_2^R)_q$ and $q \in U$, where \mathcal{D}_2^R is the orthogonal complement of \mathcal{D}_1^R in $\ker F_*$, see [20].

Throughout this paper, we will use the above notation.

3. H-ANTI-INVARIANT SUBMERSIONS

In this section, we introduce the notions of h-anti-invariant submersions and h-Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds and investigate their properties.

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F: (M, E, g_M) \rightarrow (N, g_N)$ be a Riemannian submersion. We call the map F an *h-anti-invariant submersion* if given a point $p \in M$ with a neighborhood U , there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$.

We call such a basis $\{I, J, K\}$ an *h-anti-invariant basis*.

Remark 3.2. As we see, an h-anti-invariant submersion is one of the particular cases of an almost h-slant submersion, an almost h-semi-invariant submersion, and an almost h-semi-slant submersion.

Remark 3.3. Let F be an h-anti-invariant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) . Then there does not exist a map F such that $\dim(\ker F_*) = \dim((\ker F_*)^\perp)$. If it did, then

given a local quaternionic Hermitian basis $\{I, J, K\}$ of E with $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$, we should have

$$R(\ker F_*) = (\ker F_*)^\perp \quad \text{for } R \in \{I, J, K\}$$

so that

$$K(\ker F_*) = IJ(\ker F_*) = I((\ker F_*)^\perp) = (\ker F_*),$$

contradiction!

Due to Remark 3.3, we need to define another type of such a map.

Definition 3.4. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F: (M, E, g_M) \rightarrow (N, g_N)$ be a Riemannian submersion. We call the map F a *h-Lagrangian submersion* if given a point $p \in M$ with a neighborhood U , there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $I(\ker F_*) = (\ker F_*)^\perp$, $J(\ker F_*) = \ker F_*$, and $K(\ker F_*) = (\ker F_*)^\perp$.

We call such a basis $\{I, J, K\}$ an *h-Lagrangian basis*.

Remark 3.5. (a) It is easy to check that $J(\ker F_*) = \ker F_*$ implies $J((\ker F_*)^\perp) = (\ker F_*)^\perp$.

(b) Let F be a Riemannian submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) such that $\dim(\ker F_*) = \dim((\ker F_*)^\perp)$. Then there does not exist a map F that for some local quaternionic Hermitian basis $\{I, J, K\}$ of E we have

$$I(\ker F_*) = \ker F_*, \quad J(\ker F_*) = \ker F_*, \quad K(\ker F_*) = (\ker F_*)^\perp.$$

If it did, then $K(\ker F_*) = IJ(\ker F_*) = I(\ker F_*) = \ker F_*$, contradiction!

Now, we give some examples. Note that given a Euclidean space \mathbb{R}^{4m} with coordinates $(x_1, x_2, \dots, x_{4m})$, we can canonically choose complex structures I, J, K on \mathbb{R}^{4m} as follows:

$$\begin{aligned} I\left(\frac{\partial}{\partial x_{4k+1}}\right) &= \frac{\partial}{\partial x_{4k+2}}, & I\left(\frac{\partial}{\partial x_{4k+2}}\right) &= -\frac{\partial}{\partial x_{4k+1}}, & I\left(\frac{\partial}{\partial x_{4k+3}}\right) &= \frac{\partial}{\partial x_{4k+4}}, \\ I\left(\frac{\partial}{\partial x_{4k+4}}\right) &= -\frac{\partial}{\partial x_{4k+3}}, & J\left(\frac{\partial}{\partial x_{4k+1}}\right) &= \frac{\partial}{\partial x_{4k+3}}, & J\left(\frac{\partial}{\partial x_{4k+2}}\right) &= -\frac{\partial}{\partial x_{4k+4}}, \\ J\left(\frac{\partial}{\partial x_{4k+3}}\right) &= -\frac{\partial}{\partial x_{4k+1}}, & J\left(\frac{\partial}{\partial x_{4k+4}}\right) &= \frac{\partial}{\partial x_{4k+2}}, & K\left(\frac{\partial}{\partial x_{4k+1}}\right) &= \frac{\partial}{\partial x_{4k+4}}, \\ K\left(\frac{\partial}{\partial x_{4k+2}}\right) &= \frac{\partial}{\partial x_{4k+3}}, & K\left(\frac{\partial}{\partial x_{4k+3}}\right) &= -\frac{\partial}{\partial x_{4k+2}}, & K\left(\frac{\partial}{\partial x_{4k+4}}\right) &= -\frac{\partial}{\partial x_{4k+1}} \end{aligned}$$

for $k \in \{0, 1, \dots, m-1\}$.

Then we easily check that $(I, J, K, \langle, \rangle)$ is a hyperkähler structure on \mathbb{R}^{4m} , where \langle, \rangle denotes the Euclidean metric on \mathbb{R}^{4m} .

Example 3.6. Define a map $F: \mathbb{R}^{12} \rightarrow \mathbb{R}^9$ by

$$F(x_1, \dots, x_{12}) = (x_{10}, x_{11}, x_{12}, x_4, x_3, x_2, x_8, x_6, x_7).$$

Then the map F is an h-anti-invariant submersion such that

$$\begin{aligned} \ker F_* &= \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_9} \right\rangle, \\ (\ker F_*)^\perp &= \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_{10}}, \frac{\partial}{\partial x_{11}}, \frac{\partial}{\partial x_{12}} \right\rangle, \\ I\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_2}, \quad I\left(\frac{\partial}{\partial x_5}\right) = \frac{\partial}{\partial x_6}, \quad I\left(\frac{\partial}{\partial x_9}\right) = \frac{\partial}{\partial x_{10}}, \\ J\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_3}, \quad J\left(\frac{\partial}{\partial x_5}\right) = \frac{\partial}{\partial x_7}, \quad J\left(\frac{\partial}{\partial x_9}\right) = \frac{\partial}{\partial x_{11}}, \\ K\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_4}, \quad K\left(\frac{\partial}{\partial x_5}\right) = \frac{\partial}{\partial x_8}, \quad K\left(\frac{\partial}{\partial x_9}\right) = \frac{\partial}{\partial x_{12}}. \end{aligned}$$

Example 3.7. Define a map $F: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ by

$$F(x_1, \dots, x_4) = \left(\frac{x_2 + x_3}{\sqrt{2}}, \frac{x_1 + x_4}{\sqrt{2}} \right).$$

Then the map F is an h-Lagrangian submersion such that

$$\begin{aligned} \ker F_* &= \left\langle V_1 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}, V_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_4} \right\rangle, \\ (\ker F_*)^\perp &= \left\langle X_1 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4} \right\rangle, \\ I(V_1) &= -X_2, \quad I(V_2) = X_1, \\ J(V_1) &= V_2, \quad J(V_2) = -V_1, \\ K(V_1) &= X_1, \quad K(V_2) = X_2. \end{aligned}$$

Let F be an h-anti-invariant submersion (or an h-Lagrangian submersion) from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) . Given a point $p \in M$ with a neighborhood U , we have an h-anti-invariant basis (or an h-Lagrangian basis, respectively) $\{I, J, K\}$ of sections of E on U .

Then given $X \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we write

$$(3.1) \quad RX = B_RX + C_RX,$$

where $B_RX \in \Gamma(\ker F_*)$ and $C_RX \in \Gamma((\ker F_*)^\perp)$.

If $F: (M, E, g_M) \rightarrow (N, g_N)$ is an h -anti-invariant submersion, then we get

$$(3.2) \quad (\ker F_*)^\perp = R(\ker F_*) \oplus \mu_R \quad \text{for } R \in \{I, J, K\}.$$

Then it is easy to check that μ_R is R -invariant for $R \in \{I, J, K\}$.

Given $X \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we have

$$(3.3) \quad X = P_RX + Q_RX,$$

where $P_RX \in \Gamma(R(\ker F_*))$ and $Q_RX \in \Gamma(\mu_R)$.

Furthermore, given $R \in \{I, J, K\}$, we obtain

$$(3.4) \quad C_RX \in \Gamma(\mu_R) \quad \text{for } X \in \Gamma((\ker F_*)^\perp)$$

and

$$(3.5) \quad g_M(C_RX, RV) = 0 \quad \text{for } V \in \Gamma(\ker F_*).$$

Then it is easy to have

Lemma 3.8. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then we get*

(1)

$$\begin{aligned} \mathcal{T}_V RW &= B_R \mathcal{T}_V W \\ \mathcal{H}\nabla_V RW &= C_R \mathcal{T}_V W + R \widehat{\nabla}_V W \end{aligned}$$

for $V, W \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$;

(2)

$$\begin{aligned} \mathcal{A}_X C_R Y + \mathcal{V}\nabla_X B_R Y &= B_R \mathcal{H}\nabla_X Y \\ \mathcal{H}\nabla_X C_R Y + \mathcal{A}_X B_R Y &= R \mathcal{A}_X Y + C_R \mathcal{H}\nabla_X Y \end{aligned}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$;

(3)

$$\begin{aligned} \mathcal{A}_X R V &= B_R \mathcal{A}_X V \\ \mathcal{H}\nabla_X R V &= C_R \mathcal{A}_X V + R \mathcal{V}\nabla_X V \end{aligned}$$

for $V \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$.

Theorem 3.9. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

- (a) *the distribution $(\ker F_*)^\perp$ is integrable.*
- (b)
$$g_M(\mathcal{A}_X B_I Y - \mathcal{A}_Y B_I X, IV) = g_M(C_I Y, I \mathcal{A}_X V) - g_M(C_I X, I \mathcal{A}_Y V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.
- (c)
$$g_M(\mathcal{A}_X B_J Y - \mathcal{A}_Y B_J X, JV) = g_M(C_J Y, J \mathcal{A}_X V) - g_M(C_J X, J \mathcal{A}_Y V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.
- (d)
$$g_M(\mathcal{A}_X B_K Y - \mathcal{A}_Y B_K X, KV) = g_M(C_K Y, K \mathcal{A}_X V) - g_M(C_K X, K \mathcal{A}_Y V)$$
for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, using (3.5) we get

$$\begin{aligned} g_M([X, Y], V) &= g_M(\nabla_X RY - \nabla_Y RX, RV) \\ &= g_M(\nabla_X B_R Y + \nabla_X C_R Y - \nabla_Y B_R X - \nabla_Y C_R X, RV) \\ &= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X, RV) - g_M(C_R Y, \nabla_X RV) + g_M(C_R X, \nabla_Y RV) \\ &= g_M(\mathcal{A}_X B_R Y - \mathcal{A}_Y B_R X, RV) - g_M(C_R Y, R \mathcal{A}_X V) + g_M(C_R X, R \mathcal{A}_Y V). \end{aligned}$$

Hence,

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. \square

Lemma 3.10. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

- (a) *The distribution $(\ker F_*)^\perp$ is integrable.*
- (b) $\mathcal{A}_X IY = \mathcal{A}_Y IX$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.
- (c) $\mathcal{A}_X KY = \mathcal{A}_Y KX$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.
- (d) $\mathcal{A}_X JY = \mathcal{A}_Y JX$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. By the proof of Theorem 3.9, we get (a) \Leftrightarrow (b) and (a) \Leftrightarrow (c).

Given $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, since $J(\ker F_*) = \ker F_*$, we obtain

$$\begin{aligned} g_M([X, Y], JV) &= -g_M(\nabla_X JY - \nabla_Y JX, V) \\ &= g_M(\mathcal{A}_Y JX - \mathcal{A}_X JY, V), \end{aligned}$$

which implies (a) \Leftrightarrow (d).

Therefore, the result follows. \square

We consider equivalent conditions for distributions to be totally geodesic.

Theorem 3.11. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M .*

(b)

$$g_M(\mathcal{A}_X B_I Y, IV) = g_M(C_I Y, I\mathcal{A}_X V)$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c)

$$g_M(\mathcal{A}_X B_J Y, JV) = g_M(C_J Y, J\mathcal{A}_X V)$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)

$$g_M(\mathcal{A}_X B_K Y, KV) = g_M(C_K Y, K\mathcal{A}_X V)$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V \in \Gamma(\ker F_*)$, $X, Y \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, using (3.5) we have

$$\begin{aligned} g_M(\nabla_X Y, V) &= g_M(\nabla_X B_R Y + \nabla_X C_R Y, RV) \\ &= g_M(\mathcal{A}_X B_R Y, RV) - g_M(C_R Y, \nabla_X RV) \\ &= g_M(\mathcal{A}_X B_R Y, RV) - g_M(C_R Y, R\mathcal{A}_X V), \end{aligned}$$

which implies (a) \Leftrightarrow (b), (a) \Leftrightarrow (c), (a) \Leftrightarrow (d).

Therefore, the result follows. \square

Lemma 3.12. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

(a) *The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M .*

(b) $\mathcal{A}_X IY = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c) $\mathcal{A}_X KY = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d) $\mathcal{A}_X JY = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. By the proof of Theorem 3.11, we get (a) \Leftrightarrow (b) and (a) \Leftrightarrow (c).

Given $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, we obtain

$$g_M(\nabla_X Y, JV) = -g_M(\nabla_X JY, V) = -g_M(\mathcal{A}_X JY, V),$$

which implies (a) \Leftrightarrow (d).

Therefore, the result follows. \square

Theorem 3.13. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

(a) *The distribution $\ker F_*$ defines a totally geodesic foliation on M .*

(b)
$$\mathcal{T}_V B_I X + \mathcal{A}_{C_I X} V \in \Gamma(\mu_I)$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(c)
$$\mathcal{T}_V B_J X + \mathcal{A}_{C_J X} V \in \Gamma(\mu_J)$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(d)
$$\mathcal{T}_V B_K X + \mathcal{A}_{C_K X} V \in \Gamma(\mu_K)$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, using (3.5) we get

$$\begin{aligned} g_M(\nabla_V W, X) &= g_M(\nabla_V RW, RX) \\ &= -g_M(RW, \nabla_V B_R X + \nabla_V C_R X) \\ &= -g_M(RW, \mathcal{T}_V B_R X) - g_M(RW, \nabla_V C_R X). \end{aligned}$$

However,

$$\begin{aligned} g_M(RW, \nabla_V C_R X) &= g_N(F_* RW, F_* \nabla_V C_R X) \quad (\text{since } RW \in \Gamma((\ker F_*)^\perp)) \\ &= -g_N(F_* RW, (\nabla F_*)(V, C_R X)) \\ &= -g_N(F_* RW, (\nabla F_*)(C_R X, V)) \quad (\text{by (2.10)}) \\ &= g_M(RW, \nabla_{C_R X} V) \\ &= g_M(RW, \mathcal{A}_{C_R X} V). \end{aligned}$$

Hence,

$$g_M(\nabla_V W, X) = -g_M(RW, \mathcal{T}_V B_R X + \mathcal{A}_{C_R X} V),$$

which implies (a) \Leftrightarrow (b), (a) \Leftrightarrow (c), (a) \Leftrightarrow (d).

Therefore, we obtain the result. \square

Lemma 3.14. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

- (a) *The distribution $\ker F_*$ defines a totally geodesic foliation on M .*
- (b) $\mathcal{T}_V IX = 0$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.
- (c) $\mathcal{T}_V KX = 0$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.
- (d) $\mathcal{T}_V JX = 0$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Proof. By the proof of Theorem 3.13, we have (a) \Leftrightarrow (b) and (a) \Leftrightarrow (c). Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned} g_M(\nabla_V W, JX) &= -g_M(W, \nabla_V JX) \\ &= -g_M(W, \mathcal{T}_V JX) \quad (\text{since } JX \in \Gamma((\ker F_*)^\perp)), \end{aligned}$$

which implies (a) \Leftrightarrow (d).

Therefore, the result follows. \square

Now, we consider equivalent conditions for such maps to be either totally geodesic or harmonic.

Theorem 3.15. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

- (a) *The map F is a totally geodesic map.*
- (b) $\mathcal{A}_X IV = 0, \quad Q_I \mathcal{H} \nabla_X IV = 0, \quad \mathcal{T}_V IW = 0, \quad Q_I \mathcal{H} \nabla_V IW = 0$
for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.
- (c) $\mathcal{A}_X JV = 0, \quad Q_J \mathcal{H} \nabla_X JV = 0, \quad \mathcal{T}_V JW = 0, \quad Q_J \mathcal{H} \nabla_V JW = 0$
for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.
- (d) $\mathcal{A}_X KV = 0, \quad Q_K \mathcal{H} \nabla_X KV = 0, \quad \mathcal{T}_V KW = 0, \quad Q_K \mathcal{H} \nabla_V KW = 0$
for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. By (2.11) we have $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, by using (3.2) and (3.3) we obtain

$$\begin{aligned} (\nabla F_*)(X, V) &= -F_*(\nabla_X V) = F_*(R \nabla_X RV) \\ &= F_*(R(\mathcal{A}_X RV + \mathcal{H} \nabla_X RV)) = 0 \end{aligned}$$

$\Leftrightarrow R(\mathcal{A}_X RV + Q_R \mathcal{H} \nabla_X RV) = 0 \Leftrightarrow \mathcal{A}_X RV = 0, Q_R \mathcal{H} \nabla_X RV = 0$ and

$$\begin{aligned} (\nabla F_*)(V, W) &= -F_*(\nabla_V W) = F_*(R \nabla_V RW) \\ &= F_*(R(\mathcal{T}_V RW + \mathcal{H} \nabla_V RW)) = 0 \end{aligned}$$

$\Leftrightarrow R(\mathcal{T}_V RW + Q_R \mathcal{H} \nabla_V RW) = 0 \Leftrightarrow \mathcal{T}_V RW = 0, Q_R \mathcal{H} \nabla_V RW = 0.$

Hence,

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. \square

Lemma 3.16. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

- (a) *The map F is a totally geodesic map.*
- (b) $\mathcal{A}_X IV = 0$ and $\mathcal{T}_V IW = 0$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.
- (c) $\mathcal{A}_X KV = 0$ and $\mathcal{T}_V KW = 0$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.
- (d) $\mathcal{A}_X JV = 0$ and $\mathcal{T}_V JW = 0$ for $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. By the proof of Theorem 3.15, we have (a) \Leftrightarrow (b) and (a) \Leftrightarrow (c).

Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned} (\nabla F_*)(X, V) &= -F_*(\nabla_X V) = F_*(J \nabla_X JV) \\ &= F_*(J(\mathcal{A}_X JV + \mathcal{V} \nabla_X JV)) = F_* J \mathcal{A}_X JV = 0 \end{aligned}$$

$\Leftrightarrow \mathcal{A}_X JV = 0$ and

$$\begin{aligned} (\nabla F_*)(V, W) &= -F_*(\nabla_V W) = F_*(J \nabla_V JW) \\ &= F_*(J(\mathcal{T}_V JW + \mathcal{V} \nabla_V JW)) \\ &= F_* J \mathcal{T}_V JW = 0 \end{aligned}$$

$\Leftrightarrow \mathcal{T}_V JW = 0$, which implies (a) \Leftrightarrow (d).

Therefore, we obtain the result. \square

Theorem 3.17. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

- (a) *The map F is harmonic.*
- (b) $Q_I(\text{trace}(\mathcal{T})) = 0$ on $\ker F_*$ and $\text{trace}(I \mathcal{T}_V) = 0$ on $\ker F_*$ for $V \in \Gamma(\ker F_*)$.

- (c) $Q_J(\text{trace}(\mathcal{T})) = 0$ on $\ker F_*$ and $\text{trace}(J\mathcal{T}_V) = 0$ on $\ker F_*$ for $V \in \Gamma(\ker F_*)$.
(d) $Q_K(\text{trace}(\mathcal{T})) = 0$ on $\ker F_*$ and $\text{trace}(K\mathcal{T}_V) = 0$ on $\ker F_*$ for $V \in \Gamma(\ker F_*)$.

P r o o f. By (2.11) we know that the map F is harmonic if and only if $\sum_{i=1}^m \mathcal{T}_{e_i} e_i = 0$ for any local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ of $\ker F_*$.

Given $V, W \in \Gamma(\ker F_*)$, $R \in \{I, J, K\}$, and a local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ of $\ker F_*$, using (3.2) and (3.3) we obtain

$$\begin{aligned} \mathcal{T}_V RW &= \mathcal{V}\nabla_V RW = \mathcal{V}R\nabla_V W \\ &= \mathcal{V}R(\mathcal{T}_V W + \mathcal{V}\nabla_V W) = \mathcal{V}R P_R \mathcal{T}_V W \end{aligned}$$

so that using (2.7) and (2.8) we get

$$\begin{aligned} g_M\left(\sum_{i=1}^m \mathcal{T}_{e_i} e_i, RV\right) &= \sum_{i=1}^m g_M(\mathcal{T}_{e_i} e_i, RV) = \sum_{i=1}^m g_M(P_R \mathcal{T}_{e_i} e_i, RV) \\ &= -\sum_{i=1}^m g_M(RP_R \mathcal{T}_{e_i} e_i, V) = -\sum_{i=1}^m g_M(\mathcal{V}R P_R \mathcal{T}_{e_i} e_i, V) \\ &= -\sum_{i=1}^m g_M(\mathcal{T}_{e_i} R e_i, V) = \sum_{i=1}^m g_M(R e_i, \mathcal{T}_{e_i} V) \\ &= \sum_{i=1}^m g_M(R e_i, \mathcal{T}_V e_i) = -\sum_{i=1}^m g_M(e_i, R \mathcal{T}_V e_i) = 0 \end{aligned}$$

$\Leftrightarrow \text{trace}(R\mathcal{T}_V) = 0$ for $V \in \Gamma(\ker F_*)$.

Hence,

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, the result follows. \square

Lemma 3.18. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the map F is harmonic.*

P r o o f. Since $J(\ker F_*) = \ker F_*$, we can choose a local orthonormal frame $\{e_1, J e_1, \dots, e_k, J e_k\}$ of $\ker F_*$.

Given $V, W \in \Gamma(\ker F_*)$, we have

$$\begin{aligned} \mathcal{T}_V JW &= \mathcal{H}\nabla_V JW = \mathcal{H}J\nabla_V W \\ &= \mathcal{H}J(\mathcal{T}_V W + \mathcal{V}\nabla_V W) = J\mathcal{T}_V W \end{aligned}$$

so that

$$\begin{aligned} \sum_{i=1}^k (\mathcal{T}_{e_i} e_i + \mathcal{T}_{J e_i} J e_i) &= \sum_{i=1}^k (\mathcal{T}_{e_i} e_i + J \mathcal{T}_{J e_i} e_i) = \sum_{i=1}^k (\mathcal{T}_{e_i} e_i + J \mathcal{T}_{e_i} J e_i) \\ &= \sum_{i=1}^k (\mathcal{T}_{e_i} e_i + J^2 \mathcal{T}_{e_i} e_i) = \sum_{i=1}^k (\mathcal{T}_{e_i} e_i - \mathcal{T}_{e_i} e_i) = 0. \end{aligned}$$

Therefore, the result follows. \square

4. DECOMPOSITION THEOREMS

First of all, we recall some notions. Let (M, g) be a Riemannian manifold and L a foliation of M . Let ξ be the tangent bundle of L considered as a subbundle of the tangent bundle TM of M .

We call L a *totally umbilic foliation*, see [21], of M if

$$(4.1) \quad h(X, Y) = g(X, Y)H \quad \text{for } X, Y \in \Gamma(\xi),$$

where h is the second fundamental form of L in M and H is the mean curvature vector field of L in M .

The foliation L is said to be a *spheric foliation*, see [21], if it is a totally umbilic foliation and

$$(4.2) \quad \nabla_X H \in \Gamma(\xi) \quad \text{for } X \in \Gamma(\xi),$$

where ∇ is the Levi-Civita connection of g .

We call L a *totally geodesic foliation*, see [21], of M if

$$(4.3) \quad \nabla_X Y \in \Gamma(\xi) \quad \text{for } X, Y \in \Gamma(\xi).$$

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, $f_i: M_1 \times M_2 \rightarrow \mathbb{R}$ a positive C^∞ -function, and $\pi_i: M_1 \times M_2 \rightarrow M_i$ the canonical projection for $i = 1, 2$.

We call $M_1 \times_{(f_1, f_2)} M_2$ a *double-twisted product manifold*, see [21], of (M_1, g_1) and (M_2, g_2) if it is the product manifold $M := M_1 \times M_2$ with the Riemannian metric g such that

$$(4.4) \quad g(X, Y) = f_1^2 g_1(\pi_{1*} X, \pi_{1*} Y) + f_2^2 g_2(\pi_{2*} X, \pi_{2*} Y) \quad \text{for } X, Y \in \Gamma(TM).$$

We call $M_1 \times_{(f_1, f_2)} M_2$ *nontrivial* if neither f_1 nor f_2 are constant functions.

The Riemannian manifold $M_1 \times_f M_2$ is said to be a *twisted product manifold*, see [21], of (M_1, g_1) and (M_2, g_2) if $M_1 \times_f M_2 = M_1 \times_{(1,f)} M_2$.

We call $M_1 \times_f M_2$ *nontrivial* if f is not a constant function.

The twisted product manifold $M_1 \times_f M_2$ is said to be a *warped product manifold*, see [21], of (M_1, g_1) and (M_2, g_2) if f depends only on the points of M_1 . (i.e., $f \in C^\infty(M_1, \mathbb{R})$)

Let M_1 and M_2 be connected C^∞ -manifolds and M the product manifold $M_1 \times M_2$. Let $\pi_i: M \rightarrow M_i$ be the canonical projection for $i = 1, 2$. Let $\xi_i := \ker \pi_{3-i*}$ and let $P_i: TM \rightarrow \xi_i$ be the vector bundle projection such that $TM = \xi_1 \oplus \xi_2$. And let L_i be the canonical foliation of M by the integral manifolds of ξ_i for $i = 1, 2$.

Proposition 4.1 ([21]). *Let g be a Riemannian metric on the product manifold $M_1 \times M_2$ and assume that the canonical foliations L_1 and L_2 intersect perpendicularly everywhere. Then g is a metric of*

- (a) *a double-twisted product manifold $M_1 \times_{(f_1, f_2)} M_2$ if and only if L_1 and L_2 are totally umbilic foliations,*
- (b) *a twisted product manifold $M_1 \times_f M_2$ if and only if L_1 is a totally geodesic foliation and L_2 is a totally umbilic foliation,*
- (c) *a warped product manifold $M_1 \times_f M_2$ if and only if L_1 is a totally geodesic foliation and L_2 is a spheric foliation,*
- (d) *a (usual) Riemannian product manifold $M_1 \times M_2$ if and only if L_1 and L_2 are totally geodesic foliations.*

Let F be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) such that the distributions $\ker F_*$ and $(\ker F_*)^\perp$ are integrable. Then we denote by $M_{\ker F_*}$ and $M_{(\ker F_*)^\perp}$ the integral manifolds of $\ker F_*$ and $(\ker F_*)^\perp$, respectively. We also denote by H and H^\perp the mean curvature vector fields of $\ker F_*$ and $(\ker F_*)^\perp$, respectively, i.e., $H = m^{-1} \sum_{i=1}^m \mathcal{T}_{e_i} e_i$ and $H^\perp = n^{-1} \sum_{i=1}^n \mathcal{A}_{v_i} v_i$ for a local orthonormal frame $\{e_1, \dots, e_m\}$ of $\ker F_*$ and a local orthonormal frame $\{v_1, \dots, v_n\}$ of $(\ker F_*)^\perp$.

Using Proposition 4.1, Theorem 3.11, and Theorem 3.13, we get

Theorem 4.2. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

- (a) *(M, g_M) is locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*

(b)

$$g_M(\mathcal{A}_X B_I Y, IV) = g_M(C_I Y, I \mathcal{A}_X V) \quad \text{and} \quad \mathcal{T}_V B_I X + \mathcal{A}_{C_I X} V \in \Gamma(\mu_I)$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(c)

$$g_M(\mathcal{A}_X B_J Y, JV) = g_M(C_J Y, J \mathcal{A}_X V) \quad \text{and} \quad \mathcal{T}_V B_J X + \mathcal{A}_{C_J X} V \in \Gamma(\mu_J)$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)

$$g_M(\mathcal{A}_X B_K Y, KV) = g_M(C_K Y, K \mathcal{A}_X V) \quad \text{and} \quad \mathcal{T}_V B_K X + \mathcal{A}_{C_K X} V \in \Gamma(\mu_K)$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

Using Proposition 4.1, Lemma 3.12, and Lemma 3.14, we obtain

Lemma 4.3. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

(a) (M, g_M) is locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.

(b)

$$\mathcal{A}_X IY = 0 \quad \text{and} \quad \mathcal{T}_V IX = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

(c)

$$\mathcal{A}_X KY = 0 \quad \text{and} \quad \mathcal{T}_V KX = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

(d)

$$\mathcal{A}_X JY = 0 \quad \text{and} \quad \mathcal{T}_V JX = 0$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Now, we deal with the geometry of distributions $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 4.4. *Let F be a Riemannian submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) . Assume that the distribution $(\ker F_*)^\perp$ defines a totally umbilic foliation on M . Then the distribution $(\ker F_*)^\perp$ also defines a totally geodesic foliation on M .*

P r o o f. Given $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, we get

$$(4.5) \quad g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X Y, V) = g_M(X, Y)g_M(H^\perp, V)$$

and

$$(4.6) \quad g_M(\nabla_X Y, V) = -g_M(Y, \nabla_X V) = -g_M(Y, \mathcal{A}_X V).$$

Comparing (4.5) and (4.6), we obtain $\mathcal{A}_X V = -g_M(H^\perp, V)X$.

Hence,

$$(4.7) \quad g_M(\mathcal{A}_X V, X) = -g_M(H^\perp, V)\|X\|^2.$$

But

$$\begin{aligned} g_M(\mathcal{A}_X V, X) &= g_M(\nabla_X V, X) = -g_M(V, \nabla_X X) \\ &= -g_M(V, \mathcal{A}_X X) = 0 \quad (\text{by (2.6)}) \end{aligned}$$

so that from (4.7), we have $H^\perp = 0$.

Therefore, the result follows. \square

Remark 4.5. From the equation $\mathcal{A}_X Y = -\mathcal{A}_Y X$ for $X, Y \in \Gamma((\ker F_*)^\perp)$, we can obtain Theorem 4.4. But the equation $\mathcal{T}_V W = \mathcal{T}_W V$ for $V, W \in \Gamma(\ker F_*)$, yields no theorems like Theorem 4.4 on $\ker F_*$.

Theorem 4.6. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

(a) *the distribution $\ker F_*$ defines a totally umbilic foliation on M .*

(b)

$$\mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X = -g_M(H, X)IV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(c)

$$\mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X = -g_M(H, X)JV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

(d)

$$\mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X = -g_M(H, X)KV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proof. Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, we obtain

$$\begin{aligned} g_M(\mathcal{T}_V W, X) &= g_M(\nabla_V RW, RX) \\ &= -g_M(RW, \nabla_V B_R X + \nabla_V C_R X) \\ &= -g_M(RW, \mathcal{T}_V B_R X + \mathcal{H}\nabla_V C_R X) \end{aligned}$$

so that it is easy to check that

$$\mathcal{T}_V W = g_M(V, W)H \Leftrightarrow \mathcal{T}_V B_R X + \mathcal{H}\nabla_V C_R X = -g_M(H, X)RV.$$

Hence,

$$(a) \Leftrightarrow (b), \quad (a) \Leftrightarrow (c), \quad (a) \Leftrightarrow (d).$$

Therefore, we get the result. \square

Lemma 4.7. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

- (a) *The distribution $\ker F_*$ defines a totally umbilic foliation on M .*
- (b) *$\mathcal{T}_V IX = -g_M(H, X)IV$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.*
- (c) *$\mathcal{T}_V KX = -g_M(H, X)KV$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.*
- (d) *$\mathcal{T}_V JX = -g_M(H, X)JV$ for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.*

Proof. By the proof of Theorem 4.6, we have (a) \Leftrightarrow (b) and (a) \Leftrightarrow (c).

Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we get

$$\begin{aligned} g_M(\mathcal{T}_V W, X) &= g_M(\nabla_V JW, JX) \\ &= -g_M(JW, \nabla_V JX) \\ &= -g_M(JW, \mathcal{T}_V JX) \end{aligned}$$

so that we easily check that

$$\mathcal{T}_V W = g_M(V, W)H \Leftrightarrow \mathcal{T}_V JX = -g_M(H, X)JV.$$

Hence, (a) \Leftrightarrow (d).

Therefore, the result follows. \square

Using Proposition 4.1, Theorem 3.11, and Theorem 4.6, we get

Theorem 4.8. *Let F be an h -anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -anti-invariant basis. Then the following conditions are equivalent:*

- (a) *(M, g_M) is locally a twisted product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*
- (b)

$$g_M(\mathcal{A}_X B_I Y, IV) = g_M(C_I Y, I\mathcal{A}_X V)$$

and

$$\mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X = -g_M(H, X)IV$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

- (c)
- $$g_M(\mathcal{A}_X B_J Y, JV) = g_M(C_J Y, J\mathcal{A}_X V)$$

and

$$\mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X = -g_M(H, X)JV$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

(d)

$$g_M(\mathcal{A}_X B_K Y, KV) = g_M(C_K Y, K\mathcal{A}_X V)$$

and

$$\mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X = -g_M(H, X)KV$$

for $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$.

Using Proposition 4.1, Lemma 3.12, and Lemma 4.7, we have

Lemma 4.9. *Let F be an h -Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h -Lagrangian basis. Then the following conditions are equivalent:*

(a) (M, g_M) is locally a twisted product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.

(b)

$$\mathcal{A}_X IY = 0 \quad \text{and} \quad \mathcal{T}_V IX = -g_M(H, X)IV$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

(c)

$$\mathcal{A}_X KY = 0 \quad \text{and} \quad \mathcal{T}_V KX = -g_M(H, X)KV$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

(d)

$$\mathcal{A}_X JY = 0 \quad \text{and} \quad \mathcal{T}_V JX = -g_M(H, X)JV$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Now, we consider the non-existence of some types of Riemannian submersions.

Using Proposition 4.1 and Theorem 4.4, we get

Theorem 4.10. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h -anti-invariant submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial double-twisted product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*

Lemma 4.11. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h -Lagrangian submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial double-twisted product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$.*

Theorem 4.12. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h -anti-invariant submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial twisted product manifold of the form $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$.*

Lemma 4.13. *Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Then there exists no h -Lagrangian submersion from $M = (M, E, g_M)$ onto (N, g_N) such that M is locally a nontrivial twisted product manifold of the form $M_{\ker F_*} \times M_{(\ker F_*)^\perp}$.*

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