J. Math. Fluid Mech. (2022) 24:109

1422-6928/22/040001-23
https://doi.org/10.1007/s00021-022-00742-y

Journal of Mathematical Fluid Mechanics

H^∞-Calculus for the Surface Stokes Operator and Applications

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Communicated by G. P. Galdi

Abstract. We consider a smooth, compact and embedded hypersurface Σ without boundary and show that the corresponding (shifted) surface Stokes operator admits a bounded H^∞-calculus with angle smaller than $\pi/2$. As an application, we consider critical spaces for the Navier–Stokes equations on the surface Σ. In case Σ is two-dimensional, we show that any solution with a divergence-free initial value in $L_2(\Sigma, \mathsf{T}\Sigma)$ exists globally and converges exponentially fast to an equilibrium, that is, to a Killing field.

Mathematics Subject Classification. 35Q35, 35Q30, 35B40.

Keywords. Surface Navier–Stokes equations, surface Stokes operator, H^∞-calculus, critical spaces, Killing vector fields, Korn's inequality, global existence.

1. Introduction

Suppose Σ is a smooth, compact, connected, embedded hypersurface in $\mathbb{R}^{d+1}$ without boundary. We then consider the motion of an incompressible viscous fluid that completely covers Σ and flows along Σ. The motion can be modeled by the *surface Navier–Stokes equations* for an incompressible viscous fluid

$$\begin{aligned} \varrho\big(\partial_t u + \mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma &= 0 \quad \text{on } \Sigma \\ \operatorname{div}_\Sigma u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma. \end{aligned} \tag{1.1}$$

Here, the density ϱ is a positive constant, $\mathcal{T}_\Sigma = 2\mu_s \mathcal{D}_\Sigma(u) - \pi\mathcal{P}_\Sigma$, and

$$\mathcal{D}_\Sigma(u) := \frac{1}{2}\mathcal{P}_\Sigma\left(\nabla_\Sigma u + [\nabla_\Sigma u]^{\mathsf{T}}\right)\mathcal{P}_\Sigma$$

is the surface rate-of-strain tensor, with u the fluid velocity and π the pressure. Moreover, $\mathcal{P}_\Sigma$ denotes the orthogonal projection onto the tangent bundle $\mathsf{T}\Sigma$ of Σ, $\operatorname{div}_\Sigma$ the surface divergence, and ∇_Σ the surface gradient. We refer to Chapter 2 in [22] and the Appendix in [24] for more information concerning these objects.

The formulation (1.1) coincides with [12, formula (3.2)]. In that paper, the equations were derived from fundamental continuum mechanical principles. The same equations were also derived in [14, formula (4.4)], based on global energy principles. We mention that the authors of [12,14] also consider material surfaces that may evolve in time.

Existence and uniqueness of solutions to the surface Navier–Stokes equations (1.1) was established in [24]. It was shown that the set of equilibria consists of all Killing vector fields on Σ, and that all of these

This work was supported by a grant from the Simons Foundation (#426729 and #853237, Gieri Simonett).

are normally stable. Moreover, it was shown that (1.1) can be reformulated as

$$\begin{aligned} \varrho\big(\partial_t u + \mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u)\big) - \mu_s(\Delta_\Sigma + \mathsf{Ric}_\Sigma)u + \nabla_\Sigma\pi &= 0 \quad \text{on } \Sigma \\ \mathrm{div}_\Sigma u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma, \end{aligned} \tag{1.2}$$

where Δ_Σ is the (negative) Bochner-Laplacian and Ric_Σ the Ricci tensor. To be more precise, in slight abuse of the usual convention, we interpret Ric_Σ here as the $(1,1)$-tensor given in local coordinates by $\mathsf{Ric}^\ell_k = g^{i\ell}R_{ik}$. We remind that in case $d=2$, $\mathsf{Ric}_\Sigma u = K_\Sigma u$, with K_Σ being the Gaussian curvature of Σ.

The formulation (1.2) shows that the surface Navier–Stokes equations can be formulated by intrinsic quantities that only depend on the geometry of the surface Σ, but not on the ambient space. In an intrinsic formulation, the surface Navier Stokes equations (1.2) can be stated as

$$\begin{aligned} \varrho\big(\partial_t u + \nabla_u u\big) - \mu_s\Delta_\Sigma u - \mu_s\mathsf{Ric}_\Sigma\, u + \mathsf{grad}\,\pi &= 0 \quad \text{on } \Sigma \\ \mathrm{div} u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma, \end{aligned} \tag{1.3}$$

where ∇ is the covariant derivative (induced by the Levi-Civita connection of Σ), and $\mathsf{grad}\,\pi = \nabla_\Sigma\pi$. An inspection of the proofs then shows that all the results in [24], and the results of this paper, are also valid for (1.3) for any smooth Riemannian manifold Σ without boundary. We remind that the Bochner Laplacian Δ_Σ is related to the Hodge Laplacian by the formula $\Delta_\Sigma = \Delta_H + \mathsf{Ric}_\Sigma$, with the usual identification of vector fields and one-forms by means of lowering or rising indices.

We would like to point out that several formulations for the surface Navier–Stokes equations have been considered in the literature, see [6] for a comprehensive discussion, and also [12, Section 3.2]. It has been advocated in [8, Note added to Proof], and also in the more recent publications [6,30], that the surface Navier–Stokes equations on a Riemannian manifold ought to be modeled by the system (1.3).

In Theorem 3.1, we show that *the surface Stokes operator*

$$A_{S,\Sigma}u := -2\mu_s P_{H,\Sigma}\mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) = -\mu_s P_{H,\Sigma}\big(\Delta_\Sigma + \mathsf{Ric}_\Sigma\big),$$

with $P_{\Sigma,H}$ being the *surface Helmholtz projection*, has the property that $(\omega + A_{S,\Sigma})$ admits a bounded H^∞-calculus in $L_{q,\sigma}(\Sigma,\mathsf{T}\Sigma)$ with H^∞-angle $\phi^\infty_{A_{S,\Sigma}} < \pi/2$, provided ω is larger than the spectral bound of $-A_{S,\Sigma}$ and $1<q<\infty$.

For the Stokes operator on *domains in Euclidean space* under various boundary conditions, the existence of an H^∞-calculus (or the related property of bounded imaginary powers) has been obtained by Giga [10], Abels [1], Noll and Saal [17], Saal [29], Prüss and Wilke [26], and Prüss [20]. We also refer to the survey article by Hieber and Saal [11] for additional references and information concerning the Stokes operator on domains in Euclidean space.

Having established the existence of a bounded H^∞-calculus allows us to employ the results in Prüss, Simonett, and Wilke [23,25] to establish existence and uniqueness of solutions to the system (1.1), or (1.3), for initial values u_0 in the *critical spaces* $B^{d/q-1}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma)$, see Theorem 4.1 and Theorem 4.3.

In particular, our results imply existence and uniqueness of solutions for initial values $u_0 \in L_{q,\sigma}(\Sigma,\mathsf{T}\Sigma)$ for $q=d$, see Corollary 4.4. Hence, the celebrated result of Kato [13] is also valid for the surface Navier–Stokes equations.

For $d=2$, we show in Theorem 4.9 that *any* solution to (1.1) with initial value $u_0 \in L_{2,\sigma}(\Sigma,\mathsf{T}\Sigma)$ exists globally and converges exponentially fast to an equilibrium, that is, to a Killing field. The proof is based on an abstract result in [23], Korn's inequality (established in Theorem A.3), and an energy estimate. Moreover, in Remark 4.10 we show that in case $d\ge 3$, any *global* solution converges to an equilibrium.

We refer to [23,26] for background information on critical spaces and for a discussion of the existing literature concerning critical spaces for the Navier–Stokes equations (and other equations) for domains in Euclidean space.

We would now like to briefly compare the results of this paper with previous results by other authors. Existence and uniqueness of solutions for the Navier–Stokes equations (1.3) for initial data in Morrey and Besov-Morrey spaces was established by Taylor [32] and Mazzucato [16], respectively; see also [6] for a comprehensive list of references. The authors in [16,32] employ techniques of pseudo-differential operators and they make use of the property that the Hodge Laplacian commutes with the Helmholtz projection. In case $d = 2$, global existence is proved in [32, Proposition 6.5], but that result does not establish convergence of solutions.

The Boussinesq-Scriven surface stress tensor $\mathcal{T}_\Sigma$ has also been employed in the situation where two incompressible fluids which are separated by a free surface, where surface viscosity (accounting for internal friction within the interface) is included in the model, see for instance [5,22]. Finally, we mention [12,18, 27,28] and the references contained therein for interesting numerical investigations.

Notation: We now introduce some notation and some auxiliary results that will be used in the sequel. It follows from the considerations in [24, Lemma A.1 and Remarks A.3] that

$$(\mathcal{P}_\Sigma \nabla_\Sigma u)^{\mathsf{T}} = \nabla u \quad \text{and} \quad \mathcal{P}_\Sigma (u \cdot \nabla_\Sigma v) = \nabla_u v \tag{1.4}$$

for tangential vector fields u, v, where ∇ denotes the covariant derivative (or the Levi-Civita connection) on Σ. In the following, we will occasionally take the liberty to use the shorter notation $\nabla_u v$ and ∇u mentioned in (1.4). We recall that for sufficiently smooth vectors fields u, say $u \in C^1(\Sigma, \mathsf{T}\Sigma)$, one has $\nabla u \in C(\Sigma, \mathsf{T}^1_1\Sigma)$, the space of all $(1,1)$-tensors on Σ. As the Levi-Civita connection ∇ is a metric connection, we have

$$\nabla_w (u|v) = (\nabla_w u|v) + (u|\nabla_w u) \tag{1.5}$$

for tangential vector fields u, v, w on Σ, where $(u|v) := g(u, v)$ is the Riemannian metric (induced by the Euclidean inner product of $\mathbb{R}^{d+1}$ in case Σ is embedded in $\mathbb{R}^{d+1}$). Occasionally, we also write $\mathsf{grad}\,\varphi$ in lieu of $\nabla_\Sigma \varphi$ for scalar functions φ.
We use the notation

$$(u|v)_\Sigma = \int_\Sigma (u|v)\, d\Sigma,$$

whenever the right hand side exists, say for $u \in L_q(\Sigma, \mathsf{T}\Sigma)$ and $v \in L_{q'}(\Sigma, \mathsf{T}\Sigma)$, where $1/q + 1/q' = 1$. For $k \in \mathbb{N}$ and $q \in (1, \infty)$, the space $H^k_q(\Sigma, \mathsf{T}\Sigma)$ is defined as the completion of $C^\infty(\Sigma, \mathsf{T}\Sigma)$, the space of all smooth vector fields, in $L_{1,\mathrm{loc}}(\Sigma, \mathsf{T}\Sigma)$ with respect to the norm

$$|u|_{H^k_q(\Sigma)} = \left(\sum_{i=0}^{k} |\nabla^i u|^q_{L_q(\Sigma)} \right)^{1/q}.$$

The Bessel potential spaces $H^s_q(\Sigma, \mathsf{T}\Sigma)$ and the Besov spaces $B^s_{qp}(\Sigma, \mathsf{T}\Sigma)$ can then be defined through interpolation, see for instance [3, Section 7]. It is well-known that these spaces can be given equivalent norms by means of local coordinates, see for instance [3, Theorem 7.3] (for the more general context of singular and uniformly regular manifolds).

2. The Surface Stokes Operator

In [24, Corollary 3.4] we showed that there exists a number $\omega_0 > 0$ such that for $\omega > \omega_0$, the system

$$\begin{aligned} \partial_t u + \omega u - 2\mu_s \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) + \nabla_\Sigma \pi &= f \quad \text{on } \Sigma \\ \mathrm{div}_\Sigma u &= 0 \quad \text{on } \Sigma \\ u(0) &= u_0 \quad \text{on } \Sigma \end{aligned} \tag{2.1}$$

admits a unique solution

$$u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\Sigma, \mathsf{T}\Sigma)), \quad \pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\Sigma)),$$

if and only if

$$f \in L_{p,\mu}(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)),\ u_0 \in B^{2\mu-2/p}_{qp}(\Sigma, \mathsf{T}\Sigma) \quad \text{and} \quad \mathrm{div}_\Sigma u_0 = 0.$$

Moreover, the solution (u, π) depends continuously on the given data (f, u_0) in the corresponding spaces.

Let $P_{H,\Sigma}$ denote the *surface Helmholtz projection*, defined by

$$P_{H,\Sigma} v := v - \nabla_\Sigma \psi_v, \quad v \in L_q(\Sigma, \mathsf{T}\Sigma),$$

where $\nabla_\Sigma \psi_v \in L_q(\Sigma, \mathsf{T}\Sigma)$ is the unique solution of

$$(\nabla_\Sigma \psi_v | \nabla_\Sigma \phi)_\Sigma = (v | \nabla_\Sigma \phi)_\Sigma, \quad \phi \in \dot{H}^1_{q'}(\Sigma),$$

see Lemma A.1. We note that $(P_{H,\Sigma} u | v)_\Sigma = (u | P_{H,\Sigma} v)_\Sigma$ for all $u \in L_q(\Sigma, \mathsf{T}\Sigma)$, $v \in L_{q'}(\Sigma, \mathsf{T}\Sigma)$, which follows directly from the definition of $P_{H,\Sigma}$ (and for smooth functions from the surface divergence theorem). Indeed,

$$\begin{aligned}(P_{H,\Sigma} u | v)_\Sigma &= (u - \nabla_\Sigma \psi_u | v)_\Sigma = (u|v)_\Sigma - (\nabla_\Sigma \psi_u | v)_\Sigma \\ &= (u|v)_\Sigma - (\nabla_\Sigma \psi_v | \nabla_\Sigma \psi_u)_\Sigma \\ &= (u|v)_\Sigma - (u | \nabla_\Sigma \psi_v)_\Sigma \\ &= (u | v - \nabla_\Sigma \psi_v)_\Sigma = (u | P_{H,\Sigma} v)_\Sigma,\end{aligned}$$

as $\psi_u \in \dot{H}^1_q(\Sigma)$, $\psi_v \in \dot{H}^1_{q'}(\Sigma)$. Let

$$X_0 := L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) := P_{H,\Sigma} L_q(\Sigma, \mathsf{T}\Sigma), \quad X_1 := H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) := H^2_q(\Sigma, \mathsf{T}\Sigma) \cap X_0.$$

The *surface Stokes operator* is defined by

$$A_{S,\Sigma} u := -2\mu_s P_{H,\Sigma} \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u), \quad u \in D(A_{S,\Sigma}) := X_1. \tag{2.2}$$

Making use of the projection $P_{H,\Sigma}$, (2.1) with $\mathrm{div}_\Sigma f = 0$ is equivalent to the equation

$$\partial_t u + \omega u + A_{S,\Sigma} u = f, \quad t > 0, \quad u(0) = u_0. \tag{2.3}$$

Indeed, if (u, π) is a solution to (2.1), then $u = P_{H,\Sigma} u$ solves (2.3) as can be seen by applying $P_{H,\Sigma}$ to the first equation in (2.1). Conversely, let u be a solution of (2.3). Then, by definition of $P_{H,\Sigma}$,

$$A_{S,\Sigma} u = -2\mu_s P_{H,\Sigma} \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) = -2\mu_s \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) + 2\mu_s \nabla_\Sigma \psi_v,$$

where $\psi_v \in \dot{H}^1_q(\Sigma)$ solves

$$(\nabla_\Sigma \psi_v | \nabla_\Sigma \phi)_\Sigma = (v | \nabla_\Sigma \phi)_\Sigma, \quad \phi \in \dot{H}^1_{q'}(\Sigma),$$

with $v := \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) \in L_q(\Sigma)$. Defining $\pi := 2\mu_s \psi_v$, we see that (u, π) is a solution of (2.1).

In particular, the operator $A_{S,\Sigma}$ has L_p-maximal regularity, hence $-(\omega + A_{S,\Sigma})$ generates an analytic C_0-semigroup in X_0 (see for instance [22, Proposition 3.5.2]) which is exponentially stable provided $\omega > s(-A_{S,\Sigma})$, where $s(\cdot)$ denotes the spectral bound of $-A_{S,\Sigma}$. This readily implies that the operator $\omega + A_{S,\Sigma}$ is sectorial with spectral angle $\phi_{\omega + A_{S,\Sigma}} < \pi/2$.

3. H^∞-calculus

In this section, we are going to prove the following result.

Theorem 3.1. *Let $q \in (1, \infty)$ and Σ be a smooth, compact, connected, embedded hypersurface in $\mathbb{R}^{d+1}$ without boundary. Let $A_{S,\Sigma}$ be the surface Stokes operator in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ defined in (2.2).*

Then $\omega + A_{S,\Sigma}$ admits a bounded H^∞-calculus with H^∞-angle $\phi^\infty_{A_{S,\Sigma}} < \pi/2$ for each $\omega > s(-A_{S,\Sigma})$, the spectral bound of $-A_{S,\Sigma}$.

Remark 3.2. Let $\mathcal{E}$ denote the set of equilibria. We have shown in [24], Proposition 4.1, that $s(-A_{S,\Sigma}) = 0$ provided $\mathcal{E} \neq \{0\}$ und $s(-A_{S,\Sigma}) < 0$ if $\mathcal{E} = \{0\}$.

Therefore, it follows from Theorem 3.1 that for each $\omega > 0$ the operator $\omega + A_{S,\Sigma}$ admits a bounded H^∞-calculus with H^∞-angle $\phi^\infty_{A_{S,\Sigma}} < \pi/2$. In case $\mathcal{E} = \{0\}$ one may set $\omega = 0$.

3.1. Resolvent and Pressure Estimates

We consider the following resolvent problem

$$\begin{aligned} \lambda u + (\omega - 2\mu_s \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma) u + \nabla_\Sigma \pi &= f \quad \text{on } \Sigma \\ \mathrm{div}_\Sigma u &= 0 \quad \text{on } \Sigma, \end{aligned} \tag{3.1}$$

where $\omega > s(-A_{S,\Sigma})$ and $\lambda \in \Sigma_{\pi-\phi}$, $\phi > \phi_{\omega+A_{S,\Sigma}}$. By sectoriality of the operator $\omega + A_{S,\Sigma}$ (and since $0 \in \rho(\omega + A_{S,\Sigma})$), it follows that for given $f \in L_q(\Sigma, \mathsf{T}\Sigma)$ there exists a unique solution

$$u \in H^2_q(\Sigma, \mathsf{T}\Sigma), \quad \pi \in \dot{H}^1_q(\Sigma)$$

of (3.1) and there is a constant $C > 0$ such that

$$(|\lambda| + 1)|u|_{L_q(\Sigma)} + |u|_{H^2_q(\Sigma)} + |\nabla \pi|_{L_q(\Sigma)} \le C|f|_{L_q(\Sigma)}, \tag{3.2}$$

for all $\lambda \in \Sigma_{\pi-\phi}$. Note that without loss of generality, we may assume that $P_0 \pi = \pi$, where

$$P_0 v := v - \frac{1}{|\Sigma|} \int_\Sigma v \, d\Sigma$$

for $v \in L_1(\Sigma)$. Furthermore, if $\mathrm{div}_\Sigma f = 0$, the pressure π satisfies the estimate

$$|\pi|_{L_q(\Sigma)} \le C|u|_{H^1_q(\Sigma)} \tag{3.3}$$

for some constant $C > 0$. The proof of estimate (3.3) follows exactly the lines of the proof of [24, Proposition 3.3].

3.2. Localization

By compactness of Σ, there exists a family of charts $\{(U_k, \varphi_k) : k \in \{1, \dots, N\}\}$ such that $\{U_k\}_{k=1}^N$ is an open covering of Σ. Let $\{\psi_k^2\}_{k=1}^N \subset C^\infty(\Sigma)$ be a partition of unity subordinate to the open covering $\{U_k\}_{k=1}^N$. Note that without loss of generality, we may assume that $\varphi_k(U_k) = B_{\mathbb{R}^d}(0, r)$. We call $\{(U_k, \varphi_k, \psi_k) : k \in \{1, \dots, N\}\}$ a *localization system* for Σ.

Let $\{\tau_{(k)j}(p)\}_{j=1}^d$ denote a local basis of the tangent space $\mathsf{T}_p\Sigma$ of Σ at $p \in U_k$ and denote by $\{\tau^j_{(k)}(p)\}_{j=1}^d$ the corresponding dual basis of the cotangent space $\mathsf{T}^*_p\Sigma$ at $p \in U_k$. Accordingly, we define $g^{ij}_{(k)} = (\tau^i_{(k)}|\tau^j_{(k)})$ and $g_{(k)ij}$ is defined in a very similar way, see the Appendix in [24]. Then, with $\bar{u} = u \circ \varphi_k^{-1}$, $\bar{\pi} = \pi \circ \varphi_k^{-1}$ and so on, the system (3.1) with respect to the local charts (U_k, φ_k), $k \in \{1, \dots, N\}$, reads as follows.

$$\begin{aligned} \lambda \bar{u}^\ell_{(k)} + (\omega - \mu_s \bar{g}^{ij}_{(k)} \partial_i \partial_j) \bar{u}^\ell_{(k)} + \bar{g}^{i\ell}_{(k)} \partial_i \bar{\pi}_{(k)} &= \bar{f}^\ell_{(k)} + F^\ell_{(k)}(\bar{u}, \bar{\pi}) \quad \text{in } \mathbb{R}^d \\ \partial_i \bar{u}^i_{(k)} &= H_{(k)}(\bar{u}) \qquad\qquad\quad \text{in } \mathbb{R}^d, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &\bar{u}^\ell_{(k)} = (\bar{u}\bar{\psi}_k|\bar{\tau}^\ell_{(k)}),\ \bar{\pi}_{(k)} = \bar{\pi}\bar{\psi}_k, \\ &\bar{f}^\ell_{(k)} = (\bar{f}\bar{\psi}_k|\bar{\tau}^\ell_{(k)}), \quad F^\ell_{(k)}(\bar{u}, \bar{\pi}) = \bar{\pi}\bar{g}^{i\ell}_{(k)} \partial_i \bar{\psi}_k + (B_{(k)}\bar{u}|\bar{\tau}^\ell_{(k)}), \end{aligned}$$

$\ell \in \{1, \dots, d\}$, $B_{(k)}$ collects all terms of order at most one and

$$H_{(k)}(\bar{u}) = \bar{u}^i \partial_i \bar{\psi}_k - \bar{u}^j_{(k)} (\bar{\tau}^i_{(k)} | \partial_i \bar{\tau}_{(k)j}).$$

Here, upon translation and rotation, $\bar{g}^{ij}_{(k)}(0) = \delta^i_j$ and the coefficients have been extended in such a way that $\bar{g}^{ij}_{(k)} \in W^2_\infty(\mathbb{R}^d)$ and $|\bar{g}^{ij}_{(k)} - \delta^i_j|_{L_\infty(\mathbb{R}^d)} \le \eta$, where $\eta > 0$ can be made as small as we wish, by decreasing the radius $r > 0$ of the ball $B_{\mathbb{R}^d}(0, r)$.

In order to handle system (3.4), we define vectors in $\mathbb{R}^d$ as follows:

$$\bar{u}_{(k)} := (\bar{u}^1_{(k)}, \dots, \bar{u}^d_{(k)}), \quad \bar{f}_{(k)} := (\bar{f}^1_{(k)}, \dots, \bar{f}^d_{(k)})$$

and

$$F_{(k)}(\bar{u}, \bar{\pi}) := (F^1_{(k)}(\bar{u}, \bar{\pi}), \dots, F^d_{(k)}(\bar{u}, \bar{\pi})).$$

Moreover, we define the matrix $G_{(k)} = (\bar{g}^{ij}_{(k)})^d_{i,j=1} \in \mathbb{R}^{d\times d}$. With these notations, system (3.4) reads as

$$\begin{aligned} \lambda \bar{u}_{(k)} + (\omega - \mu_s (G_{(k)}\nabla|\nabla))\bar{u}_{(k)} + G_{(k)}\nabla\bar{\pi}_{(k)} &= \bar{f}_{(k)} + F_{(k)}(\bar{u}, \bar{\pi}) && \text{in } \mathbb{R}^d \\ \operatorname{div}\, \bar{u}_{(k)} &= H_{(k)}(\bar{u}) && \text{in } \mathbb{R}^d. \end{aligned} \tag{3.5}$$

Let us remove the term $H_{(k)}(\bar{u})$, since it is not of lower order. For that purpose, we solve the equation $\operatorname{div}(G_{(k)}\nabla\phi_k) = H_{(k)}(\bar{u})$ by Lemma A.2 to obtain a unique solution $\nabla\phi_k \in H^2_q(\mathbb{R}^d)^d$ with

$$|\nabla\phi_k|_{H^2_q(\mathbb{R}^d)} \le C|H_{(k)}(\bar{u})|_{H^1_q(\mathbb{R}^d)}, \tag{3.6}$$

where C is a positive constant. For this, observe that $H_{(k)}(\bar{u})$ is compactly supported and we have $\int_{\mathbb{R}^d} H_{(k)}(\bar{u})dx = 0$. Therefore, $H_{(k)}(\bar{u})$ induces a functional on $\dot{H}^1_{q'}(\mathbb{R}^d)$ with norm bounded by $C|H_{(k)}(\bar{u})|_{L_q(\mathbb{R}^d)}$. To see this, choose $R > 0$ such that $\operatorname{supp}(H_{(k)}(\bar{u})) \subset B(0, R)$ and let $\phi \in \dot{H}^1_{q'}(\mathbb{R}^d)$. Then we have

$$\int_{\mathbb{R}^d} H_{(k)}(\bar{u})\phi\, dx = \int_{B(0,R)} H_{(k)}(\bar{u})(\phi - \hat{\phi})\, dx,$$

where $\hat{\phi} = |B(0, R)|^{-1} \int_{B(0,R)} \phi\, dx$. By the Poincaré-Wirtinger inequality,

$$\begin{aligned} \Big|\int_{B(0,R)} H_{(k)}(\bar{u})(\phi - \hat{\phi})\, dx\Big| &\le C|H_{(k)}(\bar{u})|_{L_q} |\nabla\phi|_{L_{q'}(B(0,R))} \\ &\le C|H_{(k)}(\bar{u})|_{L_q} |\nabla\phi|_{L_{q'}(\mathbb{R}^d)}. \end{aligned}$$

Let

$$\tilde{u}_{(k)} = \bar{u}_{(k)} - G_{(k)}\nabla\phi_k \quad \text{and} \quad \tilde{\pi}_{(k)} = \bar{\pi}_{(k)} + (\lambda + \omega)\phi_k.$$

It follows from (3.5) that the functions $(\tilde{u}_{(k)}, \tilde{\pi}_{(k)})$ then solve the system

$$\begin{aligned} \lambda \tilde{u}_{(k)} + (\omega - \mu_s (G_{(k)}\nabla|\nabla))\tilde{u}_{(k)} + G_{(k)}\nabla\tilde{\pi}_{(k)} &= \bar{f}_{(k)} + \tilde{F}_{(k)}(\bar{u}, \bar{\pi}) && \text{in } \mathbb{R}^d \\ \operatorname{div}\, \tilde{u}_{(k)} &= 0 && \text{in } \mathbb{R}^d, \end{aligned} \tag{3.7}$$

where $\tilde{F}_{(k)}(\bar{u}, \bar{\pi}) := F_{(k)}(\bar{u}, \bar{\pi}) + \mu_s(G_{(k)}\nabla|\nabla)(G_{(k)}\nabla\phi_k)$. In order to remove the pressure term in (3.7) we introduce the projections P^G_k, defined by

$$P^G_k v := v - G_{(k)}\nabla\Phi_k, \quad v \in L_q(\mathbb{R}^d).$$

Here, $\nabla\Phi_k \in L_q(\mathbb{R}^d)$ is the unique solution of $\operatorname{div}(G_{(k)}\nabla\Phi_k) = \operatorname{div} v$ in $\dot{H}^{-1}_q(\mathbb{R}^d)$, established in Lemma A.2. It is readily seen that $P^G_k v = v$ if $\operatorname{div} v = 0$ and $P^G_k(G_{(k)}\nabla\tilde{\pi}_{(k)}) = 0$. Applying the projection P^G_k to equation (3.7) leads to

$$\lambda \tilde{u}_{(k)} + (\omega - \mu_s P^G_k (G_{(k)}\nabla|\nabla))\tilde{u}_{(k)} = P^G_k(\bar{f}_{(k)} + \tilde{F}_{(k)}(\bar{u}, \bar{\pi})) \quad \text{in } \mathbb{R}^d. \tag{3.8}$$

We claim that each of the operators $\omega + A^G_{S,k} := \omega - \mu_s P^G_k(G_{(k)}\nabla|\nabla)$ in (3.8) admits a bounded H^∞-calculus in $P^G_k L_q(\mathbb{R}^d) = L_{q,\sigma}(\mathbb{R}^d)$, provided ω is sufficiently large. To see this, we write

$$-A^G_{S,k}u = -A_S u + \mu_s((G_{(k)} - I)\nabla|\nabla)u - \mu_s G_{(k)}\nabla\Phi_k,$$

where $A_S = -\mu_s P_H \Delta u = -\mu_s \Delta u$ is the Stokes operator in $L_{q,\sigma}(\mathbb{R}^d)$ and P_H is the Helmholtz projection in $\mathbb{R}^d$.

Recall that each matrix $G_{(k)}$ is a perturbation of the identity in $\mathbb{R}^{d\times d}$. Therefore,

$$|\mu_s((G_{(k)} - I)\nabla|\nabla)u|_{L_q(\mathbb{R}^d)} \le \eta |u|_{H^2_q(\mathbb{R}^d)},$$

where we may choose $\eta > 0$ as small as we wish. Furthermore,

$$|G_{(k)}\nabla\Phi_k|_{L_q(\mathbb{R}^d)} \le C|(G_{(k)}\nabla|\nabla)u - \Delta u|_{L_q(\mathbb{R}^d)} \le \eta|u|_{H^2_q(\mathbb{R}^d)},$$

by Lemma A.2, since

$$\operatorname{div}(G_{(k)}\nabla\Phi_k) = \operatorname{div}(G_{(k)}\nabla|\nabla)u = \operatorname{div}(G_{(k)}\nabla|\nabla)u - \operatorname{div}\Delta u$$

as $\operatorname{div}\Delta u = \Delta \operatorname{div} u = 0$ in $\mathbb{R}^d$. As before, we may choose $\eta > 0$ as small as we wish.

Note that the shifted Stokes operator $\omega + A_S$ admits a bounded H^∞-calculus in $L_{q,\sigma}(\mathbb{R}^d)$ with angle $\phi^\infty_{A_S} < \pi/2$, see e.g. [22, Theorem 7.1.2]. By the abstract perturbation result [7, Theorem 3.2] (see also [20, Section 3]), there exists $\omega_0 > 0$ such that each of the operators $\omega + A^G_{S,k}$ admits a bounded H^∞-calculus in $L_{q,\sigma}(\mathbb{R}^d)$ if $\omega \ge \omega_0$. Moreover, for any given $\phi_0 > \phi^\infty_{A_S}$ we may assume $\phi^\infty_{A^G_{S,k}} \le \phi_0$, provided that $|G_{(k)} - I|_\infty$ is sufficiently small.

This yields the following representation of the resolvent $u = (\lambda + \omega + A_{S,\Sigma})^{-1} f$.

$$\begin{aligned}
u &= P_{H,\Sigma}\sum_{k=1}^N \psi_k^2 u = P_{H,\Sigma}\sum_{k=1}^N \psi_k(\bar{u}^\ell_{(k)}\bar{\tau}_{(k)\ell})\circ\varphi_k \\
&= P_{H,\Sigma}\sum_{k=1}^N \psi_k((\tilde{u}_{(k)}|e_\ell)\bar{\tau}_{(k)\ell})\circ\varphi_k + P_{H,\Sigma}\sum_{k=1}^N \psi_k((G_{(k)}\nabla\phi_k|e_\ell)\bar{\tau}_{(k)\ell})\circ\varphi_k \\
&= Tu + S(\lambda)f + R(\lambda)f,
\end{aligned}$$

where $\{e_\ell\}_{\ell=1}^d$ is the standard basis in $\mathbb{R}^d$ and

$$\begin{aligned}
Tu &:= P_{H,\Sigma}\sum_{k=1}^N \psi_k((G_{(k)}\nabla\phi_k|e_\ell)\bar{\tau}_{(k)\ell})\circ\varphi_k, \\
S(\lambda)f &:= P_{H,\Sigma}\sum_{k=1}^N \psi_k(((\lambda+\omega+A^G_{S,k})^{-1}P^G_k\bar{f}_{(k)}|e_\ell)\bar{\tau}_{(k)\ell})\circ\varphi_k \\
R(\lambda)f &:= P_{H,\Sigma}\sum_{k=1}^N \psi_k(((\lambda+\omega+A^G_{S,k})^{-1}P^G_k\tilde{F}_{(k)}(\bar{u}(f),\bar{\pi}(f))|e_\ell)\bar{\tau}_{(k)\ell})\circ\varphi_k.
\end{aligned}$$

In a next step, we will estimate the term $R(\lambda)f$ in $L_q(\Sigma, \mathsf{T}\Sigma)$. To this end, observe that the operators $P_{H,\Sigma}$ and P^G_k are bounded in $L_q(\Sigma, \mathsf{T}\Sigma)$ and $L_q(\mathbb{R}^d)$, respectively. This, together with (3.2), (3.3), (3.6) and the fact that each of the operators $\omega + A^G_{S,k}$ is sectorial in $L_{q,\sigma}(\mathbb{R}^d)$ for ω sufficiently large, yields the estimate

$$|R(\lambda)f|_{L_q(\Sigma)} \le \frac{C}{(|\lambda|+1)^{3/2}}|f|_{L_q(\Sigma)} \tag{3.9}$$

for some constant $C > 0$. Indeed, by (3.2), (3.3), (3.6), we obtain

$$\begin{aligned}
|\tilde{F}_{(k)}(\bar{u},\bar{\pi})|_{L_q(\mathbb{R}^d)} &\le C\left(|F_{(k)}(\bar{u},\bar{\pi})|_{L_q(\mathbb{R}^d)} + |\nabla\phi_k|_{H^2_q(\mathbb{R}^d)}\right) \\
&\le C\left(|\pi|_{L_q(\Sigma)} + |u|_{H^1_q(\Sigma)}\right) \\
&\le C|u|_{H^1_q(\Sigma)} \\
&\le C(1+|\lambda|)^{-1/2}((1+|\lambda|)|u|_{L_q(\Sigma)} + |u|_{H^2_q(\Sigma)})
\end{aligned}$$

$$\leq C(1+|\lambda|)^{-1/2}|f|_{L_q(\Sigma)}.$$

Here we have also used complex interpolation $H^1_q(\Sigma) = [L_q(\Sigma), H^2_q(\Sigma))]_{1/2}$.

Next, observe that $T : L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) \to H^1_q(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$, since $G_{(k)} \in W^2_\infty(\mathbb{R}^d)^{d\times d}$ and $\nabla\phi_k \in H^1_q(\mathbb{R}^d)^d$ for each $k \in \{1, \ldots, N\}$, hence T is compact in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. Consequently, $I - T$ is a Fredholm operator with index 0. In particular, $\ker(I - T)$ is finite dimensional and the range $\operatorname{ran}(I - T)$ is closed in $L_q(\Sigma)$. Let $\{v_1, \ldots, v_m\}$ be an orthonormal basis of $\ker(I - T)$ and define

$$Q : L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) \to \ker(I - T)$$

by

$$Qu = \sum_{k=1}^{m} (u|v_k)_\Sigma v_k.$$

Then it can be readily checked that Q is a projection onto $\ker(I - T)$ and it is continuous in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ for *any* $q \in (1,\infty)$, since $v_k \in H^1_q(\Sigma, \mathsf{T}\Sigma)$ for any $q \in (1,\infty)$ (using a bootstrap argument). Consequently, the operator

$$I - T : \operatorname{ran}(I - Q) \to \operatorname{ran}(I - T)$$

is invertible with bounded inverse.

We use the resolvent representation

$$(I - T)u = S(\lambda)f + R(\lambda)f,$$

to conclude that the right hand side belongs to $\operatorname{ran}(I - T)$, hence

$$(I - Q)u = (I - T)^{-1} \left(S(\lambda)f + R(\lambda)f\right).$$

Let $\phi_0 \in (\phi^\infty_{A_S}, \pi/2)$ such that $\phi^\infty_{A^G_{S,k}} \leq \phi_0$ for each $k \in \{1, \ldots, N\}$. For $h \in H_0(\Sigma_\phi)$, $\phi \in (\phi_0, \pi/2)$, we then obtain

$$\begin{aligned}(I - Q)h(\omega + A_{S,\Sigma})f = (I - T)^{-1} \Bigg(& P_{H,\Sigma} \sum_{k=1}^{N} \psi_k((h(\omega + A^G_{S,k})P^G_k \bar{f}_{(k)}|e_\ell)\bar{\tau}_{(k)\ell}) \circ \varphi_k \\ & + \frac{1}{2\pi i}\int_\Gamma h(-\lambda)R(\lambda)f d\lambda\Bigg),\end{aligned}$$

with $\Gamma = (\infty, 0]e^{-i(\pi-\theta)} \cup [0,\infty)e^{i(\pi-\theta)}$, $\theta \in (\phi_0, \phi)$. Estimate (3.9) then yields

$$|(I - Q)h(\omega + A_{S,\Sigma})|_{\mathcal{B}(L_q(\Sigma))} \leq C|h|_\infty,$$

since each of the operators $\omega + A^G_{S,k}$ has a bounded H^∞-calculus. The remaining part $Qh(\omega + A_{S,\Sigma})$ may be treated as follows.

$$\begin{aligned}Qh(\omega + A_{S,\Sigma})f &= Q\frac{1}{2\pi i}\int_\Gamma h(-\lambda)(\lambda + \omega)(\lambda + \omega + A_{S,\Sigma})^{-1} f \frac{d\lambda}{\lambda + \omega} \\ &= Q\frac{1}{2\pi i}\int_\Gamma h(-\lambda)(I - A_{S,\Sigma}(\lambda + \omega + A_{S,\Sigma})^{-1}) f \frac{d\lambda}{\lambda + \omega} \\ &= Qh(\omega)f - Q\frac{1}{2\pi i}\int_\Gamma h(-\lambda)A_{S,\Sigma}(\lambda + \omega + A_{S,\Sigma})^{-1} f \frac{d\lambda}{\lambda + \omega}.\end{aligned}$$

For the last integral, we employ the definition of the projection Q from above to obtain

$$\begin{aligned}& Q\frac{1}{2\pi i}\int_\Gamma h(-\lambda)A_{S,\Sigma}(\lambda + \omega + A_{S,\Sigma})^{-1} f \frac{d\lambda}{\lambda + \omega} \\ & \quad = \frac{1}{2\pi i}\sum_{k=1}^{m} v_k \int_\Gamma h(-\lambda)(A_{S,\Sigma}(\lambda + \omega + A_{S,\Sigma})^{-1} f|v_k)_\Sigma \frac{d\lambda}{\lambda + \omega}\end{aligned}$$

$$= 2\mu_s \frac{1}{2\pi i}\sum_{k=1}^{m} v_k \int_\Gamma h(-\lambda)(\mathcal{D}_\Sigma(\lambda+\omega+A_{S,\Sigma})^{-1}f : \nabla_\Sigma v_k)\frac{d\lambda}{\lambda+\omega}.$$

By (3.2), we therefore obtain

$$|Qh(\omega + A_{S,\Sigma})|_{\mathcal{B}(L_q(\Sigma))} \le C|h|_\infty,$$

for some constant $C > 0$. Consequently, the operator $\omega + A_{S,\Sigma}$ admits a bounded H^∞-calculus in $L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ with angle $\phi^\infty_{A_{S,\Sigma}} < \pi/2$ provided $\omega > 0$ is large enough. An application of [22, Corollary 3.3.15] finally yields that it is enough to require $\omega > s(-A_{S,\Sigma})$. This completes the proof of Theorem 3.1.

4. Critical Spaces

4.1. Strong Setting

We consider the abstract system

$$\partial_t u + A_{S,\Sigma}u = F_\Sigma(u), \quad t > 0, \quad u(0) = u_0, \tag{4.1}$$

where $F_\Sigma(u) := -P_{H,\Sigma}\mathcal{P}_\Sigma(u\cdot\nabla_\Sigma u) = -P_{H,\Sigma}\nabla_u u$. Let $q \in (1,\infty)$,

$$X_0 := L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) \quad \text{and} \quad X_1 := D(A_{S,\Sigma}) = H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

Furthermore, let $X_\beta = [X_0, X_1]_\beta$ for some $\beta \in (0,1)$, where $[\cdot,\cdot]_\beta$ denotes the complex interpolation functor. Then, by Theorem 3.1, it holds that $X_\beta = D((\omega + A_{S,\Sigma})^\beta)$ for $\omega > s(-A_{S,\Sigma})$. In [24, Section 3.5], we determined the real and complex interpolation spaces $(X_0, X_1)_{\alpha,p}$ and $[X_0, X_1]_\alpha$ as

$$[X_0, X_1]_\alpha = H^{2\alpha}_{q,\sigma}(\Sigma, \mathsf{T}\Sigma),$$
$$(X_0, X_1)_{\alpha,p} = B^{2\alpha}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma),$$

for $\alpha \in (0,1)$ and $p \in (1,\infty)$, where $H^s_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) := H^s_q(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ and $B^s_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma) := B^s_{qp}(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ for $s \in (0,2)$.

By Hölder's inequality, the estimate

$$|F_\Sigma(u)|_{L_q(\Sigma)} \le C|u|_{L_{qr'}(\Sigma)}|u|_{H^1_{qr}(\Sigma)}$$

holds. We choose $r, r' \in (1,\infty)$, $1/r + 1/r' = 1$, in such a way that

$$1 - \frac{d}{qr} = -\frac{d}{qr'}, \quad \text{or equivalently,} \quad \frac{d}{qr} = \frac{1}{2}\left(1 + \frac{d}{q}\right),$$

which is feasible if $q \in (1,d)$. Next, by Sobolev embedding, we have

$$[X_0, X_1]_\beta \subset H^{2\beta}_q(\Sigma, \mathsf{T}\Sigma) \hookrightarrow H^1_{qr}(\Sigma, \mathsf{T}\Sigma) \cap L_{qr'}(\Sigma, \mathsf{T}\Sigma),$$

provided

$$2\beta - \frac{d}{q} = 1 - \frac{d}{qr}, \quad \text{or equivalently,} \quad \beta = \frac{1}{4}\left(\frac{d}{q} + 1\right).$$

The condition $\beta < 1$ requires $q > d/3$, hence $q \in (d/3, d)$. For $q \in (d/3, d)$ we define the *critical weight* by

$$\mu_c := \frac{1}{2}\left(\frac{d}{q} - 1\right) + \frac{1}{p},$$

with $2/p + d/q \le 3$, so that $\mu_c \in (1/p, 1]$. We consider now the problem

$$\partial_t u + \omega u + A_{S,\Sigma}u = F_\Sigma(u) + \omega u, \quad t > 0, \quad u(0) = u_0, \tag{4.2}$$

where $\omega > s(-A_{S,\Sigma})$. It is clear that u is a solution of (4.1) if and only if u is a solution to (4.2). Note that for each $\omega > s(-A_{S,\Sigma})$, the operator $\omega + A_{S,\Sigma}$ admits a bounded H^∞-calculus in X_0 with H^∞-angle $\phi^\infty_{A_{S,\Sigma}} < \pi/2$. We may therefore apply [23, Theorem 1.2] to (4.2) which yields the following result.

Theorem 4.1. *Let $p \in (1,\infty)$, $q \in (d/3, d)$ such that $\frac{2}{p} + \frac{d}{q} \leq 3$. Then for any initial value $u_0 \in B^{d/q-1}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma)$ there exists a unique strong solution*

$$u \in H^1_{p,\mu_c}((0,a); L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu_c}((0,a); H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma))$$

of (4.1) for some $a = a(u_0) > 0$, with $\mu_c = 1/p + d/(2q) - 1/2$. The solution exists on a maximal time interval $[0, t_+(u_0))$ and depends continuously on u_0. In addition, we have

$$u \in C([0,t_+); B^{d/q-1}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap C((0,t_+); B^{2-2/p}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma))$$

which means that the solution regularizes instantaneously if $2/p + d/q < 3$.

Remark 4.2. Note that in case $d = 3$ and $p = q = 2$, the initial value belongs to

$$B^{1/2}_{22,\sigma}(\Sigma, \mathsf{T}\Sigma) = H^{1/2}_{2,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

Hence, the celebrated result of Fujita & Kato [9] holds true for the surface Navier–Stokes equations.

4.2. Weak Setting

In order to cover the case $q \geq d$, we proceed as follows. Let $A_0 = \omega + A_{S,\Sigma}$, $\omega > s(-A_{S,\Sigma})$ and recall that $X_0 = L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. By [2, Theorems V.1.5.1 & V.1.5.4], the pair (X_0, A_0) generates an interpolation-extrapolation scale (X_α, A_α), $\alpha \in \mathbb{R}$, with respect to the complex interpolation functor. Note that for $\alpha \in (0,1)$, A_α is the X_α-realization of A_0 (the restriction of A_0 to X_α) and

$$X_\alpha = D(A_0^\alpha) = [X_0, X_1]_\alpha = H^{2\alpha}_{q,\sigma}(\Sigma, \mathsf{T}\Sigma),$$

since A_0 admits a bounded H^∞-calculus.

Let $X_0^\sharp := (X_0)'$ and $A_0^\sharp := (A_0)'$ with $D(A_0^\sharp) =: X_1^\sharp$. Then $(X_0^\sharp, A_0^\sharp)$ generates an interpolation-extrapolation scale $(X_\alpha^\sharp, A_\alpha^\sharp)$, the dual scale, and by [2, Theorem V.1.5.12], it holds that

$$(X_\alpha)' = X^\sharp_{-\alpha} \quad \text{and} \quad (A_\alpha)' = A^\sharp_{-\alpha}$$

for $\alpha \in \mathbb{R}$. Choosing $\alpha = 1/2$ in the scale (X_α, A_α), we obtain an operator

$$A_{-1/2} : X_{1/2} \to X_{-1/2},$$

where $X_{-1/2} = (X^\sharp_{1/2})'$ (by reflexivity) and, since also $A_0^\sharp$ has a bounded H^∞-calculus,

$$X^\sharp_{1/2} = D((A_0^\sharp)^{1/2}) = [X_0^\sharp, X_1^\sharp]_{1/2} = H^1_{q',\sigma}(\Sigma, \mathsf{T}\Sigma),$$

with $p' = p/(p-1)$ being the conjugate exponent to $p \in (1,\infty)$. Moreover, we have $A_{-1/2} = (A^\sharp_{1/2})'$ and $A^\sharp_{1/2}$ is the restriction of $A_0^\sharp$ to $X^\sharp_{1/2}$. Thus, the operator $A_{-1/2} : X_{1/2} \to X_{-1/2}$ inherits the property of a bounded H^∞-calculus with H^∞-angle $\phi^\infty_{A_{-1/2}} < \pi/2$ from the operator A_0.

Since $A_{-1/2}$ is the closure of A_0 in $X_{-1/2}$ it follows that $A_{-1/2}u = A_0 u$ for $u \in X_1 = D(A_0) = H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ and thus, for all $v \in X^\sharp_{1/2}$, it holds that

$$\langle A_{-1/2}u, v\rangle = (A_0 u|v)_\Sigma = 2\mu_s \int_\Sigma \mathcal{D}_\Sigma(u) : \mathcal{D}_\Sigma(v)\, d\Sigma + \omega(u|v)_\Sigma,$$

where we made use of the surface divergence theorem. Using that X_1 is dense in $X_{1/2}$, we obtain the identity

$$\langle A_{-1/2}u, v\rangle = 2\mu_s \int_\Sigma \mathcal{D}_\Sigma(u) : \mathcal{D}_\Sigma(v)\, d\Sigma + \omega(u|v)_\Sigma,$$

valid for all $(u,v) \in X_{1/2} \times X^\sharp_{1/2}$. We call the operator $A^{\mathsf{w}}_{S,\Sigma} := A_{-1/2} - \omega$ the **weak surface Stokes operator**, given by its representation

$$\langle A^{\mathsf{w}}_{S,\Sigma} u, v\rangle = 2\mu_s \int_\Sigma \mathcal{D}_\Sigma(u) : \mathcal{D}_\Sigma(v)\, d\Sigma, \quad (u,v) \in X_{1/2} \times X^\sharp_{1/2}.$$

Multiplying (4.1) by a function $\phi \in X^\sharp_{1/2} = H^1_{q',\sigma}(\Sigma, \mathsf{T}\Sigma)$ and using the surface divergence theorem, we obtain the weak formulation

$$\partial_t u + A^{\mathsf{w}}_{S,\Sigma} u = F^{\mathsf{w}}_\Sigma(u), \quad u(0) = u_0, \tag{4.3}$$

in $X_{-1/2}$, where

$$\langle F^{\mathsf{w}}_\Sigma(u), \phi\rangle := (\nabla_u \phi | u)_\Sigma.$$

Note that u is a solution of (4.3) if and only if u solves

$$\partial_t u + \omega u + A^{\mathsf{w}}_{S,\Sigma} u = F^{\mathsf{w}}_\Sigma(u) + \omega u, \quad u(0) = u_0, \tag{4.4}$$

in $X_{-1/2}$. We will apply Theorem 1.2 in [23] to (4.4) with the choice

$$X^{\mathsf{w}}_0 = X_{-1/2} \quad \text{and} \quad X^{\mathsf{w}}_1 = X_{1/2}.$$

For that purpose, we will first characterize some relevant interpolation spaces. Let

$$\begin{gathered} H^s_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) := \begin{cases} H^s_q(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma), & 0 \le s \le 1, \\ \big(H^{-s}_{q',\sigma}(\Sigma, \mathsf{T}\Sigma)\big)', & -1 \le s < 0, \end{cases} \\ B^s_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma) := \begin{cases} B^s_{qp}(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma), & 0 < s \le 1, \\ (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{1/2,p}, & s = 0, \\ \big(B^{-s}_{q'p',\sigma}(\Sigma, \mathsf{T}\Sigma)\big)', & -1 \le s < 0. \end{cases} \end{gathered} \tag{4.5}$$

By similar arguments as in [26, Section 2.3] we obtain

$$\begin{aligned} [X_{-1/2}, X_{1/2}]_\theta &= H^{2\theta-1}_{q,\sigma}(\Sigma, \mathsf{T}\Sigma), \qquad \theta \in (0,1), \\ (X_{-1/2}, X_{1/2})_{\theta,p} &= B^{2\theta-1}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma), \qquad \theta \in (0,1). \end{aligned} \tag{4.6}$$

Next we show that the nonlinearity $F^{\mathsf{w}}_\Sigma : X^{\mathsf{w}}_\beta \to X^{\mathsf{w}}_0$ is well defined, where X^{w}_β denotes the complex interpolation space, that is, $X^{\mathsf{w}}_\beta := [X^{\mathsf{w}}_0, X^{\mathsf{w}}_1]_\beta$ for $\beta \in (1/2, 1)$. By (4.5), (4.6) and Sobolev embedding, we have

$$X^{\mathsf{w}}_\beta \hookrightarrow H^{2\beta-1}_q(\Sigma, \mathsf{T}\Sigma) \hookrightarrow L_{2q}(\Sigma, \mathsf{T}\Sigma), \tag{4.7}$$

provided that $2\beta - 1 \ge \frac{d}{2q}$. From now on, we assume $2\beta - 1 = \frac{d}{2q}$, which means $q > d/2$ as $\beta < 1$. Then, by Hölder's inequality and (4.7), we obtain

$$|\langle F^{\mathsf{w}}_\Sigma(u), \phi\rangle| \le |u|^2_{L_{2q}(\Sigma)} |\phi|_{H^1_{q'}(\Sigma)} \le C|u|^2_{X^{\mathsf{w}}_\beta} |\phi|_{H^1_{q'}(\Sigma)},$$

showing that

$$F^{\mathsf{w}}_\Sigma : X^{\mathsf{w}}_\beta \to X^{\mathsf{w}}_0 \quad \text{with} \quad |F^{\mathsf{w}}_\Sigma(u)|_{X^{\mathsf{w}}_0} \le C|u|^2_{X^{\mathsf{w}}_\beta}.$$

For $2\beta - 1 = d/2q$, the critical weight $\mu_c \in (1/p, 1]$ is given by $\mu_c := \mu^{\mathsf{w}}_c = 1/p + d/2q$ and the corresponding critical trace space in the weak setting reads

$$X^{\mathsf{w}}_{\gamma,\mu_c} = (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{\mu_c - 1/p, p} = B^{d/q-1}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

Note that in case $q \in (d/2, d)$, the critical spaces in the weak and strong setting coincide.

The existence and uniqueness result for (4.3) in critical spaces reads as follows.

Theorem 4.3. *Let $p \in (1,\infty)$ and $q \in (d/2,\infty)$ such that $\frac{2}{p}+\frac{d}{q} \le 2$. Then for any initial value $u_0 \in B^{d/q-1}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma)$ there exists a unique solution*

$$u \in H^1_{p,\mu_c}((0,a);H^{-1}_{q,\sigma}(\Sigma,\mathsf{T}\Sigma)) \cap L_{p,\mu_c}((0,a);H^1_{q,\sigma}(\Sigma,\mathsf{T}\Sigma))$$

of (4.3) for some $a = a(u_0) > 0$, with $\mu_c = 1/p + d/2q$. The solution exists on a maximal time interval $[0,t_+(u_0))$ and depends continuously on u_0. Moreover,

$$u \in C([0,t_+);B^{d/q-1}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma)).$$

Suppose $p > 2$ and $q \ge d$. Then each solution with $u_0 \in B^{d/q-1}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma)$ satisfies in addition

$$u \in H^1_{p,\mathrm{loc}}((0,t^+);L_{q,\sigma}(\Sigma,\mathsf{T}\Sigma)) \cap L_{p,\mathrm{loc}}((0,t^+);H^2_{q,\sigma}(\Sigma,\mathsf{T}\Sigma)).$$

Hence, in this case, each solution regularizes instantaneously and becomes a strong solution.

Proof. Since $A_{-1/2} = \omega + A^{\mathsf{w}}_{S,\Sigma}$ admits a bounded H^∞-calculus in $X_{-1/2}$ with H^∞-angle $\phi^\infty_{A_{-1/2}} < \pi/2$, the first assertion follows readily from [23, Theorem 1.2].
Suppose that $p > 2$ and $q \ge d$. Then $\mu_c = 1/p + d/2q < 1$ and [23, Theorem 1.2] yields $u \in C((0,t_+);B^{1-2/p}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma))$. Therefore,

$$u(t) \in X^{\mathsf{w}}_{\gamma,1} = (X_{-1/2},X_{1/2})_{1-1/p,p} = B^{1-2/p}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma),$$

for $t \in (0,t_+(u_0))$. Noting that for $\mu \in (1/p,1/2]$ we have the embedding

$$B^{1-2/p}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma) \hookrightarrow B^{2\mu-2/p}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma)$$

at our disposal, we may now solve (4.1) by [24, Theorem 3.5] with initial value $u(t_0)$, $t_0 \in (0,t_+(u_0))$, to obtain a strong solution. The assertion of the theorem follows now from uniqueness, see [15, Theorem 3.4(c)]. □

Corollary 4.4. *Suppose $d \ge 2$. Then equation (4.3) has for each initial value $u_0 \in L_{d,\sigma}(\Sigma,\mathsf{T}\Sigma)$ a unique solution which enjoys the regularity properties of Theorem 4.3 with $q = d$ for each fixed p with $p > 2$ and $p \ge d$.*

Proof. Suppose $p > 2$ and $p \ge d$. Then the embedding $L_{d,\sigma}(\Sigma,\mathsf{T}\Sigma) \hookrightarrow B^0_{dp,\sigma}(\Sigma,\mathsf{T}\Sigma)$ holds, see for instance [4, Theorem 6.2.4 and Theorem 6.4.4]. The assertion follows now from Theorem 4.3 with $q = d$. □

We now consider the limiting case $p = q = d = 2$. In this case, we have $\mu_c = 1$ and the corresponding critical trace space is given by

$$X^{\mathsf{w}}_{\gamma,\mu_c} = (X_{-1/2},X_{1/2})_{1/2,2} = [X_{-1/2},X_{1/2}]_{1/2} = L_{2,\sigma}(\Sigma,\mathsf{T}\Sigma),$$

see for example [33, Remark 1.18.10.3].

We may now also extend each weak solution to a strong solution. To this end, observe that

$$X^{\mathsf{w}}_{\gamma,\mu_c} = (X_{-1/2},X_{1/2})_{1/2,2} \hookrightarrow (X_{-1/2},X_{1/2})_{1/2,r}$$

for any $r \in (2,\infty)$. Then we solve (4.3) by Theorem 4.3 with $u_0 \in (X_{-1/2},X_{1/2})_{1/2,r}$ and $\mu_c = 1/r + 1/2$ (choosing $p = r$ and $q = d = 2$). This yields

$$u \in H^1_{r,\mu_c}((0,a);X^{\mathsf{w}}_0) \cap L_{r,\mu_c}((0,a);X^{\mathsf{w}}_1)$$

with $X^{\mathsf{w}}_1 = H^1_{2,\sigma}(\Sigma,\mathsf{T}\Sigma)$ and $X^{\mathsf{w}}_0 = H^{-1}_{2,\sigma}(\Sigma,\mathsf{T}\Sigma)$. Since now $u(t) \in B^{1-2/r}_{2r,\sigma}(\Sigma,\mathsf{T}\Sigma)$ for $t \in (0,a)$, we may argue as above to conclude that the weak solution regularizes to a strong solution

$$u \in H^1_{r,\mathrm{loc}}((0,a);L_{2,\sigma}(\Sigma,\mathsf{T}\Sigma)) \cap L_{r,\mathrm{loc}}((0,a);H^2_{2,\sigma}(\Sigma,\mathsf{T}\Sigma)).$$

Moreover,

$$u(t) \in (L_{2,\sigma}(\Sigma,\mathsf{T}\Sigma),H^2_{2,\sigma}(\Sigma,\mathsf{T}\Sigma))_{1-1/r,r} = B^{2-2/r}_{2r,\sigma}(\Sigma,\mathsf{T}\Sigma) \hookrightarrow B^{2\mu-2/p}_{qp,\sigma}(\Sigma,\mathsf{T}\Sigma)$$

for $t \in (0,a)$, provided $q \geq 2$, $p \geq r$ and $2-2/r-d/2 \geq 2\mu - 2/p - d/q$. For $d=2$ this means

$$\mu \leq 1/2 + 1/p + 1/q - 1/r.$$

We now solve (4.1) with initial value $u(t_0) \in B^{2\mu-2/p}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma)$ by [24, Theorem 3.5] to obtain a solution

$$v \in H^1_{p,\mu}((0,a); L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}((0,a); H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)).$$

As in the proof of [15, Theorem 3.4(c)] we conclude that uniqueness holds, that is, $v(t) = u(t_0+t)$. As t_0 can be chosen arbitrarily small, this implies that u shares the regularity properties of v for $t>0$. We have shown the following result.

Theorem 4.5. *Let $d=2$. Then for any $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$, problem (4.3) admits a unique solution*

$$u \in H^1_2((0,a), H^{-1}_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_2((0,a); H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma))$$

for some $a=a(u_0)>0$. The solution exists on a maximal time interval $[0,t^+(u_0))$. In addition, we have

$$u \in C([0,t^+); L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)). \tag{4.8}$$

Furthermore, each solution satisfies

$$u \in H^1_{p,\mathrm{loc}}((0,t^+); L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mathrm{loc}}((0,t^+); H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma))$$

for any fixed $q \in (1,\infty)$ and any fixed $p \in (2,\infty)$. Therefore, any solution with initial value $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$ regularizes instantaneously and becomes a strong L_p-L_q solution.

Proof. According to the considerations preceding the Theorem, the assertions hold true for a fixed $q \geq 2$ and $p \in (2,\infty)$. The case $q \in (1,2)$ follows readily from the result in the L_2-case and the embedding $L_2(\Sigma, \mathsf{T}\Sigma) \hookrightarrow L_q(\Sigma, \mathsf{T}\Sigma)$. □

4.3. Energy Estimates and Global Existence

In [24] we showed that the set of equilibria $\mathcal{E}$ for (1.1), respectively (4.1), consists exactly of the Killing vector fields on Σ, that is,

$$\mathcal{E} = \{u \in C^\infty(\Sigma, \mathsf{T}\Sigma) \mid \mathcal{D}_\Sigma(u) = 0\}.$$

We recall that the condition $\mathcal{D}_\Sigma(u)=0$ implies that u is divergence free (which follows from the relation $\mathrm{div}_\Sigma\, u = \mathsf{tr}\, \mathcal{D}_\Sigma(u)$). Moreover, one can show that any vector field $u \in H^1_q(\Sigma, \mathsf{T}\Sigma)$ satisfying $\mathcal{D}_\Sigma(u)=0$ is already smooth, see for instance [19, Lemma 3]. Lastly, we recall that $\mathcal{E}$ is a finite dimensional vector space. If fact, $\dim \mathcal{E} \leq d(d+1)/2$, with equal sign for the case where Σ is isometric to a Euclidean sphere, see for instance the remarks in Section 4.1 of [24].

Let us define the space

$$V^j_2(\Sigma) := \{v \in H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma) \mid (v|z)_\Sigma = 0 \text{ for all } z \in \mathcal{E}\}, \quad j \in \{0,1\}. \tag{4.9}$$

Note that $V^j_2(\Sigma)$ is a closed subspace of $H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$, and hence is a Banach space. Moreover, $H^j_{2,\sigma}(\Sigma, \mathsf{T}\Sigma) = \mathcal{E} \oplus V^j_2(\Sigma)$, see Remark 4.10(a).

From now on we assume that $d=2$, and we show that any solution of (4.3) with initial value $v_0 \in L_{2,\sigma}(\Sigma)$ being orthogonal to $\mathcal{E}$ will remain orthogonal for all later times. Moreover, we establish an energy estimate for such solutions.

Proposition 4.6. *Let $d=2$. Suppose $v_0 \in V^0_2(\Sigma)$ and let v be the solution of (4.3) established in Theorem 4.5. Then*

(a) $v(t) \in V^0_2(\Sigma)$ for $t \in [0,t^+(v_0))$ and $v(t) \in V^1_2(\Sigma)$ for $t \in (0,t^+(v_0))$.

(b) There exists a universal constant $M>0$ such that

$$|v(t)|^2_{L_2(\Sigma)} + \int_0^t |v(s)|^2_{H^1_2(\Sigma)}\, ds \leq M |v_0|^2_{L_2(\Sigma)}, \quad t \in (0,t^+(v_0)).$$

Proof. (a) According to Theorem 4.5, we know that

$$v \in H^1_{p,\mathrm{loc}}((0,t^+); L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mathrm{loc}}((0,t^+); H^2_{2,\sigma}(\Sigma, \mathsf{T}\Sigma))$$

for $p > 2$. Pick any $z \in \mathcal{E}$. Then

$$\frac{d}{dt}(v(t)|z)_\Sigma = -(\nabla_{v(t)} v(t)|z)_\Sigma - (A_{S,\Sigma} v(t)|z)_\Sigma = -(\nabla_{v(t)} v(t)|z)_\Sigma,$$

where the time derivative exists for almost all $t \in (0, t^+(v_0))$. For the last equal sign we employed the property that $A_{S,\Sigma}$ is symmetric on $L_{2,\sigma}(\Sigma)$ and $N(A_{S,\Sigma}) = \mathcal{E}$, see [24, Proposition 4.1]. In a next step we show that

$$(\nabla_v v|z)_\Sigma = 0 \quad \text{for all} \quad v \in H^1_{2,\sigma}(\Sigma).$$

Indeed, this follows from

$$\begin{aligned}(\nabla_v v|z)_\Sigma &= \int_\Sigma (\nabla_v v|z)\, d\Sigma = \int_\Sigma \big(\nabla_v (v|z) - (v|\nabla_v z)\big)\, d\Sigma \\ &= \int_\Sigma \Big((v|\mathrm{grad}(v|z)) - (v|\nabla_v z)\Big)\, d\Sigma = 0,\end{aligned}$$

where we used (1.5), the surface divergence theorem and the property that z is a Killing vector field, (which implies $(\nabla_v z|v) + (v|\nabla_v z) = 0$). Hence, we have shown that $\frac{d}{dt}(v(t)|z)_\Sigma = 0$ for almost all $t \in (0, t^+(v_0))$. (4.8) now implies

$$(v(t)|z)_\Sigma = (v_0|z)_\Sigma = 0 \quad \text{for all} \quad t \in [0, t^+(v_0)).$$

(b) Similarly as in part (a), one shows (suppressing the variable t) that

$$\frac{d}{dt}\frac{1}{2}|v(t)|^2_{L_2(\Sigma)} = -\int_\Sigma \big((\nabla_v v|v) + (A_{S,\Sigma} v|v)\big)\, d\Sigma = -2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(v)|^2\, d\Sigma.$$

The assertion in part (a) and Korn's inequality (A. 2) readily imply

$$\frac{d}{dt}|v(t)|^2_{L_2(\Sigma)} + \alpha\, |v(t)|^2_{H^1_2(\Sigma)} \le 0, \quad t \in (0, t^+(v_0)), \tag{4.10}$$

with an appropriate constant $\alpha > 0$. Integration yields the assertion in (b), as $|v|^2_{L_2(\Sigma)}$ is absolutely continuous on $[0, t^+(v_0))$. □

Proposition 4.7. *Suppose that $d = 2$ and $v_0 \in V^0_2(\Sigma)$.*

Then problem (4.3) admits a unique global solution v enjoying the regularity properties stated in Theorem (4.5), with $t^+(v_0) = \infty$.

Moreover, there exists a constant $\alpha > 0$ such that

$$|v(t)|_{L_2(\Sigma)} \le e^{-\alpha t}|v_0|_{L_2(\Sigma)}, \quad t \ge 0. \tag{4.11}$$

Proof. By the abstract result [23, Theorem 2.4] on global-in-time existence, the maximal time of existence $t_+(u_0)$ satisfies the following property:

$$t_+(u_0) < \infty \Longrightarrow u \notin L_p((0,t_+); [X^{\mathsf{w}}_0, X^{\mathsf{w}}_1]_{\mu_c}).$$

Observe that in case $p = q = d = 2$, it holds $\mu_c = 1/p + d/(2q) = 1$, hence if

$$u \in L_2((0,t_+); H^1_2(\Sigma, \mathsf{T}\Sigma)),$$

then the weak solution exists globally in time. Proposition 4.6 guarantees that any solution v with initial value $v_0 \in V^0_2(\Sigma)$ satisfies

$$v \in L_2((0, t^+(v_0)), H^1_2(\Sigma, \mathsf{T}\Sigma))$$

and, hence, global existence of the weak solution follows. Since we know that the weak solution in this case regularizes to a strong solution, we obtain global in time existence of strong solutions for $q = d = 2$ as well.

Finally, we conclude from (4.10) that $\frac{d}{dt}|v(t)|^2_{L_2(\Sigma)} + \alpha\,|v(t)|^2_{L_2(\Sigma)} \le 0$ for any $t > 0$ and this implies the estimate in (4.11). □

Remark 4.8. Suppose $u_* \in \mathcal{E}$ and $v_0 \in V_2^0(\Sigma)$. Then the assertions of Theorem 4.1, Theorem 4.5 as well as Propositions 4.6 and 4.7 hold true for solutions of

$$\partial_t v + A_{S,\Sigma} v = -P_{H,\Sigma}(\nabla_v v + \nabla_{u_*} v + \nabla_v u_*), \quad v(0) = v_0, \tag{4.12}$$

respectively its weak formulation

$$\partial_t v + A^{\mathsf{w}}_{S,\Sigma} v = F^{\mathsf{w}}_\Sigma(v), \quad v(0) = v_0, \tag{4.13}$$

where $\langle F^{\mathsf{w}}_\Sigma(v), \phi\rangle = (v|\nabla_v \phi)_\Sigma + (v|\nabla_{u_*}\phi)_\Sigma + (u_*|\nabla_v\phi)_\Sigma$ for $\phi \in H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

In particular, each solution of(4.13) with initial value $v_0 \in V_2^0(\Sigma)$ exists globally and there exists a positive constant α such that

$$|v(t)|_{L_2(\Sigma)} \le e^{-\alpha t}|v_0|_{L_2(\Sigma)}, \quad t \ge 0.$$

Proof. One readily verifies that the assertions of Theorems 4.1 and 4.5 remain valid for problem (4.12) and (4.13), respectively. In fact, one only needs to verify that the terms on the right hand side can be estimated in the same way as in the proof of Theorems 4.1 and 4.5.

Next we show that $v(t) \in V_2^0(\Sigma)$ for $t \in [0, t^+(v_0))$. Let $z \in \mathcal{E}$. Following the proof of Proposition 4.6(a), we obtain

$$\frac{d}{dt}(v(t)|z)_\Sigma = -(\nabla_{v(t)} v(t) + \nabla_{u_*} v(t) + \nabla_{v(t)} u_*|z)_\Sigma, \quad t \in (0, t^+(v_0)).$$

According to the proof of Proposition 4.6(a), $(\nabla_v v|z) = 0$ and it remains to show that $(\nabla_{u_*} v + \nabla_v u_*|z)_\Sigma = 0$ for any $v \in H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$. This follows from

$$(\nabla_{u_*} v + \nabla_v u_*|z)_\Sigma = \int_\Sigma \big(\nabla_{u_*}(v|z) + \nabla_v(u_*|z) - (u_*|\nabla_v z) - (v|\nabla_{u_*} z)\big) = 0$$

where we used (1.5), the surface divergence theorem and the property that z is a Killing vector field. The same arguments as in the proof of Propositions 4.6 and 4.7 yield the remaining assertions. □

Theorem 4.9 (Global existence). *Suppose $d = 2$.*
Then any solution of (4.3) with initial value $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$ exists globally, has the regularity properties listed in Theorem 4.5, and converges at an exponential rate to the equilibrium $u_* = P_{\mathcal{E}} u_0$ in the topology of $H^2_q(\Sigma, \mathsf{T}\Sigma)$ for any fixed $q \in (1, \infty)$, where $P_{\mathcal{E}}$ is the orthogonal projection of u_0 onto $\mathcal{E}$ with respect to the $L_2(\Sigma, \mathsf{T}\Sigma)$ inner product.

Proof. Let $u_0 \in L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$ be given. Then there exist unique elements $u_* \in \mathcal{E}$ and $v_0 \in V_2^0(\Sigma)$ such that $u_0 = u_* + v_0$. Let v be the unique (global) solution of problem (4.13), respectively (4.12), whose existence has been asserted in Remark 4.8 Then

$$u(t) := u_* + v(t), \quad t > 0,$$

yields a (unique) global solution of (4.1), respectively (4.3), with initial value u_0. For this, we just need to observe that

$$\partial_t u_* + A_{S,\Sigma} u_* = -P_{H,\Sigma}\nabla_{u_*} u_*.$$

Indeed, this follows from the relations $N(A_{S,\Sigma}) = \mathcal{E}$ and $\nabla_{u_*} u_* = \frac{1}{2}\mathrm{grad}(u_*|u_*)$, with the latter assertion implying $P_{H,\Sigma}\nabla_{u_*} u_* = 0$. It follows readily that $u(t) \to u_*$ as $t \to \infty$ at an exponential rate in $L_2(\Sigma, \mathsf{T}\Sigma)$. To prove convergence in the stronger topology $H^2_q(\Sigma, \mathsf{T}\Sigma)$, we proceed as follows.

(i) First, we note that $L_2(\Sigma, \mathsf{T}\Sigma) \hookrightarrow B^0_{2r}(\Sigma, \mathsf{T}\Sigma)$ for any fixed $r \geq 2$. For fixed, but arbitrary $t_1 > 0$, we solve (4.3) with initial value $u_1 := u(t_1) \in B^0_{2r,\sigma}(\Sigma, \mathsf{T}\Sigma)$. Choosing $\mu_c = 1/r + 1/2$, it follows from [23, Theorem 1.2] that there exist positive numbers $\tau = \tau(u_*)$, $\varepsilon = \varepsilon(u_*)$ and $C_1 = C_1(u_*)$ such that

$$|u(\cdot, \hat{u}_1) - u(\cdot, u_*)|_{\mathbb{E}_{1,\mu_c}(0,2\tau)} \leq C_1 |\hat{u}_1 - u_*|_{B^0_{2r,\sigma}(\Sigma)} \tag{4.14}$$

for all $\hat{u}_1 \in B^0_{2r,\sigma}(\Sigma, \mathsf{T}\Sigma)$ with $|\hat{u}_1 - u_*|_{B^0_{2r,\sigma}(\Sigma)} < \varepsilon$, where

$$\mathbb{E}_{1,\mu_c}(0, 2\tau) = H^1_{r,\mu_c}((0, 2\tau); H^{-1}_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{r,\mu_c}((0, 2\tau); H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)).$$

It should be observed here that $u(t, u_*) = u_*$, as u_* is an equilibrium.
By following the arguments in [22, page 228] and employing

$$X^{\mathsf{w}}_\gamma = (X^{\mathsf{w}}_0, X^{\mathsf{w}}_1)_{1-1/r,r} = (H^{-1}_{2,\sigma}(\Sigma, \mathsf{T}\Sigma), H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma))_{1-1/r,r} = B^{1-2/r}_{2r,\sigma}(\Sigma, \mathsf{T}\Sigma),$$

see (4.6), we conclude with (4.14) that there is a constant $C_2 = C_2(u_*)$ such that

$$|u(\cdot, \hat{u}_1) - u(\cdot, u_*)|_{C([\tau,2\tau], B^{1-2/r}_{2r,\sigma}(\Sigma))} \leq C_2 |\hat{u}_1 - u_*|_{B^0_{2r}(\Sigma)}.$$

Letting C_3 be the embedding constant of $L_2(\Sigma, \mathsf{T}\Sigma) \hookrightarrow B^0_{2r}(\Sigma, \mathsf{T}\Sigma)$ we choose t_1 large enough such that $|u(t_1) - u_*|_{L_2(\Sigma)} < \varepsilon/C_3$. Setting $t_2 = \tau$, we may deduce from the estimates above that

$$|u(t_2, u_1) - u_*|_{B^{1-2/r}_{2r}(\Sigma)} \leq C |u_1 - u_*|_{L_2(\Sigma)}, \tag{4.15}$$

with $C = C(u_*) = C_2 C_3$. Here we have also used the uniqueness of solutions, cf. [15, Theorem 3.4(c)].
(ii) For fixed, but arbitrary $r > 2$, we now choose a weight $\mu \in (\frac{1}{r}, \frac{1}{2}]$ so that

$$B^{1-2/r}_{2r}(\Sigma, \mathsf{T}\Sigma) \hookrightarrow B^{2\mu-2/r}_{2r}(\Sigma, \mathsf{T}\Sigma).$$

Solving (4.1) with initial value $u_2 := u(t_2, u_1) \in B^{2\mu-2/r}_{2r,\sigma}(\Sigma, \mathsf{T}\Sigma)$ and repeating the above procedure in the 'strong' spaces (X_0, X_1), we obtain the estimate

$$|u(t_3, u_2) - u_*|_{B^{2-2/r}_{2r}(\Sigma)} \leq C |u_2 - u_*|_{B^{1-2/r}_{2r}(\Sigma)}, \tag{4.16}$$

for some $t_3 = t_3(u_*) > 0$.

(iii) Next, we use the Sobolev embedding

$$B^{2-2/r}_{2r}(\Sigma, \mathsf{T}\Sigma) \hookrightarrow B^{2\mu-2/r}_{sr}(\Sigma, \mathsf{T}\Sigma),$$

valid for $s \geq 2$ and $\mu \in (\frac{1}{r}, \frac{1}{2} + \frac{1}{s}]$. Choosing $t_1 > 0$ from above sufficiently large, we infer from (4.16) that $u(t_3, u_2)$ is close to u_* in the topology of $B^{2\mu-2/r}_{sr}(\Sigma, \mathsf{T}\Sigma)$. Solving (4.1) with initial value $u_3 := u(t_3, u_2) \in B^{2\mu-2/r}_{sr,\sigma}(\Sigma, \mathsf{T}\Sigma)$ and repeating the above procedure in the 'strong' spaces (X_0, X_1), we obtain the estimate

$$|u(t_4, u_3) - u_*|_{B^{2-2/r}_{sr}(\Sigma)} \leq C |u_3 - u_*|_{B^{2-2/r}_{2r}(\Sigma)}, \tag{4.17}$$

for some $t_4 = t_4(u_*) > 0$.

(iv) We will now consider (4.1) in the spaces

$$(X_{1/2}, X_{1+1/2}) = (H^1_{s,\sigma}(\Sigma, \mathsf{T}\Sigma), H^3_{s,\sigma}(\Sigma, \mathsf{T}\Sigma)),$$

where (X_α, A_α) is the interpolation-extrapolation scale with respect to the complex interpolation functor, based on $X_0 = L_{s,\sigma}(\Sigma, \mathsf{T}\Sigma)$, $s \in (1, \infty)$, introduced at the beginning of Section 4.2. We note that $A_{1/2}$, the realization of $A_0 = \omega + A_{S,\Sigma}$ in $X_{1/2}$, has exactly the same properties as A_0. For $\beta = 1/2$ we obtain

$$[X_{1/2}, X_{1+1/2}]_\beta = [H^1_{s,\sigma}(\Sigma, \mathsf{T}\Sigma), H^3_{s,\sigma}(\Sigma, \mathsf{T}\Sigma)]_{1/2} = H^2_{s,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

It is easy to see that the nonlinearity

$$F_\Sigma : H^2_{s,\sigma}(\Sigma, \mathsf{T}\Sigma) \times H^2_{s,\sigma}(\Sigma, \mathsf{T}\Sigma) \to H^1_{s,\sigma}(\Sigma, \mathsf{T}\Sigma)$$

is bilinear and bounded. We can employ [23, Theorem 1.2 or 2.1] for problem (4.1) to obtain a solution $u(\cdot, u_4)$ with initial value $u_4 = u(t_4, u_3) \in B^{2-2/r}_{sr}(\Sigma, \mathsf{T}\Sigma)$. To do so, we choose $\mu = 1/2$ and verify that

$$X_{\gamma,\mu} := (X_{1/2}, X_{1+1/2})_{1/2-1/r,r} = B^{2-2/r}_{sr,\sigma}(\Sigma, \mathsf{T}\Sigma),$$
$$X_{\gamma,1} := (X_{1/2}, X_{1+1/2})_{1-1/r,r} = B^{3-2/r}_{sr,\sigma}(\Sigma, \mathsf{T}\Sigma).$$

Noting that the numbers $\mu = 1/2$ and $\beta = 1/2$ satisfy the assumptions of [23] Theorem 1.2 or 2.1, we can once more repeat the procedure outlined in step (i) to obtain

$$|u(t_5, u_4) - u_*|_{B^{3-2/r}_{sr}(\Sigma)} \le C|u_4 - u_*|_{B^{2-2/r}_{sr}(\Sigma)}, \tag{4.18}$$

for some $t_5 = t_5(u_*) > 0$, provided t_1 is chosen sufficiently large. Combining the estimates (4.15)–(4.18) and using the semiflow property, we obtain the estimate

$$|u(t_5 + t_4 + t_3 + t_2 + t_1, u_0) - u_*|_{B^{3-2/r}_{sr}(\Sigma)} \le C|u(t_1, u_0) - u_*|_{L_2(\Sigma)}.$$

Since $|u(t_1, u_0) - u_*|_{L_2(\Sigma)} \to 0$ at an exponential rate as $t_1 \to \infty$, we conclude that $u(t, u_0)$ converges to u_* at an exponential rate in the topology of $B^{3-2/r}_{sr,\sigma}(\Sigma, \mathsf{T}\Sigma)$ as well, as $t \to \infty$.

(iv) Finally, given $q \in (1, \infty)$, the embedding $B^{3-2/r}_{sr,\sigma}(\Sigma, \mathsf{T}\Sigma) \hookrightarrow H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$, being valid for sufficiently large parameters s and r, yields the last assertion of the theorem. □

Remarks 4.10. (a) Let $d \ge 2$ and let $\mathfrak{F}^s(\Sigma)$ denote any of the spaces

$$H^s_{q,\sigma}(\Sigma, \mathsf{T}\Sigma),\ B^s_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma), \quad \text{where } s \in \mathbb{R},\ 1 < p, q < \infty.$$

Then we have

$$\mathfrak{F}^s(\Sigma) = \mathcal{E} \oplus \{v \in \mathfrak{F}^s(\Sigma) : \langle v, z\rangle_\Sigma = 0,\ z \in \mathcal{E}\}. \tag{4.19}$$

Here, $\langle v, z\rangle_\Sigma := (v|z)_\Sigma$ if $s > 0$. In case $s \le 0$,

$$\langle \cdot, \cdot\rangle_\Sigma : \mathfrak{F}^s(\Sigma) \times (\mathfrak{F}')^{-s}(\Sigma) \to \mathbb{R}$$

denotes the duality pairing, induced by $(\cdot|\cdot)_\Sigma$, where

$$(\mathfrak{F}')^s(\Sigma) \in \{H^s_{q',\sigma}(\Sigma, \mathsf{T}\Sigma), B^s_{q'p',\sigma}(\Sigma, \mathsf{T}\Sigma)\}.$$

Since we can identify $\mathcal{E} \subset C^\infty(\Sigma, \mathsf{T}\Sigma)$ as a subspace of $\mathfrak{F}^s(\Sigma)$, the expression $\langle v, z\rangle_\Sigma$ is defined for every $(v, z) \in \mathfrak{F}^s(\Sigma) \times \mathcal{E}$ and (4.19) is, therefore, meaningful.

Proof. We will provide a proof of (4.19). As $\mathcal{E}$ is finite dimensional, we can find a basis $\{z_1, \ldots, z_m\}$ for $\mathcal{E}$ which has the property that $(z_i|z_j)_\Sigma = \delta_{ij}$. With this at hand, we define the projection

$$P^s_{\mathcal{E}} : \mathfrak{F}^s(\Sigma) \to \mathcal{E}, \quad P^s_{\mathcal{E}} v := \sum_{j=1}^m \langle v, z_j\rangle z_j, \tag{4.20}$$

onto $\mathcal{E}$. This yields the direct topological decomposition $\mathfrak{F}^s(\Sigma) = \mathcal{E} \oplus V^s(\Sigma)$, where $V^s(\Sigma) = (I - P^s_{\mathcal{E}})\mathfrak{F}^s(\Sigma)$. In order to justify (4.19), it suffices to show that

$$V^s(\Sigma) = \hat{V}^s(\Sigma) := \{v \in \mathfrak{F}^s(\Sigma) \mid \langle v, z\rangle_\Sigma = 0,\ z \in \mathcal{E}\}.$$

Suppose $v \in V^s(\Sigma)$. Then $v = (I - P^s_{\mathcal{E}})v$ and we obtain

$$\langle v, z_j\rangle_\Sigma = \langle (I - P^s_{\mathcal{E}})v, z_j\rangle_\Sigma = \langle v, z_j\rangle_\Sigma - \langle P^s_{\mathcal{E}} v, z_j\rangle_\Sigma = 0, \quad j = 1, \ldots, m,$$

showing that $v \in \hat{V}^s(\Sigma)$. Suppose now that $v \in \hat{V}^s(\Sigma)$. Since $\mathfrak{F}^s(\Sigma) = \mathcal{E} \oplus V^s(\Sigma)$, there are unique elements $(z, w) \in \mathcal{E} \times V^s(\Sigma)$ such that $v = z + w$. Then, by the first step,

$$0 = \langle v, z\rangle_\Sigma = \langle z + w, z\rangle_\Sigma = \langle z, z\rangle_\Sigma = |z|^2_{L_2(\Sigma)},$$

hence $z = 0$ and therefore $v = w \in V^s(\Sigma)$. □

(b) It is interesting to note that the assertions of Propositions 4.6 and 4.7 remain valid in case $d > 2$, with the following modifications:

Let $p > 2$ and $q \geq d > 2$. Suppose $v_0 \in B^{d/q-1}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma)$ satisfies $\langle v_0, z\rangle_\Sigma = 0$ for every $z \in \mathcal{E}$, where $\langle v_0, z\rangle_\Sigma$ has the same meaning as in (a). Let v be the unique solution of (4.3) with initial value v_0.

Then there exists a constant $\alpha > 0$ such that

$$|v(t)|_{L_2(\Sigma)} \leq e^{-\alpha(t-\tau)}|v(\tau)|_{L_2(\Sigma)}, \quad t \in [\tau, t^+(v_0)), \tag{4.21}$$

for fixed $\tau \in [0, t^+(v_0))$, where $\tau \in (0, t^+(v_0))$ in case $v_0 \notin L_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$.

Proof. According to Theorem 4.3, there exists a unique solution v to problem (4.3) with regularity

$$\begin{aligned} v \in\, & H^1_{p,\mu_c}((0,a); H^{-1}_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu_c}((0,a); H^1_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \\ & \cap H^1_{p,\mathrm{loc}}((0,t^+); L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mathrm{loc}}((0,t^+); H^2_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)) \end{aligned} \tag{4.22}$$

for each fixed $a \in (0, t^+)$. For $z \in \mathcal{E}$ we obtain, as in the proof of Theorem 4.6,

$$\frac{d}{dt}\langle v(t), z\rangle_\Sigma = -(\nabla_{v(t)}v(t)|z)_\Sigma - (A_{S,\Sigma}v(t)|z)_\Sigma = -(\nabla_{v(t)}v(t)|z)_\Sigma = 0,$$

for all $t \in (0, t^+(v_0))$. Therefore, $\langle v(\cdot), z\rangle_\Sigma \in H^1_{p,\mu_c}(0,a)$ and $\frac{d}{dt}\langle v(t)|z\rangle_\Sigma = 0$ for $t \in (0, t^+(v_0))$. By [21, Lemma 2.1(b)], or [22, Lemma 3.2.5(b)], we infer that $\langle v(\cdot), z\rangle_\Sigma \in H^1_{1,\mathrm{loc}}([0,a])$ for any fixed $a \in (0, t^+(v_0))$. We can now conclude from the fundamental theorem that

$$\langle v(t), z\rangle_\Sigma = 0, \quad t \in [0, t^+(v_0)). \tag{4.23}$$

Let $\tau \in (0, t^+(v_0))$ be fixed. As $p, q > 2$, we conclude from (4.22) and (4.23) that $v(t) \in V^1_2(\Sigma) = \{u \in H^1_2(\Sigma, \mathsf{T}\Sigma) \mid (u|z)_\Sigma = 0,\ z \in \mathcal{E}\}$ for any $t \in [\tau, t^+(v_0))$. Hence, Korn's inequality (A. 2) holds true for $v(t)$ with $t \in [\tau, t^+(v_0))$, and we can now follow the proof of Propositions 4.6(b) and 4.7 to obtain the assertion in (4.21). □

(c) Suppose $d > 2$ and $q \geq d$. Then (4.21) holds true with $\tau = 0$ for initial values $v_0 \in B^{2\mu-2/p}_{qp,\sigma}(\Sigma, \mathsf{T}\Sigma)$ satisfying the assumptions of [24, Theorem 3.5(b)] and $(v_0|z)_\Sigma = 0$ for all $z \in \mathcal{E}$.

(d) Suppose $p > 2$ and $q \geq d > 2$. Then every **global** solution of (4.3), respectively (4.1), converges exponentially fast to an equilibrium, namely to $P_\mathcal{E}u_0$ (where $P_\mathcal{E}$ is the projection defined in (4.20)), provided the initial value u_0 satisfies the assumptions of Theorem 4.3 or [24, Theorem 3.5(b)].

Proof. This follows by similar arguments as in the proof of Remark 4.8 and Theorem 4.9. □

(e) According to Theorem 4.3 in [24], any solution with initial value u_0 sufficiently close to an equilibrium exists globally and, hence, converges to $P_\mathcal{E}u_0$ at an exponential rate.

Funding Open Access funding enabled and organized by Projekt DEAL.

Appendix A

A.1. Auxilliary Results

We first consider a weak elliptic problem on compact manifolds without boundary.

Lemma A.1. *Let $1 < q < \infty$. For each $v \in L_q(\Sigma, \mathsf{T}\Sigma)$ there exists a unique solution $\nabla_\Sigma \psi \in L_q(\Sigma, \mathsf{T}\Sigma)$ of*

$$(\nabla_\Sigma \psi | \nabla_\Sigma \phi)_\Sigma = (v | \nabla_\Sigma \phi)_\Sigma, \quad \phi \in \dot{H}^1_{q'}(\Sigma).$$

Proof. For $\lambda > 0$, let $A_0 := \lambda - \Delta_\Sigma$ in $X_0 = L_q(\Sigma)$ with domain $X_1 = H^2_q(\Sigma)$. By [2, Theorem V.1.5.1], the pair (X_0, A_0) generates an interpolation-extrapolation scale (X_α, A_α), $\alpha \in \mathbb{R}$, with respect to the complex interpolation functor $[\cdot,\cdot]_\theta$, $\theta \in (0, 1)$. Let $X_0^\sharp = (X_0)' = L_{q'}(\Sigma)$ and denote by $A_0^\sharp$ the dual operator of A_0 in $X_0^\sharp$ with domain $X_1^\sharp := H^2_{q'}(\Sigma)$. We write $(X_\alpha^\sharp, A_\alpha^\sharp)$, $\alpha \in \mathbb{R}$, for the dual interpolation-extrapolation scale generated by $(X_0^\sharp, A_0^\sharp)$. Then, $A_{-1/2} : X_{1/2} \to X_{-1/2}$ is a linear isomorphism, where

$$X_{1/2} = [X_0, X_1]_{1/2} = H^1_q(\Sigma)$$

and

$$X_{-1/2} = \left(X^\sharp_{1/2}\right)' = \left([X_0^\sharp, X_1^\sharp]_{1/2}\right)' = \left(H^1_{q'}(\Sigma)\right)' =: H^{-1}_q(\Sigma).$$

We claim

$$\langle A_{-1/2} u, \phi \rangle = \int_\Sigma (\nabla_\Sigma u | \nabla_\Sigma \phi)\, d\Sigma + \lambda (u|\phi)_\Sigma,$$

for all $(u, \phi) \in X_{1/2} \times X^\sharp_{1/2}$. Indeed, for $u \in X_1$, it holds that $A_{-1/2} u = A_0 u$, hence

$$\langle A_{-1/2} u, \phi \rangle = (A_0 u | \phi)_\Sigma = -(\Delta_\Sigma u | \phi)_\Sigma + \lambda (u|\phi)_\Sigma.$$

The surface divergence theorem as well as the density of X_1 in $X_{1/2}$ yield the claim.

Define an operator $B : X_{1/2} \to X_{-1/2}$ by $Bu = A_{-1/2} u - \lambda u$. Since the embedding $X_{1/2} \hookrightarrow X_{-1/2}$ is compact, the spectrum $\sigma(B)$ consists solely of eigenvalues with finite multiplicity. Furthermore, $\sigma(B)$ is invariant with respect to q and for each eigenfunction u of B it holds that $u \in H^1_r(\Sigma)$ for *any* $r \in (1, \infty)$.

We show that $\mu = 0$ is a semi-simple eigenvalue of B. The equation $Bu = 0$ in $X_{-1/2}$ is equivalent to

$$0 = \int_\Sigma (\nabla_\Sigma u | \nabla_\Sigma \phi)\, d\Sigma$$

for all $\phi \in X^\sharp_{1/2} = H^1_{q'}(\Sigma)$. Choosing $\phi = u$, we obtain $\nabla_\Sigma u = 0$, hence u is constant. This shows

$$N(B) = \{u \in H^1_q(\Sigma) \mid u \text{ is constant}\}.$$

We show $N(B^2) = N(B)$. For that purpose, let $u \in N(B^2)$ and define $v := Bu$. Then $v \in N(B)$, hence v is constant. For $\lambda > 0$ we have

$$A_{-1/2} u = \lambda u + Bu = \lambda u + v \in L_q(\Sigma).$$

Solve $A_0 w = \lambda u + v$ to obtain a unique solution $w \in H^2_q(\Sigma)$. Since $A_0 w = A_{-1/2} w$, this yields $A_{-1/2} u = A_{-1/2} w$ and therefore $u = w$ by injectivity of $A_{-1/2}$. This proves $u \in H^2_q(\Sigma)$ which in turn yields $A_0 u = \lambda u + v$ or equivalently $-\Delta_\Sigma u = v$. Integrating the last equation over Σ, yields $v = 0$ as v is constant. This shows $u \in N(B)$, hence $N(B^2) \subset N(B)$. Since the converse inclusion is obvious, we obtain the assertion.

We have shown so far that $\mu = 0$ is a semi-simple eigenvalue of B. In particular, this implies

$$X_{-1/2} = N(B) \oplus R(B).$$

Consequently, the restricted operator $B : X_{1/2} \cap R(B) \to R(B)$ is invertible. Note that

$$R(B) = \{f \in X_{-1/2} \mid \langle f, 1 \rangle = 0\}.$$

For $v \in L_q(\mathsf{T}\Sigma)$, define $f \in X_{-1/2}$ by $\langle f, \phi\rangle = (v|\nabla_\Sigma \phi)_\Sigma$. Then obviously $f \in R(B)$, hence there exists a unique solution $u \in X_{1/2} \cap R(B)$ of the equation $Bu = f$. The proof is completed. □

Next we study existence and uniqueness as well as regularity properties of solutions to some second order differential equations on $\mathbb{R}^d$.

For that purpose, we set

$$\langle F_f, \phi\rangle := \int_{\mathbb{R}^d} f\phi\, dx, \quad \phi \in C_c^\infty(\mathbb{R}^d),$$

for functions $f \in L_q(\mathbb{R}^d)$. For $\lambda > 0$, $k \in \{-1, 0, 1\}$ we then define the function spaces

$$\mathbb{F}^k_\lambda := (\mathbb{F}^k, |\cdot|_{\mathbb{F}^k_\lambda}), \quad \mathbb{F}^k := \begin{cases} \dot H^{-1}_q(\mathbb{R}^d), & k = -1, \\ \{f \in H^k_q(\mathbb{R}^d) \mid F_f \in \dot H^{-1}_q(\mathbb{R}^d)\}, & k \in \{0, 1\}, \end{cases}$$

$$\mathbb{E}^k_\lambda := (\mathbb{E}^k, |\cdot|_{\mathbb{E}^k_\lambda}), \quad \mathbb{E}^k := \bigcap_{j=1}^{k+2} \dot H^j_q(\mathbb{R}^d),$$

equipped with the parameter-dependent norms

$$|f|_{\mathbb{F}^{-1}_\lambda} := |f|_{\dot H^{-1}_q(\mathbb{R}^d)},$$

$$|f|_{\mathbb{F}^k_\lambda} := \sum_{j=0}^{k} \lambda^{(k-j)/2} |\nabla^j f|_{L_q(\mathbb{R}^d)} + \lambda^{(k+1)/2} |F_f|_{\dot H^{-1}_q(\mathbb{R}^d)}, \quad k \in \{0, 1\}$$

$$|u|_{\mathbb{E}^k_\lambda} := \sum_{j=1}^{k+2} \lambda^{(k+2-j)/2} |\nabla^j u|_{L_q(\mathbb{R}^d)}, \quad k \in \{-1, 0, 1\}.$$

We are ready to prove the following result.

Lemma A.2. *Let $1 < q < \infty$, $k \in \{-1, 0, 1\}$ and $G \in W^{1+k}_\infty(\mathbb{R}^d)^{d\times d}$. Then there exist $\eta, \lambda_0 > 0$ such that*

$$\operatorname{div}(G\nabla\cdot) : \mathbb{E}^k_\lambda \to \mathbb{F}^k_\lambda$$

is an isomorphism, provided $|G - I|_{L_\infty(\mathbb{R}^d)} \le \eta$ and $\lambda \ge \lambda_0$.

Moreover, in this case there exists a constant $c(\lambda_0) > 0$ such that the unique solution of $\operatorname{div}(G\nabla u) = f$ *satisfies the estimate*

$$|\nabla u|_{H^{k+1}_q(\mathbb{R}^d)} \le c(\lambda_0)|f|_{\mathbb{F}^k_\lambda}, \quad k \in \{-1, 0, 1\}, \quad \lambda \ge \lambda_0, \tag{A. 1}$$

for any $f \in \mathbb{F}^k_\lambda$.

Proof. (i) We start with the case $k = -1$. For $u \in \mathbb{E}^{-1}_\lambda$ we write

$$\langle \operatorname{div}(G\nabla u), \phi\rangle = \langle \Delta u, \phi\rangle + \langle \operatorname{div}((G - I)\nabla u), \phi\rangle,$$

where

$$\langle \operatorname{div} v, \phi\rangle := -\int_{\mathbb{R}^d} v \cdot \nabla\phi\, dx$$

for $(v, \phi) \in L_q(\mathbb{R}^d)^d \times \dot H^1_{q'}(\mathbb{R}^d)$.

It is known that the operator $\Delta : \mathbb{E}^{-1}_\lambda \to \mathbb{F}^{-1}_\lambda$ is an isomorphism, see for instance [34], Theorem 5.2.3.1(i) and the remarks in Section 5.2.5 concerning duality, or [31, Lemma 3.3]. Furthermore, the estimate

$$|\langle \operatorname{div}((G - I)\nabla u), \phi\rangle| \le \eta |u|_{\dot H^1_q(\mathbb{R}^d)} |\phi|_{\dot H^1_{q'}(\mathbb{R}^d)}$$

holds. A Neumann series argument yields that the operator $\operatorname{div}(G\nabla\cdot) : \mathbb{E}^{-1}_\lambda \to \mathbb{F}^{-1}_\lambda$ is an isomorphism as well, provided $\eta \in (0, 1)$.

(ii) Let $k=0$. Then we have

$$\operatorname{div}(G\nabla u) = \Delta u + \operatorname{div}((G-I)\nabla u).$$

Also in this case, $\Delta : \mathbb{E}^0_\lambda \to \mathbb{F}^0_\lambda$ is an isomorphism. Furthermore, there exists a constant $C>0$ such that $|\Delta^{-1} f|_{\mathbb{E}^0_\lambda} \le C|f|_{\mathbb{F}^0_\lambda}$ for all $f \in \mathbb{F}^0_\lambda$ and all $\lambda \in (0,\infty)$. To see this, note that

$$|\nabla(\Delta^{-1}F_f)|_{L_q(\mathbb{R}^d)} \le C|F_f|_{\dot{H}^{-1}_q(\mathbb{R}^d)}$$

and

$$|\nabla^2(\Delta^{-1}f)|_{L_q(\mathbb{R}^d)} \le C|f|_{L_q(\mathbb{R}^d)}.$$

The definition of the norms in $\mathbb{E}^0_\lambda$ and $\mathbb{F}^0_\lambda$ yields the claim. We then estimate as follows

$$\begin{aligned}|\operatorname{div}((G-I)\nabla u)|_{L_q(\mathbb{R}^d)} &\le |G|_{W^1_\infty(\mathbb{R}^d)}|\nabla u|_{L_q(\mathbb{R}^d)} + |G-I|_{L_\infty(\mathbb{R}^d)}|\nabla^2 u|_{L_q(\mathbb{R}^d)}\\ &\le C(\lambda^{-1/2}+\eta)|u|_{\mathbb{E}^0_\lambda}.\end{aligned}$$

Furthermore, as in the case $k=-1$, we have

$$\lambda^{1/2}|\operatorname{div}((G-I)\nabla u)|_{\dot{H}^{-1}_q(\mathbb{R}^d)} \le \eta|u|_{\mathbb{E}^0_\lambda}.$$

Once again, a Neumann series argument shows that $\operatorname{div}(G\nabla\cdot) : \mathbb{E}^0_\lambda \to \mathbb{F}^0_\lambda$ is an isomorphism, provided η and $\lambda^{-1/2}$ are chosen sufficiently small.

(iii) The proof for the remaining case $k=1$ follows literally the same strategy and is therefore omitted.

(iv) The assertion in (A. 1) follows from the steps above and the definition of the norm of $\mathbb{E}^k_\lambda$. □

A.2. Korn's Inequality

We will establish an appropriate version of Korn's inequality for compact surfaces Σ without boundary. For this, we use the notation from Section 4.3.

Theorem A.3 (Korn's inequality). *There exists a constant $C>0$ such that*

$$|v|_{H^1_2(\Sigma)} \le C|\mathcal{D}_\Sigma(v)|_{L_2(\Sigma)} \quad \textit{for all } v \in V^1_2(\Sigma). \tag{A. 2}$$

Proof. Let $Lu := 2\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u)$ for $u \in H^2_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$. We know from Proposition A.2 in [24] that

$$Lu = \Delta_\Sigma u + \mathsf{Ric}_\Sigma\, u,$$

where Δ_Σ is the Bochner (the connection) Laplacian and $\mathsf{Ric}_\Sigma\, u$ is the Ricci $(1,1)$-tensor, given in local coordinates by $(\mathsf{Ric}_\Sigma)^\ell_k = g^{i\ell}R_{ik}$, so that $\mathsf{Ric}_\Sigma\, u = R^\ell_k u^k \tau_\ell$. It is well-known that $(\Delta_\Sigma u|u)_\Sigma = -|\nabla u|^2_{L_2(\Sigma)}$ for $u \in H^2_2(\Sigma, \mathsf{T}\Sigma)$, see for instance [30, Lemma 3.5]. Let $u \in H^2_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$. Then

$$\begin{aligned}2\int_\Sigma |\mathcal{D}_\Sigma(u)|^2\, d\Sigma &= -\int_\Sigma (Lu|u)\, d\Sigma = -\int_\Sigma \Big((\Delta_\Sigma u|u) + (\mathsf{Ric}_\Sigma\, u|u)\Big)\, d\Sigma\\ &= \int_\Sigma |\nabla u|^2\, d\Sigma - \int_\Sigma (\mathsf{Ric}_\Sigma\, u|u)\, d\Sigma \ge |\nabla u|^2_{L_2(\Sigma)} - C_1|u|^2_{L_2(\Sigma)}.\end{aligned}$$

Here we used that on the compact surface Σ, the Ricci tensor Ric_Σ can be bounded uniformly, yielding

$$\int_\Sigma (\mathsf{Ric}_\Sigma\, u|u)\, d\Sigma \le C_1|u|^2_{L_2(\Sigma)}$$

with an appropriate positive constant C_1. By density of the space $H^2_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$ in $H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma)$ we obtain

$$|u|^2_{H^1_2(\Sigma)} \le C_2\Big(|\mathcal{D}_\Sigma(u)|^2_{L_2(\Sigma)} + |u|^2_{L_2(\Sigma)}\Big), \quad u \in H^1_{2,\sigma}(\Sigma, \mathsf{T}\Sigma), \tag{A. 3}$$

for an appropriate constant C_2. The assertion in (A. 2) now follows from the Petree-Tartar Lemma. For the readers' convenience, we include the proof here.

Suppose (A. 2) does not hold. Then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $V_2^1(\Sigma)$, see (4.9), such that $|v_n|_{H_2^1(\Sigma)} = 1$ for $n \in \mathbb{N}$ and $|\mathcal{D}_\Sigma(v_n)|_{L_2(\Sigma)} \to 0$ as $n \to \infty$. Since $H_2^1(\Sigma, \mathsf{T}\Sigma)$ is compactly embedded in $L_2(\Sigma, \mathsf{T}\Sigma)$, there exists a subsequence, still denoted by $(v_n)_{n\in\mathbb{N}}$, which converges to an element $v \in L_2(\Sigma, \mathsf{T}\Sigma)$. From (A. 3) follows that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $V_2^1(\Sigma)$. Completeness of $V_2^1(\Sigma)$ shows that $v \in V_2^1(\Sigma)$ and $v_n \to v$ in $V_2^1(\Sigma)$. Consequently, $\mathcal{D}_\Sigma(v_n) \to \mathcal{D}_\Sigma(v)$ as well as $\mathcal{D}_\Sigma(v_n) \to 0$ in $L_2(\Sigma, \mathsf{T}_1^1\Sigma)$. This implies $\mathcal{D}_\Sigma(v) = 0$, and then $v \in \mathcal{E} \cap V_2^1(\Sigma) = \{0\}$, in contradiction to the assumption $|v|_{H_2^1(\Sigma)} = 1$. □

Remark A.4. Korn's inequality for embedded surfaces has also been established in [12, Lemma 4.1]. The proof given here is considerably shorter.

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(accepted: October 8, 2022; published online: October 25, 2022)