H_∞ Sliding Mode Observer Design for a Class of Nonlinear Discrete Time-Delay Systems: A Delay-Fractioning Approach

Jun Hu, Zidong Wang, Yugang Niu and Lampros K. Stergioulas

Abstract

In this paper, the H_∞ sliding mode observer (SMO) design problem is investigated for a class of nonlinear discrete time-delay systems. The nonlinear descriptions quantify the maximum possible derivations from a linear model and the system states are allowed to be immeasurable. Attention is focused on the design of a discrete-time SMO such that the asymptotic stability as well as the H_∞ performance requirement of the error dynamics can be guaranteed in the presence of nonlinearities, time-delay and external disturbances. Firstly, a discrete-time discontinuous switched term is proposed to make sure that the reaching condition holds. Then, by constructing a new Lyapunov-Krasovskii functional based on the idea of "delay-fractioning" and introducing some appropriate free-weighting matrices, a sufficient condition is established to guarantee the desired performance of the error dynamics in the specified sliding mode surface by solving a minimization problem. Finally, an illustrative example is given to show the effectiveness of the designed SMO design scheme.

Keywords

Sliding mode observer, discrete-time systems, nonlinear systems, time-delay, H_∞ performance.

I. Introduction

During the past decades, the sliding mode control (SMC) theory has proven to be an effective tool for coping with the model uncertainties and nonlinearities by taking advantage of the concepts of sliding mode surface design and equivalent control. A variety of SMC approaches have been developed in the literature with respect to various types of systems, see e.g. [11,13,20,21,25,34,37–39]. Comparing to the large amount of publications on SMC problems for continuous-time systems, the reported results for corresponding discrete-time systems have been relatively few. In the context of SMC for discrete-time systems, the quasi-sliding mode concept has been proposed in [11] and the discrete-time sliding mode reaching condition has been thoroughly studied based on a reaching law approach. Such a reaching condition has recently been shown in [18, 38, 39] to be a popular and convenient way of addressing the SMC problems for a class of discrete-time systems.

It is well known that system states are not always available due mainly to the limit of physical conditions or expense for measuring in reality. Therefore, the state estimation problem has received a great deal of research attention [42]. In recent years, the sliding mode observer (SMO) theory has been successfully applied to a wide range of areas such as induction motor drives, n-degree-of-freedom mechanical systems and single-link

This work was supported in part by the National Natural Science Foundation of China under Grants 61074041, 61028008 and 10771047, the Engineering and Physical Sciences Research Council (EPSRC) of the U.K. under Grant GR/S27658/01, the Royal Society of the U.K., and the Alexander von Humboldt Foundation of Germany.

J. Hu is with the Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China.

Z. Wang and L. K. Stergioulas are with the Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom. (Email: Zidong.Wang@brunel.ac.uk)

Y. Niu is with the School of Information Science and Engineering, East China University of Science and Technology, Shanghai 200237, China.

flexible joint robot systems [12,17,28,40]. When designing sliding mode observers, a suitable nonlinear output injection is usually introduced to guarantee finite time convergence and induce a sliding motion. Most research on SMO design has been carried out along this line, see e.g. [1,6,7,12,14–17,24,26–30,35,40]. To be specific, by constructing an appropriate SMO, the fault reconstruction and estimation problems have been extensively studied in [7,14,27,30,40] for uncertain systems. It should be pointed out that almost all results mentioned above have been concerned with continuous-time systems, and the relevant results for *discrete-time* systems have been very few despite the fact that nowadays digitalized control systems are inherently discrete-time ones.

In reality, time-delays and nonlinearities are inevitably encountered in various industrial systems. The occurrence of time-delays and nonlinearities would cause great degradation of the system performance. Accordingly, the SMO problem for nonlinear and/or time-delay systems has gained considerable research interest and a variety of important results have been published in the literature, see [24,27,29,30,40]. To mention a few, in [24], an H_∞ SMO problem has been investigated for uncertain nonlinear Lipschitz-type systems with fault and disturbances and a sufficient condition has been given such that the H_∞ performance requirement is satisfied. By using Taylor series expansion and employing a nonlinear transformation, the discrete-time model has been derived in [29,30] from its continuous-time counterpart and then the discrete-time sliding mode state estimation problems have been addressed for uncertain nonlinear systems. Unfortunately, to the best of the authors' knowledge, very few results have been available so far for the SMO problem of *discrete-time* systems with *time-delays*.

In recent years, various delay-dependant approaches have been proposed in the literature in order to reduce the conservatism caused by the time-delays when analyzing the stability of time-delay systems. Such approaches can be classified into four categories, i.e, bounding technique [19], descriptor system approach [9], slack matrix variables [36] and delay-fractioning approach [23]. Generally speaking, the objective of investigating of the delay-dependent stability condition involves two aspects: 1) (conservatism) development of delay-dependent criteria to provide a maximal allowable delay; 2) (complexity) development of delay-dependent criteria by using as few decision variables as possible while guaranteeing the same maximal allowable delay. When comparing between different approaches, both the conservatism and complexity serve as the criteria. In fact, there does exist a tradeoff between the conservatism and the computational complexity. In other words, it's hard to find a globally best approach that is least conservative yet with least computational burden. Compared with the bounding technique [19], descriptor system approach [9], slack matrix variables [36], the delay-fractioning approach adopted in this paper is most efficient in reducing the conservatism caused by the time-delays at the cost of introducing more computational complexity especially when the number of fractions goes up. From a practical point of view, however, it is not difficult to handle the computational complexity issue nowadays because of the rapid development of computing techniques. Therefore, we choose to use the delay-fractioning approach which is arguably the up-to-date delay-dependence analysis method [23]. The main purpose of this paper is to establish a unified framework for discrete time-delay systems by using the SMO scheme based on delay-fractioning approach.

Motivated by the above discussion, in this paper, we aim to deal with the H_∞ SMO design problem for a class of uncertain nonlinear discrete-time systems with time-delays. Firstly, a new nonlinear SMO is presented to estimate the unmeasured states where a discontinuous switched term is introduced to account for the sliding mode strategy. Secondly, a linear discrete-time switching function is constructed to describe the sliding mode surface, and a discontinuous switched term is synthesized to drive the state trajectories of the error dynamics system onto the band of pre-specified sliding mode surface. Moreover, by constructing a new

Lyapunov-Krasovskii functional associated with the delay-fractioning idea and introducing some appropriate free-weighting matrices, a sufficient condition is presented to ensure the asymptotic stability as well as the H_∞ performance of the overall error dynamics by means of the feasibility of a certain semidefinite programming problem with an equality constraint. Then, a minimization problem is presented to solve the original nonconvex problem. Finally, an illustrative example is used to show the effectiveness of the proposed discrete-time H_∞ SMO design scheme.

The main contribution of this paper lies primarily in two aspects: 1) a new design scheme of the discrete-time H_∞ SMO design is presented for nonlinear discrete-time systems with time-delay and external disturbances; 2) the delay-fractioning approach is applied, for the first time, to design the SMO with hope to reduce the possible conservatism caused by the time-delays. To the best of authors' knowledge, the *discrete-time* SMO design problem for nonlinear systems with *time-delays* has never been investigated in the literature. Our research represents the one of the very first attempts in dealing with SMO problems for time-delay nonlinear systems, where our aim is to present easy-to-verify conditions by taking advantage of the delay-fractioning approach with hope to reduce the conservatism caused by the time-delay. The rest of this paper is organized as follows. Section II briefly introduces the problem under consideration and presents a new discrete-time SMO scheme. The reachability analysis is firstly conducted and the discontinuous switched term is synthesized in Section III. Then, in the same section, the asymptotic stability as well as H_∞ performance of the error dynamics are given and, moreover, a minimization algorithm is presented to address the non-convex problem. An illustrative example is given in Section IV and the paper is concluded in Section V.

Notations. The notations in this paper are quite standard except where otherwise stated. The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ ($\mathbb{R}^{n\times m}$) denote, respectively, the n-dimensional Euclidean space, the set of all $n \times m$ matrices ; the notation $P > 0$ ($P \geq 0$) means that matrix P is real symmetric and positive definite (positive semi-definite); $l_2[0,\infty)$ is the space of square summable vectors; I and 0 represent the identity matrix and a zero matrix with appropriate dimension, respectively; diag$\{\cdots\}$ stands for a block-diagonal matrix, col$\{\cdots\}$ denotes a vector column with blocks given by the vectors in $\{\cdots\}$; $\|\cdot\|$ denotes the Euclidean norm of a vector and its induced norm of a matrix. In symmetric block matrices or long matrix expressions, we use a star "$*$" to represent a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminaries

In this paper, we consider the following nonlinear discrete time-delay system:

$$\begin{cases} \bar{x}(k+1) = \bar{f}(\bar{x}(k)) + \bar{g}(\bar{x}(k-d)) + \bar{B}h(k) + \bar{D}\omega(k) \\ \quad\;\; \bar{y}(k) = \bar{C}\bar{x}(k) \\ \quad\;\; \bar{x}(k) = \phi(k), \quad \forall k \in [-d, 0] \end{cases} \tag{1}$$

where $\bar{x}(k) \in \mathbb{R}^n$ is the state vector, $\bar{y}(k) \in \mathbb{R}^p$ is the measurement output, $h(k) \in \mathbb{R}^q$ denotes the unknown input that is bounded in terms of Euclidean norm, $\omega(k) : \mathbb{R}^+ \to \mathbb{R}^r \in l_2[0,\infty)$ represents the exogenous disturbances, $\bar{f}(\cdot) \in \mathbb{R}^n$ and $\bar{g}(\cdot) \in \mathbb{R}^n$ are known nonlinear functions. d denotes the known state delay which can always be described by $d = \tau m$ with τ and m are integers. The parameters $\bar{B} \in \mathbb{R}^{n\times q}$, $\bar{C} \in \mathbb{R}^{p\times n}$ ($p < n$) and $\bar{D} \in \mathbb{R}^{n\times r}$ are known real matrices, $\bar{B}$ and $\bar{C}$ are assumed to be full rank, and $\phi(k)$ is a given initial condition.

The nonlinear functions $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$ are assumed to satisfy $\bar{f}(0)=0$, $\bar{g}(0)=0$ and

$$\left\|\bar{f}(\bar{x}(k)+\eta(k))-\bar{f}(\bar{x}(k))-\bar{A}\eta(k)\right\| \leq \varepsilon_1 \left\|\eta(k)\right\| \tag{2}$$

$$\left\|\bar{g}(\bar{x}(k-d)+\eta(k))-\bar{g}(\bar{x}(k-d))-\bar{A}_d\eta(k)\right\| \leq \varepsilon_2 \left\|\eta(k)\right\| \tag{3}$$

where $\bar{A}\in\mathbb{R}^{n\times n}$ and $\bar{A}_d\in\mathbb{R}^{n\times n}$ are known constant matrices, $\eta(k)\in\mathbb{R}^n$ is a vector, and ε_1 and ε_2 are known positive scalars.

Remark 1: The nonlinear descriptions in (2) and (3) have been extensively applied (see e.g. [31,32,41]) to quantify the maximum possible derivations from a linear model with $(\bar{A},\ \bar{A}_d)$. Such nonlinear descriptions, though similar to the commonly used Lipschitz conditions on the nonlinear functions $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$, give clearer engineering insight from the mathematical modeling viewpoint.

Assumption 1: rank($\bar{C}\bar{B}$)=rank($\bar{B}$).

Based on Assumption 1, we have the following easily accessible result.

Proposition 1: It follows from Assumption 1 that there exists a transformation such that

$$(\bar{A},\bar{A}_d,\bar{B},\bar{C},\bar{D},\bar{f},\bar{g},\varepsilon_1,\varepsilon_2)$$

can be transformed into the following structure:

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix},\begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22}\end{bmatrix},\begin{bmatrix} B_1 \\ 0\end{bmatrix},\begin{bmatrix} C_1 & 0\end{bmatrix},\begin{bmatrix} D_1 \\ D_2\end{bmatrix},f,g,\begin{bmatrix} \epsilon_{11} \\ \epsilon_{12}\end{bmatrix},\begin{bmatrix} \epsilon_{21} \\ \epsilon_{22}\end{bmatrix}\right) \tag{4}$$

where $A_{11}\in\mathbb{R}^{p\times p}$, $A_{d11}\in\mathbb{R}^{p\times p}$, $B_1\in\mathbb{R}^{p\times q}$, $D_1\in\mathbb{R}^{p\times r}$, $C_1\in\mathbb{R}^{p\times p}$ with C_1 being nonsingular and B_1 being of full column rank. Moreover, the nonlinear functions $f(\cdot)$ and $g(\cdot)$ correspond to $\bar{f}(\cdot)$ and $\bar{g}(\cdot)$, and the positive scalars ϵ_{ij} $(i,j=1,2)$ correspond to the scalars ε_1 and ε_2.

Remark 2: As pointed out in [7,40], Assumption 1 is a constraint on the input matrix and implies that rank($\bar{B}$)≤rank($\bar{C}$).

From Proposition 1, system (1) has the following form:

$$\begin{cases} x_1(k+1) = f_1(x(k)) + g_1(x(k-d)) + B_1 h(k) + D_1\omega(k) \\ x_2(k+1) = f_2(x(k)) + g_2(x(k-d)) + D_2\omega(k) \\ \quad\quad y(k) = C_1 x_1(k) \end{cases} \tag{5}$$

where $x(k)=\mathrm{col}(x_1(k),x_2(k))$ with $x_1(k)\in\mathbb{R}^p$, $f_1(x(k))$ and $g_1(x(k-d))$ are the first p components of $f(x(k))$ and $g(x(k-d))$, $f_2(x(k))$ and $g_2(x(k-d))$ are the last $n-p$ components of $f(x(k))$ and $g(x(k-d))$. It is not difficult to verify from (2)-(3) that $f_i(x(k))$ and $g_i(x(k-d))$ satisfy $f_i(0)=0$ and $g_i(0)=0$ $(i=1,2)$, respectively, and

$$\left\|f_i(x(k)+\beta(k))-f_i(x(k))-\begin{bmatrix} A_{i1} & A_{i2}\end{bmatrix}\beta(k)\right\| \leq \epsilon_{1i}\left\|\beta(k)\right\| \tag{6}$$

$$\left\|g_i(x(k-d)+\beta(k))-g_i(x(k-d))-\begin{bmatrix} A_{di1} & A_{di2}\end{bmatrix}\beta(k)\right\| \leq \epsilon_{2i}\left\|\beta(k)\right\|, \quad i=1,2 \tag{7}$$

In this paper, the SMO under consideration is of the following structure

$$\begin{cases} \hat{x}_1(k+1) = f_1(\hat{x}(k)) + g_1(\hat{x}(k-d)) + L_1[y(k)-\hat{y}(k)] + B_1 v(k) \\ \hat{x}_2(k+1) = f_2(\hat{x}(k)) + g_2(\hat{x}(k-d)) + L_2[y(k)-\hat{y}(k)] \\ \quad\quad \hat{y}(k) = C_1\hat{x}_1(k) \end{cases} \tag{8}$$

where $\hat{x}(k) = \mathrm{col}(\hat{x}_1(k), \hat{x}_2(k))$ with $\hat{x}_1(k) \in \mathbb{R}^p$, L_1 and L_2 are the observer gains to be designed later. Moreover, the discontinuous switched term $v(k)$ is introduced to reject the effect of system unknown input $h(k)$ and also drive the trajectories of the estimation error onto the specified sliding surface. Noting that $x_1(k)$ is observable due to the non-singularity of C_1, we only need to estimate $x_2(k)$. Unfortunately, in the nonlinearities $f(\cdot)$ and $g(\cdot)$, $x_1(k)$ and $x_2(k)$ are tightly coupled and we are unable to separate $x_2(k)$ from $x(k)$. As such, for mathematical convenience, we use $\hat{x}_1(k)$ as an auxiliary variable to facilitate the estimate of $x_2(k)$.

Letting the error state be $e(k) = x(k) - \hat{x}(k)$, it follows from (5) and (8) that

$$\begin{cases} e_1(k+1) = f_1(x(k)) - f_1(\hat{x}(k)) + g_1(x(k-d)) - g_1(\hat{x}(k-d)) \\ \qquad\qquad - L_1[y(k) - \hat{y}(k)] + B_1(h(k) - v(k)) + D_1\omega(k) \\ e_2(k+1) = f_2(x(k)) - f_2(\hat{x}(k)) + g_2(x(k-d)) - g_2(\hat{x}(k-d)) \\ \qquad\qquad - L_2[y(k) - \hat{y}(k)] + D_2\omega(k) \end{cases} \tag{9}$$

where $e_1(k)$ and $e_2(k)$ are the first p and the last $n-p$ components of $e(k)$.

For notational convenience, set

$$l(k) = f(x(k)) - f(\hat{x}(k)) - Ae(k) \tag{10}$$

$$m(k-d) = g(x(k-d)) - g(\hat{x}(k-d)) - A_d e(k-d) \tag{11}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}.$$

Then, taking (10) and (11) into account yields

$$\begin{cases} e_1(k+1) = (A_{11} - L_1C_1)e_1(k) + A_{12}e_2(k) + A_{d11}e_1(k-d) + A_{d12}e_2(k-d) \\ \qquad\qquad + [l_1(k) + m_1(k-d)] + B_1(h(k) - v(k)) + D_1\omega(k) \\ e_2(k+1) = (A_{21} - L_2C_1)e_1(k) + A_{22}e_2(k) + A_{d21}e_1(k-d) + A_{d22}e_2(k-d) \\ \qquad\qquad + [l_2(k) + m_2(k-d)] + D_2\omega(k) \end{cases} \tag{12}$$

where $l_1(k)$ and $m_1(k-d)$ are the first p components of $l(k)$ and $m(k-d)$, and $l_2(k)$ and $m_2(k-d)$ are the last $n-p$ components of $l(k)$ and $m(k-d)$, respectively.

The purpose of this paper is to design a discrete-time SMO of form (8) for the nonlinear discrete time-delay system (5). More specifically, we are interested in looking for the observer gains L_1 and L_2 so as to synthesize the discontinuous switched term $v(k)$ such that the following requirements are simultaneously satisfied:

(Q1) The error system (12) is globally driven onto the pre-specified sliding mode surface and, in subsequent time, the sliding motion is asymptotically stable.

(Q2) For a given scalar $\gamma > 0$ with $\omega(k) \neq 0$, the error signal $e(k)$ satisfies

$$\sum_{k=0}^{\infty} \|e(k)\|^2 \leq \gamma^2 \sum_{k=0}^{\infty} \|\omega(k)\|^2 \tag{13}$$

under the zero initial condition.

Before proceeding further, we introduce the following lemmas that will be frequently used in the proofs of our main results.

Lemma 1: For any real vectors a, b and matrix $P > 0$ of appropriate dimensions, the following inequality holds

$$a^T b + b^T a \leq a^T P a + b^T P^{-1} b. \tag{14}$$

Lemma 2: (Schur Complement) Given constant matrices $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ where $\mathcal{Q}_1 = \mathcal{Q}_1^T$ and $\mathcal{Q}_2 = \mathcal{Q}_2^T > 0$. Then, $\mathcal{Q}_1 + \mathcal{Q}_3^T \mathcal{Q}_2^{-1} \mathcal{Q}_3 < 0$ if and only if

$$\begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_3^T \\ * & -\mathcal{Q}_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\mathcal{Q}_2 & \mathcal{Q}_3 \\ * & \mathcal{Q}_1 \end{bmatrix} < 0. \tag{15}$$

III. Design of the SMO

In this section, we aim to establish a unified framework to solve the addressed H_∞ SMO design problem in the simultaneous presence of nonlinearities, time-delay and disturbances. A design scheme of the discontinuous switched term is firstly proposed to guarantee the reachability of the specified sliding surface. Then, a sufficient condition is derived such that the asymptotic stability as well as the H_∞ performance requirement of the error dynamics can be guaranteed.

A. Reachability Analysis

Let us first synthesize the discontinuous switched term $v(k)$ in (8) such that the reachability of the specified sliding surface is ensured. To begin with, we define the switching function in the space of estimation error as

$$s(k) = G e_1(k), \tag{16}$$

where G is a constant matrix to be designed such that GB_1 is nonsingular and $GD_1 = 0$. By considering the discrete-time reaching condition given in [11], we only need to show that the following inequalities hold:

$$\begin{cases} \Delta s(k) = s(k+1) - s(k) \leq -\kappa U \text{sgn}[s(k)] - \kappa V s(k) & \text{if} \quad s(k) > 0 \\ \Delta s(k) = s(k+1) - s(k) \geq -\kappa U \text{sgn}[s(k)] - \kappa V s(k) & \text{if} \quad s(k) < 0 \end{cases} \tag{17}$$

where κ denotes the sampling period, $U = \text{diag}\{\mu_1, \mu_2, \cdots, \mu_q\} \in \mathbb{R}^{q \times q}$, $V = \text{diag}\{\nu_1, \nu_2, \cdots, \nu_q\} \in \mathbb{R}^{q \times q}$, and $\mu_i > 0$, $\nu_i > 0$ are properly chosen constants satisfying $0 < 1 - \kappa\nu_i < 1$ $(i = 1, 2, \cdots, q)$.

Notice that the unknown input $h(k)$ is bounded in terms of Euclidean norm, and let $\Delta_e(k) := G[(A_{11} - L_1 C_1)]e_1(k) + GA_{12}e_2(k) + GA_{d11}e_1(k-d) + GA_{d12}e_2(k-d) + G[l_1(k) + m_1(k-d)]$ and $\Delta_h(k) := GB_1 h(k)$, then there exist $\underline{\delta}_e^i(k)$, $\overline{\delta}_e^i(k)$, $\underline{\delta}_h^i(k)$, and $\overline{\delta}_h^i(k)$ $(i = 1, 2, \cdots, q)$ satisfying

$$\underline{\delta}_e^i(k) \leq \delta_e^i(k) \leq \overline{\delta}_e^i(k), \quad \underline{\delta}_h^i(k) \leq \delta_h^i(k) \leq \overline{\delta}_h^i(k) \tag{18}$$

where $\delta_e^i(k)$ and $\delta_h^i(k)$ are the ith elements in $\Delta_e(k)$ and $\Delta_h(k)$, respectively. It should be pointed out that the assumptions on the upper and lower bounds of $\Delta_e(k)$ and $\Delta_h(k)$ are standard for dealing with discrete-time sliding mode problems, see e.g. [11, 18, 39]. In addition, the bounds of $\Delta_e(k)$ and $\Delta_h(k)$ are allowed to be time-varying.

By defining

$$\begin{aligned}
\widehat{\Delta}_e(k) &= \begin{bmatrix} \widehat{\delta}_e^1(k) & \widehat{\delta}_e^2(k) & \cdots & \widehat{\delta}_e^q(k) \end{bmatrix}^T, & \widehat{\delta}_e^i(k) = \frac{\overline{\delta}_e^i(k) + \underline{\delta}_e^i(k)}{2}, \\
\widetilde{\Delta}_e(k) &= \operatorname{diag}\left\{\widetilde{\delta}_e^1(k), \widetilde{\delta}_e^2(k), \cdots, \widetilde{\delta}_e^q(k)\right\}, & \widetilde{\delta}_e^i(k) = \frac{\overline{\delta}_e^i(k) - \underline{\delta}_e^i(k)}{2}, \\
\widehat{\Delta}_h(k) &= \begin{bmatrix} \widehat{\delta}_h^1(k) & \widehat{\delta}_h^2(k) & \cdots & \widehat{\delta}_h^q(k) \end{bmatrix}^T, & \widehat{\delta}_h^i(k) = \frac{\overline{\delta}_h^i(k) + \underline{\delta}_h^i(k)}{2}, \\
\widetilde{\Delta}_h(k) &= \operatorname{diag}\left\{\widetilde{\delta}_h^1(k), \widetilde{\delta}_h^2(k), \cdots, \widetilde{\delta}_h^q(k)\right\}, & \widetilde{\delta}_h^i(k) = \frac{\overline{\delta}_h^i(k) - \underline{\delta}_h^i(k)}{2},
\end{aligned} \tag{19}$$

we are in a position to present the design technique of the discontinuous switched term $v(k)$.

Theorem 1: Assume that the switching function (16) is given with G satisfying the nonsingularity of GB_1 and $GD_1 = 0$. If the discontinuous switched term $v(k)$ is given by

$$\begin{aligned}
v(k) &= (GB_1)^{-1}(\kappa U \operatorname{sgn}[s(k)] + \kappa V s(k) - s(k) \\
&\quad +(\widehat{\Delta}_e(k) + \widetilde{\Delta}_e(k)\operatorname{sgn}[s(k)]) + (\widehat{\Delta}_h(k) + \widetilde{\Delta}_h(k)\operatorname{sgn}[s(k)])),
\end{aligned} \tag{20}$$

then the discrete-time sliding mode reaching condition of the error system (12) with specified sliding mode surface (16) is satisfied.

Proof: Taking (12), (16) and (20) into consideration, we have

$$\begin{aligned}
\Delta s(k) &= s(k+1) - s(k) \\
&= G[(A_{11} - L_1 C_1)]e_1(k) + GA_{12}e_2(k) + GA_{d11}e_1(k-d) + GA_{d12}e_2(k-d) \\
&\quad +G[l_1(k) + m_1(k-d)] + GB_1(h(k) - v(k)) - s(k) \\
&= -\kappa U \operatorname{sgn}[s(k)] - \kappa V s(k) + \Delta_e(k) - (\widehat{\Delta}_e(k) + \widetilde{\Delta}_e(k)\operatorname{sgn}[s(k)]) \\
&\quad +\Delta_h(k) - (\widehat{\Delta}_h(k) + \widetilde{\Delta}_h(k)\operatorname{sgn}[s(k)]).
\end{aligned} \tag{21}$$

It follows from (19) that (17) holds and then the discrete-time sliding mode reaching condition is satisfied. The proof is now complete. ■

B. Performance analysis of the sliding motion

It is noted that the ideal quasi-sliding mode satisfies

$$s(k+1) = s(k) = 0. \tag{22}$$

Then, when the error trajectories of the system (12) enter into the sliding surface, the equivalent discontinuous switched term $v_{eq}(k)$ can be obtained from (12), (16) and (22) as follows:

$$\begin{aligned}
v_{eq}(k) &= (GB_1)^{-1}G(A_{11} - L_1C_1)e_1(k) + (GB_1)^{-1}GA_{12}e_2(k) + (GB_1)^{-1}GA_{d11}e_1(k-d) \\
&\quad +(GB_1)^{-1}GA_{d12}e_2(k-d) + (GB_1)^{-1}G[l_1(k) + m_1(k-d)] + h(k).
\end{aligned} \tag{23}$$

Substituting (23) as $v(k)$ into (12), we obtain the error dynamics in the specified sliding surface $s(k) = 0$ as follows:

$$e_1(k+1) = \sum_{i=1}^{6} \mathscr{A}_i(k), \tag{24}$$

where

$$\begin{aligned}
\mathscr{A}_1(k) &= (A_{11}-L_1C_1)e_1(k)+A_{12}e_2(k)+A_{d11}e_1(k-d)+A_{d12}e_2(k-d)+D_1\omega(k),\\
\mathscr{A}_2(k) &= -B_1(GB_1)^{-1}G(A_{11}-L_1C_1)e_1(k),\\
\mathscr{A}_3(k) &= -B_1(GB_1)^{-1}GA_{12}e_2(k),\\
\mathscr{A}_4(k) &= -B_1(GB_1)^{-1}GA_{d11}e_1(k-d),\\
\mathscr{A}_5(k) &= -B_1(GB_1)^{-1}GA_{d12}e_2(k-d),\\
\mathscr{A}_6(k) &= (I-B_1(GB_1)^{-1}G)[l_1(k)+m_1(k-d)].
\end{aligned}$$

In the following, a sufficient condition will be established such that the overall error dynamics composed of (24) and the second equation of (12) is asymptotically stable with H_∞ disturbance attenuation level γ in the specified sliding surface (16).

Theorem 2: Let the reachability condition be satisfied. Consider the nonlinear time-delay system (5), the SMO (8) and the sliding surface (16). For the given scalars $\alpha_i \in (0,1)$ $(i=1,2)$ and $\gamma>0$, assume that there exist matrices $P_i>0$, $Q_i>0$, $R_i>0$, Y_i, $M_i=M_i^T\geq 0$ $(i=1,2)$, X, Z and positive scalars λ_1, λ_2, λ_3 satisfying

$$P_1 \leq \lambda_1 I, \tag{25}$$

$$P_2 \leq \lambda_3 I, \tag{26}$$

$$B_1^T P_1 D_1 = 0, \tag{27}$$

$$\begin{bmatrix} -\lambda_2 I & P_1B_1 \\ * & -B_1^TP_1B_1 \end{bmatrix} \leq 0, \tag{28}$$

$$\Phi = \begin{bmatrix} M_1+M_2 & X & Z \\ * & \alpha_1 P_1 & 0 \\ * & * & \alpha_2 P_2 \end{bmatrix} \geq 0, \tag{29}$$

$$\Psi = \begin{bmatrix} \Psi_{11} & \vartheta\Sigma_1^T & \nu\Sigma_2^T & \Psi_{14} \\ * & -P_1 & 0 & 0 \\ * & * & -P_2 & 0 \\ * & * & * & \Psi_{44} \end{bmatrix} < 0, \tag{30}$$

where

$$\begin{aligned}
\Psi_{11} &= \Omega_1+\Omega_1^T+\Omega_2+\Omega_3+\Omega_4+\Omega_5,\\
\Sigma_1 &= \begin{bmatrix} P_1A_{11}-Y_1C_1 & 0_{p\times(m-1)p} & P_1A_{d11} & P_1A_{12} & 0_{p\times(m-1)(n-p)} & P_1A_{d12} & P_1D_1 \end{bmatrix},\\
\Sigma_2 &= \begin{bmatrix} P_2A_{21}-Y_2C_1 & 0_{(n-p)\times(m-1)p} & P_2A_{d21} & P_2A_{22} & 0_{(n-p)\times(m-1)(n-p)} & P_2A_{d22} & P_2D_2 \end{bmatrix},\\
\Psi_{14} &= \begin{bmatrix} \vartheta\Xi_2^T(P_1A_{11}-Y_1C_1)^TB_1 & \vartheta\Xi_3^TA_{12}^TP_1B_1 & \vartheta\Xi_4^TA_{d11}^TP_1B_1 & \vartheta\Xi_5^TA_{d12}^TP_1B_1 \end{bmatrix},\\
\Psi_{44} &= \mathrm{diag}\{-B_1^TP_1B_1,-B_1^TP_1B_1,-B_1^TP_1B_1,-B_1^TP_1B_1\},\\
\vartheta &= \sqrt{6(1+2\alpha_1\tau)},\quad \nu=\sqrt{2(1+2\alpha_2\tau)},\\
\Omega_1 &= \begin{bmatrix} X & Z \end{bmatrix}\left[\begin{array}{cccc} \multicolumn{4}{c}{I_{p\times p} \quad -I_{p\times p} \quad 0_{p\times((m+1)n-2p+r)}} \\ \hline 0_{(n-p)\times(m+1)p} & I_{(n-p)\times(n-p)} & -I_{(n-p)\times(n-p)} & 0_{(n-p)\times((m-1)(n-p)+r)} \end{array}\right],\\
\Omega_2 &= \tau(M_1+M_2),\\
\Omega_3 &= W_R^TRW_R,
\end{aligned}$$

$$
\begin{aligned}
\Omega_4 &= \sum_{i=2}^{5} \Xi_i^T \Theta_i \Xi_i, \\
\Omega_5 &= \Xi_2^T \Xi_2 + \Xi_3^T \Xi_3 - \gamma^2 \Xi_7^T \Xi_7, \\
\Xi_2 &= \left[\begin{array}{cc} I_{p\times p} & 0_{p\times(mn+n-p+r)} \end{array} \right], \\
\Xi_3 &= \left[\begin{array}{ccc} 0_{(n-p)\times(m+1)p} & I_{(n-p)\times(n-p)} & 0_{(n-p)\times(m(n-p)+r)} \end{array} \right], \\
\Xi_4 &= \left[\begin{array}{ccc} 0_{p\times mp} & I_{p\times p} & 0_{p\times((m+1)(n-p)+r)} \end{array} \right], \\
\Xi_5 &= \left[\begin{array}{ccc} 0_{(n-p)\times(mn+p)} & I_{(n-p)\times(n-p)} & 0_{(n-p)\times r} \end{array} \right], \\
\Xi_7 &= \left[\begin{array}{cc} 0_{r\times(m+1)n} & I_{r\times r} \end{array} \right], \\
W_R &= \left[\begin{array}{c} \begin{array}{cc} I_{mp\times mp} & 0_{mp\times(m(n-p)+n+r)} \end{array} \\ \hline \begin{array}{ccc} 0_{mp\times p} & I_{mp\times mp} & 0_{mp\times((m+1)(n-p)+r)} \end{array} \\ \hline \begin{array}{ccc} 0_{m(n-p)\times(m+1)p} & I_{m(n-p)\times m(n-p)} & 0_{m(n-p)\times(n-p+r)} \end{array} \\ \hline \begin{array}{ccc} 0_{m(n-p)\times(mp+n)} & I_{m(n-p)\times m(n-p)} & 0_{m(n-p)\times r} \end{array} \end{array} \right], \\
R &= \mathrm{diag}\{R_1, -R_1, R_2, -R_2\}, \\
\Theta_2 &= (2\alpha_1\tau - 1)P_1 + 24\epsilon_{11}^2(1+2\alpha_1\tau)(\lambda_1+\lambda_2)I + Q_1 + 4(1+2\alpha_2\tau)\lambda_3\epsilon_{12}^2 I, \\
\Theta_3 &= (2\alpha_2\tau - 1)P_2 + 24\epsilon_{11}^2(1+2\alpha_1\tau)(\lambda_1+\lambda_2)I + Q_2 + 4(1+2\alpha_2\tau)\lambda_3\epsilon_{12}^2 I, \\
\Theta_4 &= 24\epsilon_{21}^2(1+2\alpha_1\tau)(\lambda_1+\lambda_2)I - Q_1 + 4(1+2\alpha_2\tau)\lambda_3\epsilon_{22}^2 I, \\
\Theta_5 &= 24\epsilon_{21}^2(1+2\alpha_1\tau)(\lambda_1+\lambda_2)I - Q_2 + 4(1+2\alpha_2\tau)\lambda_3\epsilon_{22}^2 I.
\end{aligned} \tag{31}
$$

By choosing $G = B_1^T P_1$, the overall error dynamics is asymptotically stable with H_∞ disturbance attenuation level γ in the specified sliding surface (16). Moreover, the observer gains are given by $L_1 = P_1^{-1}Y_1$ and $L_2 = P_2^{-1}Y_2$.

Proof: Please see the Appendix. ■

Remark 3: In the derivation of Theorem 2, we apply the "delay-fractioning" approach and construct a more general Lyapunov-Krasovskii functional for addressing the discrete-time H_∞ SMO problem. Specifically, the so-called "weighting" scalar parameters $\alpha_i \in (0,1)$ $(i = 1,2)$ are introduced to fit both the delay-fractioning idea and the sliding mode approach, and its value can be determined *a priori* to facilitate the design of the SMO scheme. It is possible to conduct a linear search for the α_i $(i = 1,2)$ to help enhance the solvability of (25)-(30) in Theorem 2.

C. Computational Algorithm

Notice that there exists a matrix equation constraint (i.e. $B_1^T P_1 D_1 = 0$) in Theorem 2, which can be equivalently converted into

$$
\mathrm{trace}[(B_1^T P_1 D_1)^T B_1^T P_1 D_1] = 0.
$$

Based on the algorithm presented in [20], by introducing $(B_1^T P_1 D_1)^T B_1^T P_1 D_1 \le \mu I$ with $\mu > 0$ being a sufficiently small scalar, it follows from Lemma 2 that

$$
\left[\begin{array}{cc} -\mu I & D_1^T P_1 B_1 \\ * & -I \end{array} \right] \le 0. \tag{32}
$$

Hence, the H_∞ SMO problem is now transformed to an equivalent problem of finding a global solution of the following minimization problem

$$\begin{aligned}&\min\ \ \mu\\&\text{subject to (25)-(26), (28)-(30) and (32).}\end{aligned}\tag{33}$$

Remark 4: The minimization problem (33) is a convex optimization one that can be easily solved by using standard numerical software. If the solution of the minimization problem (33) equals zero, the sufficient conditions in Theorem 2 are satisfied and then the asymptotic stability as well as the H_∞ performance of the error dynamics can be guaranteed. In the implementation, we can always enhance the feasibility of the addressed minimization problem by 1) increasing the disturbance attenuation level γ; 2) decreasing the "weighting" scalar parameters α_i $(i = 1, 2)$; and 3) removing some terms in the Lyapunov-Krasovskii functional (37) at the expense of introducing some possible conservatism.

IV. An Illustrative Example

In this section, we aim to demonstrate the effectiveness and applicability of the proposed scheme. Following [8,33], we consider the SMO problem for an F-404 aircraft engine system, where the nominal system matrix A_c is given as follows

$$A_c = \begin{bmatrix} -1.4600 & 0 & 2.4280 \\ 0.1643 & -0.4000 & -0.3788 \\ 0.3107 & 0 & -2.2300 \end{bmatrix}.$$

As analyzed in [33], virtually all aircraft engine systems are in some way disturbed by external forces. The disturbances may assume a myriad of forms, such as wind gusts, gravity gradients, structural vibrations, and may enter the systems in many different ways. These perturbations generally degrade the performance of the system and, in some cases, may jeopardize the outcome of the engineering task. By doing so, the accurate fatigue life can be computed in a more reliable way and the engine design could be changed early and inexpensively if necessary. As in [10], let the motion of the F-404 aircraft engine be determined by the system of differential equations derived from the basic aerodynamics.

Therefore, when modeling the aircraft engine system, the time delay, linearization errors (nonlinear disturbances) and the external disturbances should all be taken into account. After discretization, we obtain the following nonlinear discrete time-delay system:

$$\begin{cases} x_{11}(k+1) = 0.2504x_{11}(k) + 0.3919x_2(k) + 0.015\sin(x_{11}(k) + x_2(k)) - 0.02x_{11}(k-4) \\ \qquad\qquad - 0.02x_2(k-4) + 0.01\sin(x_{11}(k-4)) - 0.1\sin(0.05k) + 0.025\omega(k) \\ x_{12}(k+1) = 0.057x_{11}(k) + 0.6188x_{12}(k) - 0.0616x_2(k) + 0.013\sin(0.8x_{12}(k)) + 0.04x_{11}(k-4) \\ \qquad\qquad - 0.106x_{12}(k-4) + 0.01\sin(x_{12}(k-4)) - 0.15\sin(0.05k) - 0.03\omega(k) \\ x_2(k+1) = 0.0502x_{11}(k) + 0.1262x_2(k) + 0.016\sin(x_{12}(k) + x_2(k)) - 0.068x_{11}(k-4) \\ \qquad\qquad + 0.10x_{12}(k-4) - 0.034x_2(k-4) + 0.011\sin(x_2(k-4)) - 0.013\omega(k) \\ y_{11}(k) = 0.34x_{11}(k) + 0.15x_{12}(k) \\ y_{12}(k) = 0.23x_{11}(k) - 0.1x_{12}(k) \end{cases}\tag{34}$$

where $x_1(k) = \begin{bmatrix} x_{11}(k) & x_{12}(k) \end{bmatrix}^T$ and $y(k) = \begin{bmatrix} y_{11}(k) & y_{12}(k) \end{bmatrix}^T$.

Considering the system (5), we have the system parameters as follows:

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0.2504 & 0 \\ 0.0570 & 0.6188 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 0.3919 \\ -0.0616 \end{bmatrix}, \; A_{d11} = \begin{bmatrix} -0.02 & 0 \\ 0.04 & -0.106 \end{bmatrix}, \; A_{d12} = \begin{bmatrix} -0.02 \\ 0 \end{bmatrix}, \\ A_{21} &= \begin{bmatrix} 0.0502 & 0 \end{bmatrix}, \; A_{22} = \begin{bmatrix} 0.1262 \end{bmatrix}, \; A_{d21} = \begin{bmatrix} -0.068 & 0.10 \end{bmatrix}, \; A_{d22} = \begin{bmatrix} -0.034 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -0.10 \\ -0.15 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0.025 \\ -0.03 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.34 & 0.15 \\ 0.23 & -0.1 \end{bmatrix}, \; D_2 = \begin{bmatrix} -0.013 \end{bmatrix}, \end{aligned}$$

and $h(k) = \sin(0.05k)$, $d = 4$, $\epsilon_{11} = \epsilon_{12} = 0.018$, $\epsilon_{21} = \epsilon_{22} = 0.012$.

Our aim is to design a discrete-time SMO in the form of (8) such that the error dynamics is asymptotically stable with a guaranteed H_∞ noise attenuation level. By setting $\gamma = 0.15$, $\alpha_1 = 0.002$, $\alpha_2 = 0.0035$, $m = 1$, and solving the minimization problem (33) in the Matlab environment, we obtain $\mu = 7.6291 \times 10^{-7}$ (hence the equality constraint is considered to be achieved) and

$$\begin{aligned} P_1 &= \begin{bmatrix} 2.1356 & 0.1461 \\ 0.1461 & 1.8246 \end{bmatrix}, \; P_2 = 13.3573, \\ Y_1 &= \begin{bmatrix} 1.0966 & 0.7400 \\ 3.9956 & 5.2949 \end{bmatrix}, \; Y_2 = \begin{bmatrix} 0.9846 & 1.4546 \end{bmatrix}. \end{aligned}$$

Then, the observer gains are given by

$$L_1 = \begin{bmatrix} 0.3656 & 0.5480 \\ 2.1606 & -2.9459 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0.0737 & 0.1089 \end{bmatrix},$$

and the designed SMO for the nonlinear discrete time-delay system (34) is given by

$$\begin{cases} \hat{x}_{11}(k+1) = 0.2504\hat{x}_{11}(k) + 0.3919\hat{x}_2(k) + 0.015\sin(\hat{x}_{11}(k) + \hat{x}_2(k)) - 0.02\hat{x}_{11}(k-4) \\ \qquad - 0.02\hat{x}_2(k-4) + 0.01\sin(\hat{x}_{11}(k-4)) + E_1L_1[y(k) - \hat{y}(k)] - 0.1v(k) \\ \hat{x}_{12}(k+1) = 0.057\hat{x}_{11}(k) + 0.6188\hat{x}_{12}(k) - 0.0616\hat{x}_2(k) + 0.013\sin(0.8\hat{x}_{12}(k)) + 0.04\hat{x}_{11}(k-4) \\ \qquad - 0.106\hat{x}_{12}(k-4) + 0.01\sin(\hat{x}_{12}(k-4)) + E_2L_1[y(k) - \hat{y}(k)] - 0.15v(k) \\ \hat{x}_2(k+1) = 0.0502\hat{x}_{11}(k) + 0.1262\hat{x}_2(k) + 0.016\sin(\hat{x}_{12}(k) + \hat{x}_2(k)) - 0.068\hat{x}_{11}(k-4) \\ \qquad + 0.10\hat{x}_{12}(k-4) - 0.034\hat{x}_2(k-4) + 0.011\sin(\hat{x}_2(k-4)) + L_2[y(k) - \hat{y}(k)] \\ \hat{y}(k) = C_1\hat{x}_1(k) \end{cases} \tag{35}$$

with $\hat{x}_1(k) = \begin{bmatrix} \hat{x}_{11}(k) & \hat{x}_{12}(k) \end{bmatrix}^T$, $E_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $v(k)$ is calculated by (20).

For the simulation purpose, the external disturbance $\omega(k)$ is described by

$$\omega(k) = \begin{cases} 4.8, & 10 \le k \le 30 \\ -1.05, & 35 \le k \le 60 \\ 0, & \text{else} \end{cases} \tag{36}$$

The simulation results are given in Figs. 1-8. Among them, Figs. 1-3 (Figs. 4-6 with small scale) show the actual states (solid line) and their estimations (dashed line) by taking $\kappa = 1.2$ and $\mu_j = \nu_j = 0.01$ $(j = 1, 2)$, which confirm that the system states are well estimated by the proposed discrete-time H_∞ SMO method. The response of error dynamics is shown in Fig. 7. The response of sliding surface is shown in Fig. 8.

From the simulation results, it can be seen that the presented scheme effectively estimates the system states and attenuates the effect of all time-delay, nonlinearities and external disturbances. Moreover, the discrete time quasi-sliding mode is well achieved in finite time. Under the zero-initial condition, the l_2 norms of the estimation error $e(k)$ and the exogenous disturbance $\omega(k)$ are computed, respectively, as 1.3750 and 22.6386. Accordingly, the actual l_2-gain from the exogenous disturbance to the estimation error can be obtained as 0.0607, which is significantly lower than the given performance level $\gamma = 0.15$. Therefore, the H_∞ performance constraint (13) is well achieved.

V. Conclusions

In this paper, we have made an attempt to investigate the discrete-time H_∞ SMO design problem for a class of nonlinear systems with time-delay. A new discrete-time SMO with a discontinuous switched term has been presented and the reachability analysis has been conducted. Moreover, by constructing a new Lyapunov-Krasovskii functional associated with delay-fractioning idea, a sufficient condition has been given such that the error dynamics is asymptotically stable and the estimation error satisfies the specified H_∞ performance requirement. A computational algorithm has been presented to make sure that the proposed scheme can be easily checked by using the standard numerical software. Finally, the effectiveness and applicability of the developed discrete-time H_∞ SMO scheme have been demonstrated by an illustrative example. One of the future research topics would be the extension of the main results obtained in this paper to networked control systems [2–5].

Appendix

Proof of Theorem 2:

We first establish the asymptotic stability of the overall error dynamics with $\omega(k) = 0$. Based on the delay-fractioning idea, we choose the following Lyapunov-Krasovskii functional candidate:

$$V(k) = \sum_{i=1}^{4} V_i(k), \tag{37}$$

where

$$\begin{aligned}
V_1(k) &= e_1^T(k)P_1e_1(k),\\
V_2(k) &= \sum_{l=k-d}^{k-1} e_1^T(l)Q_1e_1(l) + \sum_{l=k-\tau}^{k-1} \Gamma_1^T(l)R_1\Gamma_1(l) + \sum_{j=-\tau}^{-1}\sum_{l=k+j}^{k-1} \eta_1^T(l)\alpha_1P_1\eta_1(l),\\
V_3(k) &= e_2^T(k)P_2e_2(k),\\
V_4(k) &= \sum_{l=k-d}^{k-1} e_2^T(l)Q_2e_2(l) + \sum_{l=k-\tau}^{k-1} \Gamma_2^T(l)R_2\Gamma_2(l) + \sum_{j=-\tau}^{-1}\sum_{l=k+j}^{k-1} \eta_2^T(l)\alpha_2P_2\eta_2(l),\\
\eta_i(l) &= e_i(l+1) - e_i(l),\\
\Gamma_i(l) &= \mathrm{col}\{e_i(l), e_i(l-\tau), \ldots, e_i(l-(m-1)\tau)\}, \quad (i = 1, 2)
\end{aligned}$$

with $P_1 > 0$, $P_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $R_1 > 0$, $R_2 > 0$ being matrices to be determined. By calculating the difference of $V(k)$ along the trajectory of overall error dynamics, we have

$$\Delta V(k) = \sum_{i=1}^{4} \Delta V_i(k), \tag{38}$$

where

$$\begin{aligned}\Delta V_1(k) &= e_1^T(k+1)P_1e_1(k+1) - e_1^T(k)P_1e_1(k)\\ &\le 6\sum_{i=1}^{6}\mathscr{A}_i^T(k)P_1\mathscr{A}_i(k) - e_1^T(k)P_1e_1(k), \qquad (39)\end{aligned}$$

with

$$\begin{aligned}\mathscr{A}_2^T(k)P_1\mathscr{A}_2(k) &= e_1^T(k)(A_{11}-L_1C_1)^TG^T(GB_1)^{-1}G(A_{11}-L_1C_1)e_1(k),\\ \mathscr{A}_3^T(k)P_1\mathscr{A}_3(k) &= e_2^T(k)A_{12}^TG^T(GB_1)^{-1}GA_{12}e_2(k),\\ \mathscr{A}_4^T(k)P_1\mathscr{A}_4(k) &= e_1^T(k-d)A_{d11}^TG^T(GB_1)^{-1}GA_{d11}e_1(k-d),\\ \mathscr{A}_5^T(k)P_1\mathscr{A}_5(k) &= e_2^T(k-d)A_{d12}^TG^T(GB_1)^{-1}GA_{d12}e_2(k-d).\end{aligned}$$

It follows from (2)-(3) and (10)-(11) that

$$\begin{aligned}l_1^T(k)l_1(k) &\le \epsilon_{11}^2[e_1^T(k)e_1(k)+e_2^T(k)e_2(k)], \qquad (40)\\ m_1^T(k-d)m_1(k-d) &\le \epsilon_{21}^2[e_1^T(k-d)e_1(k-d)+e_2^T(k-d)e_2(k-d)], \qquad (41)\end{aligned}$$

where ϵ_{11} and ϵ_{21} are known constants.

Noting that $G = B_1^TP_1$, together with conditions (25), (28), (40) and (41), we obtain

$$\begin{aligned}\mathscr{A}_6^T(k)P_1\mathscr{A}_6(k) &= [l_1(k)+m_1(k-d)]^T(I-B_1(GB_1)^{-1})^TP_1\\ &\quad\times(I-B_1(GB_1)^{-1})[l_1(k)+m_1(k-d)]\\ &\le 2[l_1(k)+m_1(k-d)]^TP_1[l_1(k)+m_1(k-d)]\\ &\quad+2[l_1(k)+m_1(k-d)]^TG^T(GB_1)^{-1}G[l_1(k)+m_1(k-d)]\\ &\le 4(\lambda_1+\lambda_2)l_1^T(k)l_1(k)+4(\lambda_1+\lambda_2)m_1^T(k-d)m_1(k-d)\\ &\le 4(\lambda_1+\lambda_2)\epsilon_{11}^2[e_1^T(k)e_1(k)+e_2^T(k)e_2(k)]\\ &\quad+4(\lambda_1+\lambda_2)\epsilon_{21}^2[e_1^T(k-d)e_1(k-d)+e_2^T(k-d)e_2(k-d)]. \qquad (42)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Delta V_2(k) &= e_1^T(k)Q_1e_1(k)-e_1^T(k-d)Q_1e_1(k-d)+\Gamma_1^T(k)R_1\Gamma_1(k)-\Gamma_1^T(k-\tau)R_1\Gamma_1(k-\tau)\\ &\quad+\alpha_1\tau\eta_1^T(k)P_1\eta_1(k)-\alpha_1\sum_{l=k-\tau}^{k-1}\eta_1^T(l)P_1\eta_1(l). \qquad (43)\end{aligned}$$

Considering $\eta_1(l) = e_1(l+1)-e_1(l)$ and using Lemma 1, we obtain

$$\alpha_1\tau\eta_1^T(k)P_1\eta_1(k) \le 2\alpha_1\tau[e_1^T(k+1)P_1e_1(k+1)+e_1^T(k)P_1e_1(k)]. \qquad (44)$$

Hence, it follows from (39) and (42)-(44) that

$$\begin{aligned}\Delta V_1(k)+\Delta V_2(k) &\le (1+2\alpha_1\tau)e_1^T(k+1)P_1e_1(k+1)+(2\alpha_1\tau-1)e_1^T(k)P_1e_1(k)\\ &\quad+e_1^T(k)Q_1e_1(k)-e_1^T(k-d)Q_1e_1(k-d)+\Gamma_1^T(k)R_1\Gamma_1(k)\\ &\quad-\Gamma_1^T(k-\tau)R_1\Gamma_1(k-\tau)-\alpha_1\sum_{l=k-\tau}^{k-1}\eta_1^T(l)P_1\eta_1(l)\\ &\le 6(1+2\alpha_1\tau)\hat{\xi}^T(k)[\hat{\Xi}_1^TP_1\hat{\Xi}_1+\hat{\Xi}_2^T(A_{11}-L_1C_1)^TG^T(GB_1)^{-1}G(A_{11}-L_1C_1)\hat{\Xi}_2\\ &\quad+\hat{\Xi}_3^TA_{12}^TG^T(GB_1)^{-1}GA_{12}\hat{\Xi}_3+\hat{\Xi}_4^TA_{d11}^TG^T(GB_1)^{-1}GA_{d11}\hat{\Xi}_4\\ &\quad+\hat{\Xi}_5^TA_{d12}^TG^T(GB_1)^{-1}GA_{d12}\hat{\Xi}_5+4\epsilon_{11}^2(\lambda_1+\lambda_2)(\hat{\Xi}_2^T\hat{\Xi}_2+\hat{\Xi}_3^T\hat{\Xi}_3)\end{aligned}$$

$$\begin{aligned}
&+4\epsilon_{21}^2(\lambda_1+\lambda_2)(\hat{\Xi}_4^T\hat{\Xi}_4+\hat{\Xi}_5^T\hat{\Xi}_5)]\hat{\xi}(k)+(2\alpha_1\tau-1)\hat{\xi}^T(k)\hat{\Xi}_2^TP_1\hat{\Xi}_2\hat{\xi}(k)\\
&+e_1^T(k)Q_1e_1(k)-e_1^T(k-d)Q_1e_1(k-d)+\Gamma_1^T(k)R_1\Gamma_1(k)\\
&-\Gamma_1^T(k-\tau)R_1\Gamma_1(k-\tau)-\alpha_1\sum_{l=k-\tau}^{k-1}\eta_1^T(l)P_1\eta_1(l), \qquad (45)
\end{aligned}$$

where

$$\begin{aligned}
\hat{\xi}(k) &= \begin{bmatrix} \Gamma_1^T(k) & e_1^T(k-d) & \Gamma_2^T(k) & e_2^T(k-d) \end{bmatrix}^T,\\
\hat{\Xi}_1 &= \begin{bmatrix} A_{11}-L_1C_1 & 0_{p\times(m-1)p} & A_{d11} & A_{12} & 0_{p\times(m-1)(n-p)} & A_{d12} \end{bmatrix},\\
\hat{\Xi}_2 &= \begin{bmatrix} I_{p\times p} & 0_{p\times(mn+n-p)} \end{bmatrix},\\
\hat{\Xi}_3 &= \begin{bmatrix} 0_{(n-p)\times(m+1)p} & I_{(n-p)\times(n-p)} & 0_{(n-p)\times m(n-p)} \end{bmatrix},\\
\hat{\Xi}_4 &= \begin{bmatrix} 0_{p\times mp} & I_{p\times p} & 0_{p\times(m+1)(n-p)} \end{bmatrix},\\
\hat{\Xi}_5 &= \begin{bmatrix} 0_{(n-p)\times(mn+p)} & I_{(n-p)\times(n-p)} \end{bmatrix}.
\end{aligned}$$

Similarly, it can be obtained that

$$\begin{aligned}
\Delta V_3(k) &= e_2^T(k+1)P_2e_2(k+1)-e_2^T(k)P_2e_2(k)\\
&\le 2\hat{\xi}^T(k)\hat{\Xi}_6^TP_2\hat{\Xi}_6\hat{\xi}(k)+2[l_2(k)+m_2(k-d)]^TP_2[l_2(k)+m_2(k-d)]-e_2^T(k)P_2e_2(k)\\
&\le 2\hat{\xi}^T(k)\hat{\Xi}_6^TP_2\hat{\Xi}_6\hat{\xi}(k)+4\lambda_3[l_2^T(k)l_2(k)+m_2^T(k-d)m_2(k-d)]-e_2^T(k)P_2e_2(k)\\
&\le 2\hat{\xi}^T(k)\hat{\Xi}_6^TP_2\hat{\Xi}_6\hat{\xi}(k)+4\lambda_3[\epsilon_{12}^2(e_1^T(k)e_1(k)+e_2^T(k)e_2(k))+\epsilon_{22}^2(e_1^T(k-d)e_1(k-d)\\
&\quad +e_2^T(k-d)e_2(k-d))]-e_2^T(k)P_2e_2(k)\\
&= 2\hat{\xi}^T(k)[\hat{\Xi}_6^TP_2\hat{\Xi}_6+2\lambda_3\epsilon_{12}^2(\hat{\Xi}_2^T\hat{\Xi}_2+\hat{\Xi}_3^T\hat{\Xi}_3)+2\lambda_3\epsilon_{22}^2(\hat{\Xi}_4^T\hat{\Xi}_4+\hat{\Xi}_5^T\hat{\Xi}_5)-\frac{1}{2}\hat{\Xi}_3^TP_2\hat{\Xi}_3]\hat{\xi}(k), \qquad (46)
\end{aligned}$$

where

$$\hat{\Xi}_6=\begin{bmatrix} A_{21}-L_2C_1 & 0_{p\times(m-1)p} & A_{d21} & A_{22} & 0_{p\times(m-1)(n-p)} & A_{d22} \end{bmatrix},$$

and

$$\begin{aligned}
\Delta V_4(k) &= e_2^T(k)Q_2e_2(k)-e_2^T(k-d)Q_2e_2(k-d)+\Gamma_2^T(k)R_2\Gamma_2(k)-\Gamma_2^T(k-\tau)R_2\Gamma_2(k-\tau)\\
&\quad +\alpha_2\tau\eta_2^T(k)P_2\eta_2(k)-\alpha_2\sum_{l=k-\tau}^{k-1}\eta_2^T(l)P_2\eta_2(l). \qquad (47)
\end{aligned}$$

Therefore, it can be derived that

$$\begin{aligned}
\Delta V_3(k)+\Delta V_4(k) &\le (1+2\alpha_2\tau)e_2^T(k+1)P_2e_2(k+1)+(2\alpha_2\tau-1)e_2^T(k)P_2e_2(k)\\
&\quad +e_2^T(k)Q_2e_2(k)-e_2^T(k-d)Q_2e_2(k-d)+\Gamma_2^T(k)R_2\Gamma_2(k)\\
&\quad -\Gamma_2^T(k-\tau)R_2\Gamma_2(k-\tau)-\alpha_2\sum_{l=k-\tau}^{k-1}\eta_2^T(l)P_2\eta_2(l)\\
&\le \hat{\xi}^T(k)[2(1+2\alpha_2\tau)\hat{\Xi}_6^TP_2\hat{\Xi}_6+4(1+2\alpha_2\tau)\lambda_3\epsilon_{12}^2(\hat{\Xi}_2^T\hat{\Xi}_2+\hat{\Xi}_3^T\hat{\Xi}_3)\\
&\quad +4(1+2\alpha_2\tau)\lambda_3\epsilon_{22}^2(\hat{\Xi}_4^T\hat{\Xi}_4+\hat{\Xi}_5^T\hat{\Xi}_5)+(2\alpha_2\tau-1)\hat{\Xi}_3^TP_2\hat{\Xi}_3]\hat{\xi}(k)\\
&\quad +e_2^T(k)Q_2e_2(k)-e_2^T(k-d)Q_2e_2(k-d)+\Gamma_2^T(k)R_2\Gamma_2(k)\\
&\quad -\Gamma_2^T(k-\tau)R_2\Gamma_2(k-\tau)-\alpha_2\sum_{l=k-\tau}^{k-1}\eta_2^T(l)P_2\eta_2(l). \qquad (48)
\end{aligned}$$

According to the definition of $\eta_i(l)$ $(i=1,2)$, for any matrices $\hat{X}$, $\hat{Z}$ with appropriate dimensions, the following equations always hold:

$$
\begin{aligned}
0 &= 2\hat{\xi}^T(k)\hat{X}\Big[e_1(k)-e_1(k-\tau)-\sum_{l=k-\tau}^{k-1}\eta_1(l)\Big], && (49)\\
0 &= 2\hat{\xi}^T(k)\hat{Z}\Big[e_2(k)-e_2(k-\tau)-\sum_{l=k-\tau}^{k-1}\eta_2(l)\Big]. && (50)
\end{aligned}
$$

Furthermore, for any appropriately dimensioned matrices $\hat{M}_i=\hat{M}_i^T\geq 0$ $(i=1,2)$, the following equations are true:

$$
\begin{aligned}
0 &= \sum_{l=k-\tau}^{k-1}\hat{\xi}^T(k)\hat{M}_i\hat{\xi}(k)-\sum_{l=k-\tau}^{k-1}\hat{\xi}^T(k)\hat{M}_i\hat{\xi}(k)\\
&= \tau\hat{\xi}^T(k)\hat{M}_i\hat{\xi}(k)-\sum_{l=k-\tau}^{k-1}\hat{\xi}^T(k)\hat{M}_i\hat{\xi}(k). && (51)
\end{aligned}
$$

Then, substituting (45) and (48)-(51) into (40) yields

$$
\begin{aligned}
\Delta V(k) \leq\ & 6(1+2\alpha_1\tau)\hat{\xi}^T(k)[\hat{\Xi}_1^TP_1\hat{\Xi}_1+\hat{\Xi}_2^T(A_{11}-L_1C_1)^TG^T(GB_1)^{-1}G(A_{11}-L_1C_1)\hat{\Xi}_2\\
&+\hat{\Xi}_3^TA_{12}^TG^T(GB_1)^{-1}GA_{12}\hat{\Xi}_3+\hat{\Xi}_4^TA_{d11}^TG^T(GB_1)^{-1}GA_{d11}\hat{\Xi}_4\\
&+\hat{\Xi}_5^TA_{d12}^TG^T(GB_1)^{-1}GA_{d12}\hat{\Xi}_5+4\epsilon_{11}^2(\lambda_1+\lambda_2)(\hat{\Xi}_2^T\hat{\Xi}_2+\hat{\Xi}_3^T\hat{\Xi}_3)\\
&+4\epsilon_{21}^2(\lambda_1+\lambda_2)(\hat{\Xi}_4^T\hat{\Xi}_4+\hat{\Xi}_5^T\hat{\Xi}_5)]\hat{\xi}(k)+(2\alpha_1\tau-1)\hat{\xi}^T(k)\hat{\Xi}_2^TP_1\hat{\Xi}_2\hat{\xi}(k)\\
&+e_1^T(k)Q_1e_1(k)-e_1^T(k-d)Q_1e_1(k-d)+\Gamma_1^T(k)R_1\Gamma_1(k)-\Gamma_1^T(k-\tau)R_1\Gamma_1(k-\tau)\\
&+\hat{\xi}^T(k)[2(1+2\alpha_2\tau)\hat{\Xi}_6^TP_2\hat{\Xi}_6+4(1+2\alpha_2\tau)\lambda_3\epsilon_{12}^2(\hat{\Xi}_2^T\hat{\Xi}_2+\hat{\Xi}_3^T\hat{\Xi}_3)\\
&+4(1+2\alpha_2\tau)\lambda_3\epsilon_{22}^2(\hat{\Xi}_4^T\hat{\Xi}_4+\hat{\Xi}_5^T\hat{\Xi}_5)+(2\alpha_2\tau-1)\hat{\Xi}_3^TP_2\hat{\Xi}_3]\hat{\xi}(k)\\
&+e_2^T(k)Q_2e_2(k)-e_2^T(k-d)Q_2e_2(k-d)+\Gamma_2^T(k)R_2\Gamma_2(k)-\Gamma_2^T(k-\tau)R_2\Gamma_2(k-\tau)\\
&+\hat{\xi}^T(k)(\hat{\Omega}_1+\hat{\Omega}_1^T+\hat{\Omega}_2)\hat{\xi}(k)-\sum_{l=k-\tau}^{k-1}\varsigma^T(k,l)\hat{\Phi}\varsigma(k,l),\\
\leq\ & \hat{\xi}^T(k)[\hat{\Omega}_1+\hat{\Omega}_1^T+\hat{\Omega}_2+\hat{\Omega}_3+\hat{\Omega}_4+6(1+2\alpha_1\tau)\hat{\Xi}_1^TP_1\hat{\Xi}_1\\
&+6(1+2\alpha_1\tau)\hat{\Xi}_2^T(A_{11}-L_1C_1)^TG^T(GB_1)^{-1}G(A_{11}-L_1C_1)\hat{\Xi}_2\\
&+6(1+2\alpha_1\tau)\hat{\Xi}_3^TA_{12}^TG^T(GB_1)^{-1}GA_{12}\hat{\Xi}_3+6(1+2\alpha_1\tau)\hat{\Xi}_4^TA_{d11}^TG^T(GB_1)^{-1}GA_{d11}\hat{\Xi}_4\\
&+6(1+2\alpha_1\tau)\hat{\Xi}_5^TA_{d12}^TG^T(GB_1)^{-1}GA_{d12}\hat{\Xi}_5+2(1+2\alpha_2\tau)\hat{\Xi}_6^TP_2\hat{\Xi}_6]\hat{\xi}(k)\\
:=\ & \hat{\xi}^T(k)\hat{\Psi}\hat{\xi}(k), && (52)
\end{aligned}
$$

where

$$
\begin{aligned}
\varsigma(k,l) &= \begin{bmatrix}\hat{\xi}^T(k) & \eta_1^T(l) & \eta_2^T(l)\end{bmatrix}^T,\\
\hat{\Omega}_1 &= \begin{bmatrix}\hat{X} & \hat{Z}\end{bmatrix}\left[\begin{array}{c}I_{p\times p}\quad -I_{p\times p}\quad 0_{p\times((m+1)n-2p)}\\ \hline 0_{(n-p)\times(m+1)p}\quad I_{(n-p)\times(n-p)}\quad -I_{(n-p)\times(n-p)}\quad 0_{(n-p)\times(m-1)(n-p)}\end{array}\right],\\
\hat{\Omega}_2 &= \tau(\hat{M}_1+\hat{M}_2),\quad \hat{\Omega}_3=\hat{W}_R^TR\hat{W}_R,\quad \hat{\Omega}_4=\sum_{i=2}^{5}\hat{\Xi}_i^T\Theta_i\hat{\Xi}_i,
\end{aligned}
$$

$$\hat{W}_R = \begin{bmatrix} \frac{\begin{array}{cc} I_{mp\times mp} & 0_{mp\times(m(n-p)+n)} \end{array}}{\begin{array}{ccc} 0_{mp\times p} & I_{mp\times mp} & 0_{mp\times((m+1)(n-p))} \end{array}} \\ \frac{\begin{array}{ccc} 0_{m(n-p)\times(m+1)p} & I_{m(n-p)\times m(n-p)} & 0_{m(n-p)\times(n-p)} \end{array}}{\begin{array}{cc} 0_{m(n-p)\times(mp+n)} & I_{m(n-p)\times m(n-p)} \end{array}} \end{bmatrix},$$

$$\hat{\Phi} = \begin{bmatrix} \hat{M}_1+\hat{M}_2 & \hat{X} & \hat{Z} \\ * & \alpha_1 P_1 & 0 \\ * & * & \alpha_2 P_2 \end{bmatrix},$$

and R, Θ_2, Θ_3, Θ_4, Θ_5 are defined in (31). It is not difficult to see from (30) that $\hat{\Psi} < 0$ where $\hat{\Psi}$ is defined in (52). Then, it follows from the Lyapunov stability theorem that the overall error dynamics is asymptotically stable in the specified sliding surface (16).

In order to deal with the H_∞ performance of the overall error dynamics with $\omega(k) \neq 0$, we introduce the following index:

$$J(n) = \sum_{k=0}^{n} \left[e^T(k)e(k) - \gamma^2 \omega^T(k)\omega(k) \right],$$

where n is a nonnegative integer. Obviously, our aim is to show $J(n) < 0$ $(n \to \infty)$ under the zero-initial condition. Along the same line of the above proof of the stability, it is easy to obtain

$$\begin{aligned} J(n) &= \sum_{k=0}^{n} [e^T(k)e(k) - \gamma^2\omega^T(k)\omega(k) + \Delta V(k)] - V(n+1) \\ &\le \sum_{k=0}^{n} [\xi^T(k)\Lambda\xi(k)], \end{aligned}$$

where

$$\begin{aligned} \xi(k) &= \begin{bmatrix} \hat{\xi}^T(k) & \omega^T(k) \end{bmatrix}^T, \\ \Lambda &= \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + 6(1+2\alpha_1\tau)\Xi_1^T P_1 \Xi_1 \\ &\quad + 6(1+2\alpha_1\tau)\Xi_2^T (A_{11} - L_1C_1)^T G^T (GB_1)^{-1} G (A_{11} - L_1 C_1)\Xi_2 \\ &\quad + 6(1+2\alpha_1\tau)\Xi_3^T A_{12}^T G^T (GB_1)^{-1} G A_{12} \Xi_3 + 6(1+2\alpha_1\tau)\Xi_4^T A_{d11}^T G^T (GB_1)^{-1} G A_{d11}\Xi_4 \\ &\quad + 6(1+2\alpha_1\tau)\Xi_5^T A_{d12}^T G^T (GB_1)^{-1} G A_{d12} \Xi_5 + 2(1+2\alpha_2\tau)\Xi_6^T P_2 \Xi_6, \\ \Xi_1 &= \begin{bmatrix} A_{11} - L_1C_1 & 0_{p\times(m-1)p} & A_{d11} & A_{12} & 0_{p\times(m-1)(n-p)} & A_{d12} & D_1 \end{bmatrix}, \\ \Xi_6 &= \begin{bmatrix} A_{21} - L_2C_1 & 0_{p\times(m-1)p} & A_{d21} & A_{22} & 0_{p\times(m-1)(n-p)} & A_{d22} & D_2 \end{bmatrix}, \end{aligned}$$

with Ω_i $(i = 1, \ldots, 5)$ and Ξ_j $(j = 2, \ldots, 5)$ are defined in (31).

According to Lemma 2, $\Lambda < 0$ is equivalent to

$$\bar{\Lambda} = \begin{bmatrix} \bar{\Lambda}_{11} & \vartheta \Xi_1^T P_1 & \nu \Xi_6^T P_2 & \bar{\Lambda}_{14} \\ * & -P_1 & 0 & 0 \\ * & * & -P_2 & 0 \\ * & * & * & \bar{\Lambda}_{44} \end{bmatrix} < 0$$

where

$$
\begin{aligned}
\bar{\Lambda}_{11} &= \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5, \\
\bar{\Lambda}_{14} &= \left[\begin{array}{cccc} \vartheta \Xi_2^T (A_{11} - L_1 C_1)^T G^T & \vartheta \Xi_3^T A_{12}^T G^T & \vartheta \Xi_4^T A_{d11}^T G^T & \vartheta \Xi_5^T A_{d12}^T G^T \end{array} \right], \\
\bar{\Lambda}_{44} &= \mathrm{diag}\{-B_1^T P_1 B_1, -B_1^T P_1 B_1, -B_1^T P_1 B_1, -B_1^T P_1 B_1\}, \\
\vartheta &= \sqrt{6(1+2\alpha_1 \tau)}, \ \ \nu = \sqrt{2(1+2\alpha_2 \tau)}.
\end{aligned}
$$

By setting $L_i = P_i^{-1} Y_i$ $(i = 1, 2)$, condition (30) (i.e. $\Psi < 0$) implies $\bar{\Lambda} < 0$ and therefore we have $J(n) < 0$. Letting $n \to \infty$, we obtain

$$
\sum_{k=0}^{\infty} \|e(k)\|^2 \leq \gamma^2 \sum_{k=0}^{\infty} \|\omega(k)\|^2 ,
$$

which completes the proof of Theorem 2.

References

[1] F. J. Bejarano and L. Fridman, High order sliding mode observer for linear systems with unbounded unknown inputs, *International Journal of Control*, Vol. 83, No. 9, pp. 1920-1929, Sep. 2010.

[2] H. Dong, Z. Wang, D. W. C. Ho and H. Gao, Robust H_∞ fuzzy output-feedback control with multiple probabilistic delays and multiple missing measurements, *IEEE Transactions on Fuzzy Systems*, Vol. 18, No. 4, pp. 712-725, 2010.

[3] H. Dong, Z. Wang and H. Gao, Observer-based H_∞ control for systems with repeated scalar nonlinearities and multiple packet losses, *International Journal of Robust and Nonlinear Control*, Vol. 20, No. 12, pp. 1363-1378, 2010.

[4] H. Dong, Z. Wang, D. W. C. Ho and H. Gao, Variance-constrained H_∞ filtering for nonlinear time-varying stochastic systems with multiple missing measurements: the finite-horizon case, *IEEE Transactions on Signal Processing*, Vol. 58, No. 5, pp. 2534-2543, 2010.

[5] H. Dong, Z. Wang and H. Gao, Robust H_∞ filtering for a class of nonlinear networked systems with multiple stochastic communication delays and packet dropouts, *IEEE Transactions on Signal Processing*, Vol. 58, No. 4, pp. 1957-1966, 2010.

[6] C. Edwards and S. K. Spurgeon, Sliding mode stabilization of uncertain systems using only output information, *International Journal of Control*, Vol. 62, No. 5, pp. 1129-1144, Nov. 1995.

[7] C. Edwards, S. K. Spurgeon and R. Patton, Sliding mode observers for fault detection and isolation, *Automatica*, Vol. 36, No. 4, pp. 541-553, Apr. 2000.

[8] R. W. Eustace, B. A. Woodyatt, G. L. Merrington and A. Runacres, Fault signatures obtained from fault implant tests on an F404 engine, *ASME Transactions Journal of Engineering for Gas Turbines and Power*, vol. 116, no. 1, pp. 178-183, Jan. 1994.

[9] E. Fridman and U. Shaked, A descriptor system approach H_∞ control of linear time-delay systems, *IEEE Trans. Autom. Control*, Vol. 47, No. 2, pp. 253-270, Feb. 2002.

[10] W. Glover and J. Lygeros, A stochastic hybrid model for air traffic control simulation, *Seventh International Workshop on Hybrid Systems: Computation and Control*, Philadelphia, PA, U.S.A. Lecture Notes in Computer Science, Vol. 2993, pp. 372-386, 2004.

[11] W. Gao, Y. Wang and A. Homaifa, Discrete-time variable structure control systems, *IEEE Trans. Industrial Electonics*, Vol. 42, No. 2, pp. 117-122, Apr. 1995.

[12] M. Ghanes and G. Zheng, On sensorless induction motor drives: sliding-mode observer and output feedback controller, *IEEE Trans. Industrial Electonics*, Vol. 56, No. 9, pp. 3404-3413, Sep. 2009.

[13] X. Han, E. Fridman and S. K. Spurgeon, Sliding-mode control of uncertain systems in the presence of unmatched disturbances with applications, *International Journal of Control*, Vol. 83, No. 12, pp. 2413-2426, Dec. 2010.

[14] B. Jiang, M. Staroswiecki and V. Cocquempot, Fault estimation in nonlinear uncertain systems using robust/sliding-mode observers, *IEE Proceedings Control Theory and Applications*, Vol. 151, No. 1, pp. 29-37, Jan. 2004.

[15] K. Kalsi, J. Lian, S. Hui and S. Żak, Sliding-mode observers for systems with unknown inputs: A high-gain approach, *Automatica*, Vol. 46, No. 2, pp. 347-353, Feb. 2010.

[16] A. J. Koshkouei and A. S. I. Zinober, Sliding mode state observers for SISO linear discrete-time systems, in *Proc. UKACC Int. Conf. Control*, U.K., 1996, pp. 837-842.

[17] C. Lascu, I. Boldea and F. Blaabjerg, A class of speed-sensorless sliding-mode observers for high-performance induction motor drives, *IEEE Trans. Industrial Electonics*, Vol. 56, No. 9, pp. 3394-3403, Sep. 2009.

[18] L. Ma, Z. Wang and Z. Guo, Robust H_2 sliding mode control for nonlinear discrete-time stochastic systems, *IET Control Theory Appl.*, Vol. 3, No. 11, pp. 1537-1546, Nov. 2009.

[19] Y. S. Moon, P. Park, W. H. Kwon and Y. S. Lee, Delay-dependent robust stabilization of uncertain state-delayed systems, *Int. J. Control*, Vol. 74, No. 14, pp. 1447-1455, 2001.

[20] Y. Niu, D. W. C. Ho and J. Lam, Robust integral sliding mode control for uncertain stochastic systems with time-varying delay, *Automatica*, Vol. 41, No. 5, pp. 873-880, May. 2005.

[21] Y. Niu and D. W. C. Ho, Robust observer design for Itô stochastic time-delay systems via sliding mode control, *Systems & Control Letters*, Vol. 55, No. 10, pp. 781-793, Oct. 2006.

[22] Y. Niu, D. W. C. Ho and Z. Wang, Improved sliding mode control for discrete-time systems via reaching law, *IET Control Theory Appl.*, Vol. 4, No. 11, pp. 2245-2251, Nov. 2010.

[23] D. Peaucelle, D. Arzelier, D. Henrion and F. Gouaisbaut, Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation, *Automatica*, Vol. 43, No. 5, pp. 795-804, May. 2007.

[24] R. Raoufi, H. Marquez, and A. Zinober, H_∞ sliding mode observers for uncertain nonlinear Lipschitz systems with fault estimation synthesis, *International Journal of Robust and Nonlinear Control*, Vol. 20, No. 16, pp. 1785-1801, Nov. 2010.

[25] P. Shi, Y. Xia, G. Liu and D. Rees, On designing of sliding-mode control for stochastic jump systems, *IEEE Trans. Automat. Control*, Vol. 51, No. 1, pp. 97-103, Jan. 2006.

[26] S. K. Spurgeon, Sliding mode observers: a survey, *International Journal of Systems Science*, Vol. 39, No. 8, pp. 751-764, 2008.

[27] C. P. Tan and C. Edwards, Robust fault reconstruction in uncertain linear systems using multiple sliding mode observers in cascade, *IEEE Trans. Automat. Control*, Vol. 55, No. 4, pp. 855-867, Apr. 2010.

[28] C. P. Tan, X. Yu and Z. Man, Terminal sliding mode observers for a class of nonlinear systems. *Automatica*, Vol. 46, No. 8, pp. 1401-1404, Aug. 2010.

[29] K. C. Veluvolu, Y. C. Soh and W. Cao, Robust discrete-time nonlinear sliding mode state estimation of uncertain nonlinear systems, *International Journal of Robust and Nonlinear Control*, Vol. 17, No. 9, pp. 803-828, Jun. 2007.

[30] K. C. Veluvolu, and Y. C. Soh, Discrete-time sliding-mode state and unknown input estimations for nonlinear systems, *IEEE Trans. Industrial Electronics*, Vol. 56, No. 9, pp. 3443-3452, Sep. 2009.

[31] Z. Wang and D. W. C. Ho, Filtering on nonlinear time-delay stochastic systems. *Automatica*, Vol. 39, No. 1, pp. 101-109, Jan. 2003.

[32] Z. Wang, J. Lam, and X. Liu, Filtering for a class of nonlinear discrete-time stochastic systems with state delays, *Journal of Computational and Applied Mathematics*, Vol. 201, No. 1, pp. 153-163, Apr. 2007.

[33] Z. Wang, Y. Liu and X. Liu, H_∞ filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities. *Automatica*, Vol. 44, No. 5, pp. 1268-1277, May. 2008.

[34] L. Wu and D. W. C. Ho, Sliding mode control of singular stochastic hybrid systems, *Automatica*, Vol. 46, No. 4, pp. 779-783, Apr. 2010.

[35] L. Wu, P. Shi and H. Gao, State estimation and sliding-mode control of Markovian jump singular systems, *IEEE Trans. Automat. Control*, Vol. 55, No. 5, pp. 1213-1219, May. 2010.

[36] M. Wu, Y. He, J.-H. She and G.-P. Liu, Delay-dependent criteria for robust stability of time-varying delay systems, *Automatica*, Vol. 40, No. 8, pp. 1435-1439, Agu. 2004.

[37] Y. Xia, G. Liu, P. Shi, J. Chen and D. Rees, Robust delay-dependent sliding mode control for uncertain time-delay systems, *International Journal of Robust and Nonlinear Control*, Vol. 18, No. 11, pp. 1142-1161, Jul. 2008.

[38] Y. Xia, M. Fu, P. Shi and M. Wang, Robust sliding mode control for uncertain discrete-time systems with time delay, *IET Control Theory Appl.*, Vol. 4, No. 4, pp. 613-624, Apr. 2010.

[39] M. Yan and Y. Shi, Robust discrete-time sliding mode control for uncertain systems with time-varying state delay, *IET Control Theory Appl.*, Vol. 2, No. 8, pp. 662-674, Aug. 2008.

[40] X. Yan and C. Edwards, Nonlinear robust fault reconstruction and estimation using a sliding mode observer, *Automatica*, Vol. 43, No. 9, pp. 1605-1614, Sep. 2007.

[41] E. Yaz and A. Azemi, Observer design for discrete and continuous non-linear stochastic systems, *International Journal of Systems and Science*, Vol. 24, No. 12, pp. 2289-2302, Dec. 1993.

[42] L. Zhang, E. Boukas and P. Shi, Exponential H_∞ filtering for uncertain discrete-time switched linear systems with average dwell time: A μ-dependent approach, *International Journal of Robust and Nonlinear Control*, Vol. 18, No. 11, pp. 1188-1207, Jul. 2008.

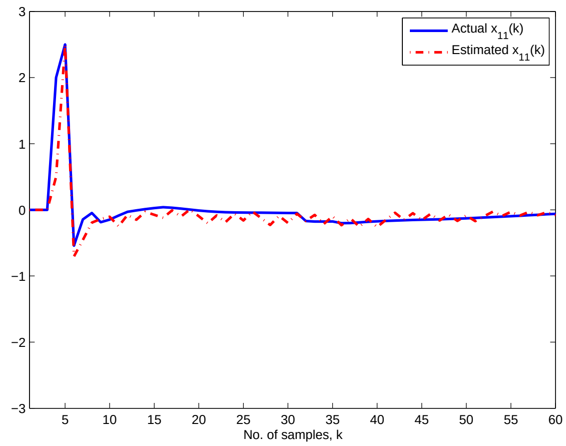

Fig. 1. The trajectories of $x_{11}(k)$ and $\hat{x}_{11}(k)$ with normal scale

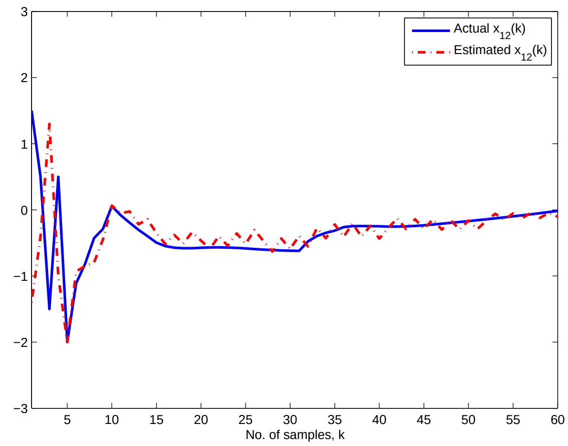

Fig. 2. The trajectories of $x_{12}(k)$ and $\hat{x}_{12}(k)$ with normal scale

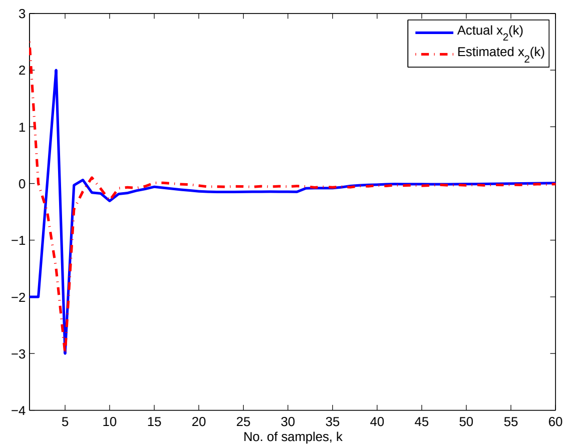

Fig. 3. The trajectories of $x_2(k)$ and $\hat{x}_2(k)$ with normal scale

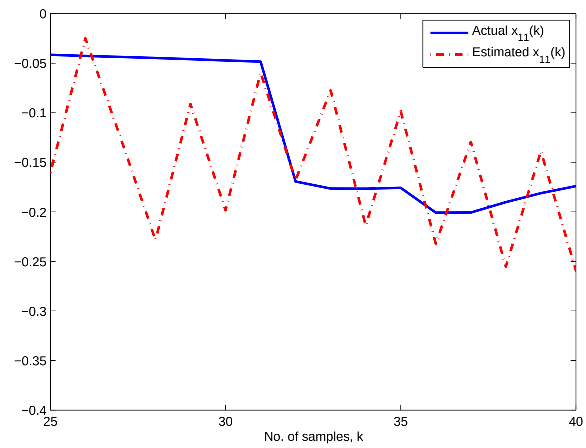

Fig. 4. The trajectories of $x_{11}(k)$ and $\hat{x}_{11}(k)$ with small scale

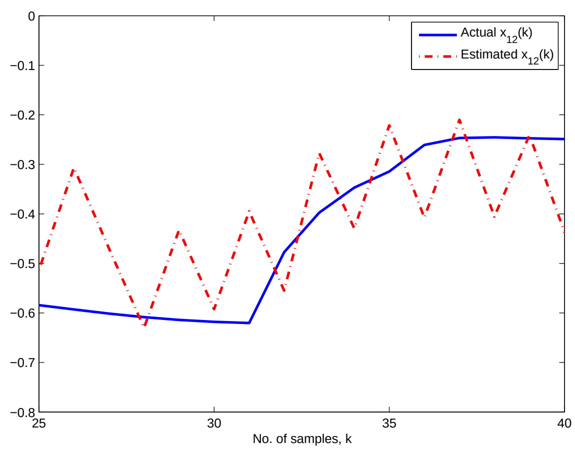

Fig. 5. The trajectories of $x_{12}(k)$ and $\hat{x}_{12}(k)$ with small scale

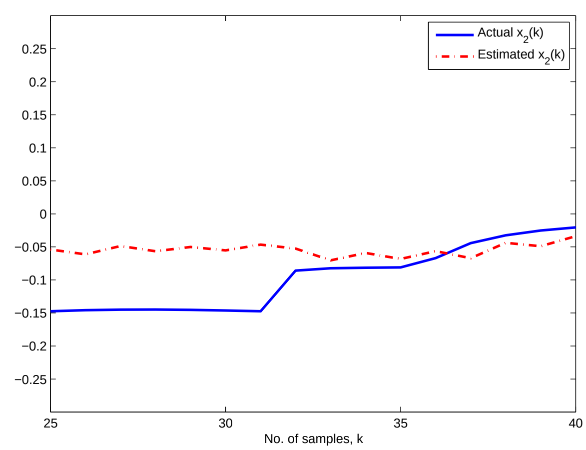

Fig. 6. The trajectories of $x_2(k)$ and $\hat{x}_2(k)$ with small scale

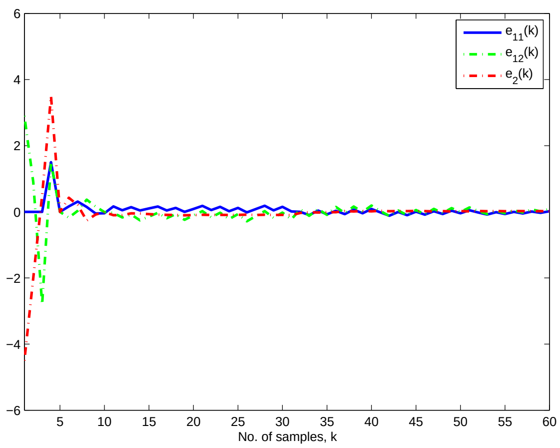

Fig. 7. The trajectory of error $e(k)$

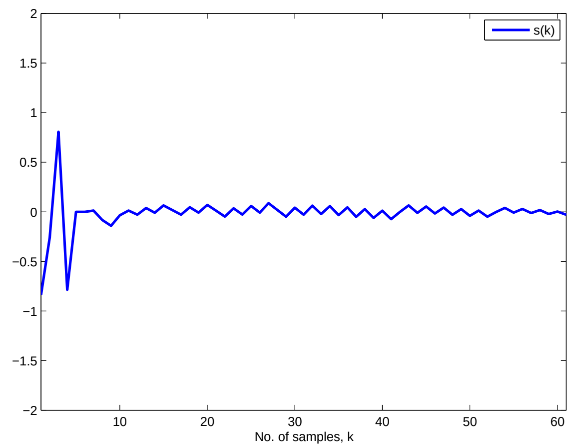

Fig. 8. The sliding surface $s(k)$