H∞ Preview Control and Fixed-Lag Smoothing I: Matrix ARE Solutions in Continuous-Time Systems

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***Abstract*— Preview control and fixed-lag smoothing allow an anti-causal component in the controller/estimator. Time domain variational analysis is used in a reduction to an open loop differential game, leading to a complete, necessary and sufficient characterization of suboptimal values and an explicit state space design, in terms of a parameterized (non-standard) algebraic matrix Riccati equation in a general continuous time linear system setting. The solution offers insight into the appropriate structure of the associated Hamiltonian, where the state and co-state are not the usual state of the original dynamic system and that of its adjoint. Rather, the state and co-state are selected to capture the respective lumped effects of initial data and future input selection in the allied game.**

I. Introduction

Preview tracking and fixed-lag smoothing are extensions of standard full-information and filtering problems, with relaxed causality constraints (preview availability). Numerous control and estimation problems fall into the category of problems with information preview. In some tracking problems, e.g., those arising in robotics [1] and vehicle suspension control [2], previewed commands or disturbances may be available. Such problems are referred to as *preview tracking*. Similarly, in many communication systems, even interactive ones, a small amount of delay or latency can be tolerated. Some delay is often permissible in speech coding [3], multirate filter banks design [4], multi-target tracking of a maneuvering target [5], etc. Such problems can be formulated as estimation problems with a constant preview window and they are referred to as *fixed-lag smoothing*.

The H_2 (LQ) theory of the preview tracking and estimation (smoothing) is currently well developed, see e.g. [2], [6], [7]. Apparently, the first mention of the preview tracking in the H_∞ (game-theoretic) context appeared in [8]. Yet in this paper, only *unmeasured* disturbances were included into the game. Thus, preview tracking is actually treated in [8] in the H_2, rather than in the H_∞ setting. The same approach was also adopted in [9].

In the pure H_∞ setting preview problems are considerably less studied. H_∞ control and estimation with preview proved to be a challenge already in the current, continuous-time setting, and even more so, in discrete-time and in sampled-data systems (stated as Open Problem 51, in [10]). Indeed, most of the existing results resort to strictly sufficient conditions, system restrictions, iterative approximations and dimension increase. For example, the solution of the continuous-time H_∞ preview tracking, in [11], is derived in terms of the standard H_∞ algebraic Riccati equation (ARE), that is associated with the tracking problem without preview. That equation, however, might not admit a stabilizing solution under some performance levels γ, for which the preview problem *is* solvable. In other words, the solvability condition in [11] is only sufficient. For some other (discrete-time) examples see the discussion in the companion paper [12]. To the best of our knowledge, the only *complete* solution of H_∞ preview problems is the solution of the continuous-time H_∞ fixed-lag smoothing problem in [13]. The approach in [13] is based on J-spectral factorization arguments (and a transformation introduced in [14]) and the solution there is formulated in terms of a modified H_∞ ARE, the Hamiltonian matrix associated with which is similar to that associated with the filtering H_∞ ARE.

The purpose of this, and of two companion papers [12], [15], is to solve H_∞ preview control and estimation problems in the current, continuous-time, the discrete-time, and the sampled-data settings. For notational simplicity, the exposition is made in a time invariant setting. Our results and the variational arguments, however, readily extend to time varying systems, which is a main advantage over transform domain methods, as those in [13]. Due to space limitations, we present here only control solutions. Smoothing results follow by applying standard duality arguments. The corresponding formulae are included to the full versions of these papers.

Linear systems with a single, pure, input lag are well recognized as the simplest among the various classes of distributed parameter systems. Custom design methods aim to exploit and match that simplicity. The Smith predictor, which reduces stabilization to an equivalent problem with no delay, is an early example. From the technical perspective of an adaptation of the variational / game theoretic analysis methods used here, the difficulty in the preview setting, is twofold. One issue is borne out by the mere fact that, with the presence of delay, this is a distributed parameter system. The *complete state*, including a relevant disturbance history, is embedded in the Hilbert space $M_2 \doteq \mathbb{R}^n \times L_2[-h, 0]$, as explained below, leading to infinite dimensionality. Following the example of earlier game theoretic solutions of H_∞ problems in systems with control delay (see the review [16]), this issue is addressed by a reduction to a non-distributed differential game and the utilization of an interplay between a distributed parameters model and the original system. The second difficulty, which is new in the current setting, concerns the utilization of Hamilton-Jacobi characterization of optimal solutions of the underlying game, and the derivation of a Riccati equation, thereof. Specifically, here the initial state of the standard Hamiltonian depends, both on initial data and on future values of the disturbance. The key solution step is a change of state, whereby the new Hamiltonian state reflects only initial data contributions, and the co-state — only future disturbance effects. This leads to a complete (necessary and sufficient) non-standard ARE solution, in a general linear system. Our solution is then stated in terms of a combination of one H_2 and one, parameterized H_∞ ARE's.

II. Main result: complete information preview control

The preview problem concerns a time invariant[1] system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t-h) + B_2 u(t) \\ z(t) &= Cx(t) + D_1 w(t-h) + D_2 u(t) \end{aligned} \tag{1}$$

[1]The restriction to time invariant systems is made for simplicity; all arguments and results in this paper readily generalize to time varying systems.

$$u(t)=-R_2\Big(\big(D_2'C+B_2'(X_\varkappa+e^{A_\varkappa' h}X_\gamma e^{A_\varkappa h})\big)x(t)+D_2'D_1w(t-h)+B_2'\int_0^h\big(e^{A_\varkappa' r}E_\varkappa'+e^{A_\varkappa' h}X_\gamma e^{A_\varkappa(h-r)}B_\tau(r)\big)w(t-h+r)dr\Big). \quad (4)$$

with the usual interpretation of $x\in\mathbb{R}^n$, w, u and z as the state, exogenous input (here, representing disturbances and tracking reference), control and controlled output. We assume that

(A1): the pair (A,B_2) is stabilizable;
(A2): the realization $[A,B_2,C,D_2]$ has no invariant zeros on the imaginary axis and is left invertible.

Note that (A2) guarantees that $D_2'D_2$ is nonsingular.

The *complete state* (in the sense of Poincaré) at the time t comprises the pair[2] $(x(t),\breve{w}_t)\in M_2$. Given initial data $(x(0),\breve{w}_0)\in M_2$ and an exogenous input $w\in L_2$, an *admissible control* is $u\in L_2$ for which $x\in L_2$. The set of admissible controls is non-empty, by assumption (A1). *Complete information* feedback controllers in (1) comprise admissible feedback control policies that determine the control value at a time t in terms of past and current values of $(x(t),\breve{w}_t)$. The delay in w reflects thus the *ability to preview* the disturbance/reference.

The notation "γ_0" will be used for the infimal attainable induced L_2 norm of the closed loop mapping $T_{wz}: w\mapsto z$, here, under complete information feedback. The definition of γ_0 presumes zero initial data or, equivalently, processes starting at $t_0=-\infty$.

In preparation for the statement of our main result, in Theorem 1 below, we introduce some notations. First, denote $R_2\doteq(D_2'D_2)^{-1}$ and define the following LQ-type control ARE

$$X_\varkappa A+A'X_\varkappa+C'C-(X_\varkappa B_2+C'D_2)R_2(B_2'X_\varkappa+D_2'C)=0, \quad (2)$$

which, by assumptions (A1) and (A2), admits a stabilizing solution $X_\varkappa\geq 0$. For future reference we denote

$$\begin{aligned}A_\varkappa &\doteq A-B_2R_2(B_2'X_\varkappa+D_2'C),\\ E_\varkappa &\doteq B_1'X_\varkappa+D_1'C-D_1'D_2R_2(B_2'X_\varkappa+D_2'C),\\ G_c(t) &\doteq \int_0^t e^{A_\varkappa r}B_2R_2B_2'e^{A_\varkappa' r}dr,\\ B_\tau(t) &\doteq B_1-B_2R_2D_2'D_1-G_c(t)E_\varkappa',\end{aligned}$$

and note that $A_\varkappa$ is Hurwitz. To simplify the exposition, in the sequel we will simply write G_c and B_τ instead of $G_c(h)$ and $B_\tau(h)$ (i.e., when $t=h$). For $\gamma>\|D_1'\big(I-D_2R_2D_2'\big)\|$ we shall also use the notations

$$\begin{aligned}\varGamma_\gamma &\doteq \gamma^2I-D_1'\big(I-D_2R_2D_2'\big)D_1,\\ A_\gamma &\doteq A_\varkappa+B_\tau\varGamma_\gamma^{-1}E_\varkappa,\\ R_\gamma &\doteq e^{A_\varkappa h}B_2R_2B_2'e^{A_\varkappa' h}-B_\tau\varGamma_\gamma^{-1}B_\tau'.\end{aligned}$$

Our main result is then formulated as follows (see Section III for the proof):

Theorem 1: The following two statements are equivalent

1) $\gamma>\gamma_0$ in the complete information preview control problem
2) $\|D_1\big(I-D_2R_2D_2'\big)\|<\gamma$ and the ARE

$$X_\gamma A_\gamma+A_\gamma'X_\gamma-X_\gamma R_\gamma X_\gamma+E_\varkappa'\varGamma_\gamma^{-1}E_\varkappa=0 \quad (3)$$

admits a stabilizing solution $X_\gamma\geq 0$ (so that $A_\gamma-R_\gamma X_\gamma$ is Hurwitz).

Furthermore, if $\gamma>\gamma_0$, then one stabilizing, strictly γ-suboptimal feedback control is given by equation (4) at the top of this page.

[2]The notation $\breve{w}_t\in L_2[-h,0]$ stands for the relevant, finite window history of an $L_{2,\text{loc}}$ trajectory w at the time t: $\breve{w}_t(r)=w(t+r)$, $r\in[-h,0]$.

Remark 2.1: The sign-indefinite ARE (3) in Theorem 1 is different from the standard H_∞ control ARE (with the stabilizing solution $\tilde{X}_\gamma$) which arises in the solution of the preview-free H_∞ problem. It can be shown (by inverting the change of variables (27) in the proof of Theorem 1, below) that, when both exist, $\tilde{X}_\gamma$ and X_γ are related by $\tilde{X}_\gamma=X_\varkappa+X_\gamma(I-G_cX_\gamma)^{-1}$ or, equivalently, by $X_\gamma=(\tilde{X}_\gamma-X_\varkappa)\big(I+G_c(\tilde{X}_\gamma-X_\varkappa)\big)^{-1}$. In particular, $I-G_cX_\gamma$ is non-singular iff $I+G_c(\tilde{X}_\gamma-X_\varkappa)$ is non-singular. Thus, in those cases where the ARE for $\tilde{X}_\gamma$ does posses a stabilizing solution $\tilde{X}_\gamma$, the characterization of the parameter γ as a suboptimal value is in terms of the conditions that $I+G_c(\tilde{X}_\gamma-X_\varkappa)$ be non-singular and that the self adjoint matrix $(\tilde{X}_\gamma-X_\varkappa)(I+G_c(\tilde{X}_\gamma-X_\varkappa))^{-1}\geq 0$. This statement, however, lacks the authority of a complete parameterization in the sense that its starting point is a strictly sufficient condition.

Comparing with the previously available solution of [11], the result of Theorem 1 has two advantages.

1) Theorem 1 offers conditions that are both necessary and sufficient. In comparison, a solution in terms of the standard H_∞ ARE rather than (3) (as in [11]) offers only a sufficient condition, as it rules out some values of $\gamma>\gamma_0$, for which a stabilizing solution of the standard ARE does not exist, but a solution based on (3) does.
2) Since the matrix $A_\varkappa$ is Hurwitz, the control law (4) involves exponentials of strictly stable matrices, which are well posed as $h\to\infty$. This is an improvement over the control law in [11], which involves the exponential of a Hamiltonian matrix, half of which eigenvalues are strictly positive.

Note also that we do not impose any simplifying assumptions on the parameters of the system (1).

III. The Proof

Throughout the rest of the paper we use the following simplifying assumption:

(A3): $D_2'\begin{bmatrix} C & D_1 & D_2\end{bmatrix}=\begin{bmatrix}0 & 0 & I\end{bmatrix}$,

which means that $\|z\|_{L_2}^2=\|Cx+D_1w\|_{L_2}^2+\|u\|_{L_2}^2$. This considerably reduces the complexity of required algebraic manipulations in the proof of Theorem 1. By standard procedure, (A3) is imposed without loss of generality, via the change of variables:

$$u(t)\to(D_2'D_2)^{-1/2}u(t)-(D_2'D_2)^{-1}D_2'\big(Cx(t)+D_1w(t-h)\big).$$

To maintain the nomenclature of Theorem 1, we nonetheless continue to use the notations of Section II, even when (A3) enables further simplifications.

A. Game theoretic analysis: necessity

Following an established practice [17]–[21], the proof of Theorem 1 is focused on the differential game

$$\max_w\Big\{\min_u\|z\|_{L_2}^2-\gamma^2\|w\|_{L_2}^2\Big\} \quad (5)$$

where the maximization is over $w\in L_2$ and then over admissible $u\in L_2$. The difference between the current version of (4), and that considered in earlier contexts, is in the data for the game, which here comprises the complete initial data $(x(0),\breve{w}_0)\in M_2$. Thus, while the analysis traces the basic steps introduced early on, our focus will be on the special features, entailed by this data structure.

The basic fact upon which the analysis of (4) hinges is the straightforward, indefinite (Krein space) variant of the projection theorem. We state it here for future reference to its notations and terms.

Theorem 2: Let U and V be Hilbert spaces with bounded linear operators $J : V \mapsto V$ and $S : U \mapsto V$. Suppose $J = J'$ and $S'JS > \epsilon I$ for some $\epsilon > 0$. Then, given any $v \in V$ there exists a unique solution to the optimization problem

$$\min_{u\in U} \|Su - v\|_J^2 = \min_{u\in U} \langle (Su - v), J(Su - v)\rangle \tag{6}$$

This solution is defined by the data, by a bounded linear operator, $u^* = (S'JS)^{-1}S'Jv$. Equivalently, u^* is completely characterized by the equality $S'J(Su^* - v) = 0$, or the fact that $\forall u \in U$, $\langle Su, J(Su^* - v)\rangle = 0$.

1) The optimal u: Given $w \in L_2$ and the initial data $(x(0), \breve{w}_0) \in M_2$, the optimization of u in (4) is a classic inhomogeneous LQ optimal control problem. For completeness, and in order to set the conceptual framework for the subsequent optimization in w, we briefly review its solution.

Let K be a stabilizing state feedback gain, as discussed above, so that the substitution $u = -Kx + \tilde{u}$, $\tilde{u} \in L_2$, provides an explicit and complete parameterization of admissible controls. Assumptions (A1) and (A2) are invariant under this substitution so they remain valid in the system $[A - B_2K, B_2, C - D_2K, D_2]$. We now appeal to Theorem 2 with the following definitions: The role of "u" is taken $\tilde{u} \in U = L_2$; the operator $S : \tilde{u} \mapsto z : L_2 \mapsto L_2$ is the input-output (I/O) mapping in $[A - B_2K, B_2, C - D_2K, D_2]$ (with the zero initial state); let $\tilde{z} \in L_2 = V$ be the response[3] in $[A - B_2K, B_1d_h, C - D_2K, D_1d_h]$ to the selected $w \in L_2$ and the initial data $(x(0), \breve{w}_0) \in M_2$; the role of "$v$" is taken by $-\tilde{z}$; finally, set $J = I$. The operator $S'S$ is uniformly positive, by assumption (A2) (actually, $S'S \geq I$, by (A3)), whereby a unique optimal $\tilde{u}$, denoted $\tilde{u}^\#$ (hence a unique optimal $u^\#$) exists, and is completely characterized by the relation $S'(S\tilde{u}^\# + \tilde{z}) = S'z^\# = 0$, where $z^\# = S\tilde{u}^\# + \tilde{z}$ is the response of (1) with prescribe initial data, w, and the optimal control. For future reference we denote by $x^\#(x(0), \breve{w}_0, w)$, $u^\#(x(0), \breve{w}_0, w)$ and $z^\#(x(0), \breve{w}_0, w)$ the bounded linear operators from $M_2 \times L_2$ to L_2, which are defined by the solution of the optimal control problem.

An explicit realization of the condition $S'z^\# = 0$ is in terms of the anti-causal system

$$\dot{p} = -(A - B_2K)'p - (C - D_2K)'z^\#, \tag{7a}$$
$$0 = B_2'p + D_2'z^\#. \tag{7b}$$

Substituting (7b) into (7a), terms in K are eliminated, showing that the arbitrary selection of K has no impact on the optimization. Leaving K has the advantage of maintaining the boundedness of the (anti-causal) mapping $z^\# \mapsto p$, hence of a linear operator $p^\#(x(0), \breve{w}_0, w)$. Using (A3), the optimal control is then of the form

$$u^\# = -B_2'p^\#. \tag{8}$$

This leads to the following Hamilton-Jacobi system, whose unique L_2 solution is the solution of the optimal control problem

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -B_2B_2' \\ -C'C & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} - \begin{bmatrix} -B_1 \\ C'D_1 \end{bmatrix} d_hw. \tag{9}$$

In the homogeneous case, where w vanishes throughout, the state and co-state of the L_2 solution are related via $p = X_\varkappa x$, where $X_\varkappa \geq 0$ is the stabilizing solution of the ARE (2). One common and very useful observation concerning the inhomogeneous case is obtained by integrating $\frac{d}{dt}\langle x, p\rangle$ under optimal control. Invoking (8) and the two equations in (9), for mutual cancellation of most terms, this leads to

$$\|z^\#\|_{L_2[t_0,t_1]}^2 = -\langle x, p\rangle\big|_{t_0}^{t_1} + \langle d_hw, B_1'p^\# + D_1'z^\#\rangle_{L_2[t_0,t_1]}. \tag{10}$$

A trick facilitating an explicit solution of the inhomogeneous (9) is based on a change of variables $\zeta = p - X_\varkappa x$. Invoking (2), it transforms (9) into a cascade of an anti-causal anti-stable system, in ζ, followed by a stable, causal system, in x:

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A_\varkappa & -B_2B_2' \\ 0 & -A_\varkappa' \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} B_1 \\ -E_\varkappa' \end{bmatrix} d_hw. \tag{11}$$

This formulation makes transparent the boundedness of the mapping from the data $(x(0), \breve{w}_0) \in M_2$ and $w \in L_2$ to the combined state in (11) (equiv., in (9)) and the decay to zero of the combined state, as $t \to \infty$.

Yet another consequence of (11) is that the mapping $w \mapsto (x, \zeta)$ is strictly proper. In particular, with the zero initial data, if $w(t)$ is selected as a sinusoidal over a compact support, increasing its frequency indefinitely will result with $\|z^\#(0, 0, w)\|_{L_2} \approx \|D_1w\|_{L_2}$. This establishes, in the current setting, the fact that $\gamma_0 \geq \|D_1\| = \sigma_{max}\{D_1\}$, which is a basic observation in the standard H_∞ problem[4]. Using the notations of Section II, $\Gamma_\gamma > 0$ for $\gamma > \gamma_0$.

2) The solution over $[0, h]$: The selection - and optimization – of w comes into effect only over the positive ray $[h, \infty)$. Therefore, there is an obvious interest in the evolution along the initial interval $[0, h]$, and its contribution to the total value of (4). The analysis begins by highlighting the distinction between the contribution of initial data and that of the selected w. Indeed, along that interval

$$\zeta(t) = e^{A_\varkappa'(h-t)}\zeta(h) + \int_t^h e^{A_\varkappa'(r-t)}E_\varkappa'w(r - h)dr \tag{12}$$

with the boundary condition $\zeta(h) = \int_0^\infty e^{A_\varkappa' r}E_\varkappa'w(r)dr$. Consequently, by direct computation

$$\begin{aligned} x(t) = {} & e^{A_\varkappa t}x(0) - G_c(t)e^{A_\varkappa'(h-t)}\zeta(h) \\ & + \int_0^t e^{A_\varkappa(t-r)}(B_1 - G_c(r)E_\varkappa')w(r - h)dr \\ & - G_c(t)\int_t^h e^{A_\varkappa'(r-t)}E_\varkappa'w(r - h)dr, \end{aligned} \tag{13}$$

where $G_c(t)$ is the Gramian defined in Ssection II. In particular

$$x(h) = \xi(h) - G_c\zeta(h), \tag{14}$$

where

$$\xi(h) \doteq e^{A_\varkappa h}x(0) + \int_0^h e^{A_\varkappa(h-r)}B_\tau(r)w(r - h)dr \tag{15}$$

captures the effect of initial data, and $\zeta(h)$ captures the contribution of the selection of w.

3) The optimal w: The computation of an optimal w in (4) is performed as part of the proof of necessity, in Theorem 1. That is, here we assume that $\gamma > \gamma_0$. This assumption means that there exists $\epsilon > 0$ and an admissible, complete information control policy, subject to which the following equality holds for all $w \in L_2$ and with the zero initial data

$$\gamma^2\|w\|_{L_2}^2 - \|z\|_{L_2}^2 \geq \epsilon\|w\|_{L_2}^2. \tag{16}$$

[3]Hereafter d_h stands for the h-delay operator: $d_h\phi(t) = \phi(t - h)$.

[4]We feel the need for this sketch of the proof since this bound does not extend from the standard to the preview problem in discrete time systems.

$$\mathcal{J}_\gamma = \langle \xi(h), \zeta(h)\rangle + \langle x(0), X_\varkappa x(0)\rangle + 2\Big\langle x(0), \int_0^h e^{A_\varkappa' r} E_\varkappa' w(r-h)dr \Big\rangle$$
$$+ 2\int_0^h \Big\langle w(r-h), E_\varkappa \int_0^r e^{A_\varkappa(r-s)}(B_1 - G_c(s)E_\varkappa')w(s-h)ds\Big\rangle dr + \int_0^h \big\langle w(r-h), D_1' D_1 w(r-h)\big\rangle dr. \quad (26)$$

This inequality can only be sharpened if the closed loop response z is replaced by the optimal response $z^\#(0,0,w)$. That is

$$\gamma^2\|w\|^2_{L_2} - \|z^\#(0,0,w)\|^2_{L_2} \geq \epsilon \|w\|^2_{L_2}. \quad (17)$$

We use this fact to appeal, once again, to Theorem 2. Here we identify $w \in L_2$ with the optimization variable "$u \in U$"; the space V will be identified with $L_2 \times L_2$, containing pairs $(w, z^\#)$; the operator $Sw = (w, z^\#(0,0,w))$ accounts for the contribution of w to such pairs; we identify "$-v$" with $(0, z^\#(x(0), \breve{w}_0, 0))$, which is the contribution of the initial data; finally, we set $J = diag\{\gamma^2 I, -I\}$. Under these definitions, (17) means that $S'JS \geq \epsilon I$, satisfying the condition of Theorem 2. The conclusion is that there is a unique optimal $w^*(x(0), \breve{w}_0)$, in (4). For later use we introduce the notations $x^*(x(0), \breve{w}_0) = x^\#(x(0), \breve{w}_0, w^*)$, $u^*(x(0), \breve{w}_0) = u^\#(x(0), \breve{w}_0, w^*)$ and $p^*(x(0), \breve{w}_0) = p^\#(x(0), \breve{w}_0, w^*)$, and note that w^*, x^*, u^*, z^* and p^* are all bounded linear operators from M_2 to L_2. By Theorem 2, w^* is completely characterized by the condition $S'J(Sw^* - v) = S'J(w^*, z^*) = 0$. Our next task is to derive an explicit interpretation of this condition.

The computation of S' is cumbersome, and we avoid the need to do so, using a trick from [17], [22]. Let $u = -Kx$ be a stabilizing feedback control, as above. Define a bounded operator $\tilde{S}w = (w, \tilde{z})$, where $\tilde{z}$ is the response to w and the zero initial data in $[A - B_2K, B_1 d_h, C - D_2K, D_1 d_h]$, and, in these notations, let $\delta z = z^\#(0,0,w) - \tilde{z}$. We return for a moment to the discussion of the optimization in u, using the fact that $z^* = z^\#(x(0), \breve{w}_0, w^*)$ is the LQ optimal output, given w^* and the initial data. By Theorem 2, this implies that z^* is orthogonal in L_2 to the response of $[A, B_2, C, D_2]$ to any admissible control and the zero initial state. One such response is δz, hence $\langle \delta z, z^*\rangle_{L_2} = 0$. Now we can interpret $S'J(w^*, z^*) = 0$ as follows: for any $w \in L_2$

$$\langle Sw, J(w^*, z^*)\rangle_{L_2\times L_2} = 0 = \langle \tilde{S}w, J(w^*, z^*)\rangle_{L_2 \times L_2}.$$

That is, the unique optimal w^* is completely characterized by the condition $\tilde{S}'J(w^*, z^*) = 0$. The explicit realization of this condition is in terms of an anti-causal system whose state equation is identical to, and thus coincides with, that in (7):

$$\dot{p} = -(A - B_2K)'p - (C - D_2K)'z^* \quad (18a)$$
$$0 = B_1'p + D_1'z^* - \gamma^2 d_h w^*, \qquad t > h. \quad (18b)$$

In particular, since (18a) coincides with (7a), the equality (7b) remains satisfied, and $(A - B_2K)'$ and $(C - D_2K)'$ can be replaced by A' and C', in the state equation.

One observation, based on (10) and (18b), provides an expression for the component of the value of the game due to the evolution over $[h, \infty)$

$$\|z^*\|^2_{L_2[h,\infty)} - \gamma^2\|w^*\|^2_{L_2} = \langle x(h), p(h)\rangle. \quad (19)$$

In closing §III-A.1 we verified that the matrix Γ_γ is positive definite, hence invertible, when $\gamma > \gamma_0$. Thus, (18b) and (A3) imply that

$$d_h w^* = \Gamma_\gamma^{-1}(D_1'Cx + B_1'p) = \Gamma_\gamma^{-1}(E_\varkappa x + B_1'\zeta), \quad (20)$$

which should hold for all $t > h$. Over the ray $[h, \infty)$, the solution of (4) is therefore characterized by the unique L_2 solution of a homogeneous Hamilton-Jacobi system of a standard form

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \tilde{A}_\gamma & B_1\Gamma_\gamma B_1' - B_2B_2' \\ -C'\Delta_\gamma C & -\tilde{A}_\gamma' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \quad (21)$$

where $\tilde{A}_\gamma \doteq A + B_1\Gamma_\gamma^{-1}D_1'C$ and $\Delta_\gamma \doteq I + D_1\Gamma_\gamma^{-1}D_1'$.

Returning to the familiar, standard case, when $h = 0$ (i.e, with no preview), the initial state $x(0) = x(h)$ in (1) can be arbitrarily assigned. In particular, each $x(0) \in \mathbb{R}^n$ is associated with a unique $p(0) \in \mathbb{R}^n$, so that the ensuing solution of (21) is in L_2. The linear dependence of $p(0)$ on $x(0)$ leads to the existence of a matrix $\tilde{X}_\gamma$, such that $p = \tilde{X}_\gamma x$. It is a simple procedure then to show that $\tilde{X}_\gamma$ is a stabilizing, self adjoint solution of the associated ARE

$$\tilde{X}_\gamma\tilde{A}_\gamma + \tilde{A}_\gamma'\tilde{X}_\gamma + C'\Delta_\gamma C + \tilde{X}_\gamma(B_1\Gamma_\gamma B_1' - B_2B_2')\tilde{X}_\gamma = 0. \quad (22)$$

Moreover, (19) then means that the optimal value of the game is $\langle x(0), \tilde{X}_\gamma x(0)\rangle$. Since $w = 0$ is one viable option in the search for an optimal w, the conclusion is then that the optimal value of (4) is non-negative, meaning that $\tilde{X}_\gamma \geq 0$. Thus, the existence of a positive semidefinite, stabilizing solution of (22) becomes a necessary condition for the sub optimality of γ.

This chain of arguments fails in the preview problem, at two critical points. First, as seen in (14), $x(h)$ is determined by *both* the initial data and the optimization of w. There is therefore no a priori assurance that any $x(h) \in \mathbb{R}^n$ is an initial state in an L_2 solution of (21). Second, even if a stabilizing solution of (22) does exist, then $\langle x(h), \tilde{X}_\gamma x(h)\rangle$ is only one component of the optimal value of the game, and the positivity requirement applies only when the contribution of the evolution along $[0, h]$ is included. The ARE (22) must therefore be replaced by an alternative equation in a characterization of $\gamma > \gamma_0$. In preparation, computing the optimal value of the game and deducing a positivity condition in that context, is our next task.

4) The optimal value of (4)*:* Our starting point remains (10) and the fact that (18a) must be satisfied for $t > h$. Setting $t_0 = 0$ and letting $t_1 \to \infty$ in (10), we have

$$\mathcal{J}_\gamma \doteq \|z^*\|^2_{L_2} - \gamma^2\|w^*\|^2_{L_2}$$
$$= \langle x(0), p(0)\rangle + \langle d_h\breve{w}_0, B_1'p^* + D_1'z^*\rangle_{L_2[0,h]}. \quad (23)$$

An explicit expression for (23) is computed, using the derivations of §III-A.2. To begin with,

$$\langle x(0), p(0)\rangle = \Big\langle x(0), X_\varkappa x(0) + e^{A_\varkappa' h}\zeta(h) + \int_0^h e^{A_\varkappa' r}E_\varkappa' w(r-h)dr\Big\rangle. \quad (24)$$

Next, in analogy to the derivation of (20), one obtains

$$\langle d_h\breve{w}_0, B_1'p + D_1'z\rangle_{L_2[0,h]} = \langle d_h\breve{w}_0, E_\varkappa x + B_1'\zeta + D_1'D_1 d_h\breve{w}_0\rangle_{L_2[0,h]}. \quad (25)$$

Using the explicit expressions for ζ and x, in (12) and (13), and also the definition of $\xi(h)$, in (15), the combined expression for the cost is now computed as shown in eqn. (26) at the top of this page.

One basic fact revealed by this equality is that the contribution of w^* to $\mathcal{J}_\gamma$ is captured by the single term $\langle\xi(h),\zeta(h)\rangle$. In particular, if $(x(0),\breve{w}_0)$ determine $\xi(h)=0$, then the contribution of w^* to the game's value is zero. That is, then the game's optimal value is also achieved with $w\equiv 0$, hence $\zeta(h)=0$, $x(h)=0$, $u^{\#}(t)=0$ for $t>h$, and finally, $x^{\#}(t)=0$ for $t>h$. The uniqueness of the optimal solution of (4) thus implies that, indeed, if $\xi(h)=0$ then $w^*\equiv 0$, $\zeta(h)=0$, and that $u^*(t)=0$ and $x^*(t)=0$, for $t>h$.

Obviously, the collection of all possible quintuples $(\xi(h),\zeta(h),x^*,w^*,u^*)$ form a linear manifold, parameterized by the initial data. The observation above thus demonstrates that $\zeta(h)$, w^*, and the restrictions of x^* and u^* to $[h,\infty)$, depend linearly on $\xi(h)$. In particular, there exists a matrix X_γ such that $\zeta(h)=X_\gamma\xi(h)$.

Should the optimization procedure be re initiated at any time $t>0$, with the initial data $(x^*(t),w^*_t)$, it is clear that the solution over $[t,\infty)$ must coincide with the solution of the original problem. In particular, the relation $\zeta(t+h)=X_\gamma\xi(t+h)$ should prevail for all $t>0$, where $\xi(t+h)$ is defined by $(x^*(t),w^*_t)$, as in (15).

5) An ARE for X_γ: Solutions of LQ optimization problems, including both traditional optimal control, H_∞ and LQ differential game problems, are characterized by Hamilton Jacobi systems. Associated Riccati equations are used to characterize L_2 solutions, as well as optimal feedback gains. The common situation is that where the state and co-state equations of the Hamilton Jacobi system are derived directly from the state equation of the original system, and that of its adjoint, as is the case in (9) and in (21). In the current problem, however, we have already noted the limited value of (21) and the associated ARE (22). Here we look for an alternative, where the roles of the state and co-state are played by the trajectories of ξ and ζ. The rationale is that the state of the Hamilton Jacobi system typically captures the contribution of past inputs / initial data, whereas the co-state captures the contributions of future inputs, which have been shown to be the respective roles of ξ and ζ.

The transformation from the state and co-state of (21) to the desired state and co-state is via

$$\begin{bmatrix}\xi\\ \zeta\end{bmatrix}=\begin{bmatrix}I & G_c\\ 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ -X_\varkappa & I\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix}. \tag{27}$$

It is a somewhat laborious but straightforward algebra that transforms (21) into the following, equivalent system

$$\begin{bmatrix}\dot\xi\\ \dot\zeta\end{bmatrix}=\begin{bmatrix}A_\gamma & -R_\gamma\\ -E'_\varkappa\Gamma_\gamma^{-1}E_\varkappa & -A'_\gamma\end{bmatrix}\begin{bmatrix}\xi\\ \zeta\end{bmatrix}, \tag{28}$$

where, again, the definitions of Section II are used. The fact that the ARE (3) is satisfied by the matrix X_γ in the relation $\zeta=X_\gamma\xi$, readily follows from equating $\dot\zeta=X_\gamma\dot\xi$ in the two equations, in (28). Since the evolution of ξ is generated by $A_X\doteq A_\gamma-R_\gamma X_\gamma$, we conclude that A_X is stable. To see that X_γ is self adjoint we rewrite (3) as a Lyapunov equation with a stable "A" matrix and a self adjoint free term

$$X_\gamma A_X+A'_XX_\gamma+X'_\gamma R_\gamma X_\gamma+E'_\varkappa\Gamma_\gamma^{-1}E_\varkappa=0. \tag{29}$$

To see that X_γ is actually positive semidefinite, we consider (4) with $\breve{w}_0=0$. Then the expression (26) for the optimal value reduces to $\langle x(0),\big(X_\varkappa+e^{A'_\varkappa}X_\gamma e^{A_\varkappa}\big)x(0)\rangle$. Since, as noted earlier, the search for w^* includes the option $w\equiv 0$, we know that the optimal value of the game is bounded below by $\|w^{\#}(x(0),0,0)\|^2_{L_2}=\langle x(0),X_\varkappa x(0)\rangle$. This necessitates $\langle x(0),e^{A'_\varkappa}X_\gamma e^{A_\varkappa}x(0)\rangle\geq 0$ for all $x(0)$, hence $X_\gamma\geq 0$.

The proof of necessity in Theorem 1 is complete. ■

B. Restatement in an abstract model

As in other delay systems, an equivalent realization, without delay, is in terms of a model for the M_2 evolution of the complete state $f(t)=(x(t),\breve{w}_t)\in M_2$. Such models have been documented extensively (see [23] and references therein), and have been used in the solution of H_∞ problems of delay systems (see, e.g., [24], [25] and the review [16]). We shall therefore be content with a brief review of the appropriate model and solution, for the preview problem. Details of technical aspects of treating a distributed system with unbounded input and output coefficients can be found in the references cited right above.

The abstract model is of the form

$$\begin{aligned}\dot f&=\mathcal{A}f+\mathcal{B}_1w+\mathcal{B}_2u,\\ z&=\mathcal{C}f+\mathcal{D}_2u,\end{aligned} \tag{30}$$

where $\mathcal{A}$ is an infinitesimal generator of a c_0-semigroup over M_2, defined by

$$\mathcal{A}(\eta,\phi)=\big(A\eta+B_1\phi(-h),\tfrac{d}{d\theta}\phi(\theta)\big),$$
$$\mathcal{D}(\mathcal{A})=\left\{(\eta,\phi)\in M_2,\phi(\theta)=\int_\theta^0\psi(\sigma)d\sigma,\psi\in L_2[-h,0]\right\}.$$

The other coefficients in (30) are defined via

$$\mathcal{B}_1w=(0,\delta_0(\cdot))w,\qquad \mathcal{B}_2u=(B_2u,0),$$
$$\mathcal{C}(\eta,\phi)=C\eta+D_1\phi(-h),\qquad \mathcal{D}_2u=D_2u,$$

where $\delta_0(\cdot)$ is Dirac's delta function, centered at $\theta=0$.

The advantage in this formulation is that (30) involves no delay, whereby the well known and relatively simple formalism of the Riccati equation based, state space solution of the standard H_∞ problem applies [22], [26]. For completeness we restate here the complete information result.

Theorem 3: The following two statements are equivalent.

1) $\gamma>\gamma_0$ in (30)
2) There exists a bounded, positive semidefinite operator $\mathcal{X}$ over M_2 such that it is a weak solution of the operator Riccati equation

$$\mathcal{X}\mathcal{A}+\mathcal{A}'\mathcal{X}+\mathcal{X}\big(\tfrac{1}{\gamma^2}\mathcal{B}_1\mathcal{B}'_1-\mathcal{B}_2\mathcal{B}'_2\big)\mathcal{X}+\mathcal{C}'\mathcal{C}=0 \tag{31}$$

and the generator $\mathcal{A}_\mathcal{X}=\mathcal{A}+(\frac{1}{\gamma^2}\mathcal{B}_1\mathcal{B}'_1-\mathcal{B}_2\mathcal{B}'_2)\mathcal{X}$ gives rise to a uniformly exponentially stable c_0-semigroup over M_2.

That solution of (31) is completely characterized by the fact that the optimal value of the game (4) is given by the quadratic form $\langle(x(0),\breve{w}_0),\mathcal{X}(x(0),\breve{w}_0)\rangle_{M_2}$. Furthermore, assume that indeed, $\gamma>\gamma_0$ and $\mathcal{X}$ is as above. Then

$$u(t)=-\mathcal{B}'_2\mathcal{X}(x(t),\breve{w}_t) \tag{32}$$

is a stabilizing, strictly γ-suboptimal control policy.

It is worth stressing that, by the structure of $\mathcal{B}_2$, the control law in (32) depends only upon the first ($\mathbb{R}^n$) "row" block of $\mathcal{X}$. This fact will be exploited in the derivation of the control law (4).

C. The proof of sufficiency

In this section it is assumed that $\Gamma_\gamma>0$ and that a stabilizing solution $X_\gamma\geq 0$ exists in the ARE (3). Our goal is to establish the fact that $\gamma>\gamma_0$ and that (4) is a stabilizing, strictly γ-suboptimal control policy. The proof utilizes the interplay between (1) and the abstract model (30).

Specifically, let $\mathcal{X}$ be defined as the self adjoint operator over M_2, that serves as the kernel for the quadratic form (26) (we leave out the obvious details). Thus, the value of (26) is

$\langle (x(0), \breve{w}_0), \mathcal{X}(x(0), \breve{w}_0)\rangle$, for any $(x(0), \breve{w}_0) \in M_2$. We intend to show that $\mathcal{X}$ is the sought positive semidefinite, stabilizing solution of the operator Riccati equation (31), and appeal to Theorem 3. Since similar arguments were made in the articles cited earlier on LQ and H_∞ solutions in systems with control delay, we shall be content with a brief outline.

Indeed, let X_γ be the assumed solution of (3), fix initial data $(x(0), \breve{w}_0) \in M_2$, let $\xi(h)$ be defined by (15), let $\xi(t) = e^{A_X(t-h)}\xi(h)$, $\zeta(t) = X_\gamma \xi(t)$, $x(t) = \xi(t) + G_c\zeta(t)$ and $p(t) = X_\varkappa x(t) + \zeta(t)$, for $t \geq h$. Define $\zeta(t)$ and $x(t)$, $t \in [0, h]$, as in (12) and (13), respectively, and $p(t) = X_\varkappa x(t) + \zeta(t)$, over that interval, as well. Finally, let w and u be defined by the feedback formulae (20) and (8). Backtracking our computations heretofore, it follows that x, w and u are L_2 trajectories, satisfying (1). Moreover, with these selections, the associated value of $\|z\|_{L_2}^2 - \gamma^2\|w\|_{L_2}^2$ is given by (26), meaning that it is equal to $\langle (x(0), \breve{w}_0), \mathcal{X}(x(0), \breve{w}_0)\rangle_{M_2}$, albeit, so far, without any claim to optimality. Deriving the explicit form of $\mathcal{X}$ from (26) (with $\zeta(h) = X_\gamma \xi(h)$), it is a matter of straightforward computation to verify that the combined equalities (20) and (8) are equivalent to $w(t) = \frac{1}{\gamma^2}\mathcal{B}_1'\mathcal{X}(x(t), \breve{w}_t)$ and $u(t) = -\mathcal{B}_2'\mathcal{X}(x(t), \breve{w}_t)$. Therefore, the trajectory $f(t) = (x(t), \breve{w}_t)$ is generated by $\mathcal{A}_\mathcal{X}$. Let $\mathcal{S}_\mathcal{X}(t)$ be the associated c_0 semigroup over M_2. That is, $\mathcal{S}_\mathcal{X}(t)$ is defined by the relation $(x(t), \breve{w}_t) = \mathcal{S}_\mathcal{X}(t)(x(0), \breve{w}_0)$, where x and w are the trajectories defined above. Thus

$$\begin{aligned}\langle f(0), \mathcal{X} f(0)\rangle &= \|z\|_{L_2}^2 - \gamma^2\|w\|_{L_2}^2 \\ &= \int_0^\infty \big\langle \mathcal{S}_\mathcal{X}(t) f(0), \big(\mathcal{C}'\mathcal{C} + \mathcal{X}(\mathcal{B}_2\mathcal{B}_2' - \tfrac{1}{\gamma^2}\mathcal{B}_1\mathcal{B}_1')\mathcal{X}\big) \\ &\qquad \times \mathcal{S}_\mathcal{X}(t) f(0)\big\rangle dt. \quad (33)\end{aligned}$$

This last equation is equivalent to the fact that $\mathcal{X}$ is a weak solution of (31). The fact that $\mathcal{X}$ is a stabilizing solution then follows directly from the fact that X_γ is a stabilizing solution of (3) and the subsequent exponential decay of all the defined trajectories, relative to $\|(x(0), \breve{w}_0)\|_{M_2}$. The fact that $\mathcal{X}$ is self adjoint follows from its definition, in (26), and the fact that $X_\gamma = X_\gamma' \geq 0$.

We still need to establish that $\mathcal{X} \geq 0$. Indeed, consider now (30) with $w(t) = 0$, $t > 0$, under the feedback policy (32). In analogy to the cited distributed parameters references, an integration by parts argument, extended to (30), yields

$$\langle f(0), \mathcal{X} f(0)\rangle = \langle f(t), \mathcal{X} f(t)\rangle + \|z\|_{L_2[0,t)}^2 + \gamma^2\|w^\triangle\|_{L_2[0,t)}^2 \quad (34)$$

for any $t \in (0, \infty)$, where $w^\triangle = \frac{1}{\gamma^2}\mathcal{B}_1'\mathcal{X} f$. Since $w \equiv 0$, for $t > h$ we have $\breve{w}_t = 0$ and $\langle f(t), \mathcal{X} f(t)\rangle_{M_2} = \big\langle x(t), \big(X_\varkappa + e^{A_\varkappa' h} X_\gamma e^{A_\varkappa h}\big)x(t)\big\rangle_{\mathbb{R}^n} \geq 0$. In particular, then the right hand side of (34) comprises three non negative terms. Thus, the left hand side is non negative, for any selection of $f(0) \in M_2$, and $\mathcal{X}$ must be positive semidefinite, as claimed.

Having established these properties of $\mathcal{X}$, the conclusion in Theorem 3 is that, indeed, $\gamma > \gamma_0$ and (32) is a stabilizing, strictly γ-suboptimal feedback. Again, it is straightforward to verify that this feedback is equivalent to (4), in (1).

The proof of Theorem 1 is complete. ■

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