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ScienceDirect
Journal of Combinatorial Theory, Series A 114 (2007) 65–76
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

h-Vectors of Gorenstein polytopes

Winfried Bruns, Tim Römer

FB Mathematik/Informatik, Universität Osnabrück, 49069 Osnabrück, Germany

Received 24 September 2005

Available online 17 April 2006

Abstract

We show that the Ehrhart h-vector of an integer Gorenstein polytope with a regular unimodular triangulation satisfies McMullen's g-theorem; in particular, it is unimodal. This result generalizes a recent theorem of Athanasiadis (conjectured by Stanley) for compressed polytopes. It is derived from a more general theorem on Gorenstein affine normal monoids M: one can factor $K[M]$ (K a field) by a "long" regular sequence in such a way that the quotient is still a normal affine monoid algebra. This technique reduces all questions about the Ehrhart h-vector of P to the Ehrhart h-vector of a Gorenstein polytope Q with exactly one interior lattice point, provided each lattice point in a multiple cP, $c \in \mathbb{N}$, can be written as the sum of c lattice points in P. (Up to a translation, the polytope Q belongs to the class of reflexive polytopes considered in connection with mirror symmetry.) If P has a regular unimodular triangulation, then it follows readily that the Ehrhart h-vector of P coincides with the combinatorial h-vector of the boundary complex of a simplicial polytope, and the g-theorem applies.
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Keywords: Lattice polytope; h-Vector; Ehrhart function; Gorenstein ring; Affine monoid; Unimodality; Triangulation; Initial ideal

1. Introduction

Let $P \subseteq \mathbb{R}^{n-1}$ be an integral convex polytope and consider the *Ehrhart function* given by $E(P,m) = |\{z \in \mathbb{Z}^{n-1}\colon \frac{z}{m} \in P\}|$ for $m > 0$ and $E(P,0) = 1$. It is well known that $E(P,m)$ is a polynomial in m of degree $\dim(P)$ and the corresponding *Ehrhart series* $E_P(t) = \sum_{m\in\mathbb{N}} E(P,m)t^m$ is a rational function

$$E_P(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{\dim(P)+1}}.$$

E-mail addresses: wbruns@uos.de (W. Bruns), troemer@uos.de (T. Römer).

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doi:10.1016/j.jcta.2006.03.003

We call $h(P) = (h_0, \dots, h_d)$ (where $h_d \neq 0$) the *(Ehrhart) h-vector* of P. This vector was intensively studied in the last decades (e.g. see [5] or [12]). In particular, the following questions are of interest:

(i) For which polytopes is $h(P)$ *symmetric*, i.e. $h_i = h_{d-i}$ for all i?
(ii) For which polytopes is $h(P)$ *unimodal*, i.e. there exists a natural number t such that $h_0 \leqslant h_1 \leqslant \cdots \leqslant h_t \geqslant h_{t+1} \geqslant \cdots \geqslant h_d$?

Let us sketch Stanley's approach to Ehrhart functions via commutative algebra. The results we are referring to can be found in [5] or [12]. The Ehrhart function of P can be interpreted as the Hilbert function of an affine monoid algebra $K[E(P)]$ (with coefficients from an arbitrary field K) and where the monoid $E(P)$ is defined as follows: one considers the cone $C(P)$ generated by $P \times \{1\}$ in $\mathbb{R}^n$, and sets $E(P) = C(P) \cap \mathbb{Z}^n$. The monomial in $K[E(P)]$ corresponding to the lattice point x is denoted by X^x where X represents a family of n indeterminates. The algebra $K[E(P)]$ is graded in such a way that the degree of X^x (or of x) is the last coordinate of x, and so the Hilbert function of $K[E(P)]$ coincides with the Ehrhart function of P. Since P is integral, $K[E(P)]$ is a finite module over its subalgebra generated by the degree 1 elements.

However, in general $K[E(P)]$ is not generated by its degree 1 elements. If it is, then we say that P is *integrally closed*, and simplify our notation by setting $K[P] = K[E(P)]$. Evidently P is integrally closed if and only if $E(P)$ is generated by the integer points in $P \times \{1\}$, or, equivalently, every integer point in cP, $c \in \mathbb{N}$, can be written as the sum of c integer points in P. Our choice of terminology coincides with that in [3]. A *unimodular triangulation* of P is a triangulation into simplices $\operatorname{conv}(s_0, \dots, s_r)$ such that $s_0, \dots, s_r \in \mathbb{Z}^{n-1}$ and $s_1 - s_0, \dots, s_r - s_0$ generate a direct summand of $\mathbb{Z}^{n-1}$. If $\dim P \geqslant 3$, P need not have a unimodular triangulation. However, if a unimodular triangulation of P exists, then P is integrally closed; this follows easily from the fact that a unimodular simplex is integrally closed.

The monoid $E(P)$ is always normal: an element x of the subgroup of $\mathbb{Z}^n$ generated by $E(P)$ such that $kx \in E(P)$ for some $k \in \mathbb{N}$, $k \geqslant 1$, belongs itself to $E(P)$. By a theorem of Hochster, $K[E(P)]$ is a Cohen–Macaulay algebra. It follows that $h_i \geqslant 0$ for all $i = 1, \dots, d$. Using Stanley's Hilbert series characterization of the Gorenstein rings among the Cohen–Macaulay domains, one sees that $h(P)$ is symmetric if and only if $K[E(P)]$ is a Gorenstein ring. In terms of the monoid $E(P)$, the Gorenstein property has a simple interpretation: it holds if and only if $E(P) \cap \operatorname{int} C(P)$ is of the form $y + E(P)$ for some $y \in E(P)$. This follows from the description of the canonical modules of normal affine monoid algebras by Danilov and Stanley.

It was conjectured by Stanley that question (ii) has a positive answer for the *Birkhoff polytope* P, whose points are the real doubly stochastic $n \times n$ matrices and for which $E(P)$ encodes the *magic squares*. This long standing conjecture was recently proved by Athanasiadis [1]. (That P is integrally closed and $K[P]$ is Gorenstein in this case is easy to see.)

Questions (i) and (ii) can be asked similarly for the combinatorial h-vector $h(\Delta(Q))$ of the boundary complex $\Delta(Q)$ of a simplicial polytope Q (derived from the f-vector of $\Delta(Q)$), and both have a positive answer. The Dehn–Sommerville equations express the symmetry, while unimodality follows from McMullen's famous g-theorem (proved by Stanley [11]): the vector $(1, h_1 - h_0, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-sequence, i.e. it represents the Hilbert function of a graded artinian K-algebra that is generated by its degree 1 elements. In particular, its entries are nonnegative, and so the h-vector is unimodal.

Athanasiadis proved Stanley's conjecture for the Birkhoff polytope P by showing that there exists a simplicial polytope P' with $h(\Delta(P')) = h(P)$. More generally, his theorem applies to compressed polytopes, i.e. integer polytopes all of whose pulling triangulations are unimodular. (The Birkhoff polytope is compressed [10,12].) In this note we generalize Athanasiadis' theorem as follows:

Theorem 1. *Let P be an integral polytope such that P has a regular unimodular triangulation and $K[P]$ is Gorenstein. Then the h-vector of P satisfies the inequalities $1 = h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{\lfloor d/2 \rfloor}$. More precisely, the vector $(1, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-sequence.*

The regular triangulations are obtained as subdivisions of P into the domains of linearity of a piecewise affine, concave and continuous function on P, provided this subdivision is really a triangulation. They can also be defined by weight vectors $(w_x\colon x \in P \cap \mathbb{Z}^{n-1})$ in the following way (with the same proviso): one takes the convex hull Q of the halflines $\{(x, z)\colon x \in P \cap \mathbb{Z}^{n-1}, z \geqslant w_x\} \subseteq \mathbb{R}^n$ and projects the "bottom" of Q onto P. (See [3] or [14] for a discussion of regular subdivisions and triangulations.)

Our strategy of proof (whose last step is Lemma 9 in Section 3) is to consider the algebra $K[M]$ of a normal affine monoid M for which $K[M]$ is Gorenstein. (An affine monoid is a finitely generated submonoid of a group $\mathbb{Z}^n$.) We relate the Hilbert series of $K[M]$ to that of a simpler affine monoid algebra $K[N]$ which we get by factoring out a suitable regular sequence of $K[M]$. In the situation of an algebra $K[P]$ for an integrally closed polytope P, the regular sequence is of degree 1, and we obtain an integrally closed and, up to a translation, reflexive polytope such that $h(P) = h(Q)$. (However, note that Mustaţă and Payne [9] have given an example of a reflexive polytope which is not integrally closed and has a nonunimodal h-vector.) If P has even a regular unimodular triangulation, we can find a simplicial polytope P' such that the combinatorial h-vector $h(\Delta(P'))$ of the boundary complex of P' coincides with $h(P)$. Then it only remains to apply the g-theorem to P'.

Without the condition on regularity of the triangulation we can only find a simplicial sphere S such that $h(P) = h(S)$. If the g-theorem can be generalized from polytopes to simplicial spheres, then our theorem holds for all polytopes with a unimodular triangulation.

As a side effect we show that the toric ideal of a Gorenstein polytope with a square-free initial ideal has also a Gorenstein square-free initial ideal.

For notions and results related to commutative algebra we refer to Bruns–Herzog [5] and Stanley [12]. For details on convex geometry we refer to the books of Bruns and Gubeladze [3] (in preparation) and Ziegler [15].

2. Gorenstein monoid algebras

We fix a field K for the rest of the paper. Let C be a pointed rational cone in $\mathbb{R}^n$, i.e. a cone generated by finitely many integral vectors that does not contain a full line. Such a cone is the irredundant intersection $\operatorname{cn}(M) = \bigcap_{i=1}^{s} H_{\sigma_i}^{+}$ of rational half-spaces. Here σ_i is a linear form with rational coefficients and $H_{\sigma_i}^{+} = \{x\colon \sigma_i(x) \geqslant 0\}$ is the positive closed halfspace associated with σ_i. The hyperplane on which σ_i vanishes is denoted by H_{σ_i}. Note that for $H_{\sigma_i}^{+}$ the form σ_i is unique up to a nonnegative factor. There is a unique multiple with coprime integral coefficients, and we call this choice of σ_i, $i = 1, \ldots, s$, the *support forms* of C. The *extreme integral generators* of C are the shortest integer vectors in the edges of C.

Let $M \subseteq \mathbb{Z}^n$ be a positive affine monoid, i.e. an affine monoid whose only invertible element is 0. Then the cone $\operatorname{cn}(M)$ generated by M is pointed and the map

$$\sigma : M \to \mathbb{Z}^s, \qquad a \mapsto \bigl(\sigma_1(a), \ldots, \sigma_s(a)\bigr),$$

is injective. It is called the *standard embedding* of M. It can be extended to the subgroup $\mathrm{gp}(M)$ of $\mathbb{Z}^n$ generated by M, and we denote the extension also by σ.

Lemma 2. *Let $M \subseteq \mathbb{Z}^n$ be a positive normal affine monoid with $\mathrm{gp}(M) = \mathbb{Z}^n$ and $R = K[M]$. Let $\sigma_1, \dots, \sigma_s$ be the support forms and $\sigma : \mathbb{Z}^n \to \mathbb{Z}^s$ the standard embedding of M. Moreover, let $\mathrm{int}(M) = M \cap \mathrm{int}(\mathrm{cn}(M))$. Then*:

(i) *The $\mathbb{Z}^n$-graded canonical module ω_R is the ideal of R generated by all X^z for $z \in \mathrm{int}(M)$.*
(ii) *R is Gorenstein if and only if there exists a (necessarily unique) $y \in \mathrm{int}(M)$ such that $\mathrm{int}(M) = y + M$, and therefore $\omega_R = (X^y)$.*
(iii) *R is Gorenstein if and only if there exists a (necessarily unique) $y \in \mathrm{int}(M)$ such that $\sigma(y) = (1, \dots, 1)$.*

Proof. (i) and (ii) are well-known results of Stanley and Danilov. A proof can be found in [5].

(iii) Assume that R is Gorenstein. By (ii) there exists $y \in \mathrm{int}(M)$ such that $\omega_R = (X^y)$. We have that $\sigma_i(y) > 0$ for $i = 1, \dots, s$ since $y \in \mathrm{int}(M)$. Fix i and choose $z \in \mathrm{int}(M)$ with $\sigma_i(z) = 1$. Such an element z can be found for the following reason. There exists an element $z' \in M$ such that $\sigma_i(z') = 0$ and $\sigma_j(z') > 0$ for $j \neq i$. Furthermore there exists $z'' \in \mathbb{Z}^n$ such that $\sigma_i(z'') = 1$ by the choice of σ_i. For $r \gg 0$ the element $z = rz' + z'' \in \mathrm{int}(M)$ will do the job.

Now $z - y \in M$ and thus $\sigma_i(z - y) \geqslant 0$. Hence $\sigma_i(y) \leqslant 1$ and therefore $\sigma_i(y) = 1$. This shows that $\sigma(y) = (1, \dots, 1)$.

Conversely, if there exists $y \in \mathrm{int}(M)$ such that $\sigma(y) = (1, \dots, 1)$ then it is easy to see that $\mathrm{int}(M) = y + M$.

In each case the uniqueness of y follows from the positivity of M. $\square$

Let $M \subseteq \mathbb{Z}^n$ be a positive affine monoid. It is well known that M has only finitely many irreducible elements which form the unique minimal system of generators of M. We call the collection of these elements the *Hilbert basis* of M, denoted $\mathrm{Hilb}(M)$. The following is our main result for monoid algebras.

Theorem 3. *Let $M \subseteq \mathbb{Z}^n$ be a positive normal affine monoid and assume that $R = K[M]$ is Gorenstein. Let $y_1, \dots, y_m \in \mathrm{Hilb}(M)$ such that $\omega_R = (X^{y_1 + \cdots + y_m})$ is the $\mathbb{Z}^n$-graded canonical module of R. Then*:

(i) *$X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m}$ is a regular sequence for R.*
(ii) *$S = R/(X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m})$ is isomorphic to a Gorenstein normal affine monoid algebra $K[N]$.*
(iii) *The canonical module ω_S is generated by the residue class of X^{y_1}.*

The reader should note that in general the elements $y_1, \dots, y_m$ are not uniquely determined; in fact, even their number m may not be unique. It is so, however, in the situation of Theorem 1.

We isolate the geometric parts of the proof in two lemmas for which the following notation is useful: for a point x in a rational cone C with support forms $\sigma_1, \dots, \sigma_s$ we denote by $\sigma^>(x)$ the set of indices i such that $\sigma_i(x) > 0$. Let F_i denote the facet of C on which σ_i vanishes. Then $i \in \sigma^>(x)$ if and only if $x \notin F_i$.

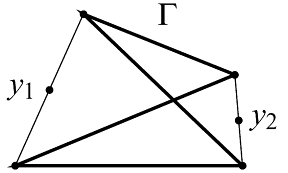

Fig. 1. The construction of Γ.

Recall that a rational cone $D \subseteq \mathbb{R}^n$ is called unimodular if it is simplicial (i.e. spanned by a linearly independent set of vectors) and its extreme integral generators form a subset of a basis of $\mathbb{Z}^n$. A unimodular triangulation of a rational cone is a triangulation into unimodular subcones.

Lemma 4. *Let $C \subseteq \mathbb{R}^n$ be a pointed rational cone with support forms $\sigma_1, \dots, \sigma_s$ and let $y_1, \dots, y_m \in C \cap \mathbb{Z}^n$ such that the sets $\sigma^>(y_i)$, $i = 1, \dots, m$, form a decomposition of $\{1, \dots, s\}$ into pairwise disjoint subsets. Furthermore let Γ be the subfan of the face lattice of C consisting of the faces*

$$F_{j_1,\dots,j_m} = \bigcap_{i=1}^{m} F_{j_i}, \quad j_i \in \sigma^>(y_i),\ i = 1, \dots, m,$$

of C and all their subfaces. Finally, let Σ be a triangulation of Γ into rational subcones.

(i) *Then*

$$\Delta = \Sigma \cup \bigcup_{j=1}^{m} \{\operatorname{cn}(G, y_{i_1}, \dots, y_{i_j})\colon\ G \in \Sigma,\ 1 \leqslant i_1 < \cdots < i_j \leqslant m\}$$

is a triangulation of C.

(ii) *If $\sigma(y_1 + \cdots + y_m) = (1, \dots, 1)$ and Σ is unimodular, then Δ is unimodular.*

We illustrate the construction of Γ by Fig. 1 for the case in which the polytope P is the join P of two line segments of length 2 (suitably embedded), y_1 and y_2 are the two midpoints (the only possible choice in this case), and $C = \operatorname{cn}(E(P))$. The bold edges then constitute a "cross-section" of Γ.

Proof of Lemma 4. We may assume that C has dimension n. Otherwise we replace $\mathbb{R}^n$ by $\mathbb{R}C$ and $\mathbb{Z}^n$ by $\mathbb{R}C \cap \mathbb{Z}^n$.

Let us first show that the cones of Δ are simplicial. It is enough to consider the maximal elements of Δ. These have the form $\operatorname{cn}(G, y_1, \dots, y_m)$ for a maximal element G of Σ. Let $v_1, \dots, v_r$ be the extreme generators of G. Since G is simplicial, $v_1, \dots, v_r$ are linearly independent. Assume that

$$0 = \sum_{i=1}^{m} \lambda_i y_i + \sum_{l=1}^{r} \mu_l v_l \quad \text{for } \lambda_k, \mu_l \in \mathbb{R}.$$

Applying σ we get $0 = \sum_{k=1}^{m} \lambda_k \sigma(y_k) + \sum_{l=1}^{r} \mu_l \sigma(v_l)$. By the definition of Γ we have $G \subseteq F_{j_1,\dots,j_m}$ for suitable $j_1, \dots, j_m$. The hypothesis on $y_1, \dots, y_m$ implies that $\sigma_{j_i}(y_k) = 0$ for $k \neq i$, and $\sigma_{j_i}(v_k) = 0$ for $k = 1, \dots, r$. It follows that $\lambda_i = 0$ for $i = 1, \dots, m$, and the linear independence of $v_1, \dots, v_r$ implies $\mu_l = 0$ for $l = 1, \dots, r$ as well. Being generated by a linearly independent subset of $\mathbb{R}^n$, $\operatorname{cn}(G, y_1, \dots, y_m)$ is simplicial.

Next we show that Δ constitutes a cover of C. Let $x \in C$ and set

$$\lambda_i = \min\left\{\frac{\sigma_j(x)}{\sigma_j(y_i)}\colon\ j \in \sigma^>(y_i)\right\} \quad \text{for } i = 1, \dots, m.$$

Consider

$$x' = x - \sum_{i=1}^{m} \lambda_i y_i. \tag{1}$$

First, $\sigma_i(x') \geqslant 0$ for $i = 1, \dots, s$, thus $x' \in C$. (Here we need the hypothesis on the sets $\sigma^>(y_i)$ in its full extent!) Second, note that there exists at least one index j_i for each $i = 1, \dots, m$ such that $x \in F_{j_i}$, but $y_i \notin F_{j_i}$. It follows that x' lies in the face $F_{j_1,\dots,j_m}$ of C, and so it belongs to one of the simplicial cones G of Σ. Clearly $x \in \operatorname{cn}(G, y_1, \dots, y_m)$.

By definition of Δ, all faces of a cone in Δ belong to Δ, too, and it remains to show that the intersection of two members of Δ is in Δ. Let Y_1 and Y_2 be subsets of $\{y_1, \dots, y_m\}$ and G_1, G_2 elements of Σ. It is clearly sufficient that

$$\operatorname{cn}(G_1, Y_1) \cap \operatorname{cn}(G_2, Y_2) = \operatorname{cn}(G_1 \cap G_2, Y_1 \cap Y_2).$$

Suppose that x lies in the intersection of the cones. The crucial point is that both Y_1 and Y_2 contain the set $Y' = \{y_i\colon\ \lambda_i > 0 \text{ in (1)}\}$: we have

$$Y' = \left\{y_i\colon\ \sigma^>(y_i) \subseteq \sigma^>(x)\right\},$$

and so $Y' \subseteq Y_1, Y_2$. We conclude that $x' \in \operatorname{cn}(G_1, Y_1) \cap \operatorname{cn}(G_2, Y_2)$ as well. But since x' belongs to one of the faces in Γ, one has $x' \in G_1 \cap G_2$ and $x \in \operatorname{cn}(G_1 \cap G_2, Y_1 \cap Y_2)$. The converse inclusion is obvious.

It remains to show that the cones in Δ are unimodular under the hypothesis of (ii). The unimodularity of Σ asserts that the extreme integral generators of G are part of a basis of $\mathbb{Z}^n$ for every cone C of Σ. By definition, G is contained in one of the submodules $L = \bigcap_{i=1}^m H_{\sigma_{j_i}} \cap \mathbb{Z}^n$ of $\mathbb{Z}^n$, which clearly is a direct summand of $\mathbb{Z}^n$ of rank $\geqslant n - m$. It is therefore enough that the residue classes of $y_1, \dots, y_m$ form a basis of $\mathbb{Z}^n/L$ (and, hence, $\operatorname{rank} L = n - m$). But this is not hard to see: first, the linear forms σ_{j_i} vanish on L, and so induce linear forms on $\mathbb{Z}^n/L$, and, second, the matrix $(\sigma_{j_i}(y_k))$ is the unit matrix. □

Lemma 5. *With the notation and hypothesis of Lemma* 4(ii) *let, in addition,* $V = \mathbb{R}^n/(y_1 - y_2, \dots, y_{m-1} - y_m)$, *and* $\pi : \mathbb{R}^n \to V$ *denote the natural projection. Then the cones* $\operatorname{cn}(\pi(G))$ *and* $\operatorname{cn}(\pi(G), \pi(y_1))$, $G \in \Sigma$, *form a triangulation* Δ' *of the cone* $\pi(C) \subseteq V$. *Moreover,* Δ' *is unimodular* (*with respect to the lattice* $U = \pi(\mathbb{Z}^n)$).

Proof. Evidently, the collection Δ' of the images (i) $\pi(\operatorname{cn}(G))$, $G \in \Sigma$, and (ii) $\pi(\operatorname{cn}(G, y_1))$, $G \in \Sigma$, covers $\pi(C)$ since all the images $\pi(y_i)$ coincide with $\pi(y_1)$. Therefore each of the cones in the triangulation Δ of C is mapped onto one of the cones in (i) or (ii).

We have seen in the proof of Lemma 4 that $y_1, \dots, y_m$ form part of a basis of $\mathbb{Z}^n$. The same holds for $y_1 - y_2, \dots, y_{m-1} - y_m$. This implies $\mathbb{Z}^n \cap \operatorname{Ker} \pi = (y_1 - y_2, \dots, y_{m-1} - y_m)$ so that $\pi(\mathbb{Z}^n)$ and $\mathbb{Z}^n/(y_1 - y_2, \dots, y_{m-1} - y_m)$ are naturally isomorphic.

Moreover, each cone in Δ' is generated by part of a basis of U, namely the residue classes of the extreme integral generators $v_1, \dots, v_r$ of G in case (i) and, in addition, $\pi(y_1)$ in case (ii). In fact, in the proof of Lemma 4 we have shown that $v_1, \dots, v_r, y_1, \dots, y_m$ are part of a basis of $\mathbb{Z}^n$. The same holds for $v_1, \dots, v_r, y_1 - y_2, \dots, y_{m-1} - y_m, y_m$, and we project onto a submodule generated by the subsystem $y_1 - y_2, \dots, y_{m-1} - y_m$.

The lemma follows once we have proved that π maps the union C' of the cones $\operatorname{cn}(G)$ and $\operatorname{cn}(G, y_1)$ bijectively onto $\pi(C)$. The surjectivity has already been shown above.

Choose x and x' in C' with $\pi(x) = \pi(x')$. Then $x - x'$ is a linear combination of the differences $y_i - y_{i+1}$. Therefore all the linear forms $\sigma_{j_1} + \dots + \sigma_{j_m}$ with $j_i \in \sigma^>(y_i)$, $i = 1, \dots, m$ vanish on $x - x'$. There exist $k_2, \dots, k_m$ and $l_2, \dots, l_m$ with $k_i, l_i \in \sigma^>(y_i)$ and $\sigma_{k_i}(x) = \sigma_{l_i}(x') = 0$, $i = 2, \dots, m$. Applying both $\sigma_{j_1} + \sigma_{k_2} + \dots + \sigma_{k_m}$ and $\sigma_{j_1} + \sigma_{l_2} + \dots + \sigma_{l_m}$ to $x - x'$, one concludes that $\sigma_{j_1}(x - x')$ must be nonnegative as well as nonpositive for $j_1 \in \sigma^>(y_1)$. So $\sigma_{j_1}(x - x') = 0$. Now the indices in $\sigma^>(y_1)$ are out of the way, and continuing in the same manner for $y_2, \dots, y_m$ one concludes that $\sigma_k(x - x') = 0$ for all k. But then $x - x' = 0$, because C is pointed. □

Proof of Theorem 3. We may assume that $\operatorname{gp}(M) = \mathbb{Z}^n$. Since M is normal in $\operatorname{gp}(M) = \mathbb{Z}^n$, there exists a unimodular triangulation of $\operatorname{cn}(M)$ such that each cone in this triangulation is a unimodular simplicial cone generated by elements of M. (See [3, Section 2.D] for a proof of this well-known result.) Restricting this triangulation to Γ we obtain a unimodular triangulation Σ of Γ.

In view of Lemma 2, the cone $C = \operatorname{cn}(M)$ and the elements $y_1, \dots, y_m$ satisfy the hypothesis of Lemma 4(ii). Therefore the triangulation Δ of Lemma 4(i) is unimodular, and Lemma 5 provides an induced unimodular triangulation of $\pi(C)$ where $\pi : \mathbb{R}^n \to V = \mathbb{R}^n/(y_1 - y_2, \dots, y_{m-1} - y_m)$ is the natural projection and the lattice of reference is $U = \pi(\mathbb{Z}^n)$.

Let $N = \pi(M)$. Since N generates $\pi(C)$ and C has a unimodular triangulation by elements of N, it follows easily that $N = U \cap \pi(C)$. Therefore N is normal.

As an auxiliary tool we introduce a positive grading on R. Let $k_i = |\sigma^>(y_i)|$, and set

$$\deg(X^a) = \left(k_2 \cdots k_m \sum_{i \in \sigma^>(y_1)} \sigma_i(a)\right) + \dots + \left(k_1 \cdots k_{m-1} \sum_{i \in \sigma^>(y_m)} \sigma_i(a)\right)$$

for $a \in \mathbb{Z}^n$. The restriction of deg to $\operatorname{gp}(M)$ is nonnegative, and for $a \in \operatorname{gp}(M)$ one has $\deg(a) = 0$ if and only if $a = 0$. It is obvious that deg extends to a positive grading of the ring R. Moreover, all preimages of an element of N have the same degree, so that we have an induced grading on $K[N]$. Finally, the elements $X^{y_i} - X^{y_{i+1}}$ are all homogeneous of degree $k_1 \cdots k_m$. Therefore, the residue class ring

$$S = R/(X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m})$$

is graded by deg, too.

We want to show that $X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m}$ is a regular R-sequence. First we prove that $X^{y_1}, \dots, X^{y_m}$ is such a sequence. This is most easily seen via the standard embedding $\sigma : R \to K[\mathbb{Z}_+^s]$ induced by the standard embedding $\sigma : M \to \mathbb{Z}^s$. Since $K[\mathbb{Z}_+^s]$ is a polynomial ring in s variables and since the sets $\sigma^>(y_i)$ are disjoint, the monomials $\sigma(X^{y_i})$ "live" in pairwise disjoint sets of variables. So they form a $K[\mathbb{Z}_+^s]$-sequence. Observe that $K[M]$ is a direct summand of $K[\mathbb{Z}_+^s]$ (as a $K[M]$-module) via σ (see [3, Section 4.D]). Thus $X^{y_1}, \dots, X^{y_m}$ is indeed a regular sequence on $K[M]$.

Since

$$(X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m}, X^{y_m}) = (X^{y_1}, \dots, X^{y_m})$$

it is not hard to show that $X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m}$ also form a regular sequence. In fact, all the ideals $I_{k-1} = (X^{y_1} - X^{y_2}, \dots, X^{y_{k-1}} - X^{y_k})$ have height $k-1$. Since R is Cohen–Macaulay, these ideals are unmixed [5, 2.1.6], and the next element cannot be a zero-divisor modulo I_{k-1}.

Evidently the ideal $(X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m})$ is contained in the kernel of the natural epimorphism $R \to K[N]$, and in order to show that these two ideals coincide, it is enough that the rings S and $K[N]$ have the same Hilbert series with respect to deg.

Recall that the maximal cones in the triangulation Δ of $\mathrm{cn}(M)$ and, therefore, all their intersections contain $y_1, \dots, y_m$. If $C_1, \dots, C_t$ are these maximal cones, then we see via inclusion–exclusion that

$$H_R(t) = \sum_{1 \leqslant i \leqslant t} \sum_{a \in \mathrm{gp}(M) \cap C_i} t^{\deg(X^a)} - \sum_{1 \leqslant i < j \leqslant t} \sum_{a \in \mathrm{gp}(M) \cap C_i \cap C_j} t^{\deg(X^a)} \pm \cdots.$$

Let $D_1, \dots, D_t$ be the images of $C_1, \dots, C_t$ with respect to π. Then

$$H_{K[N]}(t) = \sum_{1 \leqslant i \leqslant t} \sum_{a \in \mathrm{gp}(N) \cap D_i} t^{\deg(X^a)} - \sum_{1 \leqslant i < j \leqslant t} \sum_{a \in \mathrm{gp}(M) \cap D_i \cap D_j} t^{\deg(X^a)} \pm \cdots.$$

A comparison of the elements in the unimodular simplicial cones C_i and D_i yields

$$H_{K[N]}(t) = \left(1 - t^{k_1 \cdots k_m}\right)^{m-1} H_R(t).$$

But the right-hand side of the latter equation is exactly the Hilbert series of S, because $X^{y_1} - X^{y_2}, \dots, X^{y_{m-1}} - X^{y_m}$ is a regular sequence of R, and each element has degree $k_1 \cdots k_m$. Hence $H_S(t) = H_{K[N]}(t)$ and therefore $S \cong K[N]$. Since S is clearly a Gorenstein ring, its isomorphic copy $K[N]$ is Gorenstein, too.

It remains to compute the multi-graded canonical module of $S \cong K[N]$. Since $K[N]$ is Gorenstein, we have to determine the unique lattice point q in $\mathrm{int}(N)$ such that $\mathrm{int}(N) = q + N$ because then $\omega_{K[N]} = (X^q)$. By construction, q must have degree $k_1 \cdots k_m$, and the residue class of y_1 in U is an interior point of $\mathrm{cn}(N)$ of that degree. This concludes the proof. $\square$

Remark 6. The simplicial cone $\mathrm{cn}(y_1, \dots, y_m)$ is the core of the triangulation Δ in the sense of [13] where related constructions have been discussed. We are grateful to V. Batyrev for bringing this paper to our attention.

3. Gorenstein polytopes

Let $P \subseteq \mathbb{R}^{n-1}$. We set $E(P, m) = |\{z \in \mathbb{Z}^{n-1}\colon \frac{z}{m} \in P\}|$ and $E(P, 0) = 1$. In analogy to the rational function $E_P(t)$ we define

$$E_{\mathrm{int}(P)}(t) = \sum_{m \in \mathbb{N}} E\big(\mathrm{int}(P), m\big) t^m \quad \text{and} \quad E_{\partial(P)}(t) = \sum_{m \in \mathbb{N}} E\big(\partial(P), m\big) t^m.$$

Observe that $E_{\partial(P)}(t) = E_P(t) - E_{\mathrm{int}(P)}(t)$. In our situation we have that $E_P(t) = H_R(t)$ where $R = K[E(P)]$ and $E_{\mathrm{int}(P)}(t) = H_{\omega_R}(t)$. Thus these series are rational with denominator $(1-t)^{\dim(P)+1}$. Moreover, $E_{\partial(P)}(t) = E_P(t) - E_{\mathrm{int}(P)}(t) = H_{R/\omega_R}(t)$ is rational with denominator $(1-t)^{\dim(P)}$. (This follows from the fact that R/ω_R has Krull dimension equal to $\dim P$.) So it makes sense to consider the h-vectors of these series which we denote by $h(\mathrm{int}(P))$ and $h(\partial(P))$. In the following we present variations and corollaries of Theorem 3.

Corollary 7. *Let P be an integrally closed polytope such that $K[P]$ is Gorenstein. Then there exists a Gorenstein integrally closed polytope Q such that* $\operatorname{int}(Q)$ *contains a unique lattice point and*

$$h(P) = h(Q) = h\big(\partial(Q)\big).$$

Proof. Recall that $R = K[P]$ is the affine monoid ring generated by the positive normal affine monoid $M = E(P) = C \cap \mathbb{Z}^n$ where $C = \operatorname{cn}((p, 1)\colon\ p \in P)$. Observe that R is $\mathbb{Z}$-graded with respect to the exponent of the last indeterminate of a monomial, and we will use only this grading for the rest of the proof. All irreducible elements of M have degree 1, because P is integrally closed. Since R is Gorenstein, there exists a unique lattice point $y \in M$ such that $\operatorname{int}(M) = y + M$. Choosing irreducible elements $y_1, \ldots, y_m \in M$ such that $y = \sum_{i=1}^m y_i$ we are in the situation to apply Theorem 3.

In the proof of the theorem we have constructed the lattice $U = \operatorname{gp}(M)/(y_i - y_{i+1}\colon\ i = 1, \ldots, m-1)$ and the normal affine lattice monoid $N \subseteq U$ such that $K[N]$ is Gorenstein. The monoid N is also homogeneous with respect to the grading induced by that of M and generated by its degree 1 elements. Thus it is polytopal by [4, Proposition 1.1.3], and $K[N] = K[Q]$ for the polytope Q spanned by the degree 1 elements of N. It has also been shown that the canonical module of $K[Q]$ is generated by a degree 1 element, the residue class of X^{y_1}, which we denote by X^p. Thus Q can have only one interior lattice point, namely p. The h-polynomial of $K[P]$ and the one of $K[Q]$ coincide since $K[Q] \cong K[P]/(X^{y_i} - X^{y_{i+1}},\ i = 1, \ldots, m-1)$ and $X^{y_1} - X^{y_2}, \ldots, X^{y_{m-1}} - X^{y_m}$ is a regular sequence homogeneous of degree 1.

It follows from

$$E_{\partial(Q)}(t) = E_Q(t) - E_{\operatorname{int}(Q)}(t) = H_{K[Q]}(t) - H_{\omega_{K[Q]}}(t) = H_{K[Q]}(t) - t \cdot H_{K[Q]}(t)$$

that $h(Q) = h(\partial(Q))$. For the last equality we have used the fact that $\omega_{K[Q]} = (X^p) \cong K[Q](-1)$ with respect to the considered grading. This concludes the proof. □

Up to parallel translation, the Gorenstein polytopes with an interior lattice point are exactly the *reflexive polytopes* used by Batyrev in the theory of mirror symmetry; see [2]. Therefore the previous corollary reduces all questions about the h-vector of integrally closed Gorenstein polytopes to integrally closed reflexive polytopes. However, as shown by Mustaţă and Payne [9], there exist reflexive polytopes that are not integrally closed and whose h-vector is not unimodal.

If S is a simplicial sphere (or even the boundary of a simplicial polytope), then we can speak of its combinatorial h-vector (which one can read as the h-vector of the Ehrhart series of the geometric realization of S in the boundary of a suitable unit simplex).

Corollary 8. *Let P be an integer polytope such that $K[P]$ is Gorenstein.*

(i) *If P has a unimodular triangulation, then there exists a simplicial sphere S such that $h(P) = h(S)$.*
(ii) *If P has a regular unimodular triangulation, then there exists a simplicial polytope P' such that $h(P) = h(\Delta(P'))$.*

Proof. Polytopes with a unimodular triangulation are integrally closed. So we can proceed as in the proof of Corollary 7 and use the same notation. The only change is that we start with the given (regular) unimodular triangulation Ξ of P. It induces a unimodular triangulation of

$\mathrm{cn}((p,1)\colon\ p \in P)$ from which we derive the triangulation Δ of $\mathrm{cn}((p,1)\colon\ p \in P)$ as in the proof of Theorem 3. Thus the simplicial cones in Δ have generators of degree 1, and so it induces a unimodular triangulation Δ_1 of P. The restriction of Γ to P is denoted by Γ_1: the faces in Γ_1 are the intersections of the faces of Γ with P. Then Γ_1 is a subcomplex of ∂P, and $\Xi|\Gamma_1$ is a subcomplex of Δ_1. More precisely, $\Delta_1 = (\Xi|\Gamma_1) * \delta$ where δ is the simplex generated by the lattice points in P representing the irreducible elements $y_1, \dots, y_m \in E(P)$ in Theorem 3. For simplicity we denote the lattice points also by $y_1, \dots, y_m$.

With the notation of the proof of Theorem 3, Δ induces a unimodular triangulation Δ' of $\mathrm{cn}(N)$ with generators of degree 1 and thus a unimodular triangulation Δ'_1 of the (integrally closed) integer polytope Q.

Moreover, $K[Q]$ is Gorenstein, $h(P) = h(Q) = h(\partial(Q))$ and $\mathrm{int}(Q)$ contains a unique lattice point p. For (i) we simply choose $S = \partial Q$ with triangulation $\Delta'_1|\partial Q$.

For (ii) we first show that the triangulation Δ_1 is regular since this fact will be needed for an application to initial ideals. For the same reason we use Sturmfels' correspondence between monomial initial ideals and regular triangulations of P [14, Chapter 8]. Since Ξ is a regular unimodular triangulation of P, there exists a weight vector $w = (w_x\colon\ x \in P \cap \mathbb{Z}^{n-1})$ such that (i) Ξ is the regular subdivision of P induced by w, and (ii) the initial ideal $\mathrm{in}_w(I_P)$ of the toric ideal I_P is the Stanley–Reisner ideal of Ξ (as an abstract simplicial complex). By adding constants to the weights and scaling them simultaneously we can assume

$$1 \leqslant w_x < 1 + \frac{1}{n} \quad \text{for all } x \in P \cap \mathbb{Z}^{n-1}.$$

The toric ideal I_P lives in the polynomial ring $T = K[Y_x\colon\ x \in P \cap \mathbb{Z}^{n-1}]$. It is just the kernel of the natural epimorphism $\phi : T \to K[P]$, sending Y_x to the monomial $X^{(x,1)}$. It is generated by binomials that are homogeneous with respect to the standard grading on T.

We define a new weight vector w' on $P \cap \mathbb{Z}^{n-1}$ by keeping the weight w_x for $x \notin \{y_1, \dots, y_m\}$ and setting $w'_y = 0$ for $y_1, \dots, y_m$.

Let J be the Stanley–Reisner ideal of Δ_1. It is enough to show that J is contained in the initial ideal $\mathrm{in}_{w'}(I_P)$: first, $J \subseteq \mathrm{in}_{w'}(I_P)$ implies $J = \mathrm{in}_{w'}(I_P)$ since both residue class rings T/J and $T/\mathrm{in}_{w'}(I_P)$ have the same Hilbert function, namely $E(P, -)$ (we use the unimodularity of Δ_1). Second, the equality $J = \mathrm{in}_{w'}(I_P)$ implies that the regular subdivision of P induced by the weight vector w' is exactly Δ_1.

Let $M \subseteq P \cap \mathbb{Z}^{n-1}$ and denote the product of the indeterminates Y_x, $x \in M$, by Y^M. Then the ideal J is generated by all monomials Y^M such that $\mathrm{conv}(M)$ is not a face of Δ_1 and is minimal with respect to this property. In particular, Y^M has at most $n+1$ factors (for reasons of dimension). The crucial point is that no such M can contain a point $y \in P \setminus |\Gamma_1|$, as follows from the construction of Δ_1.

Suppose first that $\delta = \mathrm{conv}(M)$ is contained in one of the faces belonging to Γ_1. Then δ is a nonface of Ξ and therefore a nonface of Δ_1 since both triangulations agree on Γ_1.

Suppose second that δ is not contained in a face of Γ_1. Then the barycenter of δ lies in the interior of one of the unimodular (!) simplices in Δ_1 that have one of the lattice points y_i as a vertex. Therefore the epimorphism $\phi : T \to K[P]$ maps Y^M to a monomial that can also be represented as a monomial involving one of the variables Y_{y_i}. However, this second monomial has the same total degree and strictly smaller weight with respect to w', as the reader may check. So Y^M appears in the initial ideal.

This concludes the proof of the regularity of Δ_1. That Δ_1' is regular, is seen in the same way. The only difference is that the lattice points $y_1, \dots, y_m$ are identified to a single one. For (ii) it remains to apply the next lemma. $\square$

We include a lemma on regular triangulations that we have not found in the literature.

Lemma 9. *Let $Q \subseteq \mathbb{R}^{n-1}$ be a polytope with a regular triangulation Σ. Then there exists a simplicial polytope P' such that the boundary complex of P' is combinatorially equivalent to $\Sigma|\partial P$.*

Proof. We choose a convex, piecewise affine function $f : Q \to \mathbb{R}$ such that Σ is the subdivision of Q into the domains of linearity of f. We can assume that $f(x) > 0$ for all $x \in Q$. Consider the graph G of f in $\mathbb{R}^n$. Then G is a polytopal complex whose faces project onto the faces of Σ.

We choose a point $(x, z) \in \mathbb{R}^n$, $x \in \operatorname{int}(Q)$ and $z \ll 0$ such that (x, z) lies "below" all the hyperplanes through the facets of G. Then we form the set C as the union of all rays emanating from (x, z) and going through the points of G. It is not hard to check that C is in fact convex: let $a, b \in C$ and consider a point c on the line segment $[a, b]$; we have to show that the ray from (x, z) through c meets G. We may assume that $a, b \in G$. Let $c' = (c_1, \dots, c_n)$ be the projection of c in $\mathbb{R}^n$ along the vertical axis. By convexity of f one has $f(c') \leqslant c_{n+1}$, and, for the same reason, the graph of f over the line segment $[x, c'] \subseteq Q$ lies below the line segment $[(x, f(x)), (c', f(c'))]$. It follows that the line segment $[(x, z), c]$ intersects the graph of f.

The decomposition of ∂G (as a manifold with boundary) into maximal polytopal subsets is combinatorially equivalent to the collection of the maximal simplices in $\Sigma|\partial Q$. On the other hand, it is also combinatorially equivalent to the collection of the facets of the cone C (with apex in (x, z)). (This requires an argument very similar to the one by which we have proved the convexity.) Therefore we obtain the desired polytope P' as a cross-section of C. $\square$

With Corollary 8(ii) the proof of Theorem 1 is complete since the g-theorem applies to $h(\Delta(P'))$.

We are grateful to Ch. Haase for pointing out to us that the hypothesis of regularity cannot be omitted in Corollary 8(ii) and for suggesting the proof of Lemma 9. The assumptions of Corollary 8 appear at several places in algebraic combinatorics as has been discussed in [1].

Remark 10. In some special situations we can omit the assumption that P has a (regular) unimodular triangulation and obtain directly from Corollary 7 that the h-vector of P is unimodal. More precisely, assume that $\dim(P) \leqslant m + 4$. Then $\dim(Q) \leqslant 5$ and it follows from a result of Hibi [8] that the h-vector of Q is unimodal.

We conclude by drawing a consequence for the toric ideal I_P of P. As we have seen in the proof of Corollary 8 its initial ideal with respect to the weight vector w' is the Stanley–Reisner ideal J of the simplicial complex $\Delta_1 * \delta$ (notation as in the proof of Corollary 8). Since Δ_1 is combinatorially equivalent to the boundary of a simplicial polytope, it follows that R/J is a Gorenstein ring, and we obtain:

Corollary 11. *Let P be an integer Gorenstein polytope such that the toric ideal I_P has a square-free initial ideal. Then it also has a square-free initial ideal that is the Stanley–Reisner ideal of the join of a boundary of a simplicial polytope and a simplex, and thus defines a Gorenstein ring.*

The corollary answers a question of Conca and Welker, and the methods of this note were originally designed for its solution. See [6, Question 6] and [7] for more details related to this result.

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