

H^{1/2} MAPS WITH VALUES INTO THE CIRCLE: MINIMAL CONNECTIONS, LIFTING, AND THE GINZBURG–LANDAU EQUATION

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1. Introduction

Let $G \subset \mathbf{R}^3$ be a smooth bounded domain with $\Omega = \partial G$ simply connected. We are concerned with the properties of the space

$$H^{1/2}(\Omega; S^1) = \{g \in H^{1/2}(\Omega; \mathbf{R}^2); |g| = 1 \text{ a.e. on } \Omega\}.$$

Recall (*see* [12]) that there are functions in $H^{1/2}(\Omega; S^1)$ which cannot be written in the form $g = e^{i\varphi}$ with $\varphi \in H^{1/2}(\Omega; \mathbf{R})$. For example, we may assume that locally, near a point on Ω , say 0, Ω is a disc B_1 ; then take

$$(1.1) \quad g(x, y) = (x, y)/(x^2 + y^2)^{1/2} \quad \text{on } B_1.$$

Recall also (*see* [25]) that there are functions in $H^{1/2}(\Omega; S^1)$ which cannot be approximated in the $H^{1/2}$ -norm by functions in $C^\infty(\Omega; S^1)$. Consider, for example, again a function g which is the same as in (1.1) near 0.

It is therefore natural to introduce the classes

$$X = \{g \in H^{1/2}(\Omega; S^1); g = e^{i\varphi} \text{ for some } \varphi \in H^{1/2}(\Omega; \mathbf{R})\}$$

and

$$Y = \overline{C^\infty(\Omega; S^1)}^{H^{1/2}}.$$

Clearly, we have

$$X \subset Y \subset H^{1/2}(\Omega; S^1).$$

Moreover, these inclusions are strict. Indeed, any function $g \in H^{1/2}(\Omega; S^1)$ which satisfies (1.1) does not belong to Y . On the other hand, the function

$$g(x, y) = \begin{cases} e^{2i\pi/r^\alpha}, & \text{on } B_1 \\ 1, & \text{on } \Omega \setminus B_1 \end{cases}$$

with $r = (x^2 + y^2)^{1/2}$ and $1/2 \leq \alpha < 1$, belongs to Y , but not to X (*see* [12]).

To every map $g \in H^{1/2}(\Omega; \mathbf{R}^2)$ we associate a distribution $T = T(g) \in \mathcal{D}'(\Omega; \mathbf{R})$. When $g \in H^{1/2}(\Omega; S^1)$, the distribution $T(g)$ describes the location and the topological degree of its singularities. This is the analogue of a tool introduced by Brezis, Coron and Lieb [19] in the framework of $H^1(G; S^2)$ (*see* the discussion following Lemma 2 below). In the context of $H^{1/2}(\Omega; S^1)$, the distribution $T(g)$ and the corresponding number $L(g)$ (defined after Lemma 1) were originally introduced by the authors in 1996 and these concepts were presented in various lectures.

Given $g \in H^{1/2}(\Omega; \mathbf{R}^2)$ and $\varphi \in \text{Lip}(\Omega; \mathbf{R})$, consider any $U \in H^1(G; \mathbf{R}^2)$ and any $\Phi \in \text{Lip}(G; \mathbf{R})$ such that

$$(1.2) \quad U|_{\Omega} = g \text{ and } \Phi|_{\Omega} = \varphi.$$

Set

$$H = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y);$$

this H is independent of the choice of direct orthonormal bases in \mathbf{R}^3 (to compute derivatives) and in \mathbf{R}^2 (to compute \wedge -products). Next, consider

$$(1.3) \quad \int_G H \cdot \nabla \Phi.$$

It is not difficult to show (*see* Section 2) that (1.3) is independent of the choice of U and Φ ; it depends only on g and φ . We may thus define the distribution $T(g) \in \mathcal{D}'(\Omega; \mathbf{R})$ by

$$\langle T(g), \varphi \rangle = \int_G H \cdot \nabla \Phi.$$

If there is no ambiguity, we will simply write T instead of $T(g)$.

When g has a little more regularity, we may also express T in a simpler form:

Lemma 1. — *If $g \in H^{1/2}(\Omega; \mathbf{R}^2) \cap W^{1,1}(\Omega; \mathbf{R}^2) \cap L^\infty(\Omega; \mathbf{R}^2)$, then*

$$\langle T(g), \varphi \rangle = \int_{\Omega} ((g \wedge g_x)\varphi_y - (g \wedge g_y)\varphi_x), \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}).$$

The integrand is computed pointwise in any orthonormal frame (x, y) such that (x, y, n) is direct, where n is the outward normal to G – and the corresponding quantity is frame-invariant.

By analogy with the results of [19] and [6] we introduce, for every $g \in H^{1/2}(\Omega; \mathbf{R}^2)$, the number

$$\begin{aligned} L(g) &= \frac{1}{2\pi} \text{Sup} \{ \langle T(g), \varphi \rangle ; \varphi \in \text{Lip}(\Omega; \mathbf{R}), |\varphi|_{\text{Lip}} \leq 1 \} \\ &= \frac{1}{2\pi} \text{Max} \{ \dots \}, \end{aligned}$$

where $|\varphi|_{\text{Lip}} = \text{Sup}_{x \neq y} |\varphi(x) - \varphi(y)| / d(x, y)$ refers to a given metric d on Ω . There are three (equivalent) metrics on Ω which are of interest:

$$\begin{aligned} (1.4) \quad d_{\mathbf{R}^3}(x, y) &= |x - y|, \\ d_G(x, y) &= \text{the geodesic distance in } \bar{G}, \\ d_\Omega(x, y) &= \text{the geodesic distance in } \Omega. \end{aligned}$$

When dealing with a specified metric, we will write $L_{\mathbf{R}^3}$, L_G or L_Ω . Otherwise, we will simply write L (note that all these L 's are equivalent). It is easy to see that

$$(1.5) \quad 0 \leq L(g) \leq C \|g\|_{H^{1/2}}^2, \quad \forall g \in H^{1/2}(\Omega; \mathbf{R}^2)$$

and

$$(1.6) \quad |L(g) - L(h)| \leq C \|g - h\|_{H^{1/2}} (\|g\|_{H^{1/2}} + \|h\|_{H^{1/2}}), \quad \forall g, h \in H^{1/2}(\Omega; \mathbf{R}^2).$$

When g takes its values into S^1 and has only a finite number of singularities, there are very simple expressions for $T(g)$ and $L(g)$:

Lemma 2. — *If $g \in H^{1/2}(\Omega; S^1) \cap H_{\text{loc}}^1(\Omega \setminus \cup_{j=1}^k \{a_j\}; S^1)$, then*

$$T(g) = 2\pi \sum_{j=1}^k d_j \delta_{a_j},$$

where $d_j = \text{deg}(g, a_j)$. Moreover $L(g)$ is the length of the minimal connection associated to the configuration (a_j, d_j) and to the specific metric on Ω (in the sense of [19]; see also [27]).

Remark 1.1. — Here, $\text{deg}(g, a_j)$ denotes the topological degree of g restricted to any small circle around a_j , positively oriented with respect to the outward normal. It is well defined using the degree theory for maps in $H^{1/2}(S^1; S^1)$ (see [17] and [22]).

By the definition of $T(g)$, we see that $\langle T(g), 1 \rangle = 0$. Therefore, if g is as in Lemma 2, then $\sum d_j = 0$. Thus we may write the collection of points (a_j) , repeated with their multiplicity d_j , as $(P_1, \dots, P_k, N_1, \dots, N_k)$, where $k = 1/2 \sum |d_j|$ (we exclude from this collection the points of degree 0). A point a_j is counted among the P 's if it has positive degree and among the N 's otherwise. Then $L(g) = \text{Inf}_\sigma \sum d(P_j, N_{\sigma(j)})$. Here, the Inf is taken over all the permutations σ of $\{1, \dots, k\}$ and d is one of the metrics in (1.4).

The conclusion of Lemma 2 is reminiscent of a concept originally introduced by Brezis, Coron and Lieb [19]. There, u is a map from $G \subset \mathbf{R}^3$ into S^2 with a finite number of singularities $a_j \in G$. To such a map u , one associates a distribution $T(u)$ describing the location and the topological charge of the singular set of u . More precisely, if $u \in H^1(G; S^2)$, set

$$\mathcal{D} = (u \cdot u_y \wedge u_z, \quad u \cdot u_z \wedge u_x, \quad u \cdot u_x \wedge u_y)$$

and $T(u) = \text{div} \mathcal{D}$.

If u is smooth except at the a_j 's, it is proved in [19] that

$$T(u) = 4\pi \sum d_j \delta_{a_j}.$$

Here, d_j is the topological degree of u around a_j .

Using a density result of T. Rivière (*see* [38] and Lemma 11 in Section 2; *see* also the proof of Lemma 23, Remark 5.1 and Appendix B), we will extend Lemma 2 to general functions in $H^{1/2}(\Omega; S^1)$:

Theorem 1. — *Given any $g \in H^{1/2}(\Omega; S^1)$, there are two sequences of points (P_i) and (N_i) in Ω such that*

$$(1.7) \quad \sum_i |P_i - N_i| < \infty$$

and

$$(1.8) \quad \langle T(g), \varphi \rangle = 2\pi \sum_i (\varphi(P_i) - \varphi(N_i)), \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}).$$

In addition, for any metric d in (1.4)

$$L(g) = \text{Inf} \sum_i d(P_i, N_i),$$

where the infimum is taken over all possible sequences $(P_i), (N_i)$ satisfying (1.7), (1.8).

If the distribution T is a measure (of finite total mass), then

$$T(g) = 2\pi \sum_{\text{finite}} d_j \delta_{a_j}$$

with $d_j \in \mathbf{Z}$ and $a_j \in \Omega$.

Remark 1.2. — There are always infinitely many representations of $T(g)$ as a sum satisfying (1.7)–(1.8) and such representations need not be equivalent modulo a permutation of points. For example, a dipole $\delta_P - \delta_Q$ may be represented as $\delta_P - \delta_{Q_1} + \sum_{j \geq 1} (\delta_{Q_j} - \delta_{Q_{j+1}})$ for any sequence (Q_j) rapidly converging to Q . The last assertion in Theorem 1 is the $H^{1/2}$ -analogue of a result of Jerrard and Soner [28, 29] (see also Hang and Lin [28]) concerning maps in $W^{1,1}(\Omega; S^1)$.

Maps in Y can be characterized in terms of the distribution T :

Theorem 2 (Rivière [38]). — *Let $g \in H^{1/2}(\Omega; S^1)$. Then $T(g) = 0$ if and only if $g \in Y$.*

This result is the $H^{1/2}$ -counterpart of a well-known result of Bethuel [3] characterizing the closure of smooth maps in $H^1(B^3; S^2)$ (see also Demengel [24]).

The implication $g \in Y \implies T(g) = 0$ is trivial, using e.g. (1.6). The converse is more delicate; it uses the “dipole removing” technique of Bethuel [3] and we refer the reader to [38]; for convenience we present in Section 4 a slightly different proof.

As was mentioned earlier, functions in Y need not belong to X , i.e., they need not have a lifting in $H^{1/2}(\Omega; \mathbf{R})$. However, we have

Theorem 3. — *For every $g \in Y$ there exists $\varphi \in H^{1/2}(\Omega; \mathbf{R}) + W^{1,1}(\Omega; \mathbf{R})$, which is unique (modulo 2π), such that $g = e^{i\varphi}$. Conversely, if $g \in H^{1/2}(\Omega; S^1)$ can be written as $g = e^{i\varphi}$ with $\varphi \in H^{1/2} + W^{1,1}$, then $g \in Y$.*

The existence will be proved in Section 3 with the help of paraproducts (in the sense of J.-M. Bony and Y. Meyer). The heart of the matter is the estimate

$$(1.9) \quad \|\varphi\|_{H^{1/2}+W^{1,1}} \leq C_\Omega \|e^{i\varphi}\|_{H^{1/2}} (1 + \|e^{i\varphi}\|_{H^{1/2}}),$$

which holds for any smooth real-valued function φ ; here C_Ω depends only on Ω .

Using Theorem 3 and the basic estimate (1.9), we will prove that, for every $g \in H^{1/2}(\Omega; S^1)$, there exists $\varphi \in H^{1/2}(\Omega; \mathbf{R}) + BV(\Omega; \mathbf{R})$ such that $g = e^{i\varphi}$ (see Section 4). Of course, this φ is not unique. There is an interesting link between all possible liftings of g and the minimal connection of g :

Theorem 4. — *For every $g \in H^{1/2}(\Omega; S^1)$ we have*

$$\text{Inf} \{ |\varphi_2|_{BV}; g = e^{i(\varphi_1 + \varphi_2)}; \varphi_1 \in H^{1/2} \text{ and } \varphi_2 \in BV \} = 4\pi L_\Omega(g),$$

where $|\varphi_2|_{BV} = \int_\Omega |D\varphi_2|$.

Another useful fact about the structure of $H^{1/2}(\Omega; S^1)$ is the following factorization result:

Theorem 5. — *We have*

$$H^{1/2}(\Omega; S^1) = (\mathbf{X}) \cdot (H^{1/2} \cap W^{1,1}),$$

i.e., every $g \in H^{1/2}(\Omega; S^1)$ may be written as $g = e^{i\varphi}h$, with $\varphi \in H^{1/2}(\Omega; \mathbf{R})$ and $h \in H^{1/2}(\Omega; S^1) \cap W^{1,1}(\Omega; S^1)$. Moreover we have the control

$$\|\varphi\|_{H^{1/2}}^2 + \|h\|_{W^{1,1}} \leq C_\Omega \|g\|_{H^{1/2}}^2.$$

The interplay between the Ginzburg–Landau energy and minimal connections has been first pointed out in the important work of T. Rivière [37] (see also [34] and [38]) in the case of boundary data with a finite number of singularities. We are concerned here with a general boundary condition g in $H^{1/2}$.

Given $g \in H^{1/2}(\Omega; S^1)$, set

$$(1.10) \quad e_{\varepsilon,g} = e_\varepsilon = \operatorname{Min}_{H_g^1(G; \mathbf{R}^2)} E_\varepsilon(u),$$

where

$$E_\varepsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_G (|u|^2 - 1)^2$$

and

$$H_g^1(G; \mathbf{R}^2) = \{u \in H^1(G; \mathbf{R}^2); u = g \text{ on } \Omega\}.$$

Theorem 6. — *For every $g \in H^{1/2}(\Omega; S^1)$ we have, as $\varepsilon \rightarrow 0$,*

$$(1.11) \quad e_\varepsilon = \pi L_G(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)).$$

This result and some variants are proved in Section 5. For special g 's (namely g 's with finite number of singularities), formula (1.11) was first proved by T. Rivière in [37]. For a general $g \in H^{1/2}(\Omega; S^1)$, it was established in [12] that

$$e_\varepsilon \leq C(g) \log(1/\varepsilon)$$

where $C(g) = C(G) \|g\|_{H^{1/2}(\Omega)}^2$; another proof of the same inequality is given in [38].

Using Theorem 6, we may characterize the classes \mathbf{X} and \mathbf{Y} in terms of the behavior of the Ginzburg–Landau energy as $\varepsilon \rightarrow 0$. Indeed, Theorem 6 implies that

$$\mathbf{Y} = \{g \in H^{1/2}(\Omega; S^1); e_\varepsilon = o(\log(1/\varepsilon))\}.$$

On the other hand, it is easy to see that

$$X = \{g \in H^{1/2}(\Omega; S^1); e_\varepsilon = O(1)\}.$$

Next, we present various estimates for minimizers u_ε in (1.10). In Section 6, we discuss the following theorem (originally announced in [13] and subsequently established with a simpler proof in [5]):

Theorem 7. — For every $g \in H^{1/2}(\Omega; S^1)$ we have

$$(1.12) \quad \|u_\varepsilon\|_{W^{1,p}(G)} \leq C_p, \quad \forall 1 \leq p < 3/2.$$

In fact, we will prove the following slight generalization of Theorem 7:

Theorem 7'. — For every $g \in H^{1/2}(\Omega; S^1)$, the family (u_ε) is relatively compact in $W^{1,p}$ for every $p < 3/2$.

Remark 1.3. — It is very plausible that Theorem 7 still holds when $p = 3/2$. However, the conclusion fails for $p > 3/2$; see the discussion in Section 9.

In Section 7, we will establish stronger *interior* estimates:

Theorem 8. — For every $g \in H^{1/2}(\Omega; S^1)$, we have

$$(1.13) \quad \|u_\varepsilon\|_{W^{1,p}(K)} \leq C_{p,K}, \quad \forall 1 \leq p < 2, \quad \forall K \text{ compact in } G.$$

Consequently, (u_ε) is relatively compact in $W_{\text{loc}}^{1,p}$ for every $p < 2$.

Remark 1.4. — The conclusion of Theorem 8 fails for $p = 2$. Here is an example, with $G = B_1$, the unit ball in \mathbf{R}^3 , and $g(x_1, x_2, x_3) = (x_1, x_2)/\sqrt{x_1^2 + x_2^2}$. T. Rivière [37] (see also F.H. Lin and T. Rivière [34]) has proved that in this case $u_\varepsilon \rightarrow u = (x_1, x_2)/\sqrt{x_1^2 + x_2^2}$, and clearly this u does not belong to $H_{\text{loc}}^1(G)$.

Finally, we have a very precise result concerning the limit of u_ε when $g \in Y$:

Theorem 9. — For every $g \in Y$, write (as in Theorem 3) $g = e^{i\varphi}$, with $\varphi \in H^{1/2} + W^{1,1}$. Then we have

$$u_\varepsilon \rightarrow u_* = e^{i\tilde{\varphi}} \text{ in } W^{1,p}(G) \cap C^\infty(G), \quad \forall p < 3/2,$$

where $\tilde{\varphi}$ is the harmonic extension of φ .

Theorem 9 and some of its variants are presented in Section 8. In Section 9 we prove some partial results about estimates in $W^{1,p}$ when $p = 3/2$. In Section 10 we list some open problems.

Most of the results in this paper were announced in [13].

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2. Elementary properties of the minimal connection. Proof of Theorem 1

To every $g \in H^{1/2}(\Omega; \mathbf{R}^2)$ we associate a distribution $T(g) \in \mathcal{D}'(\Omega; \mathbf{R})$ in the following way: consider any $U \in H^1(G; \mathbf{R}^2)$ such that

$$U|_{\Omega} = g.$$

Given $\varphi \in \text{Lip}(\Omega; \mathbf{R})$, let $\Phi \in \text{Lip}(G; \mathbf{R})$ be such that

$$\Phi|_{\Omega} = \varphi.$$

Set

$$H = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y).$$

Lemma 3. — *The quantity $\int_G H \cdot \nabla \Phi$ depends only on g and φ .*

Proof. — We first claim that $\int_G H \cdot \nabla \Phi$ does not depend on the choice of Φ .

Observe that, if $U \in C^\infty(\bar{G}; \mathbf{R}^2)$, then

$$\text{div } H = 0.$$

By density, we find that

$$\operatorname{div} \mathbf{H} = 0 \text{ in } \mathcal{D}'(\mathbf{G})$$

for any $\mathbf{U} \in H^1(\mathbf{G}; \mathbf{R}^2)$. It follows easily that

$$\int_{\mathbf{G}} \mathbf{H} \cdot \nabla \Psi = 0, \quad \forall \Psi \in \operatorname{Lip}(\mathbf{G}; \mathbf{R}) \text{ with } \Psi = 0 \text{ on } \Omega.$$

This implies the above claim.

Next, we verify that $\int_{\mathbf{G}} \mathbf{H} \cdot \nabla \Phi$ does not depend on the choice of \mathbf{U} . Let \mathbf{V} be another choice in $H^1(\mathbf{G}; \mathbf{R}^2)$ such that $V|_{\Omega} = g$. Set $\mathbf{W} = \mathbf{V} - \mathbf{U} \in H_0^1(\mathbf{G}; \mathbf{R}^2)$. Then, with obvious notation,

$$\int_{\mathbf{G}} \mathbf{H}_{\mathbf{V}} \cdot \nabla \Phi = \int_{\mathbf{G}} \mathbf{H}_{\mathbf{U}} \cdot \nabla \Phi + \int_{\mathbf{G}} \mathbf{R}_1 \cdot \nabla \Phi + \int_{\mathbf{G}} \mathbf{R}_2 \cdot \nabla \Phi,$$

with $\mathbf{R}_1 = (\mathbf{W}_y \wedge \mathbf{U}_z + \mathbf{U}_y \wedge \mathbf{W}_z, \dots)$, $\mathbf{R}_2 = (\mathbf{W}_y \wedge \mathbf{W}_z, \dots)$.

We complete the proof of Lemma 3 with the help of

Lemma 4. — For each $\mathbf{U} \in H^1(\mathbf{G}; \mathbf{R}^2)$ and $\mathbf{W} \in H_0^1(\mathbf{G}; \mathbf{R}^2)$ we have

$$\int_{\mathbf{G}} \mathbf{R}_1 \cdot \nabla \Phi = 0, \quad \forall \Phi \in \operatorname{Lip}(\mathbf{G}; \mathbf{R}).$$

Proof of Lemma 4. — By density, it suffices to prove the above equality for $\mathbf{U} \in C^\infty(\bar{\mathbf{G}}; \mathbf{R}^2)$, $\mathbf{W} \in C_0^\infty(\bar{\mathbf{G}}; \mathbf{R}^2)$ and $\Phi \in C^\infty(\bar{\mathbf{G}}; \mathbf{R})$. For such \mathbf{U} and \mathbf{W} , note that

$$\mathbf{W}_y \wedge \mathbf{U}_z + \mathbf{U}_y \wedge \mathbf{W}_z = (\mathbf{W} \wedge \mathbf{U}_z)_y + (\mathbf{U}_y \wedge \mathbf{W})_z.$$

Therefore,

$$\int_{\mathbf{G}} \mathbf{R}_1 \cdot \nabla \Phi = - \int_{\mathbf{G}} [(\mathbf{W} \wedge \mathbf{U}_z)\Phi_{xy} + (\mathbf{U}_y \wedge \mathbf{W})\Phi_{xz} + \dots] = 0.$$

As a consequence of Lemma 3, the map

$$\varphi \longmapsto \int_{\mathbf{G}} \mathbf{H} \cdot \nabla \Phi$$

is a continuous linear functional on $\operatorname{Lip}(\Omega; \mathbf{R})$. In particular, it is a distribution. Again by Lemma 3, this distribution depends only on $g \in H^{1/2}(\Omega; \mathbf{R}^2)$. We will denote it $\mathbf{T}(g)$.

Remark 2.1. — It is important to note that T has a “local” character. More precisely, if $g_1, g_2 \in H^{1/2}(\Omega; \mathbf{R}^2)$ are such that $g_1 = g_2$ in ω (where ω is an open subset of Ω), then

$$\langle T(g_1), \varphi \rangle = \langle T(g_2), \varphi \rangle, \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}), \text{ with } \text{supp } \varphi \subset \omega.$$

This is an easy consequence of Lemma 3 and of the fact that, if $\text{supp } g \cap \text{supp } \varphi = \emptyset$, then one may extend g to $U \in H^1$ and φ to $\Phi \in \text{Lip}$ such that $\text{supp } U \cap \text{supp } \Phi = \emptyset$. Thus, one may define a local version of T as follows: if $g \in H_{\text{loc}}^{1/2}(\omega; \mathbf{R}^2)$, set

$$\langle T(g), \varphi \rangle = \langle T(h), \varphi \rangle, \quad \forall \varphi \in C_0^1(\omega; \mathbf{R}),$$

where h is any map in $H^{1/2}(\Omega; \mathbf{R}^2)$ such that $h = g$ in a neighborhood of $\text{supp } \varphi$.

Remark 2.2. — Another important property is the invariance under diffeomorphisms. More precisely, let Ω, G, g, φ be as above and let $\xi : \tilde{\Omega} \rightarrow \Omega$ be an orientation-preserving diffeomorphism. Then

$$\langle T(g), \varphi \rangle = \langle T(\tilde{g}), \tilde{\varphi} \rangle,$$

where $\tilde{g} = g \circ \xi$ and $\tilde{\varphi} = \varphi \circ \xi$. Clearly, ξ extends as an orientation-preserving diffeomorphism (still denoted ξ) from a small tubular neighborhood of $\tilde{\Omega}$ in \tilde{G} to a tubular neighborhood of Ω in G (as in the proof of Lemma 5 below).

We have

$$\langle T(g), \varphi \rangle = \int_G \mathbf{H} \cdot \nabla \Phi = 2 \int_G \text{Jac}(\Phi, U),$$

since

$$\mathbf{H} = 2(U_y \wedge U_z, U_z \wedge U_x, U_x \wedge U_y).$$

We may choose U and Φ supported in a small tubular neighborhood of Ω and set $\tilde{U} = U \circ \xi$ and $\tilde{\Phi} = \Phi \circ \xi$. Then, with obvious notation,

$$\begin{aligned} \langle T(\tilde{g}), \tilde{\varphi} \rangle &= \int_{\tilde{G}} \tilde{\mathbf{H}} \cdot \nabla \tilde{\Phi} = 2 \int_{\tilde{G}} \text{Jac}(\tilde{\Phi}, \tilde{U}) \\ &= 2 \int_G \text{Jac}(\Phi, U) = \langle T(g), \varphi \rangle. \end{aligned}$$

Similarly, if ω is an open subset of Ω and $\xi : \tilde{\omega} \rightarrow \omega$ is an orientation-preserving diffeomorphism, then (using Remark 2.1) we have

$$\langle T(g), \varphi \rangle = \langle T(\tilde{g}), \tilde{\varphi} \rangle$$

for every $g \in H_{\text{loc}}^{1/2}(\omega; \mathbf{R}^2)$ and $\varphi \in C_0^1(\omega; \mathbf{R})$. This is extremely useful because we can always choose a local diffeomorphism with $\tilde{\Omega}$ flat near a point. More precisely, let (ω_i) be a finite covering of Ω with each ω_i diffeomorphic to a disc D via $\xi_i : D \rightarrow \omega_i$. Let (α_i) be a corresponding partition of unity. Then, $\forall \varphi \in \text{Lip}(\Omega; \mathbf{R})$,

$$\langle T(g), \varphi \rangle = \sum \langle T(g), \alpha_i \varphi \rangle$$

and we may compute each term $\langle T(g), \alpha_i \varphi \rangle$ in D using the fact that

$$\langle T(g), \alpha_i \varphi \rangle = \langle T(g \circ \xi_i), (\alpha_i \varphi) \circ \xi_i \rangle.$$

Here is a noticeable fact about $T(g)$:

Lemma 5. — *Let $g \in H^{1/2}(\Omega; \mathbf{R}^2)$. Then there exists an L^1 -section F of the tangent bundle $T(\Omega)$ such that*

$$\langle T(g), \varphi \rangle = \int_{\Omega} F \cdot \nabla \varphi, \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}).$$

Proof of Lemma 5. — For $\beta > 0$, let

$$G_{\beta} = \{X \in G; \quad \delta(X) < \beta\}, \quad \Omega_{\beta} = \{X \in G; \quad \delta(X) = \beta\},$$

where $\delta(X) = \text{dist}(X, \Omega)$. Assuming that β is sufficiently small, say $\beta < \beta_0$, for every $X \in G_{\beta}$ there exists a unique point $\sigma(X) \in \Omega$ such that $\delta(X) = |X - \sigma(X)|$. Let $\Pi : G_{\beta} \rightarrow (0, \beta) \times \Omega$ be the mapping defined by $\Pi(X) = (\delta(X), \sigma(X))$. This mapping is a C^2 -diffeomorphism and its inverse is given by

$$\Pi^{-1}(t, \sigma) = \sigma - tn(\sigma), \quad \forall (t, \sigma) \in (0, \beta) \times \Omega,$$

where $n(\sigma)$ is the outward unit normal to Ω at σ . For $0 < t < \beta_0$, let K_t denote the mapping $\Pi^{-1}(t, \cdot)$ of Ω onto Ω_t .

Since $n(\sigma)$ is orthogonal to $\Omega_t = \Pi^{-1}(t, \Omega)$ at $\sigma - tn(\sigma)$, it follows that, for every integrable non-negative function f in G_{β} ,

$$\int_{G_{\beta}} f = \int_0^{\beta} dt \int_{\Omega_t} f d\sigma_t = \int_0^{\beta} dt \int_{\Omega} f(K_t(\sigma)) (\text{Jac } K_t) d\sigma,$$

where $d\sigma$, $d\sigma_t$ denote surface elements on Ω , Ω_t respectively.

We now make a special choice of U and Φ . Let

$$\Phi(\mathbf{X}) = \varphi(\sigma(\mathbf{X}))\zeta(\delta(\mathbf{X})),$$

where $\varphi \in C^1(\Omega; \mathbf{R})$ is the given test function and

$$\zeta(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq \beta_0/2 \\ 0, & \text{for } t \geq \beta_0. \end{cases}$$

We take U to be any H^1 extension of g such that $U(\mathbf{X}) = 0$ if $\delta(\mathbf{X}) \geq \beta_0/2$. Hence

$$\begin{aligned} \langle T(g), \varphi \rangle &= \int_G \mathbf{H} \cdot \nabla \Phi = \int_{G_{\beta_0/2}} \mathbf{H} \cdot \nabla \Phi \\ (2.1) \qquad &= \int_0^{\beta_0/2} dt \int_{\Omega} \mathbf{H} \cdot \nabla \Phi(\mathbf{K}_t(\sigma)) (\text{Jac } \mathbf{K}_t) d\sigma. \end{aligned}$$

For every $\sigma \in \Omega$, fix a frame $\mathcal{F}_\sigma = (x, y)$ as in Lemma 1. We already observed that $\mathbf{H} \cdot \nabla \Phi$ can be computed (pointwise) in any direct orthonormal frame of \mathbf{R}^3 . We choose, at any points $\mathbf{X} \in G_{\beta_0/2}$, the special frame $(\mathcal{F}_{\sigma(\mathbf{X})}, n(\sigma(\mathbf{X})))$. Then, we have, $\forall t \in (0, \beta_0/2), \forall \sigma \in \Omega$,

$$(2.2) \qquad (\mathbf{H} \cdot \nabla \Phi)(\mathbf{K}_t(\sigma)) = 2(U_y \wedge U_z)(\mathbf{K}_t(\sigma))\varphi_x(\sigma) + 2(U_z \wedge U_x)(\mathbf{K}_t(\sigma))\varphi_y(\sigma).$$

We now insert (2.2) into (2.1) and obtain the conclusion of Lemma 5 with $F(\sigma) = F_1(\sigma)\frac{\partial}{\partial x} + F_2(\sigma)\frac{\partial}{\partial y}$, where

$$F_1(\sigma) = 2 \int_0^{\beta_0/2} (U_y \wedge U_z)(\mathbf{K}_t(\sigma)) (\text{Jac } \mathbf{K}_t) dt$$

and

$$F_2(\sigma) = 2 \int_0^{\beta_0/2} (U_z \wedge U_x)(\mathbf{K}_t(\sigma)) (\text{Jac } \mathbf{K}_t) dt.$$

We now turn to the

Proof of Lemma 1. — It suffices to prove that

$$\int_G \mathbf{H} \cdot \nabla \Phi = \int_{\Omega} [(g \wedge g_x)\varphi_y - (g \wedge g_y)\varphi_x]$$

when $U \in C^\infty(\bar{G}; \mathbf{R}^2)$ and $\Phi \in C^\infty(\bar{G}; \mathbf{R})$. We write

$$H = \left((U \wedge U_z)_y + (U_y \wedge U)_z, (U \wedge U_x)_z + (U_z \wedge U)_x, \right. \\ \left. (U \wedge U_y)_x + (U_x \wedge U)_y \right).$$

Integration by parts yields

$$\int_G H \cdot \nabla \Phi = \int_\Omega U \wedge \det(\nabla U, \nabla \Phi, \vec{n}).$$

By Lemma 3, we may assume further that $\frac{\partial U}{\partial n} = 0$ and $\frac{\partial \Phi}{\partial n} = 0$.

For each $\sigma \in \Omega$, we compute $\det(\nabla U, \nabla \Phi, \vec{n})$ in the frame given by Lemma 1. We have

$$\det(\nabla U, \nabla \Phi, \vec{n}) = \frac{\partial U}{\partial x} \frac{\partial \Phi}{\partial y} - \frac{\partial U}{\partial y} \frac{\partial \Phi}{\partial x} = g_x \varphi_y - g_y \varphi_x,$$

and the conclusion follows.

Here are some straightforward variants and consequences of Lemma 1 and Remarks 2.1–2.2:

Lemma 6. — *Let ω be an open subset of Ω . Let*

$$g \in H^{1/2}(\omega; \mathbf{R}^2) \cap W^{1,1}(\omega) \cap L^\infty(\omega).$$

Then

$$(2.3) \quad \langle T(g), \varphi \rangle = \int_\omega [(g \wedge g_x) \varphi_y - (g \wedge g_y) \varphi_x], \quad \forall \varphi \in C_0^1(\omega; \mathbf{R}).$$

Lemma 7. — *Let ω be an open subset of Ω . Let $g \in H^{1/2}(\omega; S^1) \cap VMO(\omega; S^1)$. Then*

$$\langle T(g), \varphi \rangle = 0, \quad \forall \varphi \in C_0^1(\omega; \mathbf{R}).$$

Proof of Lemma 7. — In view of Remark 2.2, we may assume that ω is a disc. There is a sequence $(g_n) \in C^\infty(\omega; S^1)$ such that $g_n \rightarrow g$ in $H_{loc}^{1/2}(\omega)$ (see [22]). Hence $\langle T(g_n), \varphi \rangle \rightarrow \langle T(g), \varphi \rangle$, $\forall \varphi \in C_0^1(\omega; \mathbf{R})$, by (2.5) below. On the other hand, by Lemma 6,

$$\langle T(g_n), \varphi \rangle = \int_\omega [(g_n \wedge g_{nx}) \varphi_y - (g_n \wedge g_{ny}) \varphi_x] \\ = 2 \int_\omega (g_{nx} \wedge g_{ny}) \varphi = 0$$

since $|g_n| = 1$ on ω .

There is yet another representation formula for T :

Lemma 8. — *Let $g = (g_1, g_2) \in H^{1/2}(\Omega; \mathbf{R}^2)$. Then if $\omega \subset \Omega$ is diffeomorphic to a disc $\tilde{\omega}$ as in Remark 2.2, we have, $\forall \varphi \in C_0^\infty(\omega; \mathbf{R})$,*

$$(2.4) \quad \begin{aligned} \langle T(g), \varphi \rangle &= \langle \tilde{g}_1, (\tilde{g}_2 \tilde{\varphi}_y)_x - (\tilde{g}_2 \tilde{\varphi}_x)_y \rangle_{H^{1/2}, H^{-1/2}} \\ &\quad - \langle \tilde{g}_2, (\tilde{g}_1 \tilde{\varphi}_y)_x - (\tilde{g}_1 \tilde{\varphi}_x)_y \rangle_{H^{1/2}, H^{-1/2}}. \end{aligned}$$

Observe that, e.g. $\tilde{g}_2 \tilde{\varphi}_y \in H^{1/2}(\tilde{\omega})$, so that $(\tilde{g}_2 \tilde{\varphi}_x)_y \in H^{-1/2}(\tilde{\omega})$.

Proof of Lemma 8. — When g is smooth, (2.4) coincides with (2.3). The general case is obtained by approximation.

We now describe some elementary but useful facts about T and L :

Lemma 9. — *We have, for $g, h \in H^{1/2}(\Omega; \mathbf{R}^2)$, $\varphi \in \text{Lip}(\Omega; \mathbf{R})$,*

$$(2.5) \quad |\langle T(g) - T(h), \varphi \rangle| \leq C |g - h|_{H^{1/2}} (|g|_{H^{1/2}} + |h|_{H^{1/2}}) |\varphi|_{\text{Lip}},$$

$$(2.6) \quad |L(g) - L(h)| \leq C |g - h|_{H^{1/2}} (|g|_{H^{1/2}} + |h|_{H^{1/2}})$$

and, in particular,

$$L(g) \leq C |g|_{H^{1/2}}^2.$$

If, in addition, g and h are S^1 -valued, then

$$(2.7) \quad T(gh) = T(g) + T(h),$$

$$(2.8) \quad L(g\bar{h}) \leq C |g - h|_{H^{1/2}} (|g|_{H^{1/2}} + |h|_{H^{1/2}})$$

and

$$(2.9) \quad L(gh) \leq L(g) + L(h).$$

Here, we have identified \mathbf{R}^2 with \mathbf{C} and gh denotes complex multiplication, while $|\cdot|_{H^{1/2}}$ denotes the canonical seminorm on $H^{1/2}$:

$$|g|_{H^{1/2}}^2 = \int_{\Omega} \int_{\Omega} \frac{|g(x) - g(y)|^2}{d(x, y)^3} dx dy.$$

The constant C in this lemma depends only on Ω .

Proof. — Let $U, V \in H^1(G; \mathbf{R}^2)$ be the harmonic extensions of g , respectively h . Then clearly, $\forall \Phi \in \text{Lip}(G; \mathbf{R})$,

$$\begin{aligned} & \int_G H_U \cdot \nabla \Phi \\ & \leq \int_G H_V \cdot \nabla \Phi + C \|\nabla U - \nabla V\|_{L^2} (\|\nabla U\|_{L^2} + \|\nabla V\|_{L^2}) \|\nabla \Phi\|_{L^\infty}, \end{aligned}$$

so that (2.5) follows. Moreover, we find that

$$L(g) \leq L(h) + C|g - h|_{H^{1/2}} (|g|_{H^{1/2}} + |h|_{H^{1/2}}).$$

Reversing the roles of g and h , yields (2.6).

The proof of (2.7)–(2.9) relies on the following

Lemma 10. — For $g, h \in H^{1/2}(\Omega; \mathbf{R}^2) \cap L^\infty$, we have, $\forall \varphi \in C_0^\infty(\omega; \mathbf{R})$, with the same notation as in Lemma 8,

$$\begin{aligned} \langle T(gh), \varphi \rangle = & \langle |\tilde{h}|^2 \tilde{g}_1, (\tilde{g}_2 \tilde{\varphi}_y)_x - (\tilde{g}_2 \tilde{\varphi}_x)_y \rangle_{H^{1/2}, H^{-1/2}} \\ & - \langle |\tilde{h}|^2 \tilde{g}_2, (\tilde{g}_1 \tilde{\varphi}_y)_x - (\tilde{g}_1 \tilde{\varphi}_x)_y \rangle_{H^{1/2}, H^{-1/2}} \\ & + \langle |\tilde{g}|^2 \tilde{h}_1, (\tilde{h}_2 \varphi_y)_x - (\tilde{h}_2 \varphi_x)_y \rangle_{H^{1/2}, H^{-1/2}} \\ & - \langle |\tilde{g}|^2 \tilde{h}_2, (\tilde{h}_1 \varphi_y)_x - (\tilde{h}_1 \varphi_x)_y \rangle_{H^{1/2}, H^{-1/2}}. \end{aligned}$$

Note that the above equality makes sense since $H^{1/2} \cap L^\infty$ is an algebra.

Proof of Lemma 10. — When g and h are smooth, the above equality is clear by Lemma 8. The general case follows by approximation, using the fact that, if $g_n \rightarrow g$ in $H^{1/2}$, $h_n \rightarrow h$ in $H^{1/2}$, $\|g_n\|_{L^\infty} \leq C$, $\|h_n\|_{L^\infty} \leq C$, then $g_n h_n \rightarrow gh$ in $H^{1/2}$ (this is proved using dominated convergence).

Proof of Lemma 9 completed. — When $|g| = |h| = 1$, we find that $T(gh) = T(g) + T(h)$, by combining Lemma 8 and Lemma 10. Also in this case, we have

$$T(g\bar{h}) = T(g) + T(\bar{h}) = T(g) - T(h).$$

Using (2.5), we find that

$$L(g\bar{h}) = \sup_{|\varphi|_{\text{Lip}} \leq 1} \langle T(g) - T(h), \varphi \rangle \leq C|g - h|_{H^{1/2}} (|g|_{H^{1/2}} + |h|_{H^{1/2}}).$$

Finally, inequality (2.9) is a trivial consequence of (2.7).

Remark 2.3. — There is an alternative proof of (2.7)–(2.9), which consists of combining Lemma 2 (proved below) with the density result of T. Rivière [38]; see Lemma 11.

We now consider the special case where $g \in H^{1/2}(\Omega; S^1)$ is “smooth” except at a finite number of singularities:

Proof of Lemma 2. — The proof consists of 3 steps:

Step 1. — $\text{Supp } T(g) \subset \cup_{j=1}^k \{a_j\}$

This is a trivial consequence of Lemma 7.

Step 2. — $T(g) = \sum_{j=1}^k c_j \delta_{a_j}$.

In view of Remark 2.2 we may assume that Ω is flat near each a_j . We first note that, by a celebrated result of L. Schwartz, $T(g)$ is a finite sum of the form $T(g) = \sum_{j,\alpha} c_{j,\alpha} D^\alpha \delta_{a_j}$.

We want to prove that $c_{j,\alpha} = 0$ if $\alpha \neq 0$. For this purpose, it suffices to check that $\langle T(g), \varphi \rangle = 0$ if $\varphi(a_j) = 0, \forall j$. Let φ be any such function. Then, clearly, there is a sequence $(\varphi_n) \subset C_0^1(\Omega \setminus \cup_{j=1}^k \{a_j\})$ such that $\nabla \varphi_n \rightarrow \nabla \varphi$ a.e. and $\|\nabla \varphi_n\|_{L^\infty} \leq C$. Using Lemma 5, we obtain, by dominated convergence, that $\langle T(g), \varphi_n \rangle \rightarrow \langle T(g), \varphi \rangle$. On the other hand, $\langle T(g), \varphi_n \rangle = 0$ by Step 1.

Step 3. — We have $c_j = 2\pi d_j$ where $d_j = \text{deg}(g, a_j)$.

Let φ be a smooth function on Ω such that

$$\varphi(x) = \begin{cases} 1, & \text{for } |x - a_j| < R/2 \\ 0, & \text{for } |x - a_j| \geq R \end{cases},$$

where $R > 0$ is sufficiently small.

Note that $\nabla \varphi = 0$ outside the annulus $\mathcal{A} = \{x \in \Omega; |x - a_j| \in [R/2, R]\}$ and, moreover, that $g \in H^1$ on the same annulus. By Lemma 8 we find that

$$\langle T(g), \varphi \rangle = \int_{\mathcal{A}} g_1 [(g_2 \varphi_y)_x - (g_2 \varphi_x)_y] - \int_{\mathcal{A}} g_2 [(g_1 \varphi_y)_x - (g_1 \varphi_x)_y].$$

Integration by parts yields

$$\langle T(g), \varphi \rangle = \int_{\mathcal{A}} [(g_y \wedge g) \varphi_x + (g \wedge g_x) \varphi_y].$$

If g is smooth on \mathcal{A} , and if we integrate by parts once more, we find that

$$\langle T(g), \varphi \rangle = - \int_{\Sigma} (g_y \wedge g) \nu_x - \int_{\Sigma} (g \wedge g_x) \nu_y,$$

where $\Sigma = \{x \in \Omega; |x - a_j| = R/2\}$ and ν is the inward normal to \mathcal{A} on Σ . With τ the direct tangent vector on Σ , we have

$$-(g_y \wedge g) \nu_x - (g \wedge g_x) \nu_y = g \wedge g_\tau.$$

Since g is S^1 -valued, we find that

$$\langle T(g), \varphi \rangle = 2\pi \deg(g, a_j).$$

For a general $g \in H^1(\mathcal{A}; S^1)$, we use the fact that $C^\infty(\mathcal{A}; S^1)$ is dense in $H^1(\mathcal{A}; S^1)$ (see [41], [10] and [22]) and the stability of the degree under $H^{1/2}$ -convergence (see [17] and [22]), to conclude that $\langle T(g), \varphi \rangle = 2\pi \deg(g, a_j)$.

We now recall a useful density result due to T. Rivière, which is the $H^{1/2}$ analogue of a result of Bethuel and Zheng [10] concerning H^1 maps from B^3 to S^2 (see also a related result of Bethuel [4] concerning fractional Sobolev spaces).

Lemma 11 (Rivière [38]). — *Let \mathcal{R} denote the class of maps belonging to $W^{1,p}(\Omega; S^1)$, $\forall p < 2$, which are C^∞ on Ω except at a finite number of points. Then \mathcal{R} is dense in $H^{1/2}(\Omega; S^1)$.*

Remark 2.4. — The above assertion does not appear in Rivière [38] but it is implicit in his proof; for the convenience of the reader we present a simple proof in Remark 5.1 – see also Appendix B for a more precise statement.

Remark 2.5. — Similar density results hold in greater generality. Let $\Omega \subset \mathbf{R}^2$ be a smooth bounded domain. Let $0 < s < \infty$, $1 < p < \infty$ and

$$\mathcal{R}^{s,p} = \{u \in W^{s,p}(\Omega; S^1); u \text{ is } C^\infty \text{ except at a finite number of points}\}.$$

Then $\mathcal{R}^{s,p}$ is dense in $W^{s,p}(\Omega; S^1)$ for all values of s and p (see [16]); this extends earlier results in [10], [25] and [4].

The density result combined with Lemma 2 yields “concrete” representations of the distribution $T(g)$ and of the length of a minimal connection $L(g)$ for a general $g \in H^{1/2}(\Omega; S^1)$; this is the content of Theorem 1.

Proof of Theorem 1. — We start by recalling a result of Brezis, Coron and Lieb [19] (see also [18]).

Lemma 12 (Brezis, Coron and Lieb [19]). — Let (X, d) be a metric space. Let P_1, \dots, P_k , and N_1, \dots, N_k be two collections of k points in X . Then

$$L = \text{Min}_{\sigma \in S_k} \sum d(P_j, N_{\sigma(j)}) = \text{Max} \left\{ \sum_j (\varphi(P_j) - \varphi(N_j)); |\varphi|_{\text{Lip}} \leq 1 \right\},$$

where S_k denotes the group of permutation of $\{1, 2, \dots, k\}$.

The analogue of Lemma 12 for infinite sequences, which we need, is

Lemma 12'. — Let (X, d) be a metric space. Let $(P_i), (N_i)$ be two infinite sequences such that $\sum d(P_i, N_i) < \infty$.

Let

$$(2.10) \quad L = \text{Sup}_{\varphi} \left\{ \sum_i (\varphi(P_i) - \varphi(N_i)); |\varphi|_{\text{Lip}} \leq 1 \right\}.$$

Then

$$L = \text{Inf}_{(\tilde{N}_i)} \left\{ \sum_i d(P_i, \tilde{N}_i); \sum_i (\delta_{P_i} - \delta_{\tilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i}) \right\}.$$

Here, and throughout the rest of the paper, the equality

$$\sum_i (\delta_{\tilde{P}_i} - \delta_{\tilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i})$$

for sequences $(\tilde{P}_i), (\tilde{N}_i), (P_i), (N_i)$ such that

$$\sum_i d(\tilde{P}_i, \tilde{N}_i) < \infty \text{ and } \sum_i d(P_i, N_i) < \infty$$

means that

$$\sum_i (\varphi(\tilde{P}_i) - \varphi(\tilde{N}_i)) = \sum_i (\varphi(P_i) - \varphi(N_i)), \quad \forall \varphi \in \text{Lip}.$$

Remark 2.6. — A slightly different way of stating Lemma 12' is the following. Given sequences $(P_i), (N_i)$ in a metric space X with $\sum_i d(P_i, N_i) < \infty$, then

$$(2.10') \quad \begin{aligned} L &= \text{Inf}_{(\tilde{P}_i), (\tilde{N}_i)} \left\{ \sum_i d(\tilde{P}_i, \tilde{N}_i); \sum_i (\delta_{\tilde{P}_i} - \delta_{\tilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i}) \right\} \\ &= \text{Sup}_{\varphi} \left\{ \sum_i (\varphi(P_i) - \varphi(N_i)); \varphi \in \text{Lip}(X; \mathbf{R}) \text{ and } |\varphi|_{\text{Lip}} \leq 1 \right\}. \end{aligned}$$

It is easy to see that the supremum in (2.10') is always achieved. (Let (φ_n) be a maximizing sequence. By a diagonal process, we may assume that $\varphi_n(P_i)$ and $\varphi_n(N_i)$ con-

verge for every i to limits which define a function ψ_0 on the set $\{P_i, N_i, i = 1, 2, \dots\}$ with $|\psi_0|_{\text{Lip}} \leq 1$. Next, ψ_0 is defined on all of \mathbf{X} by a standard extension technique preserving the condition $|\psi|_{\text{Lip}} \leq 1$). A natural question is whether the infimum in (2.10') is achieved. The answer is negative. An interesting example, with $\mathbf{X} = [0, 1]$, has been constructed by A. Ponce [36].

Proof of Lemma 12'. — Let (\tilde{N}_i) be such that

$$\sum (\delta_{P_i} - \delta_{\tilde{N}_i}) = \sum (\delta_{P_i} - \delta_{N_i}).$$

Then

$$\sum_i (\varphi(P_i) - \varphi(N_i)) \leq \sum_i d(P_i, \tilde{N}_i)$$

and thus

$$L \leq \sum_i d(P_i, \tilde{N}_i).$$

Conversely, given $\varepsilon > 0$, we will construct a sequence (\tilde{N}_i) such that $\sum_i d(P_i, \tilde{N}_i) \leq L + \varepsilon$ and $\sum_i (\delta_{P_i} - \delta_{\tilde{N}_i}) = \sum_i (\delta_{P_i} - \delta_{N_i})$.

Let n_0 be such that $\sum_{j > n_0} d(P_j, N_j) < \varepsilon/2$. Let σ_0 be a permutation of the integers $\{1, 2, \dots, n_0\}$ which achieves

$$\text{Min}_{\sigma} \sum_{j=1}^{n_0} d(P_j, N_{\sigma(j)}).$$

Set

$$\tilde{N}_j = \begin{cases} N_{\sigma_0(j)}, & \text{for } 1 \leq j \leq n_0 \\ N_j, & \text{for } j > n_0 \end{cases}.$$

Clearly,

$$\sum_{j \geq 1} (\delta_{P_j} - \delta_{\tilde{N}_j}) = \sum_{j \geq 1} (\delta_{P_j} - \delta_{N_j}).$$

By definition of L , we have

$$\begin{aligned} L &= \text{Sup}_{|\varphi|_{\text{Lip}} \leq 1} \sum_{j \geq 1} (\varphi(P_j) - \varphi(N_j)) \\ &\geq \text{Max}_{|\varphi|_{\text{Lip}} \leq 1} \sum_{j=1}^{n_0} (\varphi(P_j) - \varphi(N_j)) - \varepsilon/2 \\ &= \sum_{j=1}^{n_0} d(P_j, \tilde{N}_j) - \varepsilon/2, \end{aligned}$$

by Lemma 12. Thus

$$\sum_{j \geq 1} d(\mathbf{P}_j, \tilde{\mathbf{N}}_j) \leq L + \varepsilon/2 + \varepsilon/2.$$

Proof of Theorem 1 continued. — For $g \in \mathcal{R}$ we have

$$L(g) = \sum_{j=1}^k d(\mathbf{P}_j, \mathbf{N}_j)$$

and

$$\langle T(g), \varphi \rangle = 2\pi \sum_{j=1}^k (\varphi(\mathbf{P}_j) - \varphi(\mathbf{N}_j))$$

for some suitable integer k depending on g and suitable points $\mathbf{P}_1, \dots, \mathbf{P}_k, \mathbf{N}_1, \dots, \mathbf{N}_k$ in Ω . Let now $g \in H^{1/2}(\Omega; S^1)$ and consider a sequence $(g_n) \subset \mathcal{R}$ such that $|g_n - g|_{H^{1/2}} \leq 1/2^n$.

By Lemma 2, $T(g_{n+1}) - T(g_n)$ is a finite sum of the form $2\pi \sum (\delta_{\mathbf{Q}_j} - \delta_{\mathbf{S}_j})$. By Lemma 12, after relabeling the points (\mathbf{Q}_j) and (\mathbf{S}_j) , we may assume that

$$T(g_1) = 2\pi \sum_{j=1}^{k_1} (\delta_{\mathbf{P}_j} - \delta_{\mathbf{N}_j})$$

and

$$T(g_{n+1}) - T(g_n) = 2\pi \sum_{j=k_n+1}^{k_{n+1}} (\delta_{\mathbf{P}_j} - \delta_{\mathbf{N}_j}), \forall n \geq 1$$

with

$$\begin{aligned} 2\pi \sum_{k_n+1}^{k_{n+1}} d(\mathbf{P}_j, \mathbf{N}_j) &= \text{Sup} \{ \langle T(g_{n+1}) - T(g_n), \varphi \rangle; \\ &\varphi \in \text{Lip}(\Omega; \mathbf{R}), |\varphi|_{\text{Lip}} \leq 1 \} \\ &\leq C |g_{n+1} - g_n|_{H^{1/2}} (|g_{n+1}|_{H^{1/2}} + |g_n|_{H^{1/2}}) \leq C/2^n \text{ (by (2.5)).} \end{aligned}$$

We find that $T(g_n) = 2\pi \sum_{j=1}^{k_n} (\delta_{\mathbf{P}_j} - \delta_{\mathbf{N}_j})$ and that $\sum_{j \geq 1} d(\mathbf{P}_j, \mathbf{N}_j) < \infty$.

Then for every $\varphi \in \text{Lip}(\Omega; \mathbf{R})$, the sequence $(\langle T(g_n), \varphi \rangle)$ converges to $2\pi \sum_{j \geq 1} (\varphi(\mathbf{P}_j) - \varphi(\mathbf{N}_j))$. By Lemma 9, we find that $T(g) = 2\pi \sum_{j \geq 1} (\delta_{\mathbf{P}_j} - \delta_{\mathbf{N}_j})$.

The second assertion in Theorem 1 is an immediate consequence of Lemma 12' and Remark 2.6.

The last property in Theorem 1, namely the fact that, if $T(g)$ is a measure, then $T(g)$ may be represented as a *finite* sum of the form $2\pi \sum_j (\delta_{P_j} - \delta_{N_j})$, was originally announced in [13] and established using a technique of Jerrard and Soner [31], [32], which was based on the (Jacobian) structure of $T(g)$. We do not reproduce this argument since Smets [43] has proved the following general result:

Theorem 10 (Smets [43]). — Let X be a compact metric space and let $(P_j), (N_j) \subset X$ be infinite sequences such that $\sum d(P_j, N_j) < \infty$. Assume that

$$\left| \sum_j (\varphi(P_j) - \varphi(N_j)) \right| \leq C \sup_{x \in X} |\varphi(x)|, \quad \forall \varphi \in \text{Lip}(X).$$

Then one may find two finite collections of points (Q_1, \dots, Q_k) and (M_1, \dots, M_k) , such that

$$\sum_{j=1}^{\infty} (\varphi(P_j) - \varphi(N_j)) = \sum_{i=1}^k (\varphi(Q_i) - \varphi(M_i)), \quad \forall \varphi \in \text{Lip}(X).$$

We refer to [43] and to [36] for more general results.

Remark 2.7. — A final word about the possibility of defining a minimal connection $L(g)$ when $g \in W^{s,p}(\Omega; S^1)$, for $0 < s < \infty$ and $1 \leq p < \infty$. Recall (see [16] and Remark 2.5) that $\mathcal{R}^{s,p}$ is always dense in $W^{s,p}(\Omega; S^1)$ and note that we may always define $L(g)$ for $g \in \mathcal{R}^{s,p}$. A natural question is whether there is a continuous extension of L to $W^{s,p}$:

a) When $sp < 1$, the answer is negative. Indeed, let $g \in \mathcal{R}^{s,p}$ be a map with singularities of nonzero degree, so that $L(g) > 0$. There is a sequence (g_n) in $C^\infty(\Omega; S^1)$ such that $g_n \rightarrow g$ in $W^{s,p}$ (see Escobedo [25]). Clearly, $L(g_n) = 0$, $\forall n$, and $L(g_n)$ does not converge to $L(g)$.

b) When $sp \geq 2$, the answer is positive since $L(g) = 0$, $\forall g \in \mathcal{R}^{s,p}$ (any singularity in $W^{s,p}$ must have zero degree since $W^{s,p} \subset \text{VMO}$).

c) When $1 \leq sp < 2$, the answer is positive. For $s > 1/2$ the proof is easy (indeed if $s \in (1/2, 1)$, then $W^{s,p}(\Omega; S^1) \subset H^{1/2}$, while if $s \geq 1$, then $W^{s,p} \subset W^{1,1}$ and we may apply the result of Demengel [24] which asserts the existence of a minimal connection in $W^{1,1}$). The case where $s \leq 1/2$ is delicate and studied in [16].

3. Lifting for $g \in Y$. Characterization of Y . Proof of Theorem 3

The main ingredient in this Section is the following estimate, whose proof has already been presented in Bourgain-Brezis [11]. We reproduce it here for the convenience of the reader.

Theorem 3'. — Let ψ be a smooth real-valued function on the d -dimensional torus \mathbf{T}^d and set $g = e^{i\psi}$. Then

$$(3.1) \quad |\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}} \leq C(d)(1 + |g|_{\mathbf{H}^{1/2}})|g|_{\mathbf{H}^{1/2}}.$$

Here, $|\cdot|$ denotes the canonical seminorm on $\mathbf{H}^{1/2}$ (respectively $\mathbf{H}^{1/2} + \mathbf{W}^{1,1}$).

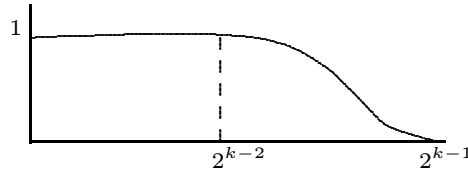
Proof of Theorem 3'. — Write $g - \mathcal{F}g$ as a Fourier series,

$$g - \mathcal{F}g = \sum_{\xi \in \mathbf{Z}^d \setminus \{0\}} \hat{g}(\xi) e^{ix \cdot \xi}.$$

The $\mathbf{H}^{1/2}$ -component in the decomposition of ψ will be obtained as a paraproduct of $g - \mathcal{F}g$ and $\bar{g} - \mathcal{F}\bar{g}$. Let

$$(3.2) \quad \mathbf{P} = \sum_k \left[\sum_{\xi_2} \lambda_k(|\xi_2|) \overline{\hat{g}(\xi_2)} e^{-ix \cdot \xi_2} \right] \left[\sum_{2^k \leq |\xi_1| < 2^{k+1}} \hat{g}(\xi_1) e^{ix \cdot \xi_1} \right],$$

where, for each k , we let $0 \leq \lambda_k \leq 1$ be a smooth function on \mathbf{R}_+ as below:



We claim that

$$(3.3) \quad |\mathbf{P}|_{\mathbf{H}^{1/2}} \leq C \|g\|_{\infty} |g|_{\mathbf{H}^{1/2}}$$

and

$$(3.4) \quad |\psi - \frac{1}{i} \mathbf{P}|_{\mathbf{W}^{1,1}} \leq C |g|_{\mathbf{H}^{1/2}}^2.$$

Proof of (3.3). — This is totally obvious from the construction since, with $\|\cdot\|_p$ standing for the L^p -norm, we have

$$(3.5) \quad \begin{aligned} |\mathbf{P}|_{\mathbf{H}^{1/2}}^2 &\sim \sum_k 2^k \left\| \left[\sum_{\xi_2} \lambda_k(|\xi_2|) \overline{\hat{g}(\xi_2)} e^{-ix \cdot \xi_2} \right] \left[\sum_{2^k \leq |\xi_1| < 2^{k+1}} \hat{g}(\xi_1) e^{ix \cdot \xi_1} \right] \right\|_2^2 \\ &\leq \sum_k 2^k \left\| \sum_{\xi} \lambda_k(|\xi|) \overline{\hat{g}(\xi)} e^{-ix \cdot \xi} \right\|_{\infty}^2 \left[\sum_{|\xi| \sim 2^k} |\hat{g}(\xi)|^2 \right] \\ &\leq C \|g\|_{\infty}^2 |g|_{\mathbf{H}^{1/2}}^2. \end{aligned}$$

Proof of (3.4). — We estimate, for instance,

$$(3.6) \quad \left\| \partial_1 \psi - \frac{1}{i} \partial_1 \mathbf{P} \right\|_{L^1}.$$

Thus, letting $\xi = (\xi^1, \dots, \xi^d) \in \mathbf{Z}^d$, we have

$$(3.7) \quad \partial_1 \psi = \frac{1}{i} \bar{g} \partial_1 g = \sum_{\xi_1, \xi_2 \in \mathbf{Z}^d} \xi_1^1 \hat{g}(\xi_1) \overline{\hat{g}(\xi_2)} e^{ix \cdot (\xi_1 - \xi_2)}$$

and, by (3.2), we find

$$(3.8) \quad \frac{1}{i} \partial_1 \mathbf{P} = \sum_k \sum_{\substack{2^k \leq |\xi_1| < 2^{k+1} \\ \xi_2 \in \mathbf{Z}^d}} (\xi_1^1 - \xi_2^1) \lambda_k(|\xi_2|) \hat{g}(\xi_1) \overline{\hat{g}(\xi_2)} e^{ix \cdot (\xi_1 - \xi_2)}$$

and

$$(3.9) \quad \partial_1 \psi - \frac{1}{i} \partial_1 \mathbf{P} = \sum_k \sum_{\substack{2^k \leq |\xi_1| < 2^{k+1} \\ \xi_2 \in \mathbf{Z}^d}} m_k(\xi_1, \xi_2) \hat{g}(\xi_1) \overline{\hat{g}(\xi_2)} e^{ix \cdot (\xi_1 - \xi_2)}.$$

Here, by definition of λ_k ,

$$(3.10) \quad m_k(\xi_1, \xi_2) = \xi_1^1 - \lambda_k(|\xi_2|) (\xi_1^1 - \xi_2^1) = \begin{cases} \xi_2^1, & \text{if } |\xi_2| \leq 2^{k-2} \\ \xi_1^1, & \text{if } |\xi_2| \geq 2^{k-1} \end{cases}.$$

Estimate

$$(3.11) \quad \left\| \partial_1 \psi - \frac{1}{i} \partial_1 \mathbf{P} \right\|_1 \leq \sum_{k_1, k_2} \left\| \sum_{|\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}} m_{k_1}(\xi_1, \xi_2) \hat{g}(\xi_1) \overline{\hat{g}(\xi_2)} e^{ix \cdot (\xi_1 - \xi_2)} \right\|_1.$$

We split the right-hand side of (3.11) as

$$\sum_{k_1 \sim k_2} + \sum_{k_1 < k_2 - 4} + \sum_{k_1 > k_2 + 4} = (3.12) + (3.13) + (3.14).$$

Clearly, $2^{-k} m_k(\xi_1, \xi_2)$ restricted to $[|\xi_1| \sim 2^k] \times [|\xi_2| \sim 2^k]$ is a smooth multiplier satisfying the usual derivative bounds. Therefore,

$$(3.15) \quad (3.12) \leq C \sum_k 2^k \left\| \sum_{|\xi_1| \sim 2^k} \hat{g}(\xi_1) e^{ix \cdot \xi_1} \right\|_2 \left\| \sum_{|\xi_2| \sim 2^k} \hat{g}(\xi_2) e^{ix \cdot \xi_2} \right\|_2 \sim |g|_{H^{1/2}}^2.$$

If $k_1 < k_2 - 4$, then $|\xi_2| > 2^{k_1}$ and $m_{k_1}(\xi_1, \xi_2) = \xi_1^1$, by (3.10). Therefore

$$\begin{aligned}
(3.13) &= \sum_{k_1 < k_2 - 4} \left\| \sum_{|\xi_1| \sim 2^{k_1}, |\xi_2| \sim 2^{k_2}} \xi_1^1 \hat{g}(\xi_1) \overline{\hat{g}(\xi_2)} e^{ix \cdot (\xi_1 - \xi_2)} \right\|_1 \\
(3.16) &\leq \sum_{k_1 < k_2 - 4} 2^{k_1} \left\| \sum_{|\xi_1| \sim 2^{k_1}} \hat{g}(\xi_1) e^{ix \cdot \xi_1} \right\|_2 \cdot \left\| \sum_{|\xi_2| \sim 2^{k_2}} \hat{g}(\xi_2) e^{ix \cdot \xi_2} \right\|_2 \\
&\leq \sum_{k_1 < k_2} 2^{k_1} \left(\sum_{|\xi_1| < 2^{k_1}} |\hat{g}(\xi_1)|^2 \right)^{1/2} \left(\sum_{|\xi_2| \sim 2^{k_2}} |\hat{g}(\xi_2)|^2 \right)^{1/2} \leq C |g|_{\mathbf{H}^{1/2}}^2.
\end{aligned}$$

If $k_1 > k_2 + 4$, then $|\xi_2| < 2^{k_1 - 2}$ and $m_{k_1}(\xi_1, \xi_2) = \xi_2^1$ and the bound on (3.14) is similar.

We now derive a consequence of Theorem 3':

Corollary 1. — Let G be a smooth bounded domain in \mathbf{R}^{d+1} such that $\Omega = \partial G$ is connected. Let ψ be a Lipschitz real-valued function on Ω and set $g = e^{i\psi}$. Then

$$|\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}} \leq C_\Omega (1 + |g|_{\mathbf{H}^{1/2}}) |g|_{\mathbf{H}^{1/2}}.$$

Proof of Corollary 1. — It is convenient to divide the argument into 4 steps.

Step 1. — The conclusion of Theorem 3' still holds if ψ is Lipschitz. This is clear by density.

Step 2. — The conclusion of Theorem 3' holds if \mathbf{T}^d is replaced by a d -dimensional cube Q and $\psi \in \text{Lip}(Q)$. This is done by standard reflections and extensions by periodicity.

As a consequence, we have

Step 3. — The conclusion of Step 2 holds when Q is replaced by a domain in Ω diffeomorphic to a cube.

Step 4. — Proof of Corollary 1. Consider a finite covering (U_α) of Ω by domains diffeomorphic to cubes. Note that, if $U_\alpha \cap U_\beta \neq \emptyset$, then

$$|\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}(U_\alpha \cup U_\beta)} \sim |\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}(U_\alpha)} + |\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}(U_\beta)}.$$

Using the connectedness of Ω , we find that

$$|\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}(\Omega)} \sim \sum_{\alpha} |\psi|_{\mathbf{H}^{1/2} + \mathbf{W}^{1,1}(U_\alpha)}.$$

The conclusion now follows from Step 3.

Proof of Theorem 3. — First, let $g \in Y$ and consider a sequence $(g_n) \subset C^\infty(\Omega; S^1)$ such that $g_n \rightarrow g$ in $H^{1/2}$. Since Ω is simply connected, we may write $g_n = e^{i\psi_n}$, with $\psi_n \in C^\infty(\Omega; \mathbf{R})$.

Applying Corollary 1 to $g_n \bar{g}_m$, we find

$$|\psi_n - \psi_m|_{H^{1/2}+W^{1,1}} \leq C(1 + |g_n \bar{g}_m|_{H^{1/2}})|g_n \bar{g}_m|_{H^{1/2}}.$$

Since $g_n \rightarrow g$ in $H^{1/2}$ and $|g_n| \equiv 1$, we have $|g_n \bar{g}_m|_{H^{1/2}} \rightarrow 0$ as $m, n \rightarrow \infty$ (see the proof of Lemma 10). Therefore, $(\psi_n - \int_\Omega \psi_n)$ converges in $H^{1/2} + W^{1,1}$ to a map ζ . Then, with C an appropriate constant, $\psi = \zeta + C \in H^{1/2} + W^{1,1}$, $g = e^{i\psi}$ and ψ satisfies the estimate

$$|\psi|_{H^{1/2}+W^{1,1}} \leq C(1 + |g|_{H^{1/2}})|g|_{H^{1/2}}.$$

The uniqueness of ψ is an immediate consequence of the following

Lemma 13. — *Let Ω be a connected open set in \mathbf{R}^d . Let $f : \Omega \rightarrow \mathbf{Z}$ be such that $f = f_0 + \sum_j f_j$, with $f_0 \in W_{\text{loc}}^{1,1}(\Omega; \mathbf{R})$ and $f_j \in W_{\text{loc}}^{s_j, p_j}(\Omega; \mathbf{R})$, where $0 < s_j < 1$, $1 < p_j < \infty$, $s_j p_j \geq 1$. Then f is a constant.*

The proof of Lemma 13 is given in [12], Appendix B, Step 2. The argument is by dimensional reduction, observing that the restriction of f to almost every line is \mathbf{Z} -valued and VMO; thus it is constant (see [22]). This implies (see e.g. Lemma 2 in [20]) that f is locally constant in Ω .

We now prove the last assertion in Theorem 3. Let $g \in H^{1/2}(\Omega; S^1)$ be such that $g = e^{i\psi}$ for some $\psi \in H^{1/2} + W^{1,1}(\Omega; \mathbf{R})$. Let $\psi = \psi_1 + \psi_2$, with $\psi_1 \in H^{1/2}$ and $\psi_2 \in W^{1,1}$. Set $g_j = e^{i\psi_j}$, $j = 1, 2$. Clearly, $g_1 \in X$, so that $g_1 \in Y$ and thus $T(g_1) = 0$. On the other hand, $g_2 \in H^{1/2} \cap W^{1,1}$, since $g_2 = g \bar{g}_1 \in H^{1/2}$. Therefore, we may use the representation of $T(g_2)$ given by Lemma 1 and find, after localization, as in Remark 2.2,

$$\langle T(g_2), \varphi \rangle = \int_\omega (\psi_{2x} \varphi_y - \psi_{2y} \varphi_x) = 0, \quad \forall \varphi \in C_0^1(\omega; \mathbf{R}).$$

Hence $T(g_2) = 0$. By (2.7) in Lemma 9, we obtain that $T(g) = 0$. Using Theorem 2, we derive that $g \in Y$.

Remark 3.1. — Theorem 3 is not fully satisfactory since, whenever $\psi \in W^{1,1}$, the function $e^{i\psi}$ need not belong to $H^{1/2}$ (but “almost”, since $e^{i\psi} \in W^{1,1} \cap L^\infty$, which is almost contained in $H^{1/2}$, but not quite). Here is an example: take some $\psi \in W^{1,1} \cap L^\infty$ with $\psi \notin H^{1/2}$. We may assume $|\psi| \leq 1$. Then

$$|e^{i\psi(x)} - e^{i\psi(y)}| \sim |\psi(x) - \psi(y)|,$$

so that

$$|e^{i\psi}|_{H^{1/2}} \sim |\psi|_{H^{1/2}} = +\infty.$$

4. Lifting for a general $g \in H^{1/2}$. Optimizing the BV part of the phase. Proof of Theorems 4 and 5

Assume g is a general element in $H^{1/2}(\Omega; S^1)$. This g need not be in Y and thus need not have a lifting in $H^{1/2} + W^{1,1}$. However, g has a lifting in the larger space $H^{1/2} + BV$. This is an immediate consequence of Theorem 3 (and estimate (1.9)) and of the following result of T. Rivière [38] (which is the analogue of a similar result of Bethuel [3] for H^1 maps from B^3 to S^2).

Lemma 14 (Rivière [38]). — *Let $g \in H^{1/2}(\Omega; S^1)$. Then there is a sequence $(g_n) \subset C^\infty(\Omega; S^1)$ such that $g_n \rightharpoonup g$ weakly in $H^{1/2}$.*

Remark 4.1. — Lemma 14 implies that $g \mapsto T(g)$ and $g \mapsto L(g)$ are not continuous under weak $H^{1/2}$ convergence.

Here is a refined version of Lemma 14 which will be proved at the end of Section 4.2:

Lemma 14'. — *Let $g \in H^{1/2}(\Omega; S^1)$. Then there is a sequence $(g_n) \subset C^\infty(\Omega; S^1)$ such that $g_n \rightharpoonup g$ weakly in $H^{1/2}$ and*

$$\limsup_{n \rightarrow \infty} |g_n|_{H^{1/2}}^2 \leq |g|_{H^{1/2}}^2 + C_\Omega L(g),$$

for some constant C_Ω depending only on Ω . Moreover, for **every** sequence (g_n) in Y such that $g_n \rightarrow g$ a.e., we have

$$\liminf_{n \rightarrow \infty} |g_n|_{H^{1/2}}^2 \geq |g|_{H^{1/2}}^2 + C'_\Omega L(g),$$

for some positive constant C'_Ω depending only on Ω .

Existence of a lifting in $H^{1/2} + BV$

Let $g \in H^{1/2}(\Omega; S^1)$. For g_n as in the above Lemma 14, write, using Corollary 1, $g_n = e^{i\varphi_n}$, with $\varphi_n \in C^\infty(\Omega; S^1)$ and

$$|\varphi_n|_{H^{1/2} + W^{1,1}} \leq C_\Omega (|g_n|_{H^{1/2}} + |g_n|_{H^{1/2}}^2).$$

Then, up to a subsequence, there is some $\zeta \in H^{1/2} + BV$ such that $\varphi_n - f \varphi_n \rightarrow \zeta$ a.e. We find that $g = e^{i\varphi}$, with $\varphi = \zeta + C$ and C some appropriate constant. Moreover, we may write $\varphi = \varphi_1 + \varphi_2$, with

$$(4.1) \quad |\varphi_1|_{H^{1/2}} + |\varphi_2|_{BV} \leq C_\Omega (|g|_{H^{1/2}} + |g|_{H^{1/2}}^2).$$

An additional information about the decomposition is contained in Theorem 4. On the other hand note that estimate (4.1) implies that every $g \in H^{1/2}$ may be written as $g = g_1 g_2$, with

$$g_1 = e^{i\varphi_1} \in X \text{ and } g_2 = e^{i\varphi_2} \in H^{1/2} \cap BV,$$

$$\text{i.e., } H^{1/2} = (X) \cdot (H^{1/2} \cap BV).$$

A finer assertion is $H^{1/2} = (X) \cdot (H^{1/2} \cap W^{1,1})$, which is the content of Theorem 5.

The proofs of Theorems 4 and 5 require a number of ingredients:

a) the dipole construction (*see* Section 4.1). This is inspired by the dipole construction in the $H^1(\mathbf{B}^3; \mathbf{S}^2)$ context (*see* [19] and [3]);

b) the construction of a map $g \in H^{1/2}(\Omega; \mathbf{S}^1) \cap W^{1,1}$ having *prescribed* singularities (with control of the norms). This is done in Section 4.2;

c) lower bound estimates for the BV part of the phase, which are presented in Section 4.3, in the spirit of [19], [2], [27]. This is a typical phenomenon in the context of relaxed energies and/or Cartesian Currents. More precisely, if one considers the Sobolev space $X = W^{s,p}(\mathbf{U}; \mathbf{S}^k)$, $\mathbf{U} \subset \mathbf{R}^N$, and if smooth maps are *not* dense in X for the strong topology, then the relaxed energy is defined by

$$E(g) = \text{Inf} \left\{ \liminf_{n \rightarrow \infty} \|g_n\|_{W^{s,p}}^p; (g_n) \subset C^\infty(\bar{\mathbf{U}}; \mathbf{S}^k), g_n \rightarrow g \text{ a.e.} \right\}.$$

The gap $E(g) - \|g\|_{W^{s,p}}^p \geq 0$ has often a geometrical interpretation in terms of the singular set of g . For example, in the $H^1(\mathbf{B}^3; \mathbf{S}^2)$ context, the gap is $8\pi L(g)$, where $L(g)$ is the length of a minimal connection associated with the singularities of g (*see* [19]). We will consider, in Section 4.3, similar lower bounds for \mathbf{S}^1 -valued maps on Ω .

4.1. The dipole construction

Throughout this section, the metric d denotes the geodesic distance d_Ω in Ω and $L(g) = L_\Omega(g)$.

Lemma 15. — *Let $P, N \in \Omega$, $P \neq N$. Given any $\varepsilon > 0$ there exists some $g(= g_\varepsilon)$ such that*

$$(4.2) \quad g \in W_{\text{loc}}^{1,\infty}(\Omega \setminus \{P, N\}; \mathbf{S}^1) \cap W^{1,p}(\Omega; \mathbf{S}^1), \forall p \in [1, 2),$$

$$(4.3) \quad T(g) = 2\pi(\delta_P - \delta_N),$$

$$(4.4) \quad |g|_{W^{1,1}} \leq 2\pi d(P, N) + \varepsilon,$$

$$(4.5) \quad |g|_{H^{1/2}}^2 \leq C_\Omega d(P, N) \quad \text{where } C_\Omega \text{ depends only on } \Omega,$$

$$(4.6) \quad \begin{cases} \text{there is a function } \psi (= \psi_\varepsilon) \in \text{BV}(\Omega; \mathbf{R}) \text{ such that } g = e^{i\psi}, \\ \text{with } \text{supp } \psi \subset \Lambda = \{x \in \Omega; d(x, \gamma) < \varepsilon\} \text{ and } |\psi|_{\text{BV}} \leq 4\pi d(P, N) + \varepsilon, \end{cases}$$

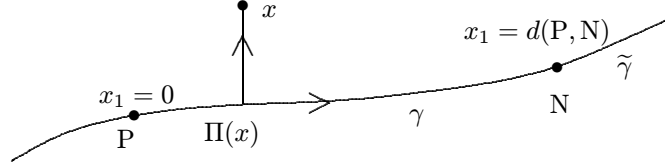
where γ is a geodesic curve joining P and N ,

$$(4.7) \quad g = 1 \text{ outside } \Lambda.$$

Proof. — Extend γ smoothly beyond P and N ; denote this extension by $\tilde{\gamma}$. For $\varepsilon_0 > 0$ sufficiently small (depending on $\tilde{\gamma}$), the projection Π of

$$\Gamma = \{x \in \Omega; d(x, \gamma) < \varepsilon_0\}$$

onto $\tilde{\gamma}$ is well-defined and smooth. Let x_1 be the arclength coordinate on $\tilde{\gamma}$, such that $x_1(P) = 0$, $x_1(N) = d(P, N) = L$.



For $x \in \Gamma$, let $x_1 = x_1(\Pi(x))$ be the arclength coordinate of $\Pi(x)$ on $\tilde{\gamma}$ and let $x_2 = \pm d(x, \tilde{\gamma})$, where we choose “+” if the basis formed by the (oriented) tangent vector at $\Pi(x)$ to $\tilde{\gamma}$, the (oriented) tangent vector at $\Pi(x)$ to the geodesic segment $[\Pi(x), x]$ and the exterior normal n at $\Pi(x)$ to G is direct in \mathbf{R}^3 ; we choose “−” otherwise. Define the mapping

$$x \in \Gamma \mapsto \Phi(x) = (x_1, x_2) \in \mathbf{R}^2.$$

Let $0 < \delta < \varepsilon_0$ and consider the domain in \mathbf{R}^2

$$\tilde{\Gamma}_\delta = \left\{ (t_1, t_2) \in \mathbf{R}^2; 0 < t_1 < L \text{ and } |t_2| < \frac{2\delta}{L} \min(t_1, L - t_1) \right\}.$$

and the corresponding domain Γ_δ in Ω ,

$$\Gamma_\delta = \{x \in \Gamma; \Phi(x) \in \tilde{\Gamma}_\delta\}.$$

Set, on \mathbf{R}^2 ,

$$\tilde{g}(t) = \tilde{g}(t_1, t_2) = \begin{cases} \exp(it\varphi(Lt_2/2\delta \min(t_1, L - t_1))), & \text{on } \tilde{\Gamma}_\delta, \\ 1, & \text{outside } \tilde{\Gamma}_\delta, \end{cases}$$

where φ is defined by $\varphi(s) = \begin{cases} \pi(s+1)^+, & \text{if } s \leq 1 \\ 2\pi, & \text{if } s > 1 \end{cases}$.

An easy computation shows that

$$\tilde{g} \in W_{\text{loc}}^{1,\infty}(\mathbf{R}^2 \setminus \{\tilde{\mathbf{P}}, \tilde{\mathbf{N}}\}; S^1) \cap W_{\text{loc}}^{1,p}(\mathbf{R}^2; S^1), \quad \forall 1 \leq p < 2,$$

where $\tilde{\mathbf{P}} = \Phi(\mathbf{P}) = (0, 0)$ and $\tilde{\mathbf{N}} = \Phi(\mathbf{N}) = (\mathbf{L}, 0)$. More precisely, we have

$$|\tilde{g}|_{W^{1,p}(\tilde{\Gamma}_\delta)}^p = 4 \int_0^{\mathbf{L}/2} \left(\frac{\mathbf{L}}{2\delta t_1} \right)^{p-1} dt_1 \int_0^{+1} \pi^p \left(\left(\frac{2\delta s}{\mathbf{L}} \right)^2 + 1 \right)^{p/2} ds.$$

In particular, we find

$$(4.8) \quad |\tilde{g}|_{W^{1,1}(\tilde{\Gamma}_\delta)} \leq 2\pi(\mathbf{L} + \delta)$$

and, for every $1 \leq p < 2$,

$$(4.9) \quad |\tilde{g}|_{W^{1,p}(\tilde{\Gamma}_\delta)} \leq C_p(\mathbf{L}\delta)^{1/p} \left(\frac{1}{\delta} + \frac{1}{\mathbf{L}} \right).$$

For later purpose, it is also convenient to observe that, for any $1 \leq q \leq \infty$,

$$(4.10) \quad \|\tilde{g} - 1\|_{L^q(\tilde{\Gamma}_\delta)} \leq 2(\mathbf{L}\delta)^{1/q}.$$

We now transport the function \tilde{g} on Ω and define

$$g(x) = \begin{cases} \tilde{g}(\Phi(x)), & \text{if } x \in \Gamma_\delta \\ 1, & \text{outside } \Gamma_\delta \end{cases}.$$

It is not difficult to see that Φ is a C²-diffeomorphism on Γ and

$$(4.11) \quad |\text{Jac } \Phi(x) - 1| \leq C_\gamma \delta \quad \text{on } \Gamma_\delta,$$

where C_γ is a constant depending on γ . Combining (4.8)–(4.11) yields

$$(4.12) \quad |g|_{W^{1,1}(\Omega)} \leq 2\pi(\mathbf{L} + \delta)(1 + C_\gamma \delta),$$

$$(4.13) \quad |g|_{W^{1,p}(\Omega)} \leq C_p(\mathbf{L}\delta)^{1/p} \left(\frac{1}{\delta} + \frac{1}{\mathbf{L}} \right) (1 + C_\gamma \delta), \quad 1 \leq p < 2,$$

and

$$(4.14) \quad \|g - 1\|_{L^q(\Omega)} \leq 2(\mathbf{L}\delta)^{1/q}(1 + C_\gamma \delta).$$

From a variant of the Gagliardo–Nirenberg inequality (*see e.g.* [21] and the references therein) we know that, if $1 < p < \infty$ and

$$(4.15) \quad \frac{1}{p} + \frac{1}{q} = 1,$$

then

$$(4.16) \quad |g|_{H^{1/2}(\Omega)}^2 \leq C(p, \Omega) |g|_{W^{1,p}(\Omega)} \|g\|_{L^q(\Omega)}.$$

We now check properties (4.2)–(4.7): (4.2), (4.3) and (4.7) are clear. Estimate (4.4) (resp. (4.5)) follows from (4.12) (resp. (4.16) applied e.g. with $p = 3/2$) provided δ is sufficiently small (depending on ε and γ).

Construction of ψ and estimate (4.6)

In the region where $\tilde{g} \equiv 1$, we take $\tilde{\psi} \equiv 0$. In the region $\tilde{\Gamma}_\delta$ where \tilde{g} lives, we take

$$\tilde{\psi}(t_1, t_2) = \begin{cases} \varphi(Lt_2/2\delta \min(t_1, L - t_1)), & \text{if } t_2 \leq 0 \\ \varphi(Lt_2/2\delta \min(t_1, L - t_1)) - 2\pi, & \text{if } t_2 > 0 \end{cases}.$$

Set

$$\psi(x) = \begin{cases} \tilde{\psi}(\Phi(x)), & \text{if } x \in \Gamma_\delta \\ 0, & \text{outside } \Gamma_\delta \end{cases}.$$

Then $|D\psi| = |Dg| + 2\pi\delta_\gamma$, where δ_γ is the $1 - d$ Hausdorff measure uniformly distributed on γ . Thus

$$|\psi|_{\text{BV}} = \int_{\Omega} |D\psi| = \int_{\Omega} |Dg| + 2\pi L \leq 4\pi L + \varepsilon.$$

4.2. Construction of a map with prescribed singularities

Let $(P_i), (N_i)$ be two sequences of points in $\Omega = \partial G$ such that $\sum d_\Omega(P_i, N_i) < \infty$. Define

$$T = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i})$$

and

$$L = L_\Omega = \frac{1}{2\pi} \sup\{\langle T, \varphi \rangle; \varphi \in \text{Lip}(\Omega; \mathbf{R}), |\varphi|_{\text{Lip}} \leq 1\}.$$

Lemma 16. — a) For every $g \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$ such that $T(g) = T$, we have

$$\int_{\Omega} |Dg| \geq 2\pi L \quad \text{and} \quad |g|_{H^{1/2}}^2 \geq C_\Omega L,$$

where C_Ω is a positive constant depending only on Ω .

b) For every $\varepsilon > 0$, there is some $g(= g_\varepsilon) \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$ such that

$$(4.17) \quad T(g) = T,$$

$$(4.18) \quad |g|_{W^{1,1}} \leq 2\pi(L + \varepsilon),$$

$$(4.19) \quad |g|_{H^{1/2}}^2 \leq C_\Omega L,$$

$$(4.20) \quad \left\{ \begin{array}{l} \text{there is a function } \psi (= \psi_\varepsilon) \in \text{BV}(\Omega; \mathbf{R}) \text{ such that} \\ g = e^{i\psi}, \text{ and } |\psi|_{\text{BV}} \leq 4\pi(L + \varepsilon) \end{array} \right. ,$$

$$(4.21) \quad \text{meas}(\text{Supp } \psi) = \text{meas}(\text{Supp}(g - 1)) \leq \varepsilon.$$

In the proof of Lemma 16 we will use:

Lemma 17. — *Let (u_n) be a bounded sequence in $H^{1/2}(\Omega; \mathbf{C}) \cap L^\infty$ such that $u_n \rightarrow 1$ a.e. Then for every $v \in H^{1/2}(\Omega; \mathbf{C}) \cap L^\infty$ we have*

$$|u_n v|_{H^{1/2}}^2 = \iint_{\Omega \times \Omega} |v(x)|^2 \frac{|u_n(x) - u_n(y)|^2}{d(x, y)^3} + |v|_{H^{1/2}}^2 + o(1) \quad \text{as } n \rightarrow \infty.$$

Proof of Lemma 17. — We have

$$\begin{aligned} |u_n v|_{H^{1/2}}^2 &= \iint_{\Omega \times \Omega} |v(x)|^2 \frac{|u_n(x) - u_n(y)|^2}{d(x, y)^3} + \iint_{\Omega \times \Omega} |u_n(y)|^2 \frac{|v(x) - v(y)|^2}{d(x, y)^3} + 2I_n \\ &= \iint_{\Omega \times \Omega} |v(x)|^2 \frac{|u_n(x) - u_n(y)|^2}{d(x, y)^3} + |v|_{H^{1/2}}^2 + 2I_n + o(1), \end{aligned}$$

where

$$I_n = \iint_{\Omega \times \Omega} \frac{(v(x)(u_n(x) - u_n(y))) \cdot (u_n(y)(v(x) - v(y)))}{d(x, y)^3},$$

so that it suffices to prove that

$$J_n = \iint_{\Omega \times \Omega} \frac{|u_n(x) - u_n(y)| |v(x) - v(y)|}{d(x, y)^3} \rightarrow 0.$$

Fix some $\varepsilon > 0$. Then

$$\begin{aligned} J_n &= \iint_{d(x, y) \geq \varepsilon} \frac{|u_n(x) - u_n(y)| |v(x) - v(y)|}{d(x, y)^3} + \iint_{d(x, y) < \varepsilon} \frac{|u_n(x) - u_n(y)| |v(x) - v(y)|}{d(x, y)^3} \\ &= o(1) + \iint_{d(x, y) < \varepsilon} \frac{|u_n(x) - u_n(y)| |v(x) - v(y)|}{d(x, y)^3} \\ &\leq o(1) + |u_n|_{H^{1/2}} \left(\iint_{d(x, y) < \varepsilon} \frac{|v(x) - v(y)|^2}{d(x, y)^3} \right)^{1/2}, \end{aligned}$$

so that $J_n \rightarrow 0$.

Proof of Lemma 16. — a) By Lemma 1, we have

$$\langle T(g), \varphi \rangle = \int_{\Omega} g \wedge (g_x \varphi_y - g_y \varphi_x), \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}),$$

so that

$$|\langle T(g), \varphi \rangle| \leq \int_{\Omega} |g| |Dg| |D\varphi| \leq \int_{\Omega} |Dg|$$

if $|\varphi|_{\text{Lip}} \leq 1$. Taking the Sup over all such φ 's yields the first inequality.

The second inequality in a), namely $L \leq C_{\Omega} |g|_{H^{1/2}}^2$, was already established in Lemma 9.

b) Let $\varepsilon < L$. By Lemma 12', we may find a sequence (\tilde{N}_j) such that

$$(4.22) \quad T = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}) = 2\pi \sum_j (\delta_{P_j} - \delta_{\tilde{N}_j})$$

and

$$(4.23) \quad \sum_j d(P_j, \tilde{N}_j) < L + \varepsilon/4\pi.$$

By the dipole construction (Lemma 15), for each j and for each $\varepsilon_j > 0$, there is some $g_j = g_{j, \varepsilon_j}$ such that

$$(4.24) \quad T(g_j) = 2\pi(\delta_{P_j} - \delta_{\tilde{N}_j}),$$

$$(4.25) \quad \int_{\Omega} |Dg_j| \leq 2\pi d(P_j, \tilde{N}_j) + \varepsilon_j,$$

$$(4.26) \quad |g_j|_{H^{1/2}}^2 \leq C_{\Omega} d(P_j, \tilde{N}_j),$$

$$(4.27) \quad \text{there is a function } \psi_j \in \text{BV} \text{ such that } g_j = e^{i\psi_j},$$

with

$$(4.28) \quad |\psi_j|_{\text{BV}} \leq 4\pi d(P_j, \tilde{N}_j) + \varepsilon_j$$

and

$$(4.29) \quad \text{meas}(\text{Supp } \psi_j) = \text{meas}(\text{Supp}(g_j - 1)) \leq \varepsilon_j.$$

We claim that $g = \prod_{j=1}^{\infty} g_j$ and $\psi = \sum_{j=1}^{\infty} \psi_j$ have all the required properties if we choose the ε_j 's appropriately.

Fix $\varepsilon_1 < \varepsilon/2$ and let $g_1 = g_{1,\varepsilon_1}$. By Lemma 17, we have

$$\limsup_{\varepsilon \rightarrow 0} |g_1 g_{2,\varepsilon}|_{H^{1/2}}^2 \leq |g_1|_{H^{1/2}}^2 + \limsup_{\varepsilon \rightarrow 0} |g_{2,\varepsilon}|_{H^{1/2}}^2.$$

Thus, we may choose $\varepsilon_2 < \varepsilon/4$ and $g_2 = g_{2,\varepsilon_2}$ such that (using (4.5))

$$|g_1 g_2|_{H^{1/2}}^2 \leq C_{\Omega}(d(P_1, \tilde{N}_1) + d(P_2, \tilde{N}_2)) + \varepsilon/2.$$

Using repeatedly Lemma 17, we choose $\varepsilon_3, \varepsilon_4, \dots$, such that

$$(4.30) \quad \varepsilon_j \leq \varepsilon 2^{-j} \quad \forall j \geq 1,$$

and, for every $k \geq 2$,

$$(4.31) \quad \begin{aligned} \left| \prod_{j=1}^k g_j \right|_{H^{1/2}}^2 &\leq C_{\Omega} \sum_{j=1}^k d(P_j, \tilde{N}_j) + \varepsilon \sum_{j=1}^{k-1} 2^{-j} \\ &\leq C_{\Omega}(L + \varepsilon) + \varepsilon \leq C'_{\Omega} L, \end{aligned}$$

since $\varepsilon < L$.

We claim that $\left(\prod_{j=1}^k g_j \right)$ converges in $W^{1,1}$. Indeed, set $H = \sum_{j \geq 1} |Dg_j|$. Then clearly $H \in L^1$ and

$$\left| D \left(\prod_{j=1}^k g_j \right) \right| \leq H.$$

On the other hand, for $k_2 \geq k_1 \geq 1$, we have, by (4.25),

$$\int_{\Omega} \left| D \left(\prod_{j=k_1}^{k_2} g_j \right) \right| \leq \sum_{j \geq k_1} \int |Dg_j| \leq 2\pi \sum_{j \geq k_1} d(P_j, \tilde{N}_j) + \varepsilon 2^{-k_1+1}.$$

Thus

$$\begin{aligned} \left| \prod_{j=1}^k g_j - \prod_{j=1}^{k+\ell} g_j \right|_{W^{1,1}} &\leq \int_{\Omega} H \left| 1 - \prod_{j=k+1}^{k+\ell} g_j \right| + 2\pi \sum_{j \geq k+1} d(P_j, \tilde{N}_j) + \varepsilon 2^{-k} \\ &\leq 2 \int_{\cup_{j>k}\{x; g_j(x) \neq 1\}} H + 2\pi \sum_{j \geq k+1} d(P_j, \tilde{N}_j) + \varepsilon 2^{-k}. \end{aligned}$$

Since $\text{meas} \left(\bigcup_{j>k} \text{Supp} (g_j - 1) \right) \leq \varepsilon 2^{-k}$ and $\sum d(\mathbf{P}_j, \tilde{\mathbf{N}}_j) < \infty$, we conclude that $\left(\prod_{j=1}^k g_j \right)$ is a Cauchy sequence in $W^{1,1}$ (note that it is clearly a Cauchy sequence in L^1 , by (4.29)).

Set $g = \prod_{j=1}^{\infty} g_j$. By construction

$$\begin{aligned} |g|_{W^{1,1}} &\leq \int_{\Omega} \mathbf{H} \leq 2\pi \sum_{j=1}^{\infty} d(\mathbf{P}_j, \tilde{\mathbf{N}}_j) + \varepsilon \\ &\leq 2\pi \left(\mathbf{L} + \frac{\varepsilon}{4\pi} \right) + \varepsilon \quad (\text{by (4.23)}) \leq 2\pi(\mathbf{L} + \varepsilon). \end{aligned}$$

This proves (4.18).

On the other hand, by (4.31), the sequence $\left(\prod_{j=1}^k g_j \right)$ is bounded in $H^{1/2}$, so that $g \in H^{1/2}$ and $|g|_{H^{1/2}}^2 \leq C'_{\Omega} \mathbf{L}$; this proves (4.19).

We now turn to (4.17). By (2.7) and (4.24), we have

$$\mathbf{T} \left(\prod_{j=1}^k g_j \right) = 2\pi \sum_{j=1}^k (\delta_{\mathbf{P}_j} - \delta_{\tilde{\mathbf{N}}_j}).$$

By Lemma 1 and the convergence of $(\prod_{j=1}^k g_j)$ to g in $W^{1,1}$ as $k \rightarrow \infty$, we have

$$\langle \mathbf{T} \left(\prod_{j=1}^k g_j \right), \varphi \rangle \rightarrow \langle \mathbf{T}(g), \varphi \rangle, \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}).$$

Thus,

$$\langle \mathbf{T}(g), \varphi \rangle = 2\pi \sum_{j=1}^{\infty} (\varphi(\mathbf{P}_j) - \varphi(\tilde{\mathbf{N}}_j)), \quad \forall \varphi \in \text{Lip}(\Omega; \mathbf{R}).$$

From (4.22) we conclude that

$$\mathbf{T}(g) = 2\pi \sum_i (\delta_{\mathbf{P}_i} - \delta_{\tilde{\mathbf{N}}_i}).$$

Properties (4.20) and (4.21) are immediate consequences of (4.23), (4.28) and (4.29).

We now derive some consequences of the above results. We start with a simple

Proof of Theorem 2. — Let $g \in H^{1/2}(\Omega; S^1)$ be such that $L(g) = 0$. We must show that $g \in Y = \overline{C^\infty(\Omega; S^1)}^{H^{1/2}}$. By Lemma 11 there exists a sequence (g_n) in \mathcal{R} such that $g_n \rightarrow g$ in $H^{1/2}$, and thus $L(g_n) \rightarrow 0$. Since each g_n has only finitely many singularities, it follows from the dipole construction there exists a sequence (h_n) such that

$$h_n \in W_{\text{loc}}^{1,\infty}(\Omega \setminus \Sigma_n; S^1) \cap W^{1,p}(\Omega; S^1), \forall p \in [1, 2), T(h_n) = T(g_n),$$

where Σ_n is the singular set of g_n (Σ_n is a finite set), and moreover

$$|h_n|_{H^{1/2}}^2 \leq C_\Omega L(h_n) \rightarrow 0,$$

$$h_n \rightarrow 1 \text{ a.e. on } \Omega.$$

Clearly $k_n = g_n \overline{h_n} \in W_{\text{loc}}^{1,\infty}(\Omega \setminus \Sigma_n; S^1) \cap W^{1,p}(\Omega; S^1)$, $\forall p \in [1, 2)$ and $T(k_n) = T(g_n) - T(h_n) = 0$. By Lemma 2, we have $\deg(k_n, a) = 0 \quad \forall a \in \Sigma_n$. Therefore k_n admits a well-defined lifting on Ω , $k_n = e^{i\varphi_n}$, with $\varphi_n \in W_{\text{loc}}^{1,\infty}(\Omega \setminus \Sigma_n; \mathbf{R}) \cap W^{1,p}(\Omega; \mathbf{R})$, $\forall p \in [1, 2)$. In particular, $k_n \in X \subset Y$. In order to prove that $g \in Y$ it suffices to check that $k_n \rightarrow g$ in $H^{1/2}$. Write

$$\begin{aligned} |k_n - g|_{H^{1/2}} &= |g_n \overline{h_n} - g|_{H^{1/2}} = |(g_n - g) \overline{h_n} + g(\overline{h_n} - 1)|_{H^{1/2}} \\ &\leq |(g_n - g) \overline{h_n}|_{H^{1/2}} + |g(\overline{h_n} - 1)|_{H^{1/2}}. \end{aligned}$$

But

$$|(g_n - g) \overline{h_n}|_{H^{1/2}} \leq |g_n - g|_{H^{1/2}} + 2|h_n|_{H^{1/2}} \rightarrow 0$$

and

$$|g(\overline{h_n} - 1)|_{H^{1/2}}^2 \leq C \int_{\Omega} \int_{\Omega} \frac{|g(x) - g(y)|^2}{d(x,y)^3} |h_n(x) - 1|^2 dx dy + C|h_n|_{H^{1/2}}^2 \rightarrow 0.$$

Corollary 2. — *Given any $g \in H^{1/2}(\Omega; S^1)$, there exist $h \in Y, k \in H^{1/2}(\Omega; S^1) \cap W^{1,1}(\Omega; S^1)$ and $\psi \in \text{BV}(\Omega; \mathbf{R})$ such that*

$$g = hk \text{ and } k = e^{i\psi}.$$

Moreover, for every $\varepsilon > 0$, one may choose h, k, ψ such that

$$\int_{\Omega} |Dk| \leq 2\pi L(g) + \varepsilon, \quad |k|_{H^{1/2}}^2 \leq C_\Omega L(g),$$

$$|h|_{H^{1/2}}^2 \leq |g|_{H^{1/2}}^2 + C_\Omega L(g)$$

and

$$|\psi|_{\text{BV}} \leq 4\pi L(g) + \varepsilon.$$

Proof. — By Lemma 16 there exists a sequence (k_n) in $H^{1/2}(\Omega; S^1) \cap W^{1,1}$ such that

$$\begin{aligned} T(k_n) &= T(g), \quad \forall n, \\ \limsup_{n \rightarrow \infty} |k_n|_{W^{1,1}} &\leq 2\pi L(g), \\ |k_n|_{H^{1/2}}^2 &\leq C_\Omega L(g), \quad \forall n, \end{aligned}$$

and

$$k_n \rightarrow 1 \quad \text{a.e. on } \Omega.$$

Set $h_n = g\bar{k}_n$, so that $T(h_n) = 0$, $\forall n$, and thus $h_n \in Y$. By Lemma 17 we have

$$\limsup_{n \rightarrow \infty} |h_n|_{H^{1/2}}^2 \leq |g|_{H^{1/2}}^2 + C_\Omega L(g).$$

The conclusion of Corollary 2 is now clear with $k = k_n$, $h = h_n$ and n sufficiently large.

Proof of Theorem 5. — As in the proof of Corollary 2 write $g = h_n k_n$. Since $h_n \in Y$, we may apply Theorem 3 and write $h_n = e^{i(\varphi_n + \psi_n)}$, with $\varphi_n \in H^{1/2}$ and $\psi_n \in W^{1,1}$. An inspection of the proof of Theorem 3 shows that

$$|\varphi_n|_{H^{1/2}} \leq C_\Omega |h_n|_{H^{1/2}} \leq C'_\Omega |g|_{H^{1/2}}$$

and

$$|\psi_n|_{W^{1,1}} \leq C_\Omega |h_n|_{H^{1/2}}^2 \leq C''_\Omega |g|_{H^{1/2}}^2.$$

Thus

$$g = e^{i\varphi_n} (e^{i\psi_n} k_n),$$

which is the desired decomposition since $e^{i\psi_n} k_n \in W^{1,1}$ and

$$|e^{i\psi_n} k_n|_{W^{1,1}} \leq |\psi_n|_{W^{1,1}} + |k_n|_{W^{1,1}} \leq C''_\Omega |g|_{H^{1/2}}^2.$$

Proof of the upper bound in Theorem 4. — We have to show that, for every $g \in H^{1/2}(\Omega; S^1)$,

$$\inf\{|\psi|_{\text{BV}}; g = e^{i(\varphi + \psi)}, \varphi \in H^{1/2}, \psi \in \text{BV}\} \leq 4\pi L(g),$$

i.e., for every $\varepsilon > 0$, we must find $\varphi_\varepsilon \in H^{1/2}$ and $\psi_\varepsilon \in \text{BV}$ such that $g = e^{i(\varphi_\varepsilon + \psi_\varepsilon)}$ and

$$|\psi_\varepsilon|_{\text{BV}} \leq 4\pi L(g) + \varepsilon.$$

Going back to the proof of Corollary 2 and Theorem 5, we may write, by (4.20), $k_n = e^{i\eta_n}$, with $\eta_n \in \text{BV}$ and

$$\limsup_{n \rightarrow \infty} |\eta_n|_{\text{BV}} \leq 4\pi L(g).$$

On the other hand, since $C^\infty(\Omega; \mathbf{R})$ is dense in $W^{1,1}(\Omega; \mathbf{R})$, we may choose $\tilde{\psi}_n \in C^\infty(\Omega; \mathbf{R})$ such that

$$\|\psi_n - \tilde{\psi}_n\|_{W^{1,1}} < 1/n.$$

Finally, we may write

$$g = h_n k_n = e^{i(\varphi_n + \psi_n + \eta_n)} = e^{i(\varphi_n + \tilde{\psi}_n) + i(\psi_n - \tilde{\psi}_n + \eta_n)},$$

with $\varphi_n + \tilde{\psi}_n \in H^{1/2}$, $\psi_n - \tilde{\psi}_n + \eta_n \in \text{BV}$ and

$$\limsup |\psi_n - \tilde{\psi}_n + \eta_n|_{\text{BV}} \leq 4\pi L(g),$$

which is the desired conclusion.

We now turn to the

Proof of Lemma 14'. — For the first assertion, we proceed as in the proof of Corollary 2. Since $h_n \in Y$, $\forall n$, we may find a sequence (\tilde{h}_n) in $C^\infty(\Omega; S^1)$ such that

$$\|\tilde{h}_n - h_n\|_{H^{1/2}}^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Recall that

$$h_n = g \bar{k}_n \longrightarrow g \text{ a.e.}$$

Thus, by Lemma 17, we find

$$\limsup |\tilde{h}_n|_{H^{1/2}}^2 \leq |g|_{H^{1/2}}^2 + C_\Omega L(g)$$

and (passing to a subsequence)

$$\tilde{h}_n \longrightarrow g \text{ a.e.}, \quad \tilde{h}_n \rightharpoonup g \text{ weakly in } H^{1/2}.$$

To prove the second assertion, let (g_n) be any sequence in Y such that $g_n \longrightarrow g$ a.e. Writing $g_n = (g_n \bar{g})g$ and observing that $g_n \bar{g} \rightarrow 1$ a.e., we deduce from Lemma 17 that

$$|g_n|_{H^{1/2}}^2 = |g|_{H^{1/2}}^2 + |g_n \bar{g}|_{H^{1/2}}^2 + o(1) \text{ as } n \rightarrow \infty.$$

On the other hand (*see* Lemma 9),

$$L(g_n \bar{g}) \leq C_\Omega |g_n \bar{g}|_{H^{1/2}}^2.$$

But $L(g_n \bar{g}) = L(\bar{g})$, since $L(g_n) = 0$, and thus

$$|g_n|_{H^{1/2}}^2 \geq |g|_{H^{1/2}}^2 + C'_\Omega L(g) + o(1).$$

Remark 4.2. — We have now at our disposal two different techniques for lifting a general $g \in H^{1/2}(\Omega; S^1)$ in the form

$$g = e^{i(\varphi+\psi)} \text{ with } \varphi \in H^{1/2} \text{ and } \psi \in \text{BV}.$$

The first method, described at the beginning of Section 4, yields some $\varphi \in H^{1/2}$ and $\psi \in \text{BV}$ such that

$$g = e^{i(\varphi+\psi)},$$

with the estimate

$$(4.32) \quad |\varphi|_{H^{1/2}} \leq C_\Omega |g|_{H^{1/2}}$$

and

$$(4.33) \quad |\psi|_{\text{BV}} \leq C_\Omega |g|_{H^{1/2}}^2.$$

The second method, described in the proof of Theorem 4 (upper bound), yields, for every $\varepsilon > 0$, some $\varphi_\varepsilon \in H^{1/2}$ and $\psi_\varepsilon \in \text{BV}$ such that

$$g = e^{i(\varphi_\varepsilon + \psi_\varepsilon)},$$

with

$$(4.34) \quad |\psi_\varepsilon|_{\text{BV}} \leq 4\pi L(g) + \varepsilon$$

and **no estimate** for φ_ε in $H^{1/2}$.

A natural question is whether one can achieve a decomposition of the phase in the form

$$g = e^{i(\varphi_\varepsilon^\# + \psi_\varepsilon^\#)}$$

with the double control

$$|\varphi_\varepsilon^\#|_{H^{1/2}} \leq C(\varepsilon, |g|_{H^{1/2}})$$

and

$$|\psi_\varepsilon^\#|_{\text{BV}} \leq 4\pi L(g) + \varepsilon ?$$

The answer is negative even with $g \in Y$. To see this, we may use an example studied in [15]. Assume that, locally, near a point of Ω , say 0, the square $Q = I^2$, with $I = (-1, +1)$, is contained in Ω . Consider the function $\gamma_\delta(x)$ defined on I by

$$\gamma_\delta(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 2\pi x/\delta, & \text{if } 0 < x < \delta \\ 2\pi, & \text{if } \delta < x < 1 \end{cases},$$

where δ is small.

On Q , set

$$g_\delta(x, y) = e^{i\gamma_\delta(x)} \text{ for } (x, y) \in Q.$$

Clearly, we have $g_\delta \in Y$, so that $L(g_\delta) = 0$. We claim that

$$(4.35) \quad \|g_\delta\|_{H^{1/2}(Q)} \leq C, \quad \forall \delta,$$

and that there exist absolute positive constants c_* and C_* such that, if

$$(4.36) \quad g_\delta = e^{i(\varphi_\delta + \psi_\delta)}, \quad \varphi_\delta \in H^{1/2}(Q), \quad \psi_\delta \in \text{BV}(Q),$$

with

$$(4.37) \quad |\psi_\delta|_{\text{BV}(Q)} \leq C_*,$$

then

$$(4.38) \quad |\varphi_\delta|_{H^{1/2}(Q)}^2 \geq c_* \log(1/\delta) \text{ as } \delta \rightarrow 0.$$

The verification of (4.35) is easy. Indeed, by scaling we have

$$|g_\delta(\cdot, y)|_{H^{1/2}(I)} \leq C, \quad \forall \delta, \forall y,$$

and recall (*see* e.g. [1], Lemma 7.44) that

$$(4.39) \quad \int_I |f(\cdot, y)|_{H^{1/2}(I)}^2 dy + \int_I |f(x, \cdot)|_{H^{1/2}(I)}^2 dx \sim |f|_{H^{1/2}(Q)}^2,$$

so that (4.35) follows.

We now turn to the proof of (4.38) under the assumptions (4.36) and (4.37). By Theorem 2 in [15] we know that, for a.e. $y \in \mathbf{I}$,

$$(4.40) \quad |\varphi_\delta(\cdot, y) + \psi_\delta(\cdot, y)|_{\mathbf{H}^1(\mathbf{I})} \geq c(\log(1/\delta))^{1/2}$$

for some absolute constant $c > 0$, where

$$(4.41) \quad 2s = 1 - (\log 1/\delta)^{-1}.$$

On the other hand, it is easy to see that

$$(4.42) \quad |f|_{\mathbf{H}^\sigma(\mathbf{I})}^2 \leq \frac{C}{1-2\sigma} |f|_{\mathbf{BV}(\mathbf{I})}^2, \quad \forall f \in \mathbf{BV}(\mathbf{I}), \forall \sigma < 1/2$$

and

$$(4.43) \quad |f|_{\mathbf{H}^\sigma(\mathbf{I})} \leq C|f|_{\mathbf{H}^{1/2}(\mathbf{I})}, \quad \forall f \in \mathbf{H}^{1/2}, \forall \sigma \leq 1/2,$$

with constants C independent of σ . Combining (4.40), (4.41), (4.42) and (4.43) yields, for a.e. $y \in \mathbf{I}$,

$$(4.44) \quad |\varphi_\delta(\cdot, y)|_{\mathbf{H}^{1/2}(\mathbf{I})} + (\log(1/\delta))^{1/2} |\psi_\delta(\cdot, y)|_{\mathbf{BV}(\mathbf{I})} \geq c(\log(1/\delta))^{1/2}.$$

Integrating (4.44) in y and using the inequalities

$$\begin{aligned} \int_{\mathbf{I}} |f(\cdot, y)|_{\mathbf{H}^{1/2}(\mathbf{I})} dy &\leq \left(2 \int_{\mathbf{I}} |f(\cdot, y)|_{\mathbf{H}^{1/2}(\mathbf{I})}^2 dy \right)^{1/2} \\ &\leq C|f|_{\mathbf{H}^{1/2}(\mathbf{Q})}, \quad \forall f \in \mathbf{H}^{1/2}(\mathbf{Q}), \end{aligned}$$

and

$$\int_{\mathbf{I}} |f(\cdot, y)|_{\mathbf{BV}(\mathbf{I})} dy \leq C|f|_{\mathbf{BV}(\mathbf{Q})}, \quad \forall f \in \mathbf{BV}(\mathbf{Q}),$$

together with (4.37), we obtain

$$|\varphi_\delta|_{\mathbf{H}^{1/2}(\mathbf{Q})} + C_*(\log 1/\delta)^{1/2} \geq c(\log 1/\delta)^{1/2},$$

and (4.38) follows, provided C_* is sufficiently small.

4.3. *Lower bound estimates for the BV part of the phase*

We start with a simple lemma about maps from S^1 into S^1 .

Lemma 18. — *Let $(g_n) \subset \text{BV}(S^1; S^1) \cap C^0(S^1; S^1)$ be such that $g_n \rightarrow g$ a.e. for some $g \in \text{BV}(S^1; S^1) \cap C^0(S^1; S^1)$ and $\|g_n\|_{\text{BV}} \leq C$. Then*

$$\liminf_{n \rightarrow \infty} \left(\int_{S^1} |\dot{g}_n| - 2\pi |\deg g_n - \deg g| \right) \geq \int_{S^1} |\dot{g}|.$$

Here, \dot{g} denotes the measure $\frac{\partial g}{\partial \theta}$.

Proof. — (We thank Augusto Ponce for simplifying our original proof). For $g \in \text{BV}(S^1; S^1) \cap C^0(S^1; S^1)$, let $f \in C^0([0, 2\pi]; \mathbf{R})$ be such that $g(\exp(i\theta)) = \exp(if(\theta))$. Then $\deg g = \frac{1}{2\pi}(f(2\pi) - f(0))$. Moreover, we have $f \in \text{BV}$ and

$$(4.45) \quad \int_0^{2\pi} |f'| = \int_{S^1} |\dot{g}|,$$

where f' is the measure $\frac{df}{dx}$. Indeed, since g is continuous, we have

$$(4.46) \quad \begin{aligned} \int_{S^1} |\dot{g}| &= \text{Sup} \left\{ \sum_{j=1}^n |g(\exp(it_{j+1})) - g(\exp(it_j))|; 0 \leq t_1 < \dots < t_n \leq 2\pi \right\} \\ &= \text{Sup} \left\{ \sum_{j=1}^{n-1} |g(\exp(it_{j+1})) - g(\exp(it_j))|; 0 \leq t_1 < \dots < t_n \leq 2\pi \right\} \end{aligned}$$

(with the convention $t_{n+1} = t_1$).

For a given $\delta > 0$, we have

$$(4.47) \quad (1 - \delta)|f(t_{j+1}) - f(t_j)| \leq |g(\exp(it_{j+1})) - g(\exp(it_j))| \leq |f(t_{j+1}) - f(t_j)|,$$

provided the partition (t_j) is sufficiently fine. We obtain (4.45) by combining (4.46) and (4.47).

Let $f_n \in \text{BV}([0, 2\pi]; \mathbf{R}) \cap C^0([0, 2\pi]; \mathbf{R})$ be such that $g_n(\exp(i\theta)) = \exp(if_n(\theta))$ and $\|f_n\|_{\text{BV}} \leq C$. Up to a subsequence, we may assume that $f_n \rightarrow h$ a.e. and in L^1 for some $h \in \text{BV}$.

Since $g = e^{ih} = e^{if}$, we find that $h = f + k$, where $k \in \text{BV}([0, 2\pi]; 2\pi\mathbf{Z})$. Thus k must be of the form

$$k = 2\pi \sum_{j=1}^p \alpha_j \chi_{I_j} \text{ a.e.},$$

where $\alpha_j \in \mathbf{Z}$, $I_j = (a_j, a_{j+1})$, $0 = a_1 < \dots < a_{p+1} = 2\pi$. Therefore

$$(4.48) \quad h' = f' + \sum_{j=2}^p \alpha_j \delta_{a_j}.$$

We have to prove that

$$(4.49) \quad \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} |f_n'| - \left| \int_0^{2\pi} (f_n' - f') \right| \right) \geq \int_0^{2\pi} |f'|.$$

It suffices to show that

$$(4.50) \quad \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} |f_n'| + \int_0^{2\pi} (f_n' - f') \right) \geq \int_0^{2\pi} |f'|.$$

Indeed, (4.50) applied to \bar{g}_n gives

$$(4.51) \quad \liminf_{n \rightarrow \infty} \left(\int_0^{2\pi} |f_n'| - \int_0^{2\pi} (f_n' - f') \right) \geq \int_0^{2\pi} |f'|$$

and the combination of (4.50) and (4.51) is equivalent to (4.49). We may rewrite (4.50) as

$$(4.52) \quad \liminf_{n \rightarrow \infty} \int_0^{2\pi} (f_n')^+ \geq \int_0^{2\pi} (f')^+.$$

Let $\varphi \in C_0^\infty(0, 2\pi)$, $0 \leq \varphi \leq 1$. Then

$$- \int_0^{2\pi} f_n' \varphi = \int_0^{2\pi} f_n' \varphi \leq \int_0^{2\pi} (f_n')^+$$

and thus

$$-\int_0^{2\pi} h\varphi' \leq \liminf_{n \rightarrow \infty} \int_0^{2\pi} (f'_n)^+.$$

Taking the supremum over such φ 's yields

$$\liminf_{n \rightarrow \infty} \int_0^{2\pi} (f'_n)^+ \geq \int_0^{2\pi} (h')^+ = \int_0^{2\pi} (f' + \sum \alpha_j \delta_{a_j})^+ \text{ by (4.48).}$$

We conclude with the help of the following elementary

Lemma 19. — *Let $f \in \text{BV}([0, 2\pi]) \cap C^0([0, 2\pi])$. Then*

$$\int_0^{2\pi} (f' + \sum_{\text{finite}} \alpha_j \delta_{a_j})^+ = \int_0^{2\pi} (f')^+ + \sum (\alpha_j)^+$$

for any choice of distinct points $a_j \in (0, 2\pi)$ and of α_j in \mathbf{R} .

Proof of Lemma 19. — It suffices to consider the case of a single point $a \in (0, 2\pi)$. Let $\zeta_n = \zeta(n(x - a))$, where ζ is a fixed cutoff function with $\zeta(0) = 1, 0 \leq \zeta \leq 1$. For any fixed $\psi \in C^1([0, 2\pi])$, we claim that

$$\int_0^{2\pi} f(\zeta_n \psi)' \rightarrow 0.$$

Indeed,

$$\int_0^{2\pi} f(\zeta_n \psi)' = \int_0^{2\pi} (f - f(a))(\zeta_n \psi)',$$

so that

$$\left| \int_0^{2\pi} f(\zeta_n \psi)' \right| \leq \int_0^{2\pi} |f - f(a)| |(\zeta_n \psi)'| \xrightarrow{n} 0,$$

since f is continuous at a .

Let $\varepsilon > 0$. Fix some $\psi \in C_0^1((0, 2\pi))$, $0 \leq \psi \leq 1$, such that

$$-\int_0^{2\pi} f\psi' \geq \int_0^{2\pi} (f')^+ - \varepsilon.$$

Then, with $0 \leq t \leq 1$,

$$\begin{aligned} \int_0^{2\pi} (f' + \alpha\delta_a)[(1 - \zeta_n)\psi + t\zeta_n] = \\ - \int_0^{2\pi} f[(1 - \zeta_n)\psi + t\zeta_n]' + t\alpha \xrightarrow{n} - \int_0^{2\pi} f\psi' + t\alpha. \end{aligned}$$

Since $0 \leq (1 - \zeta_n)\psi + t\zeta_n \leq 1$, we find that

$$\int_0^{2\pi} (f' + \alpha\delta_a)^+ \geq \int_0^{2\pi} (f')^+ + t\alpha - \varepsilon, \quad \forall \varepsilon > 0, \forall t \in [0, 1],$$

and thus

$$\int_0^{2\pi} (f' + \alpha\delta_a)^+ \geq \int_0^{2\pi} (f')^+ + \alpha^+.$$

The opposite inequality

$$\int_0^{2\pi} (f' + \alpha\delta_a)^+ \leq \int_0^{2\pi} (f')^+ + \alpha^+$$

being clear, the proof of Lemma 19 is complete.

Remark 4.3. — The assumption $\|g_n\|_{\text{BV}} \leq C$ in Lemma 18 is essential (A. Ponce, personal communication).

Corollary 3. — Let $\Gamma \subset \mathbf{R}^N$ be an oriented curve. Let $(g_n) \subset \text{BV}(\Gamma; \mathbf{S}^1) \cap C^0(\Gamma; \mathbf{S}^1)$ be such that $g_n \rightarrow g$ a.e. and $\|g_n\|_{\text{BV}} \leq C$, where $g \in \text{BV}(\Gamma; \mathbf{S}^1) \cap C^0(\Gamma; \mathbf{S}^1)$. Then

$$\liminf_{n \rightarrow \infty} \left(\int_{\Gamma} |\dot{g}_n| - 2\pi |\deg g_n - \deg g| \right) \geq \int_{\Gamma} |\dot{g}|.$$

In particular, if $\deg g_n = 0, \forall n$, then

$$\liminf_{n \rightarrow \infty} \int_{\Gamma} |\dot{g}_n| \geq 4\pi |\deg g|$$

(the assumption $\|g_n\|_{\text{BV}} \leq C$ is not required here).

Here, Γ need not be connected. If $\Gamma = \bigcup_j \gamma_j$, with each γ_j simple, we set

$$\deg g = \sum_j \deg(g; \gamma_j),$$

where γ_j has the orientation inherited from that of Γ .

Remark 4.4. — It can be easily seen that the constants 2π in Lemma 18 and 4π in Corollary 3 cannot be improved.

We now prove a coarea type formula (in the spirit of [2]) used in the proof of the lower bound in Theorem 4.

Lemma 20. — Let $g \in H^{1/2}(\Omega; S^1)$ and $\zeta \in C^\infty(\Omega; \mathbf{R})$. If $\lambda \in \mathbf{R}$ is a regular value of ζ , let

$$\Gamma_\lambda = \{x \in \Omega; \zeta(x) = \lambda\}.$$

We orient Γ_λ such that, for each $x \in \Gamma_\lambda$, the basis $(\tau(x), D\zeta(x), n(x))$ is direct, where $n(x)$ is the outward normal to Ω at x . Then

$$\langle T(g), \zeta \rangle = 2\pi \int_{\mathbf{R}} \deg(g; \Gamma_\lambda) d\lambda.$$

Remark 4.5. — For a.e. λ we have $g|_{\Gamma_\lambda} \in H^{1/2} \subset \text{VMO}$. Therefore, $\deg(g; \Gamma_\lambda)$ makes sense for a.e. λ (see [22]). In general, Γ_λ is a union of simple curves, $\Gamma_\lambda = \bigcup \gamma_j$. In this case, we set

$$\deg(g; \Gamma_\lambda) = \sum \deg(g; \gamma_j),$$

where on each γ_j we consider the orientation inherited from Γ_λ .

Proof of Lemma 20. — We write $g = g_1 h$, with $g_1 \in X$ and $h \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$. For a.e. λ , we have $h|_{\Gamma_\lambda} \in W^{1,1}$ and $g_1|_{\Gamma_\lambda} \in H^{1/2}$.

Since $g_1 = e^{i\varphi_1}$ for some $\varphi_1 \in H^{1/2}(\Omega; \mathbf{R})$, for a.e. λ we have $\deg(g_1; \Gamma_\lambda) = 0$, so that $\deg(g; \Gamma_\lambda) = \deg(h; \Gamma_\lambda)$ for a.e. λ . Moreover, we have $T(g) = T(h)$. It suffices therefore to prove the statement of the lemma for $h \in W^{1,1}(\Omega; S^1) \cap H^{1/2}(\Omega; S^1)$. In this case, we have

$$\langle T(h), \zeta \rangle = \int_{\Omega} |D\zeta| h \wedge \left(Dh \wedge \frac{D\zeta}{|D\zeta|} \right)$$

(see Lemma 1 in the introduction).

We recall the coarea formula (see, e.g., Federer [26], Simon [42])

$$(4.53) \quad \int_{\Omega} f |D\varphi| = \int_{\mathbf{R}} \left(\int_{\varphi=\lambda} f ds \right) d\lambda, \quad \varphi \in C^\infty(\Omega; \mathbf{R}), \quad f \in L^1(\Omega; \mathbf{R}).$$

Applying (4.53) with $\varphi = \zeta$, $f = h \wedge \left(Dh \wedge \frac{D\zeta}{|D\zeta|} \right) = h \wedge \frac{\partial h}{\partial \tau}$ (where τ is the oriented tangent unit vector to Γ_λ) we find

$$\langle T(h), \zeta \rangle = \int_{\mathbf{R}} \left(\int_{\Gamma_\lambda} h \wedge \frac{\partial h}{\partial \tau} ds \right) d\lambda = 2\pi \int_{\mathbf{R}} \deg(h; \Gamma_\lambda) d\lambda.$$

The final ingredient in the proof of Theorem 4 is the lower bound given by

Lemma 21. — *Let $g \in H^{1/2}(\Omega; S^1)$. If $g = e^{i(\varphi+\psi)}$ with $\varphi \in H^{1/2}(\Omega; \mathbf{R})$ and $\psi \in BV(\Omega; \mathbf{R})$, then*

$$\int_{\Omega} |D\psi| \geq 4\pi L(g).$$

Proof. — Let $h = e^{-i\varphi}g \in H^{1/2}(\Omega; S^1)$. Let (ψ_n) be a sequence of smooth real-valued functions such that $\psi_n \rightarrow \psi$ a.e. and

$$\int_{\Omega} |D\psi_n| \rightarrow \int_{\Omega} |D\psi|.$$

Fix some $\zeta \in C^\infty(\Omega; \mathbf{R})$ and let, for λ a regular value of ζ , $\Gamma_\lambda = \{x \in \Omega; \zeta(x) = \lambda\}$. Let $h_n = e^{i\psi_n}$. For a.e. λ we have $h_n|_{\Gamma_\lambda} \rightarrow h|_{\Gamma_\lambda}$ a.e. and $h|_{\Gamma_\lambda} \in H^{1/2} \cap BV$. For any such λ we have $h|_{\Gamma_\lambda} \in BV \cap C^0$. Indeed, since $k = h|_{\Gamma_\lambda} \in BV$, k has finite limits from the left and from the right at each point. These limits must coincide, since $H^{1/2} \subset VMO$ in dimension 1 (see e.g. [17] and [22]) and non-trivial characteristic functions are not in VMO.

By the second assertion in Corollary 3, we find that, for a.e. λ ,

$$\liminf_{n \rightarrow \infty} \int_{\Gamma_\lambda} |\dot{h}_n| \geq 4\pi |\deg(h; \Gamma_\lambda)|.$$

Thus, if $|\mathbf{D}\zeta| \leq 1$, we have by the coarea formula,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_{\Omega} |\mathbf{D}h_n| &\geq \liminf_{n \rightarrow \infty} \int_{\Omega} |\mathbf{D}h_n| |\mathbf{D}\zeta| = \liminf_{n \rightarrow \infty} \int_{\mathbf{R}} \left(\int_{\Gamma_\lambda} |\mathbf{D}h_n| ds \right) d\lambda \\ &\geq \liminf_{n \rightarrow \infty} \int_{\mathbf{R}} \left(\int_{\Gamma_\lambda} |\dot{h}_n| ds \right) d\lambda \geq 4\pi \int_{\mathbf{R}} |\deg(h; \Gamma_\lambda)| d\lambda \\ &\geq 4\pi \left| \int_{\mathbf{R}} \deg(h; \Gamma_\lambda) d\lambda \right|. \end{aligned}$$

On the other hand, by Lemma 20, we have

$$4\pi \left| \int_{\mathbf{R}} \deg(h; \Gamma_\lambda) d\lambda \right| = 2|\langle \mathbf{T}(h), \zeta \rangle|.$$

Thus, if $\zeta \in C^\infty(\Omega; \mathbf{R})$ is such that $|\mathbf{D}\zeta| \leq 1$, we have

$$\begin{aligned} \int_{\Omega} |\mathbf{D}\psi| &= \liminf_{n \rightarrow \infty} \int_{\Omega} |\mathbf{D}\psi_n| \\ (4.54) \quad &= \liminf_{n \rightarrow \infty} \int_{\Omega} |\mathbf{D}h_n| \geq 2|\langle \mathbf{T}(h), \zeta \rangle| = 2|\langle \mathbf{T}(g), \zeta \rangle|. \end{aligned}$$

We conclude by taking in (4.54) the supremum over all such ζ 's.

5. Minimal connection and Ginzburg–Landau energy for $g \in H^{1/2}$. Proof of Theorem 6

Throughout this section, the metric d denotes d_G , the geodesic distance (on Ω) relative to G , and $L = L_G$.

Proof of Theorem 6. — We start by deriving some elementary inequalities. For $g \in H^{1/2}(\Omega; \mathbf{R}^2)$, let

$$\ell_{\varepsilon, g} = \text{Min}\{E_\varepsilon(u); u \in H_g^1(G; \mathbf{R}^2)\}.$$

Let $g_1, g_2 \in H^{1/2}(\Omega; S^1)$ and let $u_j \in H_{g_j}^1(G; B^2)$ be such that $e_{\varepsilon, g_j} = E_\varepsilon(u_j)$, $j = 1, 2$. Then $u_1 u_2 \in H_{g_1 g_2}^1(G; \mathbf{R}^2)$. We find that, for each $\delta > 0$, we have

$$\begin{aligned}
 (5.1) \quad e_{\varepsilon, g_1 g_2} &\leq E_\varepsilon(u_1 u_2) \leq \frac{1}{2} \int_G (|\nabla u_1| + |\nabla u_2|)^2 + \frac{1}{4\varepsilon^2} \int_G (1 - |u_1 u_2|^2)^2 \\
 &\leq \frac{1+\delta}{2} \int_G |\nabla u_1|^2 + \frac{C(\delta)}{2} \int_G |\nabla u_2|^2 \\
 &\quad + \frac{1}{4\varepsilon^2} \int_G ((1 - |u_1|^2) + (1 - |u_2|^2))^2 \\
 &\leq (1+\delta)e_{\varepsilon, g_1} + C(\delta)e_{\varepsilon, g_2}.
 \end{aligned}$$

Similarly, we have

$$(5.2) \quad e_{\varepsilon, g_1 g_2} \geq (1-\delta)e_{\varepsilon, g_1} - C(\delta)e_{\varepsilon, g_2}.$$

The upper bound $e_{\varepsilon, g} \leq \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon))$

We will use Lemma A.1 in Appendix A, which asserts that, if $g \in \mathcal{R}_1$, then

$$(5.3) \quad e_{\varepsilon, g} \leq \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)) \text{ as } \varepsilon \rightarrow 0.$$

The class \mathcal{R}_1 , which is dense in $H^{1/2}(\Omega; S^1)$, is defined in Appendix A. Inequality (5.3) was essentially established by Sandier [40].

Another ingredient needed in the proof is the following upper bound, valid for $g \in H^{1/2}(\Omega; S^1)$, and already mentioned in the Introduction (*see* [12], Theorem 5 and Remark 8; *see* also [38], Proposition II.1 for a different proof):

$$(5.4) \quad e_{\varepsilon, g} \leq C|g|_{H^{1/2}}^2(1 + \log(1/\varepsilon)),$$

for some $C = C(G)$.

We now turn to the proof of the upper bound. Let $g \in H^{1/2}(\Omega; S^1)$. By Lemma B.1 in Appendix B, there is a sequence (g_k) in \mathcal{R}_1 such that $g_k \rightarrow g$ in $H^{1/2}$. On the one hand, since $H^{1/2} \cap L^\infty$ is an algebra, we find that $|g/g_k|_{H^{1/2}} \rightarrow 0$. On the other hand, recall that $L(g_k) \rightarrow L(g)$. Fix some $\tilde{\delta} > 0$. By (5.4) applied to g/g_k , we find that

$$(5.5) \quad e_{\varepsilon, g/g_k} \leq \tilde{\delta} \log(1/\varepsilon) \quad \text{for } \varepsilon \text{ sufficiently small,}$$

if k is sufficiently large. Using (5.3) for g_k , where k is sufficiently large, we obtain

$$(5.6) \quad e_{\varepsilon, g_k} \leq \pi(L(g) + \delta) \log(1/\varepsilon).$$

The upper bound follows by combining (5.1), (5.5) and (5.6).

The lower bound $e_{\varepsilon,g} \geq \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon))$

We rely on the corresponding lower bound in [40] (Theorem 3.1, part 1): if $g \in \mathcal{R}_0$ (where the class \mathcal{R}_0 , dense in $H^{1/2}(\Omega; S^1)$, is defined in Appendix A), then

$$(5.7) \quad e_{\varepsilon,g} \geq \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)) \quad \text{for } \varepsilon \text{ sufficiently small}$$

(no geometrical assumption is made on Ω or g). We fix some $\delta > 0$. Applying (5.7) to g_k for k sufficiently large, we find that

$$(5.8) \quad e_{\varepsilon,g_k} \geq \pi(L(g) - \delta) \log(1/\varepsilon) \quad \text{for } \varepsilon \text{ sufficiently small.}$$

The lower bound is a consequence of (5.2), (5.5) and (5.8).

There is a variant of Theorem 6 when the boundary condition depends on ε . Let $g \in H^{1/2}(\Omega; S^1)$ and let $g_\varepsilon \in H^{1/2}(\Omega; \mathbf{R}^2)$ be such that

$$(5.9) \quad g_\varepsilon \rightarrow g \text{ in } H^{1/2},$$

$$(5.10) \quad |g_\varepsilon| \leq 1,$$

$$(5.11) \quad \| |g_\varepsilon| - 1 \|_{L^2} \leq C\sqrt{\varepsilon}.$$

Set

$$e_{\varepsilon,g_\varepsilon} = \text{Min}\{E_\varepsilon(u); u \in H_{g_\varepsilon}^1(G; \mathbf{R}^2)\}.$$

Theorem 6'. — Assume (5.9), (5.10) and (5.11). Then we have

$$(5.12) \quad e_{\varepsilon,g_\varepsilon} = \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)) \text{ as } \varepsilon \rightarrow 0.$$

The main ingredients in the proof of (5.12) are the following Lemmas 22 and 23.

Lemma 22. — Let $\varphi \in H^{1/2}(\Omega; \mathbf{R}^2)$ and let $u(= u_\varepsilon)$ be the solution of the linear problem

$$(5.13) \quad -\Delta u + \frac{1}{\varepsilon^2} u = 0 \quad \text{in } G,$$

$$(5.14) \quad u = \varphi \quad \text{on } \Omega = \partial G.$$

Then, for sufficiently small $\varepsilon > 0$,

$$(5.15) \quad \int_G |\nabla u|^2 + \frac{1}{\varepsilon^2} \int_G |u|^2 \leq C_G \left(|\varphi|_{H^{1/2}(\Omega)}^2 + \frac{1}{\varepsilon} \int_\Omega |\varphi|^2 \right).$$

Proof of Lemma 22. — Let Φ be the harmonic extension of φ and fix some $\zeta \in C_0^\infty(\mathbf{R})$ with $\zeta(0) = 1$. Set

$$v(x) = \Phi(x)\zeta(\text{dist}(x, \Omega)/\varepsilon).$$

Using, for $0 < \delta < \delta_0(\mathbf{G})$, the standard estimate

$$\int_{\{x; \text{dist}(x, \Omega) = \delta\}} \Phi^2 \leq C \int_{\Omega} \varphi^2,$$

it is easy to see that, for $0 < \varepsilon < \varepsilon_0(\mathbf{G})$, we have

$$\int_{\mathbf{G}} |\nabla v|^2 + \frac{1}{\varepsilon^2} \int_{\mathbf{G}} |v|^2 \leq C_{\mathbf{G}} \left(|\varphi|_{\mathbf{H}^{1/2}}^2 + \frac{1}{\varepsilon} \int_{\mathbf{G}} |\varphi|^2 \right),$$

and the conclusion follows, since u is a minimizer so that,

$$\int_{\mathbf{G}} |\nabla u|^2 + \frac{1}{\varepsilon^2} \int_{\mathbf{G}} |u|^2 \leq \int_{\mathbf{G}} |\nabla v|^2 + \frac{1}{\varepsilon^2} \int_{\mathbf{G}} |v|^2.$$

For later use, we mention a related estimate, whose proof is similar and left to the reader:

Lemma 22'. — For $0 < \varepsilon < \varepsilon_0(\mathbf{G})$, set

$$\mathbf{G}_\varepsilon = \{x \in \mathbf{R}^3 \setminus \mathbf{G}; \text{dist}(x, \Omega) < \varepsilon\}.$$

Let $\varphi \in \mathbf{H}^{1/2}(\Omega; \mathbf{R}^2)$ and let $u(= u_\varepsilon)$ be the solution of the linear problem

$$(5.16) \quad -\Delta u + \frac{1}{\varepsilon^2} u = 0 \quad \text{in } \mathbf{G}_\varepsilon,$$

$$(5.17) \quad u = \varphi \quad \text{on } \Omega = \partial \mathbf{G},$$

$$(5.18) \quad u = 0 \quad \text{on } \partial \mathbf{G}_\varepsilon \setminus \partial \mathbf{G}.$$

Then

$$(5.19) \quad \int_{\mathbf{G}_\varepsilon} |\nabla u|^2 + \frac{1}{\varepsilon^2} \int_{\mathbf{G}_\varepsilon} |u|^2 \leq C_{\mathbf{G}} \left(|\varphi|_{\mathbf{H}^{1/2}}^2 + \frac{1}{\varepsilon} \int_{\Omega} |\varphi|^2 \right).$$

Lemma 23. — Let (g_ε) in $H^{1/2}(\Omega; \mathbf{R}^2)$ satisfy (5.10), (5.11) and

$$(5.20) \quad \|g_\varepsilon\|_{H^{1/2}} \leq C.$$

Then there is (h_ε) in $H^{1/2}(\Omega; S^1)$ such that

$$(5.21) \quad \|h_\varepsilon\|_{H^{1/2}} \leq C$$

and

$$(5.22) \quad \|g_\varepsilon - h_\varepsilon\|_{L^2} \leq C\sqrt{\varepsilon}.$$

Moreover if, in addition,

$$(5.23) \quad g_\varepsilon \rightarrow g \text{ in } H^{1/2},$$

then

$$(5.24) \quad h_\varepsilon \rightarrow g \text{ in } H^{1/2}.$$

Proof. — We divide the proof in 4 steps

Step 1. — Let $g_\varepsilon^1 = g_\varepsilon * P_\varepsilon$ be an ε -smoothing of g_ε .
Clearly

$$(5.25) \quad \|g_\varepsilon - g_\varepsilon^1\|_{L^2} \leq \sqrt{\varepsilon} \|g_\varepsilon\|_{H^{1/2}} \leq C\sqrt{\varepsilon}$$

and from (5.11), (5.25) we have

$$(5.26) \quad \|1 - |g_\varepsilon^1|\|_{L^2} \leq C\sqrt{\varepsilon}.$$

Also

$$(5.27) \quad \|g_\varepsilon^1\|_{H^{1/2}} \leq C,$$

and

$$(5.28) \quad \|g_\varepsilon^1\|_{H^1} \leq C\varepsilon^{-1/2} \|g_\varepsilon\|_{H^{1/2}} \leq C\varepsilon^{-1/2}.$$

Step 2. — Given a point $a \in \mathbf{R}^2$ with $|a| < 1/10$, let $\pi_a : \mathbf{R}^2 \setminus \{a\} \rightarrow S^1$ be the radial projection onto S^1 with vertex at a , i.e.,

$$\pi_a(\xi) = a + \lambda(\xi - a), \quad \xi \in \mathbf{R}^2 \setminus \{a\}$$

where $\lambda \in \mathbf{R}$ is the unique positive solution of

$$|a + \lambda(\xi - a)| = 1.$$

It is also convenient to note that

$$\pi_a(\xi) = j_a^{-1} \left(\frac{\xi - a}{|\xi - a|} \right) \text{ for } \xi \neq a$$

where $j_a : \mathbf{S}^1 \rightarrow \mathbf{S}^1, j_a(z) = \frac{z - a}{|z - a|}$, is a smooth diffeomorphism.

In particular,

$$(5.29) \quad |\mathrm{D}\pi_a(\xi)| \leq \frac{C}{|\xi - a|} \quad \forall \xi \in \mathbf{R}^2 \setminus \{a\},$$

and π_a is lipschitzian on $\{|\xi| \geq 1/2\}$ with a uniform Lipschitz constant (independent of a).

We claim that

$$(5.30) \quad h_{a,\varepsilon} = \pi_a \circ g_\varepsilon^1 : \Omega \rightarrow \mathbf{S}^1$$

satisfies all the required properties for an appropriate choice of $a = a_\varepsilon, |a_\varepsilon| < 1/10$.

For this purpose, it is useful to introduce a smooth function $\psi : [0, \infty) \rightarrow [0, 1]$ such that

$$\psi(t) = \begin{cases} 0 & \text{if } t \leq 1/4, \\ 1 & \text{if } t \geq 1/2, \end{cases}$$

and to write

$$(5.31) \quad h_{a,\varepsilon} = \pi_a(g_\varepsilon^1) \psi(|g_\varepsilon^1|) + \pi_a(g_\varepsilon^1) (1 - \psi(|g_\varepsilon^1|)) = u_{a,\varepsilon} + v_{a,\varepsilon}.$$

Note that, in general, $h_{a,\varepsilon}$ is not well-defined since g_ε^1 may take the value a on a large set. However, if a is chosen to be a *regular value* of g_ε^1 , then

$$\Sigma_\varepsilon = \{x \in \Omega; g_\varepsilon^1(x) = a\}$$

consists of a finite number of points and $h_{a,\varepsilon}$ is smooth on $\Omega \setminus \Sigma_\varepsilon$, and we have, using (5.29),

$$(5.32) \quad |\nabla(\pi_a(g_\varepsilon^1))| \leq C \frac{|\nabla g_\varepsilon^1|}{|g_\varepsilon^1 - a|} \text{ on } \Omega \setminus \Sigma_\varepsilon.$$

Moreover, near every point $\sigma \in \Sigma_\varepsilon$, we have $|g_\varepsilon^1(x) - a| \geq c|x - \sigma|$, $c > 0$, and thus

$$|\nabla(\pi_a(g_\varepsilon^1))| \leq \frac{C_\varepsilon}{|x - \sigma|}.$$

In particular $h_{a,\varepsilon} \in W^{1,p}(\Omega; \mathbf{S}^1)$, $\forall p < 2$.

Clearly, the function $\pi_a(z)\psi(|z|)$ is well-defined and lipschitzian on \mathbf{R}^2 for any a , $|a| < 1/10$, with a uniform Lipschitz constant independent of a . Therefore, (5.27) yields

$$(5.33) \quad \|u_{a,\varepsilon}\|_{H^{1/2}} \leq C \|g_\varepsilon^1\|_{H^{1/2}} \leq C,$$

where C is independent of a and ε .

Next, we turn to $v_{a,\varepsilon}$, which is well-defined only if a is a regular value of g_ε^1 . On $\Omega \setminus \Sigma_\varepsilon$, we have

$$\begin{aligned} |\nabla v_{a,\varepsilon}| &\leq C \frac{|\nabla g_\varepsilon^1|}{|g_\varepsilon^1 - a|} (1 - \psi)(|g_\varepsilon^1|) + |\psi'(|g_\varepsilon^1|)| |\nabla g_\varepsilon^1| \\ &\leq C \frac{|\nabla g_\varepsilon^1|}{|g_\varepsilon^1 - a|} \chi_{[|g_\varepsilon^1| < 1/2]}, \end{aligned}$$

with C independent of a and ε .

We now make use of an averaging device due to H. Federer and W. H. Fleming [FF] and adapted by R. Hardt, D. Kinderlehrer and F. H. Lin [29] in the context of Sobolev maps with values into spheres. Recall that, by Sard's theorem, the regular values of g_ε^1 have full measure and thus

$$(5.34) \quad \int_{B_{1/10}} \int_{\Omega} |\nabla v_{a,\varepsilon}|^p dx da \leq C_p \int_{[|g_\varepsilon^1| < 1/2]} |\nabla g_\varepsilon^1|^p dx, \text{ for any } p < 2.$$

By Hölder, (5.34), (5.26) and (5.28) we find

$$(5.35) \quad \int_{B_{1/10}} \int_{\Omega} |\nabla v_{a,\varepsilon}|^p dx da \leq \|g_\varepsilon^1\|_{H^1}^p |[|g_\varepsilon^1| < 1/2]|^{1-\frac{p}{2}} \leq C\varepsilon^{-\frac{p}{2}} \varepsilon^{1-\frac{p}{2}} \leq C\varepsilon^{1-p}.$$

Next, fix any $1 < p < 2$ and estimate (see e.g. [21])

$$(5.36) \quad \|v_{a,\varepsilon}\|_{H^{1/2}} \leq C \|v_{a,\varepsilon}\|_{L^{p'}}^{1/2} \|v_{a,\varepsilon}\|_{W^{1,p}}^{1/2}.$$

From the definition of ψ we have

$$|v_{a,\varepsilon}| \leq \chi_{[|g_\varepsilon^1| < 1/2]}$$

and, using (5.26), we obtain

$$(5.37) \quad \|v_{a,\varepsilon}\|_{L^{p'}} \leq C\varepsilon^{1/p'}.$$

Substitution of (5.37) and (5.35) in (5.36) yields

$$(5.38) \quad \int_{B_{1/10}} \|v_{a,\varepsilon}\|_{H^{1/2}}^{2p} da \leq C\varepsilon^{p-1}\varepsilon^{1-p} \leq C.$$

In view of (5.38) we may now choose $a = a_\varepsilon \in B_{1/10}$, a regular value of g_ε^1 , such that

$$(5.39) \quad \|v_{a_\varepsilon,\varepsilon}\|_{H^{1/2}} \leq C.$$

Returning to (5.31), and using (5.33) and (5.39), we obtain (5.21) with $h_\varepsilon = h_{a_\varepsilon,\varepsilon}$.

Step 3. — Write $Z_\varepsilon = [|g_\varepsilon^1| > 1/2]$. For any regular value a of g_ε^1 we have

$$\begin{aligned} \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(\Omega)}^2 &= \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(|g_\varepsilon^1| \leq 1/2)}^2 + \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(Z_\varepsilon)}^2 \\ &\leq C\varepsilon + \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(Z_\varepsilon)}^2 \text{ by (5.26)}. \end{aligned}$$

Next we estimate

$$\begin{aligned} \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(Z_\varepsilon)} &\leq \left\| h_{a,\varepsilon} - \frac{g_\varepsilon^1}{|g_\varepsilon^1|} \right\|_{L^2(Z_\varepsilon)} + \left\| \frac{g_\varepsilon^1}{|g_\varepsilon^1|} - g_\varepsilon^1 \right\|_{L^2(Z_\varepsilon)} \\ &= \left\| \pi_a(g_\varepsilon^1) - \pi_a\left(\frac{g_\varepsilon^1}{|g_\varepsilon^1|}\right) \right\|_{L^2(Z_\varepsilon)} + \left\| \frac{g_\varepsilon^1}{|g_\varepsilon^1|} - g_\varepsilon^1 \right\|_{L^2(Z_\varepsilon)}. \end{aligned}$$

Since $\pi_a(\xi)$ is lipschitzian on $[|\xi| \geq 1/2]$ we obtain

$$\begin{aligned} \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(Z_\varepsilon)} &\leq C \left\| g_\varepsilon^1 - \frac{g_\varepsilon^1}{|g_\varepsilon^1|} \right\|_{L^2(Z_\varepsilon)} \leq C \|1 - |g_\varepsilon^1|\|_{L^2(Z_\varepsilon)} \\ &\leq C\sqrt{\varepsilon}, \text{ by (5.26)}. \end{aligned}$$

Therefore

$$(5.40) \quad \|h_{a,\varepsilon} - g_\varepsilon^1\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}$$

with C independent of a and ε .

Combining (5.25) and (5.40) yields

$$\|h_{a,\varepsilon} - g_\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon},$$

which is (5.22) when choosing $a = a_\varepsilon$.

Step 4. — Suppose now, in addition, that $g_\varepsilon \rightarrow g$ in $H^{1/2}$. We claim that $h_\varepsilon \rightarrow g$ in $H^{1/2}$.

Indeed, we have

$$\begin{aligned} \|g_\varepsilon^1\|_{H^1} &\leq \|(g_\varepsilon - g) * P_\varepsilon\|_{H^1} + \|g * P_\varepsilon\|_{H^1} \\ &\leq C\varepsilon^{-1/2}\|g_\varepsilon - g\|_{H^{1/2}} + \|g * P_\varepsilon\|_{H^1} \\ &= o(\varepsilon^{-1/2}). \end{aligned}$$

Returning to (5.35) and (5.38) we now find

$$\int_{B_{1/10}} \int_{\Omega} |\nabla v_{a_\varepsilon}|^p dx da \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

and we may choose a_ε so that

$$\|v_{a_\varepsilon, \varepsilon}\|_{H^{1/2}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

It remains to show that

$$(5.41) \quad u_{a_\varepsilon, \varepsilon} \rightarrow g \text{ in } H^{1/2} \text{ as } \varepsilon \rightarrow 0.$$

Recall that

$$u_{a_\varepsilon, \varepsilon} = \pi_{a_\varepsilon}(g_\varepsilon^1)\psi(|g_\varepsilon^1|) = L_\varepsilon(g_\varepsilon^1),$$

where $L_\varepsilon : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ are lipschitzian maps with a uniform Lipschitz constant.

We have

$$\begin{aligned} \|g_\varepsilon^1 - g\|_{H^{1/2}} &= \|(g_\varepsilon - g) * P_\varepsilon + (g * P_\varepsilon) - g\|_{H^{1/2}} \\ &\leq C\|g_\varepsilon - g\|_{H^{1/2}} + \|(g * P_\varepsilon) - g\|_{H^{1/2}}, \end{aligned}$$

so that

$$(5.42) \quad \|g_\varepsilon^1 - g\|_{H^{1/2}} \rightarrow 0.$$

Finally we use the following claim:

$$(5.43) \quad \begin{cases} \text{If } (k_n) \text{ is a sequence in } H^{1/2}(\Omega; \mathbf{R}^2) \text{ such that } k_n \rightarrow k \text{ in } H^{1/2} \text{ and} \\ \text{L}_n : \mathbf{R}^2 \rightarrow \mathbf{R}^2 \text{ satisfy a uniform Lipschitz condition, then} \\ \text{L}_n(k_n) - L_n(k) \rightarrow 0 \text{ in } H^{1/2}. \end{cases}$$

Proof of (5.43). — It suffices to argue on subsequences. Since

$$|k_n - k|_{\mathbf{H}^{1/2}}^2 = \int_{\Omega} \int_{\Omega} \frac{|k_n(x) - k(x) - k_n(y) + k(y)|^2}{d(x,y)^3} dx dy \rightarrow 0,$$

there is, (modulo a subsequence), some fixed $h(x,y) \in L^1(\Omega \times \Omega)$ such that

$$\frac{|k_n(x) - k_n(y)|^2}{d(x,y)^3} \leq h(x,y), \quad \forall n.$$

We have

$$\begin{aligned} & |\mathbf{L}_n(k_n) - \mathbf{L}_n(k)|_{\mathbf{H}^{1/2}}^2 \\ &= \int_{\Omega} \int_{\Omega} \frac{|\mathbf{L}_n(k_n(x)) - \mathbf{L}_n(k(x)) - \mathbf{L}_n(k_n(y)) + \mathbf{L}_n(k(y))|^2}{d(x,y)^3} dx dy, \end{aligned}$$

and the integrand $\mathbf{I}_n(x,y)$ satisfies

$$\begin{aligned} \mathbf{I}_n(x,y) &\leq C \frac{(|k_n(x) - k_n(y)|^2 + |k(x) - k(y)|^2)}{d(x,y)^3} \\ &\leq Ch(x,y), \end{aligned}$$

and also,

$$\mathbf{I}_n(x,y) \leq C \frac{(|k_n(x) - k(x)|^2 + |k_n(y) - k(y)|^2)}{d(x,y)^3}.$$

Therefore, by dominated convergence,

$$|\mathbf{L}_n(k_n) - \mathbf{L}_n(k)|_{\mathbf{H}^{1/2}} \rightarrow 0.$$

This proves (5.43).

We now return to the proof of (5.41). Applying (5.43) to $\mathbf{L}_n(\xi) = \pi_{a_{\varepsilon_n}}(\xi)\psi(|\xi|)$ and to $k_n = g_{\varepsilon_n}^1 \rightarrow g$ in $\mathbf{H}^{1/2}$ by (5.42), we find that

$$\mathbf{L}_n(g_{\varepsilon_n}^1) - \mathbf{L}_n(g) \rightarrow 0 \text{ in } \mathbf{H}^{1/2}.$$

But $\mathbf{L}_n(g) = g \quad \forall n$ since $|g| = 1$. Thus we are led to $\mathbf{L}_n(g_{\varepsilon_n}^1) \rightarrow g$ in $\mathbf{H}^{1/2}$, which is (5.41).

This completes the proof of Lemma 23.

Remark 5.1. — It is interesting to observe that the construction used in the proof of Lemma 23 gives a simple proof of Rivière’s Lemma 11. In fact, we have a more precise statement. Fix any element $g \in H^{1/2}(\Omega; S^1)$ and apply the construction described above with $g_\varepsilon \equiv g$. The sequence

$$h_\varepsilon = \pi_{a_\varepsilon}(g * P_\varepsilon)$$

satisfies the following properties:

$$(5.44) \quad h_\varepsilon \in W^{1,p}(\Omega; S^1), \quad \forall p < 2, \forall \varepsilon,$$

$$(5.45) \quad h_\varepsilon \rightarrow g \text{ in } H^{1/2} \text{ as } \varepsilon \rightarrow 0,$$

$$(5.46) \quad \begin{cases} h_\varepsilon \text{ is smooth except on a finite set } \Sigma_\varepsilon \subset \Omega \text{ and} \\ |\nabla h_\varepsilon(x)| \leq \frac{C_\varepsilon}{\text{dist}(x, \Sigma_\varepsilon)}, \quad \forall x \in \Omega \setminus \Sigma_\varepsilon, \end{cases}$$

$$(5.47) \quad \begin{cases} \text{for each } \sigma \in \Sigma_\varepsilon, \text{ there is a smooth diffeomorphism } \gamma = \gamma_{\varepsilon, \sigma}, \\ \text{from the unit circle in } T_\sigma(\Omega) \text{ onto } S^1, \text{ such that, assuming} \\ \Omega \text{ flat near } \sigma \text{ (for simplicity), we have} \\ \left| h_\varepsilon(x) - \gamma\left(\frac{x - \sigma}{|x - \sigma|}\right) \right| \leq C_\varepsilon |x - \sigma| \text{ for } x \in \Omega \text{ near } \sigma. \end{cases}$$

Here, $T_\sigma(\Omega)$ denotes the tangent space to Ω at σ . Note that (5.47) implies that $\text{deg}(g, \sigma) = \pm 1$ for each singularity σ .

All the above properties are clear from the proof of Lemma 23, except possibly (5.47). Taylor’s expansion near $\sigma \in \Sigma_\varepsilon$ gives

$$g_\varepsilon^1(x) = g_\varepsilon^1(\sigma) + M(x - \sigma) + O(|x - \sigma|^2)$$

where $g_\varepsilon^1(\sigma) = a_\varepsilon$ and $M = M_{\varepsilon, \sigma} = Dg_\varepsilon^1(\sigma)$ is a bounded invertible linear operator from $T_\sigma(\Omega)$ onto \mathbf{R}^2 (since a_ε is a regular value of g_ε^1). Thus

$$\frac{g_\varepsilon^1(x) - a_\varepsilon}{|g_\varepsilon^1(x) - a_\varepsilon|} = \frac{M(x - \sigma)}{|M(x - \sigma)|} + O(|x - \sigma|)$$

and therefore

$$h_\varepsilon(x) = j_{a_\varepsilon}^{-1} \left(\frac{g_\varepsilon^1(x) - a_\varepsilon}{|g_\varepsilon^1(x) - a_\varepsilon|} \right) = j_{a_\varepsilon}^{-1} \left(\frac{M(x - \sigma)}{|M(x - \sigma)|} \right) + O(|x - \sigma|),$$

where $j_{a_\varepsilon}(\xi) = \frac{\xi - a_\varepsilon}{|\xi - a_\varepsilon|} : S^1 \rightarrow S^1$. This proves (5.47) with

$$\gamma(z) = j_{a_\varepsilon}^{-1} \left(\frac{Mz}{|Mz|} \right), \quad z \in T_\sigma(\Omega).$$

Clearly, γ is a smooth diffeomorphism from the unit circle in $T_\sigma(\Omega)$ onto S^1 . We will present in Appendix B a more precise statement.

Remark 5.2. — The averaging process over a in the proof of Lemma 23 can be done on any ball B_ρ , $0 < \rho \leq 1/10$, with ρ possibly depending on ε . In particular, when $g_\varepsilon \rightarrow g$ in $H^{1/2}$, one may choose some special $\rho_\varepsilon \rightarrow 0$ and obtain a corresponding a_ε with $a_\varepsilon \rightarrow 0$. Then

$$\tilde{h}_{a_\varepsilon, \varepsilon} = \frac{g_\varepsilon^1 - a_\varepsilon}{|g_\varepsilon^1 - a_\varepsilon|}$$

has all the desired properties without having to consider

$$h_{a_\varepsilon, \varepsilon} = j_{a_\varepsilon}^{-1} \tilde{h}_{a_\varepsilon, \varepsilon}.$$

The argument is similar, with a minor modification in Step 3.

Proof of Theorem 6'. — Let $k_\varepsilon \in H^{1/2}(\Omega; \mathbf{R}^2)$ with $|k_\varepsilon| \leq 1$. We claim that

$$(5.48) \quad \ell_{\varepsilon, k_\varepsilon} \leq C_\Omega \left(|k_\varepsilon|_{H^{1/2}}^2 + \frac{1}{\varepsilon} \|k_\varepsilon - 1\|_{L^2}^2 \right).$$

Indeed, let $u = u_\varepsilon$ be the solution of (5.13), (5.14) corresponding to $\varphi = k_\varepsilon - 1$. Using the function $(u_\varepsilon + 1)$ as a test function in the definition of $\ell_{\varepsilon, k_\varepsilon}$, we find

$$(5.49) \quad \ell_{\varepsilon, k_\varepsilon} \leq \frac{1}{2} \int_G |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} \int_G (|u_\varepsilon + 1|^2 - 1)^2.$$

From (5.15), we have

$$(5.50) \quad \int_G |\nabla u_\varepsilon|^2 \leq C \left(|k_\varepsilon|_{H^{1/2}}^2 + \frac{1}{\varepsilon} \|k_\varepsilon - 1\|_{L^2}^2 \right).$$

On the other hand, by the maximum principle, we have

$$\|u_\varepsilon\|_{L^\infty(G)} \leq \|k_\varepsilon - 1\|_{L^\infty(\Omega)} \leq 2,$$

and thus, by (5.15),

$$(5.51) \quad \begin{aligned} \int_G (|u_\varepsilon + 1|^2 - 1)^2 &= \int_G (|u_\varepsilon + 1| - 1)^2 (|u_\varepsilon + 1| + 1)^2 \leq 16 \int_G |u_\varepsilon|^2 \\ &\leq C\varepsilon^2 \left(|k_\varepsilon|_{H^{1/2}}^2 + \frac{1}{\varepsilon} \|k_\varepsilon - 1\|_{L^2}^2 \right). \end{aligned}$$

Combining (5.49), (5.50) and (5.51) yields (5.48).

Next, we write, using h_ε from Lemma 23,

$$g_\varepsilon = (g_\varepsilon \bar{h}_\varepsilon)(h_\varepsilon \bar{g})g$$

and apply (5.1) to find

$$(5.52) \quad \ell_{\varepsilon, g_\varepsilon} \leq (1 + \delta)\ell_{\varepsilon, g} + C(\delta)(\ell_{\varepsilon, h_\varepsilon \bar{g}} + \ell_{\varepsilon, g_\varepsilon \bar{h}_\varepsilon}).$$

We deduce from (5.48) (applied to $k_\varepsilon = g_\varepsilon \bar{h}_\varepsilon$) that

$$(5.53) \quad \begin{aligned} \ell_{\varepsilon, g_\varepsilon \bar{h}_\varepsilon} &\leq C \left(|g_\varepsilon \bar{h}_\varepsilon|_{H^{1/2}}^2 + \frac{1}{\varepsilon} \|g_\varepsilon \bar{h}_\varepsilon - 1\|_{L^2}^2 \right) \\ &\leq C \left(|g_\varepsilon|_{H^{1/2}}^2 + |h_\varepsilon|_{H^{1/2}}^2 + \frac{1}{\varepsilon} \|g_\varepsilon - h_\varepsilon\|_{L^2}^2 \right) \leq C. \end{aligned}$$

Applying (5.4) (with g replaced by $h_\varepsilon \bar{g}$) yields

$$(5.54) \quad \ell_{\varepsilon, h_\varepsilon \bar{g}} \leq C|h_\varepsilon \bar{g}|_{H^{1/2}}^2(1 + \log(1/\varepsilon)).$$

Recall that $|h_\varepsilon \bar{g}|_{H^{1/2}} \rightarrow 0$ as $\varepsilon \rightarrow 0$ (by (5.24)). By Theorem 6, we know that

$$(5.55) \quad \ell_{\varepsilon, g} = \pi L(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)).$$

Combining (5.52)–(5.55) we finally obtain

$$\limsup_{\varepsilon \rightarrow 0} \frac{\ell_{\varepsilon, g_\varepsilon}}{\log(1/\varepsilon)} \leq \pi L(g)(1 + \delta), \quad \forall \delta > 0.$$

The lower bound

$$\liminf_{\varepsilon \rightarrow 0} \frac{\ell_{\varepsilon, g_\varepsilon}}{\log(1/\varepsilon)} \geq \pi L(g)(1 - \delta), \quad \forall \delta > 0,$$

is deduced in the same way via (5.2). This completes the proof of Theorem 6'.

6. $W^{1,p}(G)$ compactness for $p < 3/2$ and $g \in H^{1/2}$. Proof of Theorem 7'

Proof of Theorem 7'. — The estimate

$$\|u_\varepsilon\|_{W^{1,p}(G)} \leq C_p, \quad \forall 1 \leq p < 3/2,$$

was established in [5]. We will now show that a simple adaptation of the argument there yields compactness. We rely on the following

Lemma 24. — *The family $(u_\varepsilon \wedge du_\varepsilon)$ is compact in $L^p(G)$, $1 \leq p < 3/2$.*

Proof of Lemma 24. — Let $X_\varepsilon = u_\varepsilon \wedge du_\varepsilon$. Since $\operatorname{div}(X_\varepsilon) = 0$, we may write $X_\varepsilon = \operatorname{curl} H_\varepsilon$. As explained in Section 3 of [5], we may choose H_ε of the form $H_\varepsilon = H_\varepsilon^1 + H^2$. Here $H^2 \in W^{1,p}(G)$, $1 \leq p < 3/2$, depends only on g , while H_ε^1 is a linear operator acting on X_ε satisfying the estimate

$$\|H_\varepsilon^1\|_{W^{1,p}(G)} \leq C_p \|dX_\varepsilon\|_{[W^{1,q}(G)]^*}, \quad 1 \leq p < 3/2, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore, it suffices to prove that (dX_ε) is relatively compact in $[W^{1,q}(G)]^*$.

For $1 \leq p < 3/2$ and $\frac{1}{p} + \frac{1}{q} = 1$, let $0 < \beta < \alpha = 1 - \frac{3}{q}$. Then the imbedding $W^{1,q}(G) \subset C^{0,\beta}(\overline{G})$ is compact. Hence the imbedding $(C^{0,\beta}(\overline{G}))^* \subset (W^{1,q}(G))^*$ is compact. The conclusion of Lemma 24 follows now easily from the bound $\|dX_\varepsilon\|_{[C^{0,\beta}(\overline{G})]^*} \leq C$ derived in [5]; see Theorem 2bis in [5].

Proof of Theorem 7' completed. — Let $A = A_\varepsilon = \{x \in G; |u_\varepsilon(x)| \leq 1/2\}$. Since $E_\varepsilon(u_\varepsilon) \leq C \log(1/\varepsilon)$, we have $|A_\varepsilon| \leq C\varepsilon^2 \log(1/\varepsilon)$. In $G \setminus A_\varepsilon$, we have

$$(6.1) \quad du_\varepsilon = \frac{iu_\varepsilon}{|u_\varepsilon|^2} u_\varepsilon \wedge du_\varepsilon + \frac{u_\varepsilon}{|u_\varepsilon|} d|u_\varepsilon|.$$

We may thus write in G

$$du_\varepsilon = \chi_{A_\varepsilon} du_\varepsilon + \chi_{G \setminus A_\varepsilon} \left(\frac{iu_\varepsilon}{|u_\varepsilon|^2} u_\varepsilon \wedge du_\varepsilon + \frac{u_\varepsilon}{|u_\varepsilon|} d|u_\varepsilon| \right).$$

Note that

$$\int_{A_\varepsilon} |du_\varepsilon|^p \leq \left(\int_{A_\varepsilon} |du_\varepsilon|^2 \right)^{p/2} |A_\varepsilon|^{1-p/2} \xrightarrow{\varepsilon} 0, \quad 1 \leq p < 2.$$

Recall the following estimate (see [9], Proposition VI. 4):

$$\int_G |d|u_\varepsilon||^p \xrightarrow{\varepsilon} 0, \quad 1 \leq p < 2.$$

Applying (6.1) and Lemma 24 we see that (u_ε) is bounded in $W^{1,p}$, $p < 3/2$. In particular, up to a subsequence, we have $u_\varepsilon \xrightarrow{\varepsilon} u_0$ a.e. for some u_0 . Moreover, we see that $|u_\varepsilon| \xrightarrow{\varepsilon} 1$ a.e., since

$$\frac{1}{\varepsilon^2} \int_G (1 - |u_\varepsilon|^2)^2 \leq C \log(1/\varepsilon),$$

so that $|u_0| = 1$. Thus, up to a subsequence, we find

$$du_\varepsilon - \iota u_0(u_\varepsilon \wedge du_\varepsilon) \xrightarrow{\varepsilon} 0 \text{ in } L^p, \quad 1 \leq p < 2.$$

Finally, Lemma 24 implies that, up to a further sequence, (du_ε) converges in $L^p(G)$, $1 \leq p < 3/2$.

The proof of Theorem 7' is complete.

As in the case of Theorem 6, Theorem 7' generalizes to the situation where the boundary data is not fixed anymore:

Theorem 7''. — Assume that the maps $g_\varepsilon \in H^{1/2}(\Omega; \mathbf{R}^2)$ are such that:

$$(6.2) \quad |g_\varepsilon|_{H^{1/2}} \leq C,$$

$$(6.3) \quad |g_\varepsilon| \leq 1 \quad \text{on } \Omega,$$

and

$$(6.4) \quad \| |g_\varepsilon| - 1 \|_{L^2} \leq C\sqrt{\varepsilon}.$$

Let u_ε be a minimizer of E_ε in $H_{g_\varepsilon}^1(G; \mathbf{R}^2)$. Then $E_\varepsilon(u_\varepsilon) \leq C \log(1/\varepsilon)$ and (u_ε) is relatively compact in $W^{1,p}(G)$, $1 \leq p < 3/2$.

An easy variant of the proof of Theorem 6' yields the bound $E_\varepsilon(u_\varepsilon) \leq C \log(1/\varepsilon)$. To establish compactness in $W^{1,p}$ we rely on the following variant of Lemma 24:

Lemma 24'. — The family $(u_\varepsilon \wedge du_\varepsilon)$ is compact in $L^p(G)$, $1 \leq p < 3/2$.

Proof of Lemma 24'. — With $X_\varepsilon = u_\varepsilon \wedge du_\varepsilon$, we may write $X_\varepsilon = \text{curl } H_\varepsilon$, where H_ε is a linear operator acting on $(X_\varepsilon, g_\varepsilon \wedge d_T g_\varepsilon)$ and satisfying the estimate

$$\begin{aligned} \|H_\varepsilon\|_{W^{1,p}} &\leq C(\|dX_\varepsilon\|_{[W^{1,q}(G)]^*} + \|g_\varepsilon \wedge d_T g_\varepsilon\|_{[W^{1-1/q,q}(\Omega)]^*}), \\ 1 \leq p < 3/2, \quad \frac{1}{p} + \frac{1}{q} &= 1 \end{aligned}$$

(see [5]). Here, d_T stands for the tangential differential operator on Ω .

The proof of Lemma 2 in [5] implies that $(g_\varepsilon \wedge d_T g_\varepsilon)$ is bounded in $[W^{\sigma,q}(\Omega)]^*$ provided $\sigma > 1/2$ and $\sigma q > 2$. If we choose $\sigma > 1/2$ such that $\frac{2}{q} < \sigma < 1 - \frac{1}{q}$, we find that $(g_\varepsilon \wedge d_T g_\varepsilon)$ is compact in $[W^{1-1/q,q}(\Omega)]^*$.

It remains to prove that (dX_ε) is compact in $[W^{1,q}(G)]^*$. As in the proof of Lemma 24, it suffices to prove that (dX_ε) is bounded in $[C^{0,\alpha}(\overline{G})]^*$ for $0 < \alpha < 1$.

For this purpose, we construct an appropriate extension of u_ε to a larger domain. Let, for $0 < \varepsilon < \varepsilon_0(\mathbf{G})$, Π_ε be the projection onto Ω of the set

$$\Omega_\varepsilon = \{x \in \mathbf{R}^3 \setminus \Omega; \text{dist}(x, \Omega) = \varepsilon\}.$$

Set $\tilde{h}_\varepsilon = h_\varepsilon \circ \Pi_\varepsilon \in \mathbf{H}^{1/2}(\Omega_\varepsilon)$ (where h_ε is defined in Lemma 23) and let \mathbf{K}_ε be the harmonic extension of \tilde{h}_ε to

$$\mathbf{G} \cup \{x \in \mathbf{R}^3; \text{dist}(x, \Omega) < \varepsilon\}.$$

By standard estimates, we have

$$\|h_\varepsilon - \mathbf{K}_{\varepsilon|\Omega}\|_{L^2} \leq C_{\mathbf{G}} |h_\varepsilon|_{\mathbf{H}^{1/2}} \varepsilon^{1/2},$$

so that

$$\|g_\varepsilon - \mathbf{K}_{\varepsilon|\Omega}\|_{L^2} \leq C\varepsilon^{1/2}.$$

By Lemma 22' applied to $\varphi = g_\varepsilon - \mathbf{K}_{\varepsilon|\Omega}$, we may find a map $v_\varepsilon : \mathbf{G}_\varepsilon \rightarrow \mathbf{C}$ such that

$$\begin{aligned} \int_{\mathbf{G}_\varepsilon} |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_{\mathbf{G}_\varepsilon} |v_\varepsilon|^2 &\leq C, \\ v_\varepsilon &= g_\varepsilon - \mathbf{K}_{\varepsilon|\Omega} \quad \text{on } \Omega, \quad v_\varepsilon = 0 \quad \text{on } \Omega_\varepsilon \end{aligned}$$

and

$$|v_\varepsilon| \leq 2 \quad \text{in } \mathbf{G}_\varepsilon.$$

Set

$$\mathbf{U}_\varepsilon = \begin{cases} u_\varepsilon, & \text{in } \mathbf{G} \\ v_\varepsilon + \mathbf{K}_\varepsilon, & \text{in } \mathbf{G}_\varepsilon \end{cases},$$

which satisfies $\mathbf{U}_\varepsilon = \tilde{h}_\varepsilon$ on Ω_ε . Since, for $0 < \delta < \varepsilon$, we have

$$\begin{aligned} \int_{\Omega_\delta} (1 - |\mathbf{U}_\varepsilon|^2)^2 &\leq \int_{\Omega_\delta} (|1 - |\mathbf{K}_\varepsilon|| + |v_\varepsilon|)^2 (1 + |\mathbf{K}_\varepsilon| + |v_\varepsilon|)^2 \\ &\leq 32 \int_{\Omega_\delta} (|h_\varepsilon \circ \Pi_\delta - \mathbf{K}_\varepsilon|^2 + |v_\varepsilon|^2), \end{aligned}$$

we find by standard estimates that

$$(6.5) \quad \int_{\Omega_\delta} (1 - |\mathbf{U}_\varepsilon|^2)^2 \leq C \left(\varepsilon |h_\varepsilon|_{\mathbf{H}^{1/2}}^2 + \int_{\Omega_\delta} |v_\varepsilon|^2 \right).$$

Integration of (6.5) over δ combined with the obvious bound

$$\|\mathbf{K}_\varepsilon\|_{H^1(G \cup G_\varepsilon)} \leq C$$

yields

$$(6.6) \quad E_\varepsilon(\mathbf{U}_\varepsilon; G_\varepsilon) \leq C.$$

As we already mentioned, an easy variant of the proof of Theorem 6' gives

$$E_\varepsilon(u_\varepsilon; G) \leq C \log(1/\varepsilon)$$

and thus

$$(6.7) \quad E_\varepsilon(\mathbf{U}_\varepsilon; G \cup G_\varepsilon) \leq C \log(1/\varepsilon).$$

Let now $R > 0$ be such that

$$\overline{G \cup G_{\varepsilon_0(G)}} \subset B_R.$$

A straightforward adaptation of Proposition 4 in [5] implies that, for $0 < \varepsilon < \varepsilon_0(G)$, there is a map $w_\varepsilon \in H^1(B_R \setminus (G \cup G_\varepsilon))$ such that

$$(6.8) \quad w_\varepsilon = \tilde{h}_\varepsilon \quad \text{on } \Omega_\varepsilon, \quad w_\varepsilon = 1 \quad \text{on } \partial B_R,$$

$$(6.9) \quad E_\varepsilon(w_\varepsilon) \leq C \log(1/\varepsilon),$$

and

$$(6.10) \quad \int_{B_R \setminus (G \cup G_\varepsilon)} |\text{Jac } w_\varepsilon| \leq C.$$

Set

$$V_\varepsilon = \begin{cases} \mathbf{U}_\varepsilon, & \text{in } G \cup G_\varepsilon \\ w_\varepsilon, & \text{in } B_R \setminus (G \cup G_\varepsilon) \end{cases}.$$

By (6.7) and (6.9), we have

$$E_\varepsilon(V_\varepsilon; B_R) \leq C \log(1/\varepsilon),$$

so that $\text{Jac } V_\varepsilon$ is bounded in $[C_{\text{loc}}^{0,\alpha}(B_R)]^*$ for $0 < \alpha < 1$ (see [33]). As in the proof of Theorem 2bis in [5], we may now establish the boundedness of dX_ε in $[C^{0,\alpha}(\overline{G})]^*$ for

$0 < \alpha < 1$. Indeed, let $\delta > 0$ be sufficiently small. For $\zeta \in C^{0,\alpha}(\overline{G}; \wedge^1(\mathbf{R}))$, let ψ be an extension of ζ to \mathbf{R}^3 such that $\|\psi\|_{C^{0,\alpha}(\mathbf{R}^3)} \leq C\|\zeta\|_{C^{0,\alpha}(\overline{G})}$ and $\text{Supp } \psi \subset \overline{B_{R-\delta}}$. Then

$$\begin{aligned} \left| \int_G dX_\varepsilon \wedge \zeta \right| &\leq \left| \int_{B_R} d(V_\varepsilon \wedge dV_\varepsilon) \wedge \psi \right| + \int_{B_R \setminus G} \left| d(V_\varepsilon \wedge dV_\varepsilon) \wedge \psi \right| \\ &\leq C_\alpha \|\psi\|_{C^{0,\alpha}(\overline{G})} + \|\psi\|_{L^\infty} \int_{B_R \setminus G} |\text{Jac } V_\varepsilon| \leq C\|\zeta\|_{C^{0,\alpha}(\overline{G})}, \end{aligned}$$

by (6.6) and (6.10).

The proof of Lemma 24' is complete.

Proof of Theorem 7'. — An inspection of the proof of Theorem 7' shows that it suffices to establish the estimate

$$(6.11) \quad \int_G |\nabla |u_\varepsilon||^p \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad \forall 1 \leq p < 2.$$

We adapt the proof of Proposition VI.4 in [9]. Set $\eta = \eta_\varepsilon = 1 - |u_\varepsilon|^2$, which satisfies

$$(6.12) \quad -\Delta \eta + \frac{2}{\varepsilon^2} |u_\varepsilon|^2 \eta = 2|\nabla u_\varepsilon|^2 \quad \text{in } G,$$

$$(6.13) \quad \eta \geq 0 \quad \text{on } \Omega.$$

Let $\tilde{\eta}$ be the solution of

$$(6.14) \quad -\Delta \tilde{\eta} + \frac{2}{\varepsilon^2} |u_\varepsilon|^2 \tilde{\eta} = 2|\nabla u_\varepsilon|^2 \quad \text{in } G,$$

$$(6.15) \quad \tilde{\eta} = 0 \quad \text{on } \Omega,$$

so that

$$(6.16) \quad 1 - |u_\varepsilon|^2 = \eta \geq \tilde{\eta} \geq 0,$$

by the maximum principle. Set $\bar{\eta} = \text{Min}(\tilde{\eta}, \varepsilon^{1/2})$. Multiplying (6.14) by $\bar{\eta}$, we find

$$(6.17) \quad \int_{\{\tilde{\eta} < \varepsilon^{1/2}\}} |\nabla \tilde{\eta}|^2 \leq 2\varepsilon^{1/2} \int_G |\nabla u_\varepsilon|^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

On the other hand, we have

$$(6.18) \quad \{x; \tilde{\eta}(x) \geq \varepsilon^{1/2}\} \subset \{x; |u_\varepsilon(x)|^2 \leq 1 - \varepsilon^{1/2}\}.$$

Set $\zeta = \eta - \tilde{\eta}$, which satisfies

$$(6.19) \quad -\Delta\zeta + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\zeta = 0 \quad \text{in } G,$$

$$(6.20) \quad \zeta = \varphi_\varepsilon \quad \text{on } \Omega,$$

where $\varphi_\varepsilon = 1 - |g_\varepsilon|^2$. Clearly, we have $|\varphi_\varepsilon|_{H^{1/2}} \leq C$ and by (6.4)

$$(6.21) \quad \|\varphi_\varepsilon\|_{L^2} \leq C\varepsilon^{1/2}.$$

By the proof of Lemma 22, we find that

$$(6.22) \quad \int_G |\nabla\zeta|^2 \leq C.$$

We claim that

$$(6.23) \quad \int_G |\nabla\zeta|^\rho \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad \forall \rho < 2.$$

Indeed, by the maximum principle, $0 \leq \zeta \leq \hat{\zeta}$ where $\hat{\zeta}$ is the solution of

$$\begin{aligned} -\Delta\hat{\zeta} &= 0 \quad \text{in } G, \\ \hat{\zeta} &= \varphi_\varepsilon \quad \text{on } \Omega. \end{aligned}$$

In particular, from (6.21) we see that

$$(6.24) \quad \int_G |\hat{\zeta}|^2 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Let $\chi \in C_0^\infty(G)$ with $0 \leq \chi \leq 1$ on G . Multiplying (6.19) by $\zeta\chi$ and integrating we obtain

$$\int_G |\nabla\zeta|^2 \chi \leq \frac{1}{2} \int_G \zeta^2 |\Delta\chi| \leq \frac{1}{2} \int_G \hat{\zeta}^2 |\Delta\chi|.$$

Combining this with (6.24) yields

$$(6.25) \quad \int_G |\nabla\zeta|^2 \chi \rightarrow 0 \quad \forall \chi \in C_0^\infty(G), 0 \leq \chi \leq 1.$$

From (6.22) and (6.25) we deduce (6.23).

We now claim that

$$(6.26) \quad \int_G |\nabla \eta|^p \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad \forall p < 2.$$

Since $\eta = \zeta + \tilde{\eta}$, in view of (6.17) and (6.23) it suffices to prove that

$$\int_{Z_\varepsilon} |\nabla \tilde{\eta}|^p \rightarrow 0.$$

where $Z_\varepsilon = \{x; |u_\varepsilon(x)|^2 \leq 1 - \varepsilon^{1/2}\}$. But

$$\int_G (1 - |u_\varepsilon|^2)^2 \leq C\varepsilon^2 \log(1/\varepsilon),$$

and thus

$$(6.27) \quad |Z_\varepsilon| \leq C\varepsilon \log(1/\varepsilon),$$

so that, by Hölder and (6.14)–(6.15),

$$(6.28) \quad \begin{aligned} \int_{Z_\varepsilon} |\nabla \tilde{\eta}|^p &\leq \|\nabla \tilde{\eta}\|_{L^2}^p |Z_\varepsilon|^{(2-p)/2} \\ &\leq C \|\nabla u_\varepsilon\|_{L^2}^p |Z_\varepsilon|^{(2-p)/2} \leq C\varepsilon^{(2-p)/2} (\log(1/\varepsilon)) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Hence we have established (6.26). Similarly,

$$(6.29) \quad \int_{Z_\varepsilon} |\nabla u_\varepsilon|^p \leq \|\nabla u_\varepsilon\|_{L^2}^p |Z_\varepsilon|^{(2-p)/2} \leq C\varepsilon^{(2-p)/2} \log(1/\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Finally, we note that, for ε sufficiently small, we have

$$(6.30) \quad |\nabla |u_\varepsilon|| \leq |\nabla u_\varepsilon| \chi_{Z_\varepsilon} + |\nabla \eta|,$$

so that (6.11) follows by combining (6.26), (6.29) and (6.30).

The proof of Theorem 7'' is complete.

7. Improved interior estimates. $W_{\text{loc}}^{1,p}(G)$ compactness for $p < 2$ and $g \in H^{1/2}$. Proof of Theorem 8

Remark 7.1. — As in the proof of Theorems 7' and 7'', it suffices to establish the estimate

$$(7.1) \quad \|u_\varepsilon \wedge du_\varepsilon\|_{L^p(K)} \leq C, \quad 3/2 \leq p < 2, \quad K \text{ compact in } G.$$

Estimate (7.1) will be proved under the following assumptions:

$$E_\varepsilon(u_\varepsilon) \leq C \log(1/\varepsilon)$$

and

$$u_\varepsilon \text{ is bounded in } W^{1,r}(G), \quad \text{for some } 4/3 < r < 3/2.$$

In view of Theorems 6, 7 and of their variants, we find that Theorem 8 extends to minimizers u_ε of E_ε when the variable boundary conditions satisfy (6.1)–(6.3).

Proof of Theorem 8. — In what follows, we establish (7.1) when K is any compact subset of the unit ball B .

Fix some $3/2 \leq p < 2$ and $0 < \gamma < 1$. Fix

$$(7.2) \quad 4/3 < r < 3/2.$$

Denote $u = u_\varepsilon$. Since, by Theorems 6 and 7, we have

$$\|u\|_{W^{1,r}(B)} \leq C \quad \text{and} \quad \|u\|_{H^1(B)} \leq C(\log(1/\varepsilon))^{1/2},$$

we may choose

$$1 - \gamma < \rho < 1 - \gamma/2$$

such that

$$(7.3) \quad \|u\|_{W^{1,r}(\partial B_\rho)} \leq C_\gamma$$

and

$$(7.4) \quad \|u\|_{H^1(B_\rho)} \leq C_\gamma(\log(1/\varepsilon))^{1/2}.$$

Set now $p = 2 - s$, so that $s > 0$ and the conjugate exponent of p is

$$(7.5) \quad 2 < q = \frac{2-s}{1-s} \leq 3.$$

Perform on B_ρ a Hodge decomposition

$$\frac{u \wedge du}{|u \wedge du|^s} = d^*k + dL,$$

where

$$(7.6) \quad L = 0\text{-form}, \quad L = 0 \text{ on } \partial B_\rho$$

and

$$(7.7) \quad k = 2\text{-form}, \quad \|k\|_{W^{1,q}} \leq C \left\| \frac{u \wedge du}{|u \wedge du|^s} \right\|_q = C \|u \wedge du\|_\rho^{1-s} \\ = C \|u \wedge du\|_\rho^{p-1};$$

here, we use the notation $\| \cdot \|_\rho = \| \cdot \|_{L^p(B_\rho)}$.

Recalling the fact that $\operatorname{div}(u \wedge du) = 0$, we find that

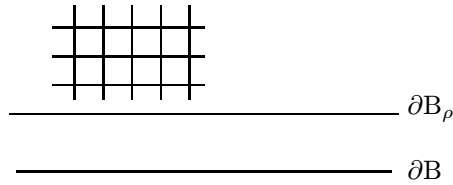
$$(7.8) \quad \|u \wedge du\|_\rho^p = \int_{B_\rho} (d^*k) \cdot (u \wedge du) + \int_{B_\rho} dL \cdot (u \wedge du) = \int_{B_\rho} (d^*k) \wedge (u \wedge du),$$

since, by (7.6), we have $L = 0$ on ∂B_ρ .

Let

$$(7.9) \quad \delta = \varepsilon^{10^{-3}}.$$

Assuming, for simplicity, ∂B to be flat near some point, consider a partition of B_ρ in δ -cubes Q



(we will average over translates of this grid in later estimates).

Define

$$\mathcal{F} = \left\{ Q | Q \cap \left[|u| < \frac{1}{2} \right] \neq \emptyset \right\}.$$

We are going to estimate the number of cubes in \mathcal{F} with the help of the η -ellipticity property of T. Rivière [37], that we state in a more precise form, proved in [8]:

Lemma 25. — *Let u_ε be a minimizer of E_ε in B_R with respect to its own boundary condition. Then there is a universal constant C such that, for every $\eta > 0$, $0 < \varepsilon < 1$ and $R > 0$ we have*

$$E_\varepsilon(u_\varepsilon; B_R) \leq \eta R \log(R/\varepsilon) \Rightarrow |u_\varepsilon(0)| \geq 1 - C\eta^{1/60}.$$

Let, for $Q \in \mathcal{F}$, \tilde{Q} be the cube having the same center as Q and the size twice the one of Q . From the η -ellipticity property, we have

$$(7.10) \quad \int_{\tilde{Q}} e_\varepsilon(u) \geq C\delta \log(\delta/\varepsilon) \sim \delta \log(1/\varepsilon), \quad \forall Q \in \mathcal{F},$$

so that

$$(7.11) \quad \#\mathcal{F} \leq C\delta^{-1} \quad \text{and} \quad \left| \bigcup_{Q \in \mathcal{F}} Q \right| \leq C\delta^2.$$

Define

$$(7.12) \quad \Omega = B_\rho \setminus \bigcup_{Q \in \mathcal{F}} Q,$$

on which $|u| > 1/2$.

We have, by (7.8),

$$(7.13) \quad \begin{aligned} \|u \wedge du\|_\rho^p &= \int_{\Omega} (d * k) \wedge (u \wedge du) + \int_{B_\rho \setminus \Omega} (d * k) \wedge (u \wedge du) \\ &\leq \int_{\Omega} (d * k) \wedge (u \wedge du) + 2\|k\|_{W^{1,q}} \|\nabla u\|_2 (B_\rho \setminus \Omega)^{1/2-1/q}. \end{aligned}$$

By (7.7) and (7.11), the second term of (7.13) is bounded by

$$(7.14) \quad C(\log(1/\varepsilon))^{1/2} \cdot \delta^{1-2/q} \|u \wedge du\|_\rho^{1-s} \leq \|u \wedge du\|_\rho^{1-s},$$

provided ε is sufficiently small.

For the first term of (7.13), we use the identity

$$u \wedge du = \frac{u}{|u|} \wedge \left(d \left(\frac{u}{|u|} \right) \right) + \left(1 - \frac{1}{|u|^2} \right) (u \wedge du) \quad \text{in } \Omega$$

and the fact that

$$d \left(\frac{u}{|u|} \wedge \left(d \left(\frac{u}{|u|} \right) \right) \right) = 0,$$

to get

$$(7.15) \quad \int_{\Omega} (d * k) \wedge (u \wedge du) = \int_{\partial\Omega} (*k) \wedge \left(\frac{u}{|u|} \wedge d \left(\frac{u}{|u|} \right) \right) \\ + O(\|k\|_{W^{1,q}} \|\nabla u\|_2 \|1 - |u|^2\|_{2q/(q-2)}).$$

Since $|u| \leq 1$ and

$$\|1 - |u|^2\|_2 \leq 2\varepsilon (E_\varepsilon(u_\varepsilon))^{1/2} \leq C\varepsilon (\log(1/\varepsilon))^{1/2},$$

the second term of (7.15) bounded by

$$(7.16) \quad C \|u \wedge du\|_p^{1-s} (\log(1/\varepsilon))^{1-1/q} \varepsilon^{1-2/q} \leq \|u \wedge du\|_p^{1-s},$$

provided ε is sufficiently small.

Let $\varphi : D = [|z| \leq 1] \rightarrow D$ be a smooth map such that $\varphi(\bar{z}) = \overline{\varphi(z)}$ and $\varphi(z) = z/|z|$ if $|z| > 1/10$. Thus

$$\int_{\partial\Omega} *k \wedge \left(\frac{u}{|u|} \wedge d \left(\frac{u}{|u|} \right) \right) = \int_{\partial B_\rho} *k \wedge (\varphi(u) \wedge d\varphi(u)) \\ - \sum_{Q \in \mathcal{F}_{\partial Q}} \int *k \wedge (\varphi(u) \wedge d\varphi(u)) \\ = (7.17) - (7.18).$$

Using (7.3) and the fact that, by (7.5), we have $q > 2$, we find that

$$(7.19) \quad (7.17) \leq C \|u\|_{W^{1,r}(\partial B_\rho)} \|k\|_{L^{r'}(\partial B_\rho)} \leq C \|k\|_{L^{r'}(\partial B_\rho)} \leq C \|k\|_{H^{1-2/r'}(\partial B_\rho)} \\ \leq C \|k\|_{H^{3/2-2/r'}(B_\rho)} \leq C \|k\|_{W^{1,q}(B_\rho)} \leq C \|u \wedge du\|_p^{1-s}.$$

In order to estimate the term (7.18) we replace, on each cube Q , k by its mean \bar{k}_Q . The error is of the order of

$$\sum_{Q \in \mathcal{F}} \int_{\partial Q} |k - \bar{k}_Q| |\nabla u| \leq \int_{\partial B_\rho} |k| \cdot |\nabla u| + \sum_{\substack{Q \in \mathcal{F} \\ Q \cap \partial B_\rho \neq \emptyset}} |\bar{k}_Q| \int_{\partial Q \cap \partial B_\rho} |\nabla u| \\ + \sum_{Q \in \mathcal{F}} \int_{\partial Q \setminus \partial B_\rho} |k - \bar{k}_Q| |\nabla u| \\ = (7.20) + (7.21) + (7.22).$$

As for (7.17), we find that

$$(7.23) \quad (7.20) \leq C \|u \wedge du\|_p^{1-s}.$$

Since

$$|\tilde{k}_Q| \leq \delta^{-3} \int_Q |k| \leq \delta^{-3/r'} \left(\int_Q |k|^{r'} \right)^{1/r'}$$

and

$$\int_{\partial Q \cap \partial B_\rho} |\nabla u| \leq \delta^{2/r'} \left(\int_{\partial Q \cap \partial B_\rho} |\nabla u|^r \right)^{1/r},$$

we have

$$\begin{aligned} (7.21) &\leq C \delta^{-1/r'} \sum_{\substack{Q \in \mathcal{F} \\ Q \cap \partial B_\rho \neq \emptyset}} \left(\int_Q |k|^{r'} \right)^{1/r'} \left(\int_{\partial Q \cap \partial B_\rho} |\nabla u|^r \right)^{1/r} \\ &\leq C \delta^{-1/r'} \|u\|_{W^{1,r}(\partial B_\rho)} \cdot \left(\int_{\bigcup_{\substack{Q \in \mathcal{F} \\ Q \cap \partial B_\rho \neq \emptyset}} Q} |k|^{r'} \right)^{1/r'} \\ &\leq C \delta^{-1/r'} \left| \bigcup_{Q \in \mathcal{F}, Q \cap \partial B_\rho \neq \emptyset} Q \right|^{1/r' - 1/6} \cdot \|k\|_6. \end{aligned}$$

In view of (7.11) one may clearly choose $1 - \gamma < \rho < 1 - \gamma/2$ such that

$$(7.24) \quad \#\{Q \in \mathcal{F} \mid Q \cap \partial B_\rho \neq \emptyset\} \lesssim 1/\gamma,$$

and therefore

$$\left| \bigcup_{Q \in \mathcal{F}, Q \cap \partial B_\rho \neq \emptyset} Q \right| \leq C \delta^3.$$

This gives

$$(7.25) \quad (7.21) \leq C \delta^{-1/r'} \delta^{3/r' - 1/2} \|k\|_{W^{1,q}} \leq C \delta^{2/r' - 1/2} \|k\|_{W^{1,q}} < \|u \wedge du\|_p^{1-s},$$

provided ε is sufficiently small.

To bound (7.22), we use averaging over the grids. For $\lambda \in \mathbf{R}^3$ with $|\lambda| < \delta$, consider the grid of δ -cubes having λ as one of the vertices and let \mathcal{F}_λ be the corresponding collection of bad cubes. Then

$$\begin{aligned}
\delta^{-3} \int_{|\lambda| < \delta} (7.22) &\leq \delta^{-3} \int_{|\lambda| < \delta} \delta^{-3} \sum_{Q \in \mathcal{F}_\lambda} \int_{\partial Q \setminus \partial B_\rho} dx \int_Q dy |k(x) - k(y)| |\nabla u(x)| \\
&\leq C \delta^{-4} \sum_{Q \in \mathcal{F}_0} \iint_{\tilde{Q} \times \tilde{Q}} dx dy |k(x) - k(y)| |\nabla u(x)| \\
&\leq C \delta^{1/2-6/q} \sum_{Q \in \mathcal{F}_0} \|\nabla u\|_{L^2(\tilde{Q})} \|k(x) - k(y)\|_{L^q(\tilde{Q} \times \tilde{Q})} \\
&\leq C \delta^{-5/q} \|\nabla u\|_{L^2(B_\rho)} \left[\sum_{Q \in \mathcal{F}_0} \int_{\tilde{Q} \times \tilde{Q}} |k(x) - k(y)|^q dx dy \right]^{1/q} \\
&\leq C \delta^{1-2/q} (\log(1/\varepsilon))^{1/2} \left[\sum_{Q \in \mathcal{F}_0} \int_{\tilde{Q}} |\nabla k|^q \right]^{1/q} \\
&\leq \|u \wedge du\|_p^{1-s},
\end{aligned}$$

provided ε is sufficiently small. Therefore, by choosing the proper grid, we may assume that

$$(7.26) \quad (7.22) \leq C \|u \wedge du\|_p^{1-s}.$$

Combining (7.23), (7.25) and (7.26), it follows that

$$(7.27) \quad (7.20) + (7.21) + (7.22) \leq C \|u \wedge du\|_p^{1-s}.$$

By (7.13), (7.14), (7.16) and (7.27), we have

$$(7.28) \quad \|u \wedge du\|_p^p = (7.29) + O(\|u \wedge du\|_p^{1-s}),$$

where

$$(7.29) = - \sum_{Q \in \mathcal{F}_{\partial Q}} \int *k_Q \wedge (\varphi(u) \wedge d\varphi(u)).$$

For $i = 1, 2, 3$, let π_i be the projection onto the axis $0x_i$. For $x_i \in \pi_i(\partial Q)$, let

$$\Gamma_{x_i} = (\pi_i)^{-1}(x_i) \cap \partial Q.$$

Then

$$(7.30) \quad |(7.29)| \leq \sum_{i=1}^3 \sum_{Q \in \mathcal{F}} |\tilde{k}_Q| \int_{\pi_i(Q)} \left| \int_{\Gamma_{x_i}} \varphi(u) \wedge \partial\varphi(u)/\partial\tau \right| dx_i.$$

Denote $\tilde{\Gamma}$ the δ -square with $\partial \tilde{\Gamma} = \Gamma$ and let

$$(7.31) \quad \delta_1 = \delta^3, \delta_2 = \delta^4.$$

Consider “good” sections Γ , i.e., such that

$$(7.32) \quad \text{dist}(\Gamma, [|u| < 1/2]) > \delta_1$$

and, with

$$e_\varepsilon(u) = e_\varepsilon(u)(x) = |\nabla u(x)|^2 + \frac{1}{\varepsilon^2}(1 - |u|^2)^2(x),$$

$$(7.33) \quad \int_{\tilde{\Gamma}} e_\varepsilon(u) < \delta_2 \varepsilon^{-1}.$$

Condition (7.33) implies that

$$(7.34) \quad \frac{1}{\varepsilon^2} \int_{\tilde{\Gamma}} (1 - |u|^2)^2 < \delta_2 \varepsilon^{-1}.$$

Since $|\nabla u| \leq C/\varepsilon$, it follows that the set $\tilde{\Gamma} \cap [|u| < 1/2]$ may be covered by a family \mathcal{G} of ε -squares such that

$$\#\mathcal{G} \leq C_0 \delta_2 / \varepsilon$$

and

$$(7.35) \quad \sum_{S \in \mathcal{G}} \text{length}(S) \leq C_0 \varepsilon \delta_2 / \varepsilon = C_0 \delta_2.$$

We next invoke the following estimate (see the proposition in Section 1 in [39]):

Lemma 26 (Sandier [39]). — *Under the assumptions (7.32) and (7.35) we have, with C_0 the constant in (7.35),*

$$\int_{\tilde{\Gamma} \cap [|u| \geq 1/2]} \left| \nabla \left(\frac{u}{|u|} \right) \right|^2 dx \geq \mathbf{K} |d| \log(\delta_1 / (2C_0 \delta_2)),$$

where d is the degree of $u|_\Gamma$ and \mathbf{K} is some universal constant.

By Lemma 26 and our choice of δ_1, δ_2 , we find that

$$(7.36) \quad \left| \int_{\Gamma} \varphi(u) \wedge d\varphi(u) \right| = \left| \deg \left(\frac{u}{|u|}, \Gamma \right) \right| \leq C \int_{\tilde{\Gamma}} |\nabla u|^2 / \log(1/\varepsilon).$$

On the other hand, recall the monotonicity formula of T. Rivière (see Lemma 2.5 in [37]):

Lemma 27 (Rivière [37]). — *Let $x \in G$. Then, for $0 < r < \text{dist}(x, \Omega)$, the map*

$$r \mapsto \frac{1}{r} \int_{B_r(x)} \left(|\nabla u_\varepsilon(x)|^2 + \frac{3}{2\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \right)$$

is non-increasing.

By combining (7.36) and Lemma 27, we see that the collected contribution of the good sections in the r.h.s. of (7.30) is bounded by

$$(7.37) \quad C \sum_{Q \in \mathcal{F}} |\tilde{k}_Q| \int_Q |\nabla u|^2 / \log(1/\varepsilon) \leq C\delta \sum_{Q \in \mathcal{F}} |\tilde{k}_Q| \lesssim \delta^{-2} \int_{B_\rho} |k| \left(\sum_{Q \in \mathcal{F}} \chi_Q \right).$$

We consider an extension, denoted by h , of $|k|$ to \mathbf{R}^3 , such that

$$\|h\|_{W^{1,q}(\mathbf{R}^3)} \leq C \| |k| \|_{W^{1,q}(B_\rho)}.$$

We estimate the integral in (7.37) using the $(B_{q,q}^1, B_{p,p}^{-1})$ -duality (for the definition of the Besov spaces $B_{p,q}^\sigma$, see e.g. H. Triebel [45]), where

$$(7.38) \quad \|f\|_{B_{r,r}^\sigma} = \left[2^{\sigma r} \|f * P_1\|_r^r + \sum_{j \geq 2} (2^{\sigma j} \|f * P_{2^{-j}} - f * P_{2^{-j+1}}\|_r)^r \right]^{1/r}.$$

We let here $P_1 \geq 0$ be a suitable L^1 -normalized smooth bump function supported in the unit cube of \mathbf{R}^3 , and denote $P_h(x) = h^{-3} P_1(h^{-1}x)$.

On the one hand, since $q > 2$ we have

$$(7.39) \quad \|h\|_{B_{q,q}^1} \leq C \|h\|_{W^{1,q}} \leq C \|k\|_{W^{1,q}} \leq C \|u \wedge du\|_p^{1-s}.$$

Letting $f = \sum_{Q \in \mathcal{F}} \chi_Q$, we estimate next $\|f\|_{B_{p,p}^{-1}}$. Without any loss of generality, we may assume that $B_6 \subset G$.

Assume first that j is such that $1 \geq 2^{-j} \geq \delta$. If $Q_1 \subset B_3$ is a 2^{-j} -cube, then

$$(7.40) \quad \int_{Q_1} e_\varepsilon(u) \leq C 2^{-j} \log(1/\varepsilon),$$

by Lemma 27. On the other hand, if $Q \in \mathcal{F}$, then (7.10) holds. Therefore

$$(7.41) \quad \#\{Q \in \mathcal{F}; Q \subset Q_1\} \leq C2^{-j}\delta^{-1}.$$

Also, if $Q_1 \cap \mathcal{F} \neq \emptyset$, the η -ellipticity lemma implies

$$(7.42) \quad \int_{\tilde{Q}_1} e_\varepsilon(u) \geq C2^{-j} \log(1/\varepsilon),$$

and hence the set $[|u| \leq 1/2]$ intersects at most $C2^j$ cubes Q_1 of size 2^{-j} . Thus

$$(7.43) \quad \begin{aligned} \|(f * P_{2^{-j}}) - (f * P_{2^{-j+1}})\|_p &\lesssim \|f * P_{2^{-j}}\|_p \\ &\lesssim \left\| \sum_{Q_1, Q_1 \cap \mathcal{F} \neq \emptyset} \frac{1}{|Q_1|} \chi_{\tilde{Q}_1} \int_{\tilde{Q}_1} f \right\|_p \\ &\lesssim \left[\sum_{Q_1, Q_1 \cap \mathcal{F} \neq \emptyset} 2^{-3j} (2^{3j} |\tilde{Q}_1 \cap \mathcal{F}|)^p \right]^{1/p} \\ &\lesssim \left[\sum_{Q_1 \cap \mathcal{F} \neq \emptyset} 2^{-3j} (2^{3j} \cdot \delta^3 \cdot 2^{-j} \delta^{-1})^p \right]^{1/p} \text{ by (7.41)} \\ &\lesssim 2^{-2j/p} 2^{2j} \delta^2 = \delta^2 4^{j/q}. \end{aligned}$$

Assume now that $2^{-j} < \delta$. Estimate then

$$|f * (P_{2^{-j}} - P_{2^{-j+1}})| \leq \sum_{Q \in \mathcal{F}} |\chi_Q * (P_{2^{-j}} - P_{2^{-j+1}})|.$$

In this case, it is easy to see that

$$|\chi_Q * (P_{2^{-j}} - P_{2^{-j+1}})| \leq C\chi_A,$$

where

$$A = \{x; \text{dist}(x, \partial Q) \leq 2^{-j}\}.$$

In particular, each point in \mathbf{R}^3 belongs to at most 8 A 's. Thus

$$(7.44) \quad \left\| \sum_{Q \in \mathcal{F}} \chi_Q * (P_{2^{-j}} - P_{2^{-j+1}}) \right\|_p^p \leq C \sum_{Q \in \mathcal{F}} \|\chi_Q * (P_{2^{-j}} - P_{2^{-j+1}})\|_p^p \leq C\delta 2^{-j}.$$

From (7.43), (7.44)

$$(7.45) \quad \begin{aligned} \|f\|_{B_{p,p}^{-1}} &\leq C \left[\sum_{2^{-j} \geq \delta} (2^{-j} \delta^2 4^{j/q})^p + \sum_{2^{-j} < \delta} (2^{-j} \delta^{1/p} 2^{-j/p})^p \right]^{1/q'} \\ &\lesssim (\delta^{2p} + \delta^{2+p})^{1/p} < \delta^2. \end{aligned}$$

Here, we have used the fact that $p < 2 < q$.

From (7.37), (7.39) and (7.45), we find that

$$(7.46) \quad (7.37) \leq C \|u \wedge du\|_p^{1-s}.$$

Next, we analyze the contribution of the “bad” sections Γ_{x_i} in (7.30). A bad section $\Gamma_{x_i} = \Gamma$ fails either (7.32) or (7.33).

Fix $i = 1, 2, 3$ and $Q \in \mathcal{F}$. Define

$$(7.47) \quad J'_Q = \{x_i \in \pi_i(Q); \Gamma_{x_i} \text{ fails (7.32)}\},$$

$$(7.48) \quad J''_Q = \{x_i \in \pi_i(Q); \Gamma_{x_i} \text{ fails (7.33)}\},$$

and the surfaces

$$(7.49) \quad \mathfrak{S}' = \mathfrak{S}'_i = \bigcup_Q \bigcup_{x_i \in J'_Q} \Gamma_{x_i}$$

$$(7.50) \quad \mathfrak{S}'' = \mathfrak{S}''_i = \bigcup_Q \bigcup_{x_i \in J''_Q} \Gamma_{x_i}.$$

Estimate the contribution of the bad sections in (7.30) by

$$(7.51) \quad \left(\max_{Q \in \mathcal{F}} |\mathfrak{k}_Q| \right) \sum_{i=1}^3 \int_{\mathfrak{S}'_i \cup \mathfrak{S}''_i} |\nabla u|.$$

Estimate

$$(7.52) \quad |\mathfrak{k}_Q| \leq \delta^{-3} \int_Q |k| \leq \delta^{-3} |Q|^{5/6} \|k\|_{L^6(B_\rho)} \lesssim \delta^{-1/2} \|k\|_{W^{1,q}(B_\rho)} \\ \lesssim \delta^{-1/2} \|u \wedge du\|_p^{1-s}.$$

Consider, for $\lambda \in \mathbf{R}^3$, the grid of δ -cubes having λ as one of the edges and let \mathcal{G}_λ be the grid defined by the boundaries of these cubes. For each λ , we have

$$(7.53) \quad \int_{\mathfrak{S}'_i \cup \mathfrak{S}''_i} |\nabla u| \leq \left(\int_{\mathcal{G}_\lambda} |\nabla u|^2 \right)^{1/2} (|\mathfrak{S}'_i| + |\mathfrak{S}''_i|)^{1/2} \\ \leq C \left(\int_{\mathcal{G}_\lambda} |\nabla u|^2 \right)^{1/2} \left(\delta \sum_{Q \in \mathcal{F}_\lambda} (|J'_Q| + |J''_Q|) \right)^{1/2}.$$

Since (7.33) fails for $x_i \in J''_Q$, we have

$$\int_Q e_\varepsilon(u) \geq \int_{\substack{\cup \tilde{\Gamma}_{x_i} \\ x_i \in J''_Q}} e_\varepsilon(u) \geq |J''_Q| \delta_2 \varepsilon^{-1}.$$

Thus

$$(7.54) \quad \sum_{Q \in \mathcal{F}_\lambda} |J''_Q| \lesssim \varepsilon \delta_2^{-1} \log(1/\varepsilon).$$

To estimate (7.53), we use again an average over the grids \mathcal{G}_λ . Denote this averaging by Av_τ (τ refers to the translation).

Thus, taking (7.54) into account, we obtain

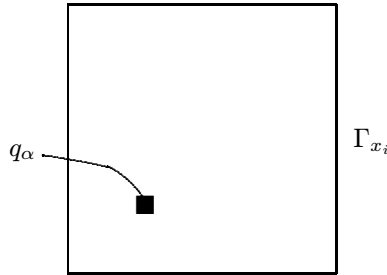
$$(7.55) \quad (7.53) \lesssim \left[Av_\tau \int_{\mathcal{G}_\lambda} |\nabla u|^2 \right]^{1/2} \left[\delta \delta_2^{-1} \varepsilon \log(1/\varepsilon) + \delta Av_\tau \left(\sum_{Q \in \mathcal{F}_\lambda} |J''_Q| \right) \right]^{1/2}.$$

Notice that the J''_Q -intervals of points x_i such that $\text{dist}(\Gamma_{x_i}, [|u| < \frac{1}{2}]) < \delta_1$ do depend on the grid translation – a fact that will be exploited next.

First, recalling (7.4), we have

$$(7.56) \quad Av_\tau \int |\nabla u|^2 \leq \int_{\partial B_\rho} |\nabla u|^2 + \frac{1}{\delta} \int_{B_\rho} |\nabla u|^2 \lesssim \frac{\log 1/\varepsilon}{\delta}.$$

By the η -ellipticity lemma, we may cover $[|u| < 1/2] \cap B$ with at most $C\delta_1^{-1}$ δ_1 -cubes q_α , $\alpha \leq C\delta_1^{-1}$. We fix such a covering (independent of λ). Fix i, Q . If $\text{dist}(\Gamma_{x_i}, [|u| < 1/2]) < \delta_1$, then clearly $x_i \in \pi_i(\tilde{q}_\alpha)$ for some $q_\alpha \subset \tilde{Q}$ with $\text{dist}(q_\alpha, \mathcal{G}_\lambda) < \delta_1$.



Hence

$$(7.57) \quad |J'_Q| \leq 2\delta_1 \cdot \#\{\alpha; q_\alpha \subset \tilde{Q}, \text{dist}(q_\alpha, \mathcal{G}_\lambda) < \delta_1\}$$

and

$$(7.58) \quad \sum_{\mathcal{Q}} |J'_{\mathcal{Q}}| \leq C\delta_1 \cdot \#\{\alpha; \text{dist}(q_\alpha, \mathcal{G}_\lambda) < \delta_1\}.$$

We now average over the grid translation. On the one hand, for fixed α , the inequality

$$\text{dist}(q_\alpha, \mathcal{G}_\lambda \setminus \partial B_\rho) < \delta_1$$

holds with τ -probability $\sim \delta_1/\delta$. On the other hand, for fixed α and $1 - \gamma < \rho < 1 - \gamma/2$, the inequality

$$\text{dist}(q_\alpha, \partial B_\rho) < \delta_1$$

holds with ρ -probability $\sim \delta_1/\gamma$.

Hence, by choosing ρ properly, we may assume that

$$\#\{\alpha; \text{dist}(q_\alpha, \partial B_\rho) < \delta_1\} \leq C.$$

For any such ρ , we have

$$(7.59) \quad Av_\tau(7.58) \lesssim \delta_1 \cdot \frac{1}{\delta_1} \cdot \frac{\delta_1}{\delta} + C \lesssim \frac{\delta_1}{\delta}.$$

Hence

$$(7.60) \quad Av_\tau\left(\sum |J'_{\mathcal{Q}}|\right) \leq C\frac{\delta_1}{\delta}.$$

Substitution of (7.56), (7.60) into (7.55) yields, for small ε ,

$$(7.61) \quad (7.55) \lesssim \left(\frac{\log(1/\varepsilon)}{\delta}\right)^{1/2} \left(\delta\delta_2^{-1}\varepsilon \log(1/\varepsilon) + \delta_1\right)^{1/2} < \delta^{3/4},$$

by (7.9) and (7.31).

From (7.52) and (7.61),

$$(7.62) \quad (7.51) \leq \delta^{3/4}\delta^{-1/2}\|u \wedge du\|_p^{1-s} \leq C\|u \wedge du\|_p^{1-s}.$$

This completes the analysis. Indeed, by collecting the estimates (7.28), (7.30), (7.37), (7.46), (7.51) and (7.62), it follows that

$$(7.63) \quad \|u \wedge du\|_{L^p(B_\rho)}^p \leq C_\gamma \|u \wedge du\|_{L^p(B_\rho)}^{1-s},$$

and thus

$$\|u \wedge du\|_{L^p(B_{1-\gamma})} \leq C_\gamma.$$

Since $0 < \gamma < 1$ and $3/2 \leq p < 2$ are arbitrary, the proof of Theorem 8 is complete.

8. Convergence for $g \in Y$. Proof of Theorem 9

Proof of Theorem 9. — We already know that a subsequence of (u_ε) converges in $W^{1,p}(G)$, $1 \leq p < 3/2$. The main novelties in Theorem 9 are:

a) the identification of the limit

$$u_* = e^{i\tilde{\varphi}},$$

where $g = e^{i\varphi}$, $\varphi \in H^{1/2} + W^{1,1}$ and $\tilde{\varphi}$ is the harmonic extension of φ ;

b) $u_\varepsilon \rightarrow u_*$ in $C^\infty(G)$.

We first discuss b), which is easier. In view of a), it suffices to prove that (u_ε) is bounded in $C^k(\mathbf{K})$ for every integer k and every compact subset \mathbf{K} of G . Since $E_\varepsilon(u_\varepsilon) = o(\log 1/\varepsilon)$, by Theorem 6, we find, with the help of the η -ellipticity Lemma 24 that, for every compact \mathbf{K} in G , we have

$$|u_\varepsilon| \geq \frac{1}{2}$$

in \mathbf{K} for small ε .

We next recall Theorem IV.1 in [9].

Lemma 28. — *Let u_ε be a solution of*

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) \text{ in } B_1$$

such that

$$(8.1) \quad E_\varepsilon(u_\varepsilon; B_1) \leq C.$$

Then (u_ε) is bounded in $C^k(B_{1/2})$, for every $k \in \mathbf{N}$.

We now complete the proof of b) by establishing (8.1) on every ball B compactly contained in G .

We write $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ in B . Let ζ be a cutoff function with $\zeta \equiv 1$ in B . We start by multiplying the equation for φ_ε

$$\operatorname{div}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 0$$

by $\zeta^2(\varphi_\varepsilon - \int_B \varphi_\varepsilon)$.

We find that

$$\begin{aligned} \int \rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2 \zeta^2 &\leq 2 \int \rho_\varepsilon^2 |\nabla \varphi_\varepsilon| |\zeta| |\nabla \zeta| |\varphi_\varepsilon - \int_B \varphi_\varepsilon| \\ &\leq C \left(\int \rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2 \zeta^2 \right)^{1/2} \left(\int |\nabla \varphi_\varepsilon|^{6/5} \right)^{5/6}, \end{aligned}$$

by the Sobolev imbedding $W^{1,6/5} \subset L^2$,

We obtain that φ_ε is bounded in H_{loc}^1 , since $|\nabla\varphi_\varepsilon| \leq 2|\nabla u_\varepsilon|$ in B and u_ε is bounded in $W^{1,6/5}$ by Theorem 7.

Next consider the equation for ρ_ε ,

$$-\Delta\rho_\varepsilon + \rho_\varepsilon|\nabla\varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho_\varepsilon(1 - \rho_\varepsilon^2).$$

Multiplying by $(1 - \rho_\varepsilon)\zeta$, we find that

$$\int |\nabla\rho_\varepsilon|^2\zeta + \frac{1}{\varepsilon^2} \int (1 - \rho_\varepsilon^2)^2\zeta \leq C \left(\int |\nabla\rho_\varepsilon| + \int |\nabla\varphi_\varepsilon|^2 \right).$$

We conclude by noting that

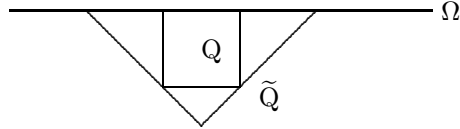
$$E_\varepsilon(u_\varepsilon; B) \leq \int_B |\nabla\rho_\varepsilon|^2 + \int_B |\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_B (1 - \rho_\varepsilon^2)^2 \leq C_B.$$

We now turn to the proof of a).

We start by constructing an appropriate domain $G_\varepsilon \subset G$ on which $|u_\varepsilon| \sim 1$. For simplicity, we assume Ω flat near some point. Fix some $0 < \delta_0 < 1$ to be determined later. Let $0 < \delta < \delta_0$ and $u = u_\varepsilon$. Set

$$(8.2) \quad A_\delta = \{x \in G; \text{dist}(x, \Omega) \geq \sqrt{\varepsilon}, |u(x)| \leq 1 - \delta\}.$$

For $x \in A_\delta$, let Q be the cube centered at x such that one of its faces is contained in Ω and let \tilde{Q} be the conical domain



Let also $Q^\#$ be the cube centered at x having the size a third the one of Q . By Vitali's lemma, we may choose a finite family $(Q_\alpha^\#)$ of disjoint cubes such that $A_\delta \subset \cup Q_\alpha$. By the η -ellipticity property, there is some $\eta(\delta) > 0$ such that we have, with δ_α the size of Q_α ,

$$(8.3) \quad E_\varepsilon(u, Q_\alpha^\#) \geq \eta(\delta)\delta_\alpha \log(\delta_\alpha/\varepsilon) \geq 1/2\eta(\delta)\delta_\alpha \log(1/\varepsilon),$$

since $\delta_\alpha \geq \sqrt{\varepsilon}$. Thus

$$(8.4) \quad \sum \delta_\alpha < \frac{2}{\eta(\delta)} \frac{E_\varepsilon(u, G)}{\log(1/\varepsilon)}.$$

Since, by Theorem 6, we have $E_\varepsilon(u, G) = o(\log(1/\varepsilon))$, we find that

$$(8.5) \quad \sum \delta_\alpha < \delta,$$

provided ε is sufficiently small.

We now set

$$G_\varepsilon = \{x \in G; \text{dist}(x, \Omega) \geq \sqrt{\varepsilon}\} \setminus \cup \tilde{Q}_\alpha,$$

so that $|u_\varepsilon| \geq 1 - \delta$ in G_ε .

By (8.5) and the construction of G_ε , there is a Lipschitz homeomorphism $\Phi_\varepsilon : G_\varepsilon \rightarrow G$ such that

$$(8.6) \quad \begin{aligned} \|D\Phi_\varepsilon\|_{L^\infty} &\leq C, \quad \|D(\Phi_\varepsilon^{-1})\|_{L^\infty} \leq C, \\ \Phi_\varepsilon|_{\partial G_\varepsilon} &= \Pi|_{\partial G_\varepsilon}, \quad \Phi_\varepsilon|_{\{x \in G; \text{dist}(x, \Omega) \geq 2\delta\}} = \text{id}, \end{aligned}$$

provided δ_0 is sufficiently small, with constants C independent of ε .

Here, Π is the projection on Ω . In particular, G_ε is simply connected. We may thus write in G_ε

$$(8.7) \quad u = \rho e^{i\psi}, \quad \rho = |u|, \quad \psi \in C^\infty.$$

Assuming further that $\delta_0 < 1/2$, we have $\rho \geq 1/2$ in G_ε and thus

$$(8.8) \quad |\psi|_{H^1(G_\varepsilon)}^2 \leq 4|u|_{H^1(G_\varepsilon)}^2 \leq 4|u|_{H^1(G)}^2 \leq \delta \log(1/\varepsilon),$$

provided ε is sufficiently small. Moreover, by Theorem 7, we have

$$(8.9) \quad |\psi|_{W^{1,p}(G_\varepsilon)} \leq 2|u|_{W^{1,p}(G_\varepsilon)} \leq 2|u|_{W^{1,p}(G)} \leq C_p, \quad 1 \leq p < 3/2.$$

We are now going to prove that $\psi|_{\partial G_\varepsilon}$ is almost equal to $\varphi \circ \Pi|_{\partial G_\varepsilon}$, where $\varphi \in H^{1/2} + W^{1,1}(\Omega; \mathbf{R})$ is such that $g = e^{i\varphi}$.

Let $\eta > 0$ be to be determined later. Since $g \in Y$, we may find some $h \in C^\infty(\Omega; S^1)$ such that $\|g - h\|_{H^{1/2}} < \eta$. Let $\zeta \in C^\infty(\Omega; \mathbf{R})$ be such that $h = e^{i\zeta}$. Let $T_\varepsilon = \Phi_\varepsilon|_{\partial G_\varepsilon}$ and $U_\varepsilon = T_\varepsilon^{-1} : \Omega \rightarrow \partial G_\varepsilon$. Fix a smooth map $\pi : \mathbf{C} \rightarrow \mathbf{C}$ such that $\pi(z) = z/|z|$ if $|z| \geq 1/2$ and let

$$\xi(x) = g(x) - e^{i\psi(U_\varepsilon(x))}, \quad x \in \Omega,$$

so that

$$(8.10) \quad \xi(x) = \pi(g(x)) - \pi(e^{i\psi(U_\varepsilon(x))}), \quad x \in \Omega \setminus \cup \tilde{Q}_\alpha.$$

Therefore, we have

$$(8.11) \quad \int_{\Omega \setminus \cup \tilde{Q}_\alpha} |\xi(x)| dx \leq C(G) \int_{\{x; \text{dist}(x, \partial\Omega) \leq \sqrt{\varepsilon}\}} |Du| \leq C \|Du\|_{L^2} \varepsilon^{1/4} \\ \leq C \varepsilon^{1/4} (\log 1/\varepsilon)^{1/2} \leq 1/2 \varepsilon^{1/5},$$

provided ε is sufficiently small. It follows that

$$(8.12) \quad \int_{\Omega \setminus \cup \tilde{Q}_\alpha} |h(x) - e^{i\psi(U_\varepsilon(x))}| dx < \varepsilon^{1/5},$$

provided η is sufficiently small. Thus, with $\lambda = \zeta - \psi \circ U_\varepsilon$, we have

$$(8.13) \quad \|e^{i\lambda} - 1\|_{L^1(\Omega \setminus \cup \tilde{Q}_\alpha)} < \varepsilon^{1/5}.$$

By combining (8.6) and (8.8) (resp. (8.6) and (8.9)), we find that

$$(8.14) \quad |\lambda|_{H^{1/2}(\Omega)} \leq \|\zeta\|_{H^{1/2}(\Omega)} + C \|\psi\|_{H^1(G_\varepsilon)} < \delta^{1/2} (\log(1/\varepsilon))^{1/2}$$

and

$$(8.15) \quad \|\lambda\|_{W^{1/4, 4/3}(\Omega)} \leq \|\zeta\|_{W^{1/4, 4/3}(\Omega)} + C \|\psi\|_{W^{1, 4/3}(G_\varepsilon)} \leq C,$$

provided ε is sufficiently small. In particular, we have

$$(8.16) \quad \|\lambda\|_{L^{4/3}(\Omega)} \leq C.$$

By Lemma C.1 in Appendix C, if δ_0 is sufficiently small and λ satisfies (8.13), (8.14) and (8.15), while the squares $\tilde{Q}_\alpha \cap \Omega$ satisfy (8.5), then there is some integer a such that

$$(8.17) \quad \|\lambda - 2\pi a\|_{L^1(\Omega)} < \delta^{1/18}.$$

Without restricting the generality, we may assume that $a = 0$, so that

$$(8.18) \quad \|\xi - \psi \circ U_\varepsilon\|_{L^1(\Omega)} < \delta^{1/18}.$$

We actually claim that

$$(8.19) \quad \|\varphi - \psi \circ U_\varepsilon\|_{L^1(\Omega)} < \delta^{1/20},$$

if we choose the lifting φ of g properly. Indeed, by estimate (1.9) in Theorem 3, the map $g\bar{h} \in Y$ has a lifting $\chi \in H^{1/2} + W^{1,1}$ such that

$$(8.20) \quad |\chi|_{H^{1/2} + W^{1,1}} \leq C(G) |g\bar{h}|_{H^{1/2}} (1 + |g\bar{h}|_{H^{1/2}}).$$

Since

$$|\bar{g}\bar{h}|_{H^{1/2}} = |\bar{h}(g-h)|_{H^{1/2}} \rightarrow 0 \text{ as } h \rightarrow g,$$

we may choose η sufficiently small in order to have

$$(8.21) \quad \|\chi - \int \chi\|_{L^1(\Omega)} < \delta^{1/18}.$$

Using the fact that

$$\|\bar{g}\bar{h} - e^{i\int \chi}\|_{L^1} = \|e^{i\chi} - e^{i\int \chi}\|_{L^1} \leq \|\chi - \int \chi\|_{L^1} < \delta^{1/18}$$

and

$$\|\bar{g}\bar{h} - 1\|_{L^1} < \delta^{1/18},$$

provided η is sufficiently small, we find that, modulo $2\pi\mathbf{Z}$, we may assume that

$$(8.22) \quad \|\int \chi\|_{L^1(\Omega)} < 2\delta^{1/18}.$$

Since $g = e^{i(\chi+\xi)}$, inequality (8.19) follows by combining (8.20)–(8.22), provided δ_0 is sufficiently small.

We now prove that ψ and $\tilde{\varphi}$ are close on compact sets of G . Set $\tilde{\psi} = \psi \circ \Phi_\varepsilon^{-1}$, $\tilde{\rho} = \rho \circ \Phi_\varepsilon^{-1}$, so that $\tilde{\psi}$, $\tilde{\rho}$ are defined on G and, in the set

$$M = \{x \in G; \text{dist}(x, \Omega) \geq 2\delta\},$$

we have $\tilde{\psi} = \psi$ and $\tilde{\rho} = \rho$.

Recall that ψ satisfies the equation $\text{div}(\rho^2 \nabla \psi) = 0$ in G_ε . Transporting this equation on G and using (8.6), we see that ψ satisfies

$$(8.23) \quad \begin{cases} \text{div}(A(x)\tilde{\rho}^2 \nabla \tilde{\psi}) = 0 & \text{in } G \\ \tilde{\psi} = \psi \circ U_\varepsilon & \text{on } \Omega \end{cases},$$

with

$$(8.24) \quad C^{-1}|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq C|\xi|^2, \quad \tilde{\rho}(x) = \rho(x) \text{ and } A(x) = I \text{ if } x \in M.$$

Therefore, the function

$$f = \tilde{\varphi} - \tilde{\psi}$$

satisfies

$$(8.25) \quad \begin{cases} \Delta f = \operatorname{div} ((I - A(x)\tilde{\rho}^2)\nabla\tilde{\psi}) & \text{in } G \\ f = \varphi - \psi \circ U_\varepsilon & \text{on } \partial G \end{cases}.$$

Thus, for $1 \leq p < 3/2$ and \mathbf{K} compact in G , we have

$$(8.26) \quad \|f\|_{W^{1,p}(\mathbf{K})} \leq C_{\mathbf{K}}(\|(I - A(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{L^p(G)} + \|\varphi - \psi \circ U_\varepsilon\|_{L^1(\Omega)}).$$

As we already observed in the proof of part b) of the theorem, we have $\rho \rightarrow 1$ uniformly on the compacts of G . Thus

$$(8.27) \quad \|(I - A(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{L^p(\mathbf{M})} \rightarrow 0.$$

as $\varepsilon \rightarrow 0$. On the other hand, we have

$$(8.28) \quad \|(I - A(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{L^p(G \setminus \mathbf{M})} \leq C\|\nabla\tilde{\psi}\|_{L^p(G \setminus \mathbf{M})} \leq C\|\nabla u\|_{L^p(G \setminus \mathbf{M})}.$$

If we choose some r with $p < r < 3/2$, we find that

$$(8.29) \quad \|(I - A(x)\tilde{\rho}^2)\nabla\tilde{\psi}\|_{L^p(G \setminus \mathbf{M})} \leq C\|\nabla u\|_{L^r(G \setminus \mathbf{M})}|G \setminus \mathbf{M}|^{\frac{r-p}{r}} \leq C\delta^{\frac{r-p}{r}},$$

by Theorem 7. By combining (8.19), (8.26), (8.27) and (8.29) we find that, for some $0 < \alpha < 1$ fixed, we have

$$(8.30) \quad \|f\|_{W^{1,p}(\mathbf{K})} \leq \delta^\alpha,$$

provided ε is sufficiently small.

Since, for $\delta_0 = \delta_0(\mathbf{K})$ sufficiently small, we have $f = \varphi - \psi$ in \mathbf{K} , we find that, as $\varepsilon \rightarrow 0$, $\tilde{\varphi} - \psi \rightarrow 0$ in $W_{\text{loc}}^{1,p}(G)$, $1 \leq p < 3/2$. Using once more the fact that $\rho \rightarrow 1$ in $C_{\text{loc}}^k(G)$, we find that $u_\varepsilon \rightarrow u_*$ in $W_{\text{loc}}^{1,p}(G)$. This proves Theorem 9.

Remark 8.1. — Under the assumptions of Theorem 9 it is not true in general that $|u_\varepsilon| \rightarrow 1$ uniformly on \bar{G} . Indeed, if this were true, then $u_\varepsilon/|u_\varepsilon|$ would belong to $H^1(G; S^1)$ for ε sufficiently small. Thus $u_\varepsilon/|u_\varepsilon|$ admits a lifting $\varphi_\varepsilon \in H^1(G; \mathbf{R})$ and $g = e^{i\varphi_\varepsilon}$. Hence g must necessarily belong to \mathbf{X} . But, even when $g \in \mathbf{X}$ it is unlikely that $|u_\varepsilon| \rightarrow 1$ uniformly on \bar{G} .

Remark 8.2. — Let $g \in H^{1/2}(\Omega; S^1)$ with $L(g) = 0$ and write $g = e^{i\varphi}$ with $\varphi \in H^{1/2} + W^{1,1}$. Let $\tilde{\varphi}$ be the harmonic extension of φ . One may wonder whether

$$(8.31) \quad \|u_\varepsilon e^{-i\tilde{\varphi}}\|_{W^{1,p}} \leq C \quad \forall p < 2 \text{ as } \varepsilon \rightarrow 0?$$

The answer is negative. The argument relies on the following

Lemma 29. — Fix ε and let u_ε be a minimizer for E_ε , with $u_\varepsilon = g$ on Ω . Then

$$(8.32) \quad u_\varepsilon = \tilde{g} + \psi$$

where \tilde{g} is the harmonic extension of g and

$$(8.33) \quad |\psi(x)| \leq C\varepsilon^{-1} \text{dist}(x, \Omega).$$

Proof. — Clearly $\psi = 0$ on Ω , $|\psi| \leq 2$, and $|\Delta\psi| \leq C\varepsilon^{-2}$ on G . By interpolation one deduces that $|\nabla\psi| \leq C\varepsilon^{-1}$ (see e.g. [7]) and the conclusion follows.

1. Using (8.32), write

$$(8.34) \quad \begin{aligned} |\nabla(u_\varepsilon e^{-i\tilde{\varphi}})| &\geq |u_\varepsilon| |\nabla\tilde{\varphi}| - |\nabla u_\varepsilon| \\ &\geq |\tilde{g}| |\nabla\tilde{\varphi}| - |\psi| |\nabla\tilde{\varphi}| - |\nabla u_\varepsilon|. \end{aligned}$$

We have

$$\|\nabla u_\varepsilon\|_{L^2(G)} \lesssim \left(\log \frac{1}{\varepsilon}\right)^{1/2} < \infty$$

and, by (8.33)

$$\begin{aligned} \int_G (|\psi| |\nabla\tilde{\varphi}|)^2 &\leq C\varepsilon^{-2} \sum_{s \geq 0} 4^{-s} \int_{\text{dist}(x, \Omega) \sim 2^{-s}} |(\nabla\tilde{\varphi})(x)|^2 \\ &\leq C\varepsilon^{-2} \sum_{s \geq 0} 4^{-s} \cdot 4^s \cdot 2^{-s} \|\varphi\|_{L^2(\Omega)}^2 \leq C\varepsilon^{-2} < \infty. \end{aligned}$$

Consequently, assuming (8.31) were true for some $p < 2$, we necessarily must have, by (8.34), that

$$(8.35) \quad |\tilde{g}| |\nabla\tilde{\varphi}| \in L^p(G)$$

whenever $g = e^{i\varphi} \in H^{1/2}(\Omega, S^1)$.

This statement relates only to g and we show next that (8.35) *cannot* hold for $p > 3/2$.

2. Let $0 < \delta < 1$ be small and take $0 \leq \varphi \leq (\frac{1}{\delta})^{1-}$ such that

$$(8.36) \quad \text{supp } \varphi \subset B(0, 2\delta) \subset \Omega \text{ (identified with the } x_1, x_2\text{-plane),}$$

$$(8.37) \quad \varphi = \left(\frac{1}{\delta}\right)^{1-} \text{ on } B(0, \delta),$$

$$(8.38) \quad |\nabla\varphi| \leq \left(\frac{1}{\delta}\right)^{2-}.$$

Hence

$$\|e^{i\varphi}\|_{H^{1/2}} < C.$$

Also, from (8.1)

$$\|1 - e^{i\varphi}\|_{L^1} \leq C\delta^2.$$

Hence for $x_3 > C\delta$

$$(8.39) \quad |1 - \tilde{g}(x_1, x_2, x_3)| \leq \int |1 - e^{i\varphi}|(x'_1, x'_2) P_x(x'_1, x'_2) dx_1 dx_2 \leq C\delta^2 \|P_x\|_\infty < \frac{1}{10}.$$

Thus from (8.39)

$$(8.40) \quad \begin{aligned} \|\tilde{g} \cdot |\nabla\tilde{\varphi}|\|_{L^p} &\gtrsim \|\nabla\tilde{\varphi}\|_{L^p(x_1, x_2; x_3 > C\delta)} \\ &\sim \left\| \int_{\mathbf{R}^2} |\xi| \hat{\varphi}(\xi) e^{i(x_1\xi_1 + x_2\xi_2)} e^{-x_3|\xi|} d\xi \right\|_{L^p(x_1, x_2; x_3 > C\delta)} \\ &\geq \left\| \|\xi| \hat{\varphi}(\xi) e^{-x_3|\xi|}\|_{L^{\frac{p'}{\xi}}}_{L^p(x_3 > C\delta)} \right\| \\ &\geq c \left[\|\xi| \hat{\varphi}(\xi)\|_{L^{\frac{p'}{|\xi| \sim \frac{1}{10\delta}}}} \right] \cdot \delta^{\frac{1}{p}} \\ &\sim \delta^{-1} \hat{\varphi}(0) \cdot \left(\frac{1}{\delta}\right)^{\frac{2}{p'}} \delta^{1/p} \\ (8.41) \quad &\sim \delta^{\frac{1}{p} - \frac{2}{p'} +}. \end{aligned}$$

In (8.40), we use Hausdorff–Young inequality and (8.41) follows from (8.36), (8.37).

Since $\frac{1}{p} - \frac{2}{p'} < 0$ for $p > 3/2$, a gluing construction with the preceding as building block and $\delta \rightarrow 0$ will clearly violate (8.35).

As in the previous sections and with some more work, we may prove the following variant of Theorem 9:

Theorem 9'. — Assume $g \in Y$, and let g_ε be as in Theorem 6' of Section 5. Let u_ε be a minimizer of E_ε in $H_{g_\varepsilon}^1$. Then

$$u_\varepsilon \rightarrow u_* \text{ in } W^{1,p}(G) \cap C^\infty(G), \quad \forall p < 3/2,$$

where u_* is the same as in Theorem 9.

9. Further thoughts about $p = 3/2$

Let $g \in H^{1/2}(\Omega; S^1)$ and let (u_ε) be a minimizer for E_ε in H_g^1 . In Section 6 we have established that (u_ε) is relatively compact in $W^{1,p}(G)$ for every $p < 3/2$. It is plausible that (u_ε) is bounded and possibly even relatively compact in $W^{1,3/2}$; see Open Problem 2 in Section 10.

There are two directions of evidence suggesting that, indeed, (u_ε) is bounded in $W^{1,3/2}$.

The first one relies on a conjectured strengthening of the Jerrard–Soner inequality mentioned below.

The second one is a complete proof of the fact that any limit (in $W^{1,p}$, $p < 3/2$) of (u_ε) belongs to $W^{1,3/2}$; see Theorem 12.

9.1. Jerrard–Soner revisited

First recall the following immediate consequence of a result in [33]:

Proposition 1 (Jerrard and Soner [33]). — Let (v_ε) be a sequence in $H^1(Q; \mathbf{R}^2)$, $Q \subset \mathbf{R}^3$ a cube, satisfying

$$(9.1) \quad E_\varepsilon(v_\varepsilon; Q) = \int_Q \left[\frac{1}{2} |\nabla v_\varepsilon|^2 + \frac{1}{4\varepsilon^2} ||v_\varepsilon|^2 - 1|^2 \right] \leq C \log 1/\varepsilon$$

for all $\varepsilon < \varepsilon_0$. Then for $\zeta \in C_0^\infty(\omega)$, $\bar{\omega} \subset Q$, we have the inequality

$$(9.2) \quad \left| \int J(v_\varepsilon) \zeta \right| \leq K \|\zeta\|_{W^{1,q}(Q)}$$

where $J(v_\varepsilon)$ is any 2×2 Jacobian determinant of v_ε , $q > 3$, and $K = K(C, q, \omega)$.

Remark 9.1. — In fact in [33] one obtains a stronger estimate with the norm $\|\zeta\|_{W^{1,q}}$ replaced by any $\|\zeta\|_{C^{0,\alpha}}$ -norm, $\alpha > 0$.

In this subsection, we will show that:

- a) The conclusion of Proposition 1 fails for any $q < 3$.
- b) The validity of Proposition 1 for $q = 3$ (which we conjecture) would imply the boundedness in $W^{1,3/2}$ of the minimizers (u_ε) of the Ginzburg–Landau problem in G with boundary data g controlled in $H^{1/2}(\Omega; S^1)$, $\Omega = \partial G$.

A basic tool is the following construction of an extension of g outside G .

Lemma 30. — Assume $\bar{G} \subset Q$ and $g \in H^{1/2}(\Omega; S^1)$. Then there is $w_\varepsilon \in H^1(Q \setminus G; \mathbf{R}^2)$ satisfying

$$(9.3) \quad w_\varepsilon = g \text{ on } \partial G \text{ and } w_\varepsilon \equiv 1 \text{ in some fixed neighborhood of } \partial Q,$$

$$(9.4) \quad E_\varepsilon(w_\varepsilon; Q \setminus G) \leq C \|g\|_{H^{1/2}} \log 1/\varepsilon,$$

$$(9.5) \quad \|w_\varepsilon\|_{W^{1,p}(Q \setminus G)} \leq C_p \|g\|_{H^{1/2}} \text{ for every } p < 2,$$

$$(9.6) \quad w_{\varepsilon_n} \rightarrow w \text{ in } W^{1,p}(Q \setminus G) \text{ for every } p < 2 \text{ with } w \in W^{1,p}(Q \setminus G), \quad \forall p < 2$$

$$(9.7) \quad |w_\varepsilon| \leq 1 \text{ in } Q \setminus G.$$

Proof. — We follow the same construction as in [5] which we briefly recall here. First, let H be any smooth function in $Q \setminus G$ with $H \in H^1(Q \setminus G; \mathbf{R}^2)$ satisfying the boundary conditions $H = g$ on $\Omega = \partial G$, $H \equiv 1$ near ∂Q , and $\|H\|_{H^1} \leq C \|g\|_{H^{1/2}}$.

Using the same notation as in the proof of Lemma 23, define

$$w_{\varepsilon,a}(x) = \psi\left(\frac{|H(x) - a|}{\varepsilon}\right) \pi_a(H(x)).$$

It may be shown as in [5] (or as in the proof of Lemma 23) that for some $a = a_\varepsilon \in \mathbf{C}$, $|a_\varepsilon| < 1/10$, the functions $(w_{\varepsilon,a_\varepsilon})$ satisfy all the required properties.

Next, we establish the following

Proposition 2. — *Assume that the conclusion of Proposition 1 is valid for some $2 < q \leq 3$. Let (u_ε) be a sequence of minimizers of E_ε in G as above. Then (u_ε) is bounded in $W^{1,q'}(G)$ with $q' = q/(q-1)$.*

Proof. — As in Section 6, it suffices to establish the boundedness of $u_\varepsilon \wedge du_\varepsilon$ in the space $L^q(G)$. Proceeding by duality, consider $\zeta \in L^q(G; \mathbf{R}^3)$, $\|\zeta\|_q \leq 1$ and take its Hodge decomposition as

$$(9.8) \quad \begin{cases} \zeta = \text{curl } k + \nabla L \text{ in } G \\ L = 0 \text{ on } \Omega, \\ \text{with } \|k\|_{W^{1,q}(G)} + \|L\|_{W^{1,q}(Q)} \leq C \end{cases}$$

(see e.g. [30] or [27]). Recall that, with the notations of differential forms we used earlier, $\text{curl} = d^*$ and $\nabla = d$. Let Q be a cube with $\overline{G} \subset Q$ and let ω be an open set such that

$$\overline{G} \subset \omega \text{ and } \overline{\omega} \subset Q.$$

Next, extend k to \tilde{k} on Q , $\tilde{k} = 0$ on $Q \setminus \omega$, with control of $\|\tilde{k}\|_{W^{1,q}(Q)}$. We extend u_ε to Q defining

$$v_\varepsilon = \begin{cases} u_\varepsilon & \text{in } G \\ w_\varepsilon & \text{in } Q \setminus G \end{cases}$$

where w_ε is provided by Lemma 30.

Recall that $\operatorname{div}(u_\varepsilon \wedge du_\varepsilon) = 0$, and thus

$$\int_G (u_\varepsilon \wedge du_\varepsilon) \cdot \zeta = \int_G (u_\varepsilon \wedge du_\varepsilon) \cdot \operatorname{curl} k.$$

Hence

$$(9.9) \quad \left| \int_G (u_\varepsilon \wedge du_\varepsilon) \cdot \zeta \right| \leq \left| \int_Q (v_\varepsilon \wedge dv_\varepsilon) \cdot \operatorname{curl} \tilde{k} \right| + \int_{Q \setminus G} |\nabla w_\varepsilon| |\nabla \tilde{k}|.$$

From (9.5), the last term in (9.9) is bounded by $C\|w_\varepsilon\|_{W^{1,q'}(Q \setminus G)}$, hence by $C\|g\|_{H^{1/2}}$, since $q' < 2$.

For the first term, perform an integration by part ($\tilde{k} = 0$ on ∂Q) to get

$$(9.10) \quad \left| \int_Q (v_\varepsilon \wedge dv_\varepsilon) \cdot \operatorname{curl} \tilde{k} \right| = 2 \left| \int_Q J(v_\varepsilon) \cdot \tilde{k} \right|$$

and this quantity is bounded, by assumption, by $C\|\tilde{k}\|_{W^{1,q}(Q)}$ (since $\operatorname{supp} \tilde{k} \subset \bar{\omega}$).

This proves Proposition 2.

Remark 9.2. — The proof of Proposition 2 also provides an alternative quick proof of Theorem 7.

Corollary 4. — *The conclusion of Proposition 1 fails for every $q < 3$.*

Proof. — By Proposition 2, one would otherwise obtain the boundedness of the Ginzburg–Landau minimizers in $W^{1,p}(G)$ for some $p > 3/2$. This is not true in general, even for certain $g \in Y$. Arguing by contradiction, one would otherwise obtain that the limit u_* obtained in Theorem 9 belongs to $W^{1,p}$ with $p > 3/2$. However, this is false. Indeed

Remark 9.3. — In general $u_* \notin W^{1,t}$ for $t > 3/2$. Here is an example (see [5]): Suppose Ω is flat near 0 and choose $g(r) = e^{t/r^\alpha}$ with $\alpha < 1$, α close to 1 and g smooth away from 0. This g belongs to Y . It is easy to see that the harmonic extension of $1/r^\alpha$ does not belong to $W^{1,t}$, for $t > 3/(\alpha + 1)$. Thus $u_* \notin W^{1,t}$.

Remark 9.4. — The preceding also shows that the improved interior estimates from Section 7 can not be established via a strengthening of Jerrard–Soner but requires additional structure (in particular the monotonicity formula).

9.2. $W^{1,3/2}$ – estimate of the limit

We start with the simple case when $g \in Y$.

Theorem 11. — Assume $g \in Y$ and let u_* be as in Theorem 9. Then $u_* \in W^{1,3/2}$.

Proof of Theorem 11. — Recall that $u_* = e^{i\tilde{\varphi}}$ where $\tilde{\varphi}$ is the harmonic extension of $\varphi \in H^{1/2} + W^{1,1}$. Therefore, it suffices to apply the following imbedding result, which is an immediate consequence of Theorem 1.5 in Cohen, Dahmen, Daubechies and DeVore [23]:

Lemma 31. — In 2-dimensions we have $W^{1,1}(\Omega) \subset W^{\frac{1}{3}, \frac{3}{2}}(\Omega)$.

For completeness we will prove a slightly more general form of this result in Appendix D.

We now turn to the case of a general $g \in H^{1/2}(\Omega; S^1)$.

Theorem 12. — Let $g \in H^{1/2}(\Omega; S^1)$ and let (u_ε) be a minimizer of E_ε in $H_g^1(G; \mathbf{R}^2)$. In view of Theorem 7' we may assume that (modulo a subsequence)

$$u_{\varepsilon_n} \rightarrow U \text{ in } W^{1,p}(G), \quad \forall p < 3/2.$$

Then

$$U \in W^{1,3/2}(G).$$

Proof of Theorem 12. — In the proof we will not fully use the fact that u_ε is a minimizer. We will only make use of the properties

$$(9.0.1) \quad \operatorname{div}(u_\varepsilon \wedge du_\varepsilon) = 0 \text{ in } G,$$

$$(9.0.2) \quad e_\varepsilon = E_\varepsilon(u_\varepsilon) \leq C \log 1/\varepsilon,$$

$$(9.0.3) \quad u_{\varepsilon_n} \rightarrow U \text{ in } W^{1,p}(G), \quad \forall p < 3/2,$$

$$(9.0.4) \quad u_{\varepsilon|\Omega} = g \in H^{1/2}(\Omega; S^1).$$

Claim.

$$(9.0.5) \quad U \wedge dU \text{ belongs to } L^{3/2}(G).$$

This implies that $U \in W^{1,3/2}$. Indeed we have

$$|b|^2 = |a \wedge b|^2 + |a \cdot b|^2$$

for any vectors a, b in \mathbf{R}^2 with $|a| = 1$; applying this with $a = U$ and $b = \frac{\partial U}{\partial x_i}$ yields $|dU| = |U \wedge dU|$ since $U \cdot \frac{\partial U}{\partial x_i} = 0$.

In order to prove the Claim (9.0.5) we will check that, for every $\vec{\zeta} \in L^3(G; \mathbf{R}^3)$, we have

$$(9.0.6) \quad \left| \int_G \vec{\zeta} \cdot (U \wedge dU) \right| \leq C \|\vec{\zeta}\|_{L^3}.$$

Clearly, it suffices to verify (9.0.6) when $\vec{\zeta} \in C_0^\infty$. Consider the Hodge decomposition of $\vec{\zeta}$ as above, i.e.,

$$(9.0.7) \quad \vec{\zeta} = \text{curl } \vec{k} + \nabla L \quad \text{in } G,$$

$$(9.0.8) \quad L = 0 \quad \text{on } \partial G,$$

$$(9.0.9) \quad \|\vec{k}\|_{W^{1,3}(G)} \leq C \|\vec{\zeta}\|_{L^3}.$$

Then, by (9.0.1) and (9.0.8),

$$\int_G \nabla L \cdot (U \wedge dU) = 0$$

and thus

$$(9.0.10) \quad \int_G \vec{\zeta} \cdot (U \wedge dU) = \int_G (\text{curl } \vec{k}) \cdot (U \wedge dU).$$

We will establish the bound

$$(9.0.11) \quad \left| \int_G (\text{curl } \vec{k}) \cdot (U \wedge dU) \right| \leq C \|\vec{k}\|_{W^{1,3}}$$

in 5 steps. The desired estimate (9.0.6) will be consequence of (9.0.10) and (9.0.11).

Step 1. — Extensions.

Let Q be a cube such that $\bar{G} \subset Q$. Let $\tilde{k} \in W^{1,3}(Q; \mathbf{R}^3)$ be such that $\text{supp } \tilde{k}$ is contained in a fixed compact subset of Q ,

$$\tilde{k} = \vec{k} \text{ in } G,$$

and

$$\|\tilde{k}\|_{W^{1,3}(Q)} \leq C \|\vec{k}\|_{W^{1,3}(G)}.$$

Next, we extend g to $Q \setminus G$ using Lemma 30. Thus, we obtain a family $w_\varepsilon \in H^1(Q \setminus G; \mathbf{R}^2)$ satisfying

$$(9.1.1) \quad w_{\varepsilon|_{\partial G}} = g,$$

$$(9.1.2) \quad w_\varepsilon \equiv 1 \text{ in some fixed neighborhood of } \partial Q,$$

$$(9.1.3) \quad E_\varepsilon(w_\varepsilon; Q \setminus G) \leq C \log 1/\varepsilon$$

$$(9.1.4) \quad \|w_\varepsilon\|_{W^{1,p}(Q \setminus G)} \leq C_p, \quad \forall p < 2$$

$$(9.1.5) \quad w_{\varepsilon_n} \longrightarrow w \text{ in } W^{1,p}(Q \setminus G), \quad \forall p < 2,$$

for some $w \in W^{1,p}(Q \setminus G; S^1)$, $\forall p < 2$.

Set

$$\tilde{u}_\varepsilon = \begin{cases} u_\varepsilon & \text{in } G \\ w_\varepsilon & \text{in } Q \setminus G, \end{cases}$$

so that $\tilde{u}_\varepsilon \in H^1(Q; \mathbf{R}^2)$ and

$$(9.1.6) \quad \tilde{u}_{\varepsilon_n} \longrightarrow \tilde{U} \text{ in } W^{1,p}(Q), \quad \forall p < 3/2,$$

where

$$\tilde{U} = \begin{cases} u & \text{in } G \\ w & \text{in } Q \setminus G \end{cases}$$

and $\tilde{U} \in W^{1,p}(Q; S^1)$, $\forall p < 3/2$.

Clearly,

$$(9.1.7) \quad E_\varepsilon(\tilde{u}_\varepsilon; Q) \leq C \log 1/\varepsilon.$$

It is convenient to introduce the following distribution denoted $\tilde{U}_{x_i} \wedge \tilde{U}_{x_j}$, $i \neq j$

$$\tilde{U}_{x_i} \wedge \tilde{U}_{x_j} = \frac{1}{2}(\tilde{U}_{x_i} \wedge \tilde{U})_{x_j} + \frac{1}{2}(\tilde{U} \wedge \tilde{U}_{x_j})_{x_i}$$

acting on functions $C_0^\infty(Q; \mathbf{R})$.

An immediate computation shows that

$$(9.1.8) \quad -\frac{1}{2} \int_Q (\text{curl } \tilde{k}) \cdot \tilde{U} \wedge d\tilde{U} = \langle \tilde{U}_{x_2} \wedge \tilde{U}_{x_3}, \tilde{k}_1 \rangle + \langle \tilde{U}_{x_3} \wedge \tilde{U}_{x_1}, \tilde{k}_2 \rangle \\ + \langle \tilde{U}_{x_1} \wedge \tilde{U}_{x_2}, \tilde{k}_3 \rangle.$$

We will prove e.g. that

$$(9.1.9) \quad \left| \langle \tilde{U}_{x_1} \wedge \tilde{U}_{x_2}, k \rangle \right| \leq C \|k\|_{W^{1,3}}.$$

for every $k \in C_0^\infty(Q; \mathbf{R})$ and similarly for the other terms.

Assuming (9.1.9) we then have

$$(9.1.10) \quad \left| \int_Q (\operatorname{curl} \tilde{k}) \cdot (\tilde{U} \wedge d\tilde{U}) \right| \leq C \|\tilde{k}\|_{W^{1,3}(Q)}$$

and thus

$$(9.1.11) \quad \begin{aligned} \left| \int_G (\operatorname{curl} \vec{k}) \cdot (U \wedge dU) \right| &\leq \left| \int_{Q \setminus G} (\operatorname{curl} \tilde{k}) \cdot w \wedge dw \right| + C \|\tilde{k}\|_{W^{1,3}(Q)} \\ &\leq \|\tilde{k}\|_{W^{1,3}(Q \setminus G)} \|w\|_{L^{3/2}(Q \setminus G)} + C \|\tilde{k}\|_{W^{1,3}(Q)}. \end{aligned}$$

Finally we obtain, by (9.1.4),

$$(9.1.12) \quad \left| \int_G (\operatorname{curl} \vec{k}) \cdot (U \wedge dU) \right| \leq C \|\vec{k}\|_{W^{1,3}(G)}$$

which is the desired estimate (9.0.11).

The rest of the argument is devoted to the proof of (9.1.9).

Step 2. — Use of a result of Jerrard–Soner.

For any $\bar{x}_3 \in \mathbf{R}$ set

$$\Sigma_{\bar{x}_3} = Q \cap (\mathbf{R}^2 \times \{\bar{x}_3\}).$$

Consider \bar{x}_3 such that

$$(9.2.1) \quad \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(\tilde{u}_\varepsilon | \Sigma_{\bar{x}_3})}{\log 1/\varepsilon} < \infty$$

and

$$(9.2.2) \quad \tilde{U}_{\varepsilon_n | \Sigma_{\bar{x}_3}} \longrightarrow \tilde{U}_{|\Sigma_{\bar{x}_3}} \text{ in } W^{1, \frac{3}{2}-}(\Sigma_{\bar{x}_3}).$$

From (9.1.6), (9.1.7), this is the case for almost all \bar{x}_3 .

It follows then from Theorem 3.1 in [33] that $(\tilde{u}_{\varepsilon_n})_{x_1} \wedge (\tilde{u}_{\varepsilon_n})_{x_2}$ converges in $\mathcal{D}'(\Sigma_{\bar{x}_3})$ to $\tilde{U}_{x_1} \wedge \tilde{U}_{x_2}$ and that

$$(9.2.3) \quad \tilde{U}_{x_1} \wedge \tilde{U}_{x_2} = \pi \sum_i d_i \delta_{a_i}$$

where $d_i = d_i(\bar{x}_3) \in \mathbf{Z}$, $a_i = a_i(\bar{x}_3) \in \Sigma_{\bar{x}_3}$ satisfy

$$(9.2.4) \quad \pi \sum_i |d_i(\bar{x}_3)| \leq \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(\tilde{u}_\varepsilon | \Sigma_{\bar{x}_3})}{\log 1/\varepsilon}.$$

Thus, from (9.1.7)

$$(9.2.5) \quad \sum_i \int |d_i(x_3)| dx_3 \leq C$$

and we may write

$$(9.2.6) \quad \langle \tilde{U}_{x_1} \wedge \tilde{U}_{x_2}, k \rangle = \pi \int dx_3 \left\{ \sum_i d_i(x_3) k(a_i(x_3)) \right\}.$$

To bound (9.2.6), we will need, besides (9.2.5), also certain cancellations that have to do with the sign of d_i 's.

Step 3. — Use of minimal connections.

Take \bar{x}_3 as in Step 2 and consider the domain

$$\Omega_{\bar{x}_3} = \mathbf{Q} \cap [x_3 \leq \bar{x}_3] \quad (\text{or } x_3 \geq \bar{x}_3).$$

Since $\tilde{u}_{\varepsilon_n} \rightarrow \tilde{U}$ in $W^{1, \frac{3}{2}-}(\partial\Omega_{\bar{x}_3})$, $\tilde{u}_{\varepsilon_n} \rightarrow \tilde{U}$ in $H^{1/2}(\partial\Omega_{\bar{x}_3})$. Remark also that, since $\tilde{U} = 1$ on $\partial\mathbf{Q}$, the singularities of \tilde{U} on $\partial\Omega_{\bar{x}_3}$ are necessarily in $\Sigma_{\bar{x}_3}$.

Invoke next Theorem 6' to claim that

$$(9.3.1) \quad \pi L(\tilde{U}|_{\Sigma_{\bar{x}_3}}) = \pi L(\tilde{U}|_{\partial\Omega_{\bar{x}_3}}) \leq \liminf_{\varepsilon \rightarrow 0} \frac{E_\varepsilon(\tilde{u}_\varepsilon | \Omega_{\bar{x}_3})}{\log 1/\varepsilon} \leq \sup \frac{E_\varepsilon(\tilde{u}_\varepsilon)}{\log 1/\varepsilon} \leq C.$$

Note that assumption (5.11) is satisfied since

$$\frac{1}{\varepsilon^2} \int_{\mathbf{Q}} (|\tilde{u}_\varepsilon|^2 - 1)^2 \leq C \log 1/\varepsilon$$

implies

$$\frac{1}{\varepsilon} \int_{\mathbf{Q}} (|\tilde{u}_\varepsilon|^2 - 1)^2 = \frac{1}{\varepsilon} \int dx_3 \int_{\Sigma_{x_3}} (|\tilde{u}_\varepsilon|^2 - 1)^2 \longrightarrow 0$$

and then

$$\frac{1}{\varepsilon_n} \int_{\Sigma_{x_3}} (|\tilde{u}_{\varepsilon_n}| - 1)^2 \leq h(x_3)$$

for some fixed function $h \in L^1$.

Thus, by (9.3.1), there is a reordering

$$\{a_i(d_i)\} = \{p_1, \dots, p_\ell\} \cup \{n_1, \dots, n_\ell\}$$

with possible repetition, such that

$$(9.3.2) \quad \sum_j |p_j(\bar{x}_3) - n_j(\bar{x}_3)| \leq C$$

and (9.2.5), (9.2.6) may be rewritten as

$$(9.3.3) \quad \int \ell(x_3) dx_3 \leq C$$

(where $2\ell(x_3) = \sum |d_i(x_3)|$)

and

$$(9.3.4) \quad \langle \tilde{U}_{x_1} \wedge \tilde{U}_{x_2}, k \rangle = \pi \int dx_3 \left\{ \sum_j [k(p_j(x_3)) - k(n_j(x_3))] \right\}.$$

We will now establish the desired bound (9.1.9) with the help of the following

Proposition 3. — Assume (9.3.3) and (9.3.4), then, for every $k \in C_0^\infty(\mathbf{Q}; \mathbf{R})$,

$$(9.3.5) \quad \left| \int dx_3 \left\{ \sum_j [k(p_j(x_3)) - k(n_j(x_3))] \right\} \right| \leq C \|k\|_{W^{1,3}(\mathbf{Q})}.$$

Step 4. — Decomposition of $W^{1,3}(\mathbf{R}^3)$ -function.

Let $k \in W^{1,3}(\mathbf{R}^3)$, $\|k\|_{W^{1,3}} \leq 1$ and let

$$k = \sum_{s \geq 0} \Delta_s k$$

be a usual Littlewood–Paley decomposition (we assume $\text{supp } k \subset \mathbf{Q}$).

Thus

$$(9.4.1) \quad \sum 8^s \|\Delta_s k\|_3^3 < C.$$

Denote

$$(9.4.2) \quad \lambda_s = 8^s \|\Delta_s k\|_3^3;$$

hence

$$(9.4.3) \quad \sum \lambda_s < C.$$

First we estimate for fixed $\rho > 0$

$$(9.4.4) \quad \text{meas } [x_3; \sup_{x_1, x_2} |\Delta_s k(x_1, x_2, x_3)| > \rho].$$

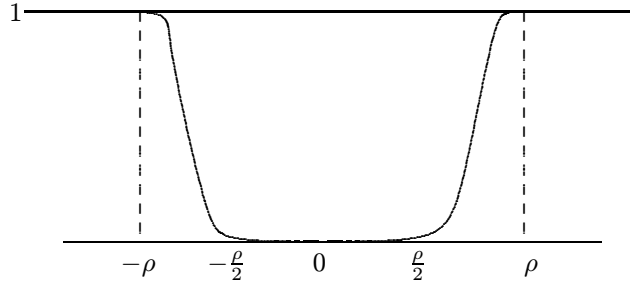
Clearly, for fixed x_3 ,

$$\|\Delta_s k(x_3)\|_{L^\infty_{x_1, x_2}} \leq C4^{s/3} \|\Delta_s k(x_3)\|_{L^3_{x_1, x_2}}$$

so that

$$(9.4.5) \quad (9.4.4) \leq \rho^{-3} \int (\|\Delta_s k(x_3)\|_{L^\infty_{x_1, x_2}})^3 dx_3 \leq C\rho^{-3} 4^s \|\Delta_s k\|_3^3 \leq C\rho^{-3} 2^{-s} \lambda_s.$$

Denote ζ_ρ the function on \mathbf{R}



Fix s_0 and decompose for $s \geq s_0 + 1$

$$\Delta_s k = k_{s, s_0}^1 + k_{s, s_0}^2 \quad \text{with} \quad k_{s, s_0}^1 = \Delta_s k (1 - \zeta_{1/(s-s_0)^2})(\Delta_s k).$$

Hence

$$\begin{aligned} |k_{s, s_0}^1| &\leq |\Delta_s k| \chi_{[|\Delta_s k| < (s-s_0)^{-2}]} \\ |k_{s, s_0}^2| &\leq |\Delta_s k| \chi_{[|\Delta_s k| > \frac{1}{2}(s-s_0)^{-2}]}. \end{aligned}$$

Therefore

$$(9.4.6) \quad \sum_{s \geq s_0+1} |k_{s, s_0}^1| < C$$

and by (9.4.5)

$$(9.4.7) \quad \text{meas }_{x_3} (\text{Proj}_{x_3} (\text{supp } k_{s, s_0}^2)) \leq C(s - s_0)^6 2^{-s} \lambda_s.$$

Step 5. — Estimation of (9.3.5).

Using the decomposition of Step 4, estimate

$$(9.5.0) \quad (9.3.5) \leq \int dx_3 \left\{ \sum_{s_0} \sum_{j | |p_j - n_j| \sim 2^{-s_0}} |k(p_j(x_3)) - k(n_j(x_3))| \right\}$$

and

$$(9.5.1) \quad |k(p_j) - k(n_j)| \leq \sum_{s \leq s_0} |\Delta_s k(p_j) - \Delta_s k(n_j)|$$

$$(9.5.2) \quad + \sum_{s > s_0} (|k_{s,s_0}^1(p_j)| + |k_{s,s_0}^1(n_j)|)$$

$$(9.5.3) \quad + \sum_{s > s_0} (|k_{s,s_0}^2(p_j)| + |k_{s,s_0}^2(n_j)|).$$

Contribution of (9.5.1)

Estimate

$$|\Delta_s k(p_j) - \Delta_s k(n_j)| \leq \|\Delta_s k\|_{\text{Lip}} |p_j - n_j| \leq C 2^{s-s_0}.$$

Thus the contribution in (9.5.0) is bounded by

$$\begin{aligned} & \int dx_3 \left[\sum_{s_0, s \leq s_0} 2^{s-s_0} (\#\{j \mid |p_j(x_3) - n_j(x_3)| \sim 2^{-s_0}\}) \right] \\ & \leq \int \ell(x_3) dx_3 < C \end{aligned}$$

by (9.3.3).

Contribution of (9.5.2)

Same, since (9.5.2) < C from (9.4.6).

Contribution of (9.5.3)

This is the crux of the argument.

Estimate, using (9.3.2) and the fact that $|k_{s,s_0}^2| \leq C$,

$$\begin{aligned} \sum_{j \mid |p_j - n_j| \sim 2^{-s_0}} |k_{s,s_0}^2(p_j(x_3))| & \leq \|k_{s,s_0}^2\|_{\infty} \cdot \chi_{\text{Proj}_{x_3}(\text{supp } k_{s,s_0}^2)}(x_3) \\ & \quad \cdot [\#\{j \mid |p_j(x_3) - n_j(x_3)| \sim 2^{-s_0}\}] \\ & < C 2^{s_0} \chi_{\text{Proj}_{x_3}(\text{supp } k_{s,s_0}^2)}(x_3). \end{aligned}$$

Integration in x_3 gives therefore, using (9.4.7),

$$(9.5.4) \quad C(s - s_0)^6 2^{-(s-s_0)} \lambda_s$$

which, by (9.4.3), is summable in $\sum_{s_0, s > s_0}$.

This completes the proof of (9.3.5), and thus of Theorem 12.

9.3. A geometric estimate related to Proposition 3

With the same technique as in the proof of Proposition 3 we may derive the following estimate which has an interesting geometric flavour. It may be used to provide an alternative proof of Theorem 12 as in [BOS1].

Proposition 4. — Let Γ be a closed, oriented, rectifiable curve in \mathbf{R}^3 , and denote by \vec{t} the unit tangent vector along Γ ; let $\vec{k} \in W^{1,3}(\mathbf{R}^3; \mathbf{R}^3)$. Then

$$\left| \int_{\Gamma} \vec{k} \cdot \vec{t} \right| \leq C \|\vec{k}\|_{W^{1,3}} |\Gamma|.$$

Proof. — Part of the argument is a repetition of the proof of Proposition 3, but we have kept it for the convenience of the reader who wishes to concentrate on Proposition 4 independently of the rest of the paper. Assume $|\Gamma| = 1$ and let $\gamma : [0, 1] \rightarrow \Gamma$ be the arclength parametrization ($|\dot{\gamma}| = 1$).

We need to bound

$$(9.6.1) \quad \int_{\Gamma} k_3(\gamma(s)) \dot{\gamma}_3(s) ds = \int dx_3 \left[\sum_{x \in \Gamma_{x_3}} \sigma(x) k_3(x) \right],$$

where $\Gamma_{x_3} = \Gamma \cap [x = x_3]$ is assumed finite (by choice of coordinate system) and $\sigma(\gamma(s)) = \text{sign} \dot{\gamma}_3(s)$.

Thus $\Gamma_{x_3} = \{P_1, \dots, P_r\} \cup \{N_1, \dots, N_r\}$, where $\sigma(P_i) = 1$ and $\sigma(Q_i) = -1$. Also,

$$r = r(x_3) = \frac{1}{2} \text{card}(\Gamma_{x_3})$$

and

$$\int r(x_3) dx_3 = \frac{1}{2} \int |\dot{\gamma}_3(s)| ds < 1,$$

$$(9.6.3) \quad \sum_i |P_i - N_i| \leq |\Gamma| = 1.$$

Write k for k_3 and assume $\|k\|_{W^{1,3}} \leq 1$. Write, for fixed x_3 ,

$$(9.6.4) \quad \left| \sum_{x \in \Gamma_{x_3}} \sigma(x) k(x) \right| \leq \sum_{i=1}^{r(x_3)} |k(\mathbf{P}_i) - k(\mathbf{N}_i)| \\ = \sum_{s_0} \sum_{|\mathbf{P}_i - \mathbf{N}_i| \sim 2^{-s_0}} |k(\mathbf{P}_i) - k(\mathbf{N}_i)|.$$

To estimate (9.6.4), we perform again the same decomposition of $k \in W^{1,3}$. Thus, for fixed s_0 ,

$$k = k_{s_0} + \sum_{s > s_0} k_{s_0,s}^1 + \sum_{s > s_0} k_{s_0,s}^2$$

satisfying

$$(9.6.5) \quad |\nabla k_{s_0}| \lesssim 2^{s_0}$$

$$(9.6.6) \quad |k_{s_0,s}^1| \lesssim (s - s_0)^{-2}$$

$$(9.6.7) \quad \begin{cases} |k_{s_0,s}^2| \lesssim 1 \text{ and} \\ \text{supp } k_{s_0,s}^2 \text{ contained in the union of } \lesssim \sigma_s (s - s_0)^6 \text{ cubes of size } 2^{-s} \end{cases}$$

with

$$(9.6.8) \quad \sum \sigma_s < C$$

(in fact $\sigma_s^{1/3} = \|\Delta_s k\|_{W^{1,3}}$, $k = \sum \Delta_s k$, Littlewood-Paley decomposition).

Returning to (9.6.4), we get for fixed s_0 ,

$$(9.6.9) \quad \sum_{|\mathbf{P}_i - \mathbf{N}_i| \sim 2^{-s_0}} |k_{s_0}(\mathbf{P}_i) - k_{s_0}(\mathbf{N}_i)| \\ +$$

$$(9.6.10) \quad \sum_{s > s_0} \sum_{|\mathbf{P}_i - \mathbf{N}_i| \sim 2^{-s_0}} |k_{s_0,s}^1(\mathbf{P}_i)| + |k_{s_0,s}^1(\mathbf{N}_i)| \\ +$$

$$(9.6.11) \quad \sum_{s > s_0} \sum_{|\mathbf{P}_i - \mathbf{N}_i| \sim 2^{-s_0}} |k_{s_0,s}^2(\mathbf{P}_i)| + |k_{s_0,s}^2(\mathbf{N}_i)|.$$

Contribution of (9.6.9)

$$(9.6.5) \Rightarrow (9.6.9) \lesssim \#\{i \mid |\mathbf{P}_i - \mathbf{N}_i| \sim 2^{-s_0}\}.$$

Sum in $s_0 \Rightarrow r(x_3)$ satisfying (9.6.2).

Contribution of (9.6.10)

$$(9.6.6) \Rightarrow \sum_{s>s_0} |k_{s_0,s}^1| < C.$$

Hence

$$(9.6.10) \lesssim \#\{i \mid |P_i - N_i| \sim 2^{-s_0}\}.$$

Contribution of (9.6.11)

For fixed $s > s_0$, we need to restrict x_3 to $\text{Proj}_{x_3}(\text{supp } k_{s_0,s}^2) \subset \mathbf{R}$ of measure $\lesssim \sigma_s(s-s_0)^6 2^{-s}$ by (9.6.7).

By (9.6.3), $\#\{i \mid |P_i - N_i| \sim 2^{-s_0}\} \leq 2^{s_0}$, $\forall x_3$.

Thus,

$$\int dx_3 \left[\sum_{|P_i - N_i| \sim 2^{-s_0}} |k_{s_0,s}^2(P_i)| + \dots \right] \leq \sigma_s(s-s_0)^6 2^{-(s-s_0)},$$

summable in $s, s_0, s > s_0$, taking also (9.6.8) into account.

10. Open problems

OP 1. — Let u_ε be a minimizer of E_ε in H_g^1 with $g \in H^{1/2}(\Omega; S^1)$. Is it true that

$$\int_G |u_{\varepsilon x_i} \wedge u_{\varepsilon x_j}| \leq C \quad \forall i, j \text{ as } \varepsilon \rightarrow 0?$$

OP 2. — Let u_ε be a minimizer of E_ε in H_g^1 with $g \in H^{1/2}(\Omega; S^1)$. Is it true that

$$\|u_\varepsilon\|_{W^{1,3/2}(G)} \leq C \text{ as } \varepsilon \rightarrow 0?$$

Is (u_ε) relatively compact in $W^{1,3/2}$?

OP 3. — Assume $u_\varepsilon : B \rightarrow \mathbf{R}^2$ (B unit ball in \mathbf{R}^3) is smooth and satisfies

$$\int_B |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} \int_B (|u_\varepsilon|^2 - 1)^2 \leq C \log(1/\varepsilon).$$

Is it true that for every compact subset $K \subset B$,

$$\left| \int_B (u_{\varepsilon x} \wedge u_{\varepsilon y}) \varphi \right| \leq C_K \|\varphi\|_{W^{1,3}} \quad \forall \varphi \in C_0^\infty(K)?$$

(As explained in Section 9.1 a positive solution of OP3 yields a positive answer to the first question in OP2)

OP 4. — Let u_ε be a minimizer of E_ε in H_g^1 with $g \in H^{1/2}(\Omega; S^1)$.

Is it true that

$$|u_\varepsilon| \text{ is bounded in } H^1(G)?$$

11. Appendices

Appendix A. The upper bound for the energy

With G and $\Omega = \partial G$ as in Section 1, consider the following distinguished classes in $H^{1/2}(\Omega; S^1)$:

$$\mathcal{R} = \left\{ \begin{array}{l} g \in W^{1,p}(\Omega; S^1), \forall p < 2; g \text{ is smooth away from} \\ \text{a finite set } \Sigma \text{ of singularities} \end{array} \right\},$$

$$\mathcal{R}_0 = \left\{ \begin{array}{l} g \in \mathcal{R}; |\nabla g(x)| \leq C/|x - \sigma| \text{ near each } \sigma \in \Sigma \\ \text{and } \deg(g, \sigma) = \pm 1, \forall \sigma \in \Sigma \end{array} \right\},$$

$$\mathcal{R}_1 = \left\{ \begin{array}{l} g \in \mathcal{R}_0 \left| \begin{array}{l} \text{for each } \sigma \in \Sigma, \text{ there is some } R \in \mathcal{O}(3) \text{ such that} \\ |g(x) - R\left(\frac{x-\sigma}{|x-\sigma|}\right)| \leq C|x - \sigma| \text{ for } x \text{ near } \sigma \end{array} \right. \end{array} \right\},$$

where $\mathcal{O}(3)$ denotes the group of linear isometries of \mathbf{R}^3 . Here, we identify $S^1 \subset \mathbf{R}^2$ with $S^1 \times \{0\}$ viewed as a subset of \mathbf{R}^3 . From the definition of \mathcal{R}_1 we see that R must map the tangent plane $T_\sigma(\Omega)$ into $\mathbf{R}^2 \times \{0\}$ and thus $R(n(\sigma)) = (0, 0, \pm 1)$, where $n(\sigma)$ is the outward unit normal to Ω . Clearly, $\deg(g, \sigma) = +1$ if R is orientation-preserving and -1 otherwise.

This appendix is devoted to the proof of the following

Lemma A.1. — *Let $g \in \mathcal{R}_1$ and let L_G be the length of a minimal connection corresponding to the geodesic distance in G . Then*

$$\begin{aligned} \text{(A.1)} \quad & \text{Min } \{E_\varepsilon(u); u \in H_g^1(G; \mathbf{R}^2)\} \\ & \leq \pi L_G(g) \log(1/\varepsilon) + o(\log(1/\varepsilon)) \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

The proof we present below uses some arguments from [40], Section 1.

Proof. — Given $\delta > 0$ small, we first construct a domain G_δ and a diffeomorphism $\xi_\delta: G \rightarrow G_\delta$ (with $\xi_\delta: \partial G \rightarrow \partial G_\delta$) such that

$$(A.2) \quad \|D\xi_\delta - I\| \leq C\delta \text{ on } G$$

and ∂G_δ is flat in a δ -neighborhood of each singularity $\xi_\delta(a_j)$ of $g_\delta = g \circ \xi_\delta^{-1}$.

The construction of ξ_δ is standard. Assume, for simplicity, that 0 is a singular point of g on Ω and that, near 0, the graph of Ω is given by $x_3 = \psi(x_1, x_2)$ with ψ smooth and $\nabla\psi(0) = 0$. Set

$$\eta(x_1, x_2, x_3) = (x_1, x_2, x_3 - \psi(x_1, x_2))$$

so that $\|D\eta(x) - I\| \leq C|x|$ near 0. Let $\zeta \in C_0^\infty(B_1)$ with $\zeta = 1$ on $B_{1/2}$. Then

$$\xi_\delta(x) = x + \zeta(x/\delta)(\eta(x) - x), \quad x \in G$$

has all the required properties relative to one singularity. We proceed similarly for the other singularities.

We now write G and g instead of G_δ and g_δ , so that we may assume that Ω is flat in a δ -neighborhood of each singularity.

After relabeling the singularities of g , we may assume that $L_G(g) = \sum_{j=1}^k \text{length}(\gamma_j)$, where γ_j connects (in G) P_j and N_j . We now introduce a second parameter λ , $0 < \lambda < \delta$, and we choose some disjoint smooth curves Γ_j having the following properties:

- a) $\sum_{j=1}^k \text{length}(\Gamma_j) \leq L_G(g) + \lambda$;
- b) Γ_j is a simple curve;
- c) Γ_j is contained in G except for its endpoints P_j and N_j ;
- d) the curve Γ_j is orthogonal to Ω in a λ -neighborhood of its endpoints.

Moreover, we may assume that Γ_j is parametrized in such a way that the tangent vector at P_j is outward and the one at N_j is inward. We take the arclength as parameter. We may thus write $\Gamma_j = \{X_j(t); t \in [0, T_j]\}$, with $X_j(0) = N_j$, $X_j(T_j) = P_j$, where X_j is smooth, into and an immersion, and $T_j = \text{length}(\Gamma_j)$.

We consider the unit tangent vector to Γ_j , $e(X_j(t)) = X_j'(t)$. We may find two smooth vector fields f, g on Γ_j such that $\{f(X_j(t)), g(X_j(t)), e(X_j(t))\}$ is a direct orthonormal basis for each t .

We now define the map $\Phi_j: [0, T_j] \times \bar{B}_\lambda \rightarrow \mathbf{R}^3$ by

$$\Phi_j(t, u, v) = X_j(t) + uf(X_j(t)) + vg(X_j(t)),$$

where $B_\lambda = \{(u, v) \in \mathbf{R}^2; u^2 + v^2 \leq \lambda^2\}$.

Clearly,

$$(A.3) \quad \|D\Phi_j(t, u, v) - M(t)\| \leq C\lambda \text{ on } [0, T_j] \times B_\lambda,$$

where $M(t) \in \mathcal{O}(3)$. Thus, for λ sufficiently small, Φ_j is a diffeomorphism from $[0, T_j] \times \overline{B}_\lambda$ onto a λ -tubular neighborhood U_j of Γ_j . Moreover $U_j \subset \overline{G}$ for λ small.

It is easy to see that the restriction of g to $\Omega \setminus \cup_j U_j$ has a smooth S^1 -valued extension, \tilde{g} , to $\overline{G} \setminus \cup_j U_j$. Indeed, let $\zeta_j : G \rightarrow \mathbf{R}^3$ be a diffeomorphism onto $\zeta_j(G)$ with $\zeta_j(G) \subset B_R \times [0, T_j]$ and $\zeta_j(U_j) = \overline{B}_\lambda \times [0, T_j]$. Consider the function $k : \mathbf{R}^3 \rightarrow S^1$ defined by

$$k(x, y, z) = (x, y) / (x^2 + y^2)^{1/2}.$$

Then

$$k_j = k \circ \zeta_j : G \setminus U_j \rightarrow S^1$$

is smooth and

$$q = \prod_{j=1}^k k_j : G \setminus \cup_j U_j \rightarrow S^1$$

is also smooth. Moreover

$$\deg(q, C_j^\pm) = \pm 1 \quad \forall j$$

where $C_j^+ = \{x \in \Omega; |x - P_j| = \lambda\}$ and $C_j^- = \{x \in \Omega; |x - N_j| = \lambda\}$. Therefore

$$\deg(g/q, C_j^\pm) = 0 \quad \forall j.$$

Hence the function g/q restricted to $\Omega \setminus \cup_j U_j$ admits a smooth extension $f : \Omega \rightarrow S^1$.

Then f extends to a smooth map $\tilde{f} : \overline{G} \rightarrow S^1$. Finally, the map $\tilde{g} = \tilde{f}q$ has the desired properties.

Clearly we have

$$(A.4) \quad E_\epsilon(\tilde{g}; G \setminus \cup_j U_j) \leq C_\lambda.$$

Consider the map $h_j : \partial([0, T_j] \times \overline{B}_\lambda) \rightarrow S^1$ defined by

$$h_j = \begin{cases} \tilde{g} \circ \Phi_j, & \text{on } [0, T_j] \times \partial\overline{B}_\lambda \\ g \circ \Phi_j, & \text{on } \{0\} \times \overline{B}_\lambda \text{ and on } \{T_j\} \times \overline{B}_\lambda \end{cases}.$$

Then h_j is smooth on $\partial([0, T_j] \times B_\lambda)$ except at the points $(0, 0, 0)$ and $(T_j, 0, 0)$. From the construction in [40] we know that

$$(A.5) \quad \begin{aligned} & \text{Min } \{E_\varepsilon(u; (0, T_j) \times B_\lambda); u \in H_{h_j}^1((0, T_j) \times B_\lambda; \mathbf{R}^2)\} \\ & \leq \pi T_j \log(1/\varepsilon) + C_\lambda. \end{aligned}$$

Using (A.5) and (A.3) we return to U_j via Φ_j and obtain a map

$$v = v_{j,\varepsilon,\lambda} : U_j \rightarrow \mathbf{R}^2$$

such that $v = g$ on $(\partial U_j) \cap \Omega$ and

$$(A.6) \quad E_\varepsilon(v; U_j) \leq (\pi T_j \log(1/\varepsilon) + C_\lambda)(1 + C\lambda).$$

Gluing the maps $v_{j,\varepsilon,\lambda}$ defined above with the map $\tilde{g}|_{\bar{G} \setminus \cup_j U_j}$, we obtain a map $w_{\varepsilon,\lambda} : G \rightarrow \mathbf{R}^2$ satisfying

$$w_{\varepsilon,\lambda} = g \text{ on } \Omega$$

and (by (A.4) and (A.6)),

$$(A.7) \quad E_\varepsilon(w_{\varepsilon,\lambda}; G) \leq \left(\pi \left(\sum T_j \right) \log(1/\varepsilon) + C_\lambda \right) (1 + C\lambda) + C_\lambda.$$

Returning to the original notation G_δ and $\Omega_\delta = \partial G_\delta$, we have just constructed a map $w_{\varepsilon,\lambda} : G_\delta \rightarrow \mathbf{R}^2$ satisfying

$$w_{\varepsilon,\lambda} = g_\delta = g \circ \xi_\delta^{-1} \text{ on } \Omega_\delta$$

and

$$(A.8) \quad E_\varepsilon(w_{\varepsilon,\lambda}; G_\delta) \leq \pi(L_{G_\delta}(g_\delta) + \lambda) \log(1/\varepsilon)(1 + C\lambda) + C'_\lambda.$$

Finally, coming back to the original domain G via ξ_δ , we obtain some $\tilde{w}_{\varepsilon,\lambda,\delta} \in H_g^1(G; \mathbf{R}^2)$ such that

$$(A.9) \quad E_\varepsilon(\tilde{w}_{\varepsilon,\lambda,\delta}; G) \leq \left[\pi(L_{G_\delta}(g_\delta) + \lambda) \log(1/\varepsilon)(1 + C\lambda) + C'_\lambda \right] (1 + C\delta).$$

It is easy to see that

$$|L_{G_\delta}(g_\delta) - L_G(g)| \leq C\delta$$

and thus we arrive at

$$(A.10) \quad E_\varepsilon(\tilde{w}_{\varepsilon,\lambda,\delta}; G) \leq \pi L_G(g) \log(1/\varepsilon)(1 + C\lambda + C\delta) + C'_{\lambda,\delta},$$

which yields the desired conclusion (A.1) since $\lambda < \delta$ are arbitrarily small.

Appendix B. A variant of the density result of T. Rivière

We use the same notation as in Appendix A for \mathcal{R} , \mathcal{R}_0 and \mathcal{R}_1 . Recall that \mathcal{R}_0 is dense in $H^{1/2}(\Omega; S^1)$; see Rivière [38], quoted as Lemma 11, and see Remark 5.1 for a proof. This appendix is devoted to the following improvement:

Lemma B.1. — *The class \mathcal{R}_1 is dense in $H^{1/2}(\Omega; S^1)$.*

Proof. — Given $g \in H^{1/2}(\Omega; S^1)$ and $\varepsilon > 0$ we first use the density of \mathcal{R}_0 to construct a map $h \in \mathcal{R}_0$ such that $\|h - g\|_{H^{1/2}} < \varepsilon$.

Next, write, as usual, the singular set Σ of h as

$$\Sigma = \{P_1, P_2, \dots, P_k, N_1, N_2, \dots, N_k\}.$$

For every $\sigma \in \Omega$, let $T_\sigma(\Omega)$ denote the tangent plane to Ω at σ ; we orient it using the outward normal $n(\sigma)$ to G . Let P_Ω denote the projection onto Ω defined in a tubular neighborhood of Ω in \mathbf{R}^3 .

For each $i = 1, 2, \dots, k$, fix two smooth maps:

$$\begin{aligned} \gamma_i^+ &: \{\xi \in T_{P_i}(\Omega); |\xi| = 1\} \rightarrow S^1, \\ \gamma_i^- &: \{\xi \in T_{N_i}(\Omega); |\xi| = 1\} \rightarrow S^1, \end{aligned}$$

such that

$$\mathbf{(B.1)} \quad \deg(\gamma_i^+) = +1 \text{ and } \deg(\gamma_i^-) = -1.$$

The conclusion of Lemma B.1 is an immediate consequence of the following more general:

Claim. — With h as above, there is a sequence (h_n) in $H^{1/2}(\Omega; S^1)$ such that:

$$\mathbf{(B.2)} \quad h_n \rightarrow h \text{ in } H^{1/2}$$

$$\mathbf{(B.3)} \quad h_n \in C^\infty(\Omega \setminus \Sigma; S^1), \quad \forall n,$$

$$\mathbf{(B.4)} \quad h_n \in W^{1,p}(\Omega \setminus \Sigma; S^1), \quad \forall n, \quad \forall p < 2,$$

$$\mathbf{(B.5)} \quad |\nabla h_n(x)| \leq C_n / \text{dist}(x, \Sigma), \quad \forall n, \quad \forall x \in \Omega \setminus \Sigma,$$

for all $0 < t < t_0$ (sufficiently small, depending only on Ω) and all $i = 1, 2, \dots, k$, we have:

$$\mathbf{(B.6)} \quad |h_n(P_\Omega(P_i + t\xi)) - \gamma_i^+(\xi)| \leq C_n t, \quad \forall n, \forall \xi \in T_{P_i}(\Omega), |\xi| = 1,$$

$$\mathbf{(B.7)} \quad |h_n(P_\Omega(N_i + t\xi)) - \gamma_i^-(\xi)| \leq C_n t, \quad \forall n, \forall \xi \in T_{N_i}(\Omega), |\xi| = 1.$$

Proof of the Claim. — Fix an arbitrary function $k \in C^\infty(\Omega \setminus \Sigma; \mathbb{S}^1) \cap W^{1,p}(\Omega, \mathbb{S}^1)$, $\forall p < 2$ satisfying

$$(B.8) \quad |\nabla k(x)| \leq C \operatorname{dist}(x, \Sigma), \quad \forall x \in \Omega \setminus \Sigma,$$

$$(B.9) \quad |k(P_\Omega(P_i + t\xi)) - \gamma_i^+(\xi)| \leq Ct,$$

$$(B.10) \quad |k(P_\Omega(N_i + t\xi)) - \gamma_i^-(\xi)| \leq Ct,$$

for all t, i, ξ as in (B.6)–(B.7).

The existence of k is proved as in Appendix A. First we define it on $\partial B_1 \times [0, T]$ using the parameter t to homotopy γ_i^+ to the complex conjugate of γ_i^- . We then extend it to $B_1 \times [0, T]$ by homogeneity of degree 0 and transfer it to a “tube-like” region U_i in G connecting P_i to N_i . Finally, we extend these functions smoothly to $G \setminus U_i$, take their complex product, and restrict it to Ω .

To complete the proof of the claim, note that $T(h) = T(k) = 2\pi \sum_{i=1}^k (\delta_{P_i} - \delta_{N_i})$. Thus $T(h\bar{k}) = 0$ and, by Theorem 2, there exists a sequence $r_n \in C^\infty(\Omega; \mathbb{S}^1)$ such that $r_n \rightarrow h\bar{k}$ in $H^{1/2}$. Using the fact that points have zero H^1 -capacity in $2 - d$ (and thus zero $H^{1/2}$ -capacity), we may also assume that $r_n(P_i) = r_n(N_i) = 1, \forall n, \forall i$. Clearly, the sequence $h_n = kr_n$ has all the desired properties (B.2)–(B.7).

Lemma B.1 is obtained by choosing, in the claim, as γ_i^+ and γ_i^- any isometries from $T_{P_i}(\Omega)$ and $T_{N_i}(\Omega)$ onto \mathbf{R}^2 .

Appendix C: Almost \mathbf{Z} -valued functions

The purpose of this section is to prove the following fact used earlier in Section 8.

Lemma C.1. — *Assume $\varphi \in H^{1/2}((0, 1) \times (0, 1))$ and $\{Q_\alpha\}$ a collection of squares in $(0, 1)^2$ such that*

$$(C.1) \quad \|\varphi\|_{L^{4/3}} \leq C$$

$$(C.2) \quad \|e^{i\varphi} - 1\|_{L^1([0,1]^2 \setminus \cup Q_\alpha)} \leq \varepsilon$$

$$(C.3) \quad |\varphi|_{H^{1/2}} \leq \delta(\log(1/\varepsilon))^{1/2}$$

$$(C.4) \quad \sum_{\alpha} \sigma_{\alpha} \leq \delta,$$

where $\varepsilon < \delta \ll 1$ and σ_{α} denotes the size of Q_{α} .

Then there is some $a \in \mathbf{Z}$ such that

$$(C.5) \quad \|\varphi - 2\pi a\|_{L^1} \leq C\delta^{1/8}.$$

The proof will rely on the following inequality (*see* also [15] and [35] for related results).

Lemma C.2. — *Let* $Q = (0, 1)^2, f \in L^1(Q)$. *Then for all* $0 < \rho < \rho_0, \rho_0$ *sufficiently small,*

$$(C.6) \quad \left\| f - \int f \right\|_{L^1} \leq C |\log \rho|^{-1} \iint_{Q \times Q} \frac{|f(x) - f(y)|}{|x - y|(|x - y| + \rho)^2} dx dy$$

with C some constant.

Proof of Lemma C.1. — It follows from (C2) that we may write Q as a disjoint union

$$Q = \bigcup Q_\alpha \cup Z_0 \cup \bigcup_{j \in \mathbf{Z}} A_j.$$

where

$$(C.7) \quad A_j \subset [|\varphi - 2\pi j| < \varepsilon^{1/8}]$$

$$(C.8) \quad |Z_0| < \varepsilon^{3/4}.$$

Apply Lemma C.2 to $f = \chi_{A_j}$ with $\rho = \varepsilon^{1/20}$. Hence, denoting $Z = Z_0 \cup \bigcup_\alpha Q_\alpha$,

$$\begin{aligned} |A_j|(1 - |A_j|) &\leq C |\log \varepsilon|^{-1} \iint_{A_j \times (Q \setminus A_j)} |x - y|^{-1} (|x - y| + \rho)^{-2} \\ &\leq C |\log \varepsilon|^{-1} \sum_{k \neq j} \iint_{A_j \times A_k} |x - y|^{-3} + C |\log \varepsilon|^{-1} \\ &\quad \times \iint_{A_j \times Z} |x - y|^{-1} (|x - y| + \rho)^{-2} \\ &\leq C |\log \varepsilon|^{-1} \iint_{\substack{A_j \times \cup A_k \\ k \neq j}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^3} + C |\log \varepsilon|^{-1} \\ &\quad \times \iint_{A_j \times Z} |x - y|^{-1} (|x - y| + \rho)^{-2}. \end{aligned}$$

Summation over j gives

$$\begin{aligned}
\sum_j |A_j|(1 - |A_j|) &\leq C |\log \varepsilon|^{-1} \|\varphi\|_{\mathbb{H}^{1/2}}^2 \\
&\quad + C |\log \varepsilon|^{-1} \iint_{Z \times (\mathbb{Q} \setminus Z)} |x - y|^{-1} (|x - y| + \rho)^{-2} \\
\text{(C.9)} \quad &\stackrel{\text{by (C.3)}}{\leq} C \delta^2 + C |\log \varepsilon|^{-1} \\
&\quad \times \left[\sum_{\alpha} \iint_{\mathbb{Q}_\alpha \times (\mathbb{Q} \setminus \mathbb{Q}_\alpha)} |x - y|^{-1} (|x - y| + \rho)^{-2} \right] \\
&\quad + C |Z_0| \cdot \varepsilon^{-\frac{1}{10}}.
\end{aligned}$$

For fixed α , estimate

$$\text{(C.10)} \quad \iint_{\mathbb{Q}_\alpha \times (\mathbb{Q} \setminus \mathbb{Q}_\alpha)} |x - y|^{-1} (|x - y| + \rho)^{-2}.$$

Since for fixed $x \in \mathbb{Q}_\alpha$, $|x - y| > \text{dist}(x, \partial \mathbb{Q}_\alpha)$, we get easily

$$\text{(C.10)} \leq C \int_{\mathbb{Q}_\alpha} [\text{dist}(x, \partial \mathbb{Q}_\alpha) + \rho]^{-1} dx < C |\log \varepsilon| \sigma_\alpha$$

with σ_α the size of \mathbb{Q}_α .

Substitute in (C.9) and use (C.4), (C.8) to bound

$$\text{(C.11)} \quad \sum_j |A_j|(1 - |A_j|) \leq C \delta^2 + C \sum \sigma_\alpha + \varepsilon^{\frac{3}{4} - \frac{1}{10}} \leq C \delta + \varepsilon^{3/5}.$$

Take j_0 with $|A_{j_0}| = \max |A_j|$. Thus $|A_j| \leq \frac{1}{2}$ for $j \neq j_0$ and by (C.11)

$$\text{(C.12)} \quad \sum_{j \neq j_0} |A_j| \leq C(\delta + \varepsilon^{3/5}).$$

Taking $a = j_0$, finally estimate using (C.1), (C.7)

$$\begin{aligned}
\|\varphi - 2\pi a\|_1 &\leq \|\varphi - 2\pi j_0\|_{L^1(A_{j_0})} + \|\varphi\|_{L^1(\mathbb{Q} \setminus A_{j_0})} + 2\pi |a| |\mathbb{Q} \setminus A_{j_0}| \\
&\leq \varepsilon^{\frac{1}{8}} + C |\mathbb{Q} \setminus A_{j_0}|^{\frac{1}{4}} + 2\pi |a| |\mathbb{Q} \setminus A_{j_0}|
\end{aligned}$$

where, by (C.4), (C.8), (C.12)

$$\begin{aligned}
|\mathbb{Q} \setminus A_{j_0}| &\leq \sum |\mathbb{Q}_\alpha| + |Z_0| + \sum_{j \neq j_0} |A_j| \leq \sum \sigma_\alpha^2 + \varepsilon^{3/4} + C(\delta + \varepsilon^{3/5}) \\
&\leq C(\delta + \varepsilon^{3/5}).
\end{aligned}$$

Hence

$$\|\varphi - 2\pi a\|_1 \leq C(\varepsilon^{1/8} + \delta^{1/4}) + C|a|(\delta + \varepsilon^{3/5})$$

implying

$$2\pi|a| \leq \|\varphi\|_1 + 1 + |a|$$

so that

$$|a| \leq C \text{ and } \|\varphi - 2\pi a\|_1 \leq C(\delta^{1/4} + \varepsilon^{1/8}) \leq C\delta^{1/8}$$

which is (C.5).

Proof of Lemma C.2. — We will derive the inequality by contradiction, using Theorem 4 in [14]. Let thus (f_n) be a sequence in $L^1(\mathbf{Q})$ and $(\varepsilon_n) \downarrow 0$ such that

$$(C.13) \quad |\log \varepsilon_n|^{-1} \iint_{\mathbf{Q} \times \mathbf{Q}} \frac{|f_n(x) - f_n(y)|}{|x - y|(|x - y| + \varepsilon_n)^2} dx dy \leq 1$$

and

$$(C.14) \quad \|f_n - \int f_n\|_{L^1} \rightarrow \infty.$$

Denote by ρ_n the radial mollifier on \mathbf{R}^2

$$(C.15) \quad \rho_n(x) = c_n |\log \varepsilon_n|^{-1} (|x| + \varepsilon_n)^{-2}$$

with c_n such that $\int \rho_n = 1$ (hence $c_n \sim 1$). Applying Theorem 4 from [14], with $p = 1$, it follows that (f_n) is relatively compact in $L^1(\mathbf{Q})$, contradicting (C.14). This proves (C.6).

Appendix D. Sobolev imbeddings for BV

It is well-known that, if $p > 1$ and $0 < s < 1$, then

$$W^{1,p}(\Omega) \subset W^{s,q}(\Omega), \quad \Omega \subset \mathbf{R}^d$$

with

$$\frac{1}{q} = \frac{1}{p} - \frac{(1-s)}{d}.$$

This imbedding fails for $p = 1$ and $d = 1$, i.e., $W^{1,1}$ is *not* contained in $W^{1/q,q}$ for $q > 1$. Surprisingly, the imbedding holds when $p = 1$ and $d \geq 2$.

Lemma D.1. — Assume $d \geq 2$ and $0 < s < 1$. Then

$$\text{BV}(\mathbf{R}^d) \subset W^{s,p}(\mathbf{R}^d)$$

with

$$(D.1) \quad \frac{1}{p} = 1 - \frac{1-s}{d}.$$

When $d = 2$, this result is an immediate consequence of an interpolation result of Cohen, Dahmen, Daubechies and DeVore [23]. It also seems to be contained in an earlier work of V. A. Solonnikov [44] although the condition $d \geq 2$ does not appear in his paper. We thank V. Maz'ya and T. Shaposhnikova for calling our attention to the paper of Solonnikov and for confirming that the assumption $d \geq 2$ is indeed used there implicitly; they have also devised another proof of Solonnikov's inequality (personal communication).

Our proof relies on the following one-dimensional elementary inequality:

Lemma D.2. — Let $1 < p < \infty$ and $0 < s < 1/p$. Then, for every $f \in C_0^\infty(\mathbf{R})$,

$$(D.2) \quad |f|_{W^{s,p}(\mathbf{R})}^p \leq C \|f\|_{L^p(\mathbf{R})}^{p(1-sp)} \|f'\|_{L^1(\mathbf{R})}^{sp^2},$$

where C depends only on p and s .

Here, $|\cdot|_{W^{s,p}(\mathbf{R})}$ denotes the canonical semi-norm on $W^{s,p}(\mathbf{R})$, i.e.,

$$|f|_{W^{s,p}(\mathbf{R})}^p = \int_{\mathbf{R}} dx \int_0^\infty \frac{|f(x+h) - f(x)|^p}{h^{1+sp}} dh.$$

Proof. — Write, for $\lambda > 0$,

$$\begin{aligned} |f|_{W^{s,p}(\mathbf{R})}^p &= \int_{\mathbf{R}} dx \int_0^\lambda \dots dh + \int_{\mathbf{R}} dx \int_\lambda^\infty \dots dh \\ &\leq 2^{p-1} \|f\|_{L^\infty}^{p-1} \|f'\|_{L^1} \frac{\lambda^{1-sp}}{1-sp} + 2^{p-1} \|f\|_{L^p}^p \frac{\lambda^{-sp}}{sp} \\ &\leq 2^{p-1} \left(\|f'\|_{L^1}^p \frac{\lambda^{1-sp}}{1-sp} + \|f\|_{L^p}^p \frac{\lambda^{-sp}}{sp} \right), \end{aligned}$$

since $sp < 1$. Minimizing in λ yields (D.2) with $C = 2^{p-1}/sp(1-sp)$.

Proof of Lemma D.1. — Let $u \in C_0^\infty(\mathbf{R}^d)$. We will use the following equivalent norm on $W^{s,p}$ (see e.g. Adams [1], Lemma 7.44)

$$(D.3) \quad \|u\|_{W^{s,p}}^p \sim \|u\|_{L^p}^p + \sum_{j=1}^d \int_{\mathbf{R}^d} dx \int_0^\infty \frac{|u(x + he_j) - u(x)|^p}{h^{1+sp}} dh.$$

Note that $BV \subset L^1 \cap L^{d/(d-1)}$ and thus we may estimate (via Hölder)

$$\|u\|_{L^p} \leq C \|u\|_{BV},$$

since

$$(D.4) \quad \frac{1}{p} = 1 - \frac{(1-s)}{d} = \frac{s}{1} + \frac{1-s}{d/(d-1)}.$$

We now turn to the second term in (D.3); without loss of generality we may take $j = 1$. We apply Lemma D.1 to the function

$$f(\cdot) = u(\cdot, x_2, x_3, \dots, x_d)$$

(note that, by (D.4), $sp < 1$) and we obtain

$$(D.5) \quad \begin{aligned} & \int_{\mathbf{R}} dx_1 \int_0^\infty \frac{|u(x_1 + h, x_2, \dots, x_d) - u(x_1, x_2, \dots, x_d)|^p}{h^{1+sp}} dh \\ & \leq C \|f\|_{L^p(\mathbf{R})}^{p(1-sp)} \|f'\|_{L^1(\mathbf{R})}^{sp^2} \leq C \|f\|_{L^1}^{sp(1-sp)} \|f\|_{L^{d/(d-1)}}^{(1-s)p(1-sp)} \|f'\|_{L^1}^{sp^2}. \end{aligned}$$

On the one hand, we have

$$(D.6) \quad \int_{\mathbf{R}^{d-1}} \|f'\|_{L^1(\mathbf{R})} dx_2 dx_3 \dots dx_d \leq \int_{\mathbf{R}^d} |\nabla u| dx.$$

On the other hand, the imbedding $BV \subset L^{d/(d-1)}$ gives, with $q = d/(d-1)$,

$$(D.7) \quad \int_{\mathbf{R}^{d-1}} \|f\|_{L^q(\mathbf{R})}^q dx_2 dx_3 \dots dx_d = \|u\|_{L^q(\mathbf{R}^d)}^q \leq C \left(\int_{\mathbf{R}^d} |\nabla u| dx \right)^q.$$

Finally we claim that

$$(D.8) \quad \int_{\mathbf{R}^{d-1}} \|f\|_{L^1(\mathbf{R})}^{(d-1)/(d-2)} dx_2 dx_3 \dots dx_d \leq C \left(\int_{\mathbf{R}^d} |\nabla u| dx \right)^{(d-1)/(d-2)};$$

when $d = 2$, inequality (D.8) reads

$$\|f\|_{L^\infty_x(L^1_{x_1})} \leq \int_{\mathbf{R}^2} |\nabla u|.$$

To prove (D.8) we use once more the imbedding $BV \subset L^r$, but this time in \mathbf{R}^{d-1} , with $r = (d-1)/(d-2)$, and we obtain

$$(D.9) \quad \|f(x_1, \cdot)\|_{L^r(\mathbf{R}^{d-1})} \leq C \int_{\mathbf{R}^{d-1}} |\nabla u(x_1, \cdot)| dx_2 dx_3 \dots dx_d.$$

Next, we have

$$\begin{aligned} \|f\|_{L^r(\mathbf{R}^{d-1}; L^1(\mathbf{R}))} &= \left\| \int_{\mathbf{R}} |f(x_1, \cdot)| dx_1 \right\|_{L^r(\mathbf{R}^{d-1})} \\ &\leq \int_{\mathbf{R}} \|f(x_1, \cdot)\|_{L^r(\mathbf{R}^{d-1})} dx_1 \quad \text{by the triangle inequality} \\ &\leq C \int_{\mathbf{R}^d} |\nabla u(x)| dx \quad \text{by (D.9)}. \end{aligned}$$

Finally, we return to (D.5), integrate in $dx_2 dx_3 \dots dx_d$, and apply Hölder with exponents P, Q, R such that

$$\begin{aligned} Psp(1-sp) &= (d-1)/(d-2), \\ Q(1-s)p(1-sp) &= d/(d-1), \\ Rsp^2 &= 1. \end{aligned}$$

[A straightforward computation shows that $\frac{1}{P} + \frac{1}{Q} + \frac{1}{R} = 1$]. From (D.8), (D.7) and (D.6) we deduce that

$$\|u\|_{W^{s,p}(\mathbf{R}^d)}^p \leq C \left(\int_{\mathbf{R}^d} |\nabla u| dx \right)^p.$$

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Added in proof:

1) After our work was completed some of our results were generalized to higher dimensions in [ABO].

2) F. Bethuel, G. Orlandi and D. Smets have solved our Open Problem 3 (and thereby also the first part of Open Problem 2) in Section 10; see [BOS1] and [BOS2].

3) J. Van Schaftingen [VS] has given an elementary proof of our Proposition 4, which extends easily to higher dimensions. His proof follows the same strategy as ours, except that he uses the Morrey-Sobolev imbedding in place of a Littlewood Paley decomposition.

4) An alternative approach to Proposition 4 is to use a new estimate for the div-curl system (see [BB]), namely

$$\|u\|_{L^{3/2}} \leq C\|\operatorname{curl} u\|_{L^1}, \forall u \text{ with } \operatorname{div} u = 0.$$

5) An interesting extension of Lemma C.2 may be found in [P].

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