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H^1_{loc}-stress and strain regularity in Cosserat plasticity

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Dedicated to Professor Vladimir A. Palmov on the occasion of his 75th birthday.

We show H^1_{loc}-regularity of the Cauchy stress tensor and H^1_{loc}-regularity of the infinitesimal strain tensor and the plastic strain tensor in infinitesimal Cosserat plasticity with monotone flow rule. We use energy estimates for difference quotients.

1 Introduction: Plasticity and Cosserat models

The regularity question for small strain models of elasto-plastic behavior has recently found renewed interest [4, 5, 8, 9, 18–20], in part motivated by the need for qualitative statements on the rate of convergence of finite element methods in elasto-plasticity. There it is necessary to know precisely the regularity of the function to be approximated, see [17]. This article addresses the regularity question for time-continuous formulations of *geometrically linear* elasto-plasticity. As a representative model problem we consider generalized continua of *Cosserat-micropolar* type. The basic difference of a Cosserat model as compared with classical continuum models is the appearance of a nonsymmetric stress tensor which is augmented by a generalized balance of angular momentum equation allowing to model interaction of particles not only by surface forces (classical Cauchy continuum) but also through surface couples (Cosserat continuum). For an introduction to the theory of Cosserat and micropolar models we refer to [11–13, 16, 17, 21]. The second author has also proposed an elasto-plastic Cosserat model [13] in a finite strain framework, for alternative variants see [7]. A geometrical linearization of this model has been investigated in [3, 14–16] and is shown to be well-posed also in the rate-independent limit for both quasistatic and dynamic processes.

Time-incremental formulations for this and other models have already been shown to posses smooth updates, see [18] and the references therein. However, the employed method did not allow to pass to the continuous time limit and it was not clear what kind of regularity to expect for the time-continuous setting. An early statement can be found in [6]. A first major breakthrough regarding global spatial regularity was obtained recently by Alber and Nesenenko [1] where $L^\infty(0,T;H^{1/3-\delta}(\Omega))$-regularity is shown for stresses and plastic strains for classical rate-dependent viscoplasticity and rate-independent models with linear kinematic hardening. This is followed by Knees [10] where viscosity is replaced by the linear hardening assumption together with the subdifferential structure of the flow rule. She obtains the improved $L^\infty(0,T;H^{1/2-\delta}(\Omega))$-regularity.

Local regularity results for elasto-plasticity with linear hardening and variants thereof have been derived by several authors [2, 4, 5, 22, 23]. Typically, one gets $L^\infty(0,T;H^1_{\mathrm{loc}}(\Omega))$. This is also what we will obtain for the Cosserat model, however, without any hardening and for both the quasistatic and dynamic case and without using a subdifferential structure.

Our focus on Cosserat models is justified by the fact that the Cosserat type models are today increasingly advocated as a means to regularize the pathological mesh size dependence of localization computations where shear failure mechanisms play a dominant role.

This contribution is now organized as follows: first, we recall the time-continuous geometrically linear elasto-plastic Cosserat model as introduced in [13] and investigated mathematically in [14–16] together with the major statements obtained for this model. Then we prove that for initial plastic strain $\varepsilon_p^0 \in H^1_{\mathrm{loc}}(\Omega)$ and body force $f \in L^2(0,T;H^1_{\mathrm{loc}}(\Omega))$ the solution obtained in the existence theorem is more regular. In Sects. 4 and 5 we repeat the regularity procedure defined in

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Sect. 3 in the dynamical setting of the problem and for general flow rules of monotone type. Our notation is that of previous papers, e.g., [16], suffice it to say that $\mathbb{1}$ denotes the second order identity tensor and $\langle X, Y\rangle$ is the scalar product on second order tensors.

2 The infinitesimal Cosserat elasto-plastic model

We consider the infinitesimal Cosserat elasto-plastic model as introduced in [14–16]. The goal is to prove a higher regularity of the stress tensor and the strain tensors assuming that the external force and the initial plastic strain are more regular. In the quasistatic setting of the problem the system of equations is in the form

$$\begin{aligned}
\mathrm{Div}\sigma &= -f\,,\\
\sigma &= 2\mu\,(\varepsilon-\varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot\mathbb{1}\,,\\
-\mathrm{Div}(l_c\,\nabla \mathrm{axl}(A)) &= \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u) - A)\,,\\
\dot\varepsilon_p &\in \mathfrak{f}(T_E)\,,\quad T_E = 2\mu\,(\varepsilon-\varepsilon_p)\,,\\
u_{|\partial\Omega} &= u_d\,,\quad A_{|\partial\Omega} = A_d\,,\quad \varepsilon_p(0) = \varepsilon_p^0\,.
\end{aligned} \tag{2.1}$$

Here u is the displacement vector, σ is the Cauchy stress tensor, which for $\mu_c > 0$ is not necessary symmetric, $\varepsilon = \mathrm{sym}(\nabla u)$ is the infinitesimal strain tensor, ε_p is the symmetric plastic strain tensor, $A \in \mathfrak{so}(3)$ is the infinitesimal skew-symmetric microrotation matrix, $\mathrm{axl} : \mathfrak{so}(3) \mapsto \mathbb{R}^3$ is the canonical identification of the Lie-algebra of skew-symmetric real 3×3-matrices $\mathfrak{so}(3)$ and vectors in $\mathbb{R}^3$, T_E is the reduced Eshelby tensor, f is the density of the external force acting on the material, $\mathfrak{f} : D(\mathfrak{f}) \subset \mathrm{Sym}(3) \to \mathfrak{sl}(3)\cap\mathrm{Sym}(3)$ is supposed to be a maximal monotone mapping with trace free, symmetric image and $\{0\} \in \mathfrak{f}(0)$, ε_p^0 is the initial plastic strain and u_d, A_d are given boundary data. In this article, in extension of [14], we assume that the coefficients μ, λ, μ_c, l_c depend also on $x \in \Omega$. We require that μ, λ, μ_c are positive, continuous on $\overline{\Omega}$, locally Lipschitz and additionaly $l_c \in C^1(\overline{\Omega})$. The parameter l_c abbreviates $l_c = \mu\,L_c^2$ with an internal length scale L_c. In all of the following we assume $\Omega \subset \mathbb{R}^3$ is a bounded, open domain with smooth boundary $\partial\Omega$. (Note, that the regularity assumption on the boundary is necessary in the existence theorem only. In the main result of the article the smoothness of the boundary is not important.)

In [14] the following existence and uniqueness theorem for system (2.1) with constant coefficients $\mu, \lambda, \mu_c, l_c \geq c > 0$ is proved:[1]

Theorem 2.1 (Existence for the infinitesimal elasto-plastic Cosserat model). *Suppose that the given data f, u_d, A_d satisfy: for all times $T > 0$*

$$\begin{aligned}
&f \in C^1([0,T], L^2(\Omega,\mathbb{R}^3))\,, && \ddot f \in L^2((0,T)\times\Omega,\mathbb{R}^3)\,,\\
&u_d \in C^2([0,T], H^{\frac12}(\partial\Omega,\mathbb{R}^3))\,, && \ddot v_d \in L^2((0,T)\times\partial\Omega,\mathbb{R}^3)\,,\\
&A_d \in C^2([0,T], H^{\frac32}(\partial\Omega,\mathfrak{so}(3)))\,, && \ddot B_d \in L^2((0,T)\times\partial\Omega,\mathfrak{so}(3))\,,
\end{aligned}$$

where $v_d = \dot u_d$ and $B_d = \dot A_d$. Moreover, assume that the initial data $\varepsilon_p^0 \in L^2(\Omega,\mathrm{Sym}(3))$ is chosen such that the initial value of the reduced Eshelby tensor $T_E(0) = 2\mu\,(\varepsilon(0)-\varepsilon_p^0)$ defined by the initial data ε_p^0 belongs to the domain of the maximal monotone operator $\mathfrak{f}$. Then system (2.1) possesses a global in time, unique solution $(u, \varepsilon, \varepsilon_p, A)$ with the regularity: for all times $T > 0$

$$\begin{aligned}
&u \in H^{1,\infty}((0,T), H^1(\Omega,\mathbb{R}^3))\,, && \varepsilon,\varepsilon_p \in H^{1,\infty}((0,T), L^2(\Omega,\mathrm{Sym}(3)))\,,\\
&A \in H^{1,\infty}((0,T), H^2(\Omega,\mathfrak{so}(3)))\,.
\end{aligned}$$

In Theorem 2.1 $\mathrm{Sym}(3)$ denotes the set 3×3 real-valued symmetric matrices. If the coefficients of the model are locally Lipschitz, positive and $l_c \in C^1(\overline{\Omega})$ Theorem 2.1 can be proved using the same technics as in [14].

3 $H^1_{\rm loc}$-regularity in the quasistatic case

The goal of this section is to prove that for $\varepsilon_p^0 \in H^1_{\rm loc}(\Omega)$ and $f \in L^2(0,T; H^1_{\rm loc}(\Omega))$ with $f(0) \in H^1_{\rm loc}(\Omega)$ the solution of system (2.1) is more regular. We are using the difference quotient method. Let $V, U \subset \Omega$ be open sets such that $V \Subset U \Subset \Omega$.

[1] Note that in contrast to linear elasticity, $\lambda > 0$ is mandatory, but is verified for metals.

Let $\eta \in C_0^\infty(\Omega)$ be a cutoff function such that $\eta(x) \in [0,1]$ for each $x \in \Omega$, $\eta \equiv 1$ on V and $\operatorname{supp}\eta \subset \overline{U}$. Let us observe that using the standard regularity theory of linear elliptic systems if $\varepsilon_p^0 \in H^1_{\rm loc}(\Omega)$ and $f(0) \in H^1_{\rm loc}(\Omega)$ then the initial stress, the initial strain tensors and the initial microrotation are more regular as obtained in [14]. Namely, these initial functions are solutions to the system

$$\begin{aligned}
&\operatorname{Div}\sigma(0) = -f\,,\\
&\sigma(0) = 2\mu\,(\varepsilon(0) - \varepsilon_p(0)) + 2\,\mu_c\,(\operatorname{skew}(\nabla u(0)) - A(0)) + \lambda\,\operatorname{tr}[\varepsilon(0)]\cdot \mathbb{1}\,,\\
&-\operatorname{Div}(l_c \nabla \operatorname{axl}(A(0))) = -\mu_c\,\operatorname{axl}(A(0)) + \mu_c\,\operatorname{axl}(\operatorname{skew}(\nabla u(0)))\,,\\
&u(0)_{|\partial\Omega} = u_d\,,\quad A(0)_{|\partial\Omega} = A_d\,,
\end{aligned} \tag{3.1}$$

where $\varepsilon(0) = \operatorname{sym}(\nabla u(0))$. Hence, we have that $\sigma(0), \varepsilon(0) \in H^1_{\rm loc}(\Omega; \operatorname{Sym}(3))$. Moreover, if additionally $l_c \in C^2(\Omega)$ then the initial microrotation $A(0) \in H^3_{\rm loc}(\Omega; \mathfrak{so}(3))$. Let us recall the energy function associated with system (2.1)

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A)(t) = \int_\Omega \Big(\mu\|\varepsilon - \varepsilon_p\|^2 + \frac{\lambda}{2}\operatorname{tr}[\varepsilon]^2 + \mu_c\|\operatorname{skew}(\nabla u) - A\|^2 + 2l_c\,\|\nabla \operatorname{axl}(A)\|^2\Big)\,\mathrm{dx}\,.$$

The following coerciveness property of the energy function is proved in [14]

Theorem 3.1 (Coerciveness of the energy)**.** *The energy function is elastically coercive with respect to ∇u. This means that $\exists\, C_E > 0\,,\ \forall\, u \in H^1_0(\Omega)\,,\ \forall\, A \in H^1_0(\Omega)\,,\ \forall\, \varepsilon_p \in L^2(\Omega)$*

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A) \geq C_E(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)})\,.$$

Moreover, $\exists\, C_E > 0$, $\forall\, u_d, A_d \in H^{\frac{1}{2}}(\partial\Omega)$, $\exists\, C_d > 0$, $\forall\, \varepsilon_p \in L^2(\Omega)$, $\forall\, u \in H^1(\Omega)$, $\forall\, A \in H^1(\Omega)$ with $u_{|\partial\Omega} = u_d$ and $A_{|\partial\Omega} = A_d$ it holds that

$$\mathcal{E}(u, \varepsilon, \varepsilon_p, A) + C_d \geq C_E(\|u\|^2_{H^1(\Omega)} + \|A\|^2_{H^1(\Omega)})\,.$$

This theorem was proved for constant coefficients only. Nevertheless, a simple modification of the proof from [14] allows to conclude the same result for locally Lipschitz and positive coefficients.

Let us denote by $\mathcal{E}_V(u, \varepsilon, \varepsilon_p, A)$ the energy calculated on the set V only. Let us also choose a basis e_k, $k = 1, \ldots, n$ of $\mathbb{R}^n$. For $h \in \mathbb{R}$ and a fixed $k \in \{1, \ldots, n\}$ we denote by D^h_k the difference quotient in the direction e_k with the step h. This means that for a function w defined on Ω

$$D^h_k w(x) := \frac{w(x + he_k) - w(x)}{h} \qquad \text{defined for } \ x + he_k \in \Omega\,.$$

Theorem 3.2 (Main estimate)**.** *Let us assume that $\varepsilon_p^0 \in H^1_{\rm loc}(\Omega; \mathcal{P}(\operatorname{Sym}(3)))$, $f \in L^2(0,T; H^1_{\rm loc}(\Omega; \mathbb{R}^3))$ with $f(0) \in H^1_{\rm loc}(\Omega; \mathbb{R}^3)$ and μ, λ, μ_c are positive and continuous on $\overline{\Omega}$, locally Lipschitz and additionaly $l_c \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then for all $k \in \{1, \ldots, n\}$, $t \in [0,T]$ and sufficiently small $h \in \mathbb{R}$ the solution of system (2.1) satisfies*

$$\begin{aligned}
&\mathcal{E}_V(\eta D^h_k u, \eta D^h_k \varepsilon, \eta D^h_k \varepsilon_p, \eta D^h_k A)(t) \leq C\Big(\mathcal{E}_{U^*}(u_{x_k}, \varepsilon_{x_k}, \varepsilon_{p,x_k}, A_{x_k})(0)\\
&\quad + \|f\|^2_{L^2(H^1(\Omega)} + \int_0^t (\mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) + \mathcal{E}_{U^*}(\dot u, \dot\varepsilon, \dot\varepsilon_p, \dot A))\,d\tau\\
&\quad + \mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) + \|\dot u\|^2_{L^2(H^1(\Omega))} + \|\dot A\|^2_{L^2(H^1(\Omega))} + 1\Big)\,,
\end{aligned} \tag{3.2}$$

where $U^* = U + B(0,r)$ with $r = 1/2\,\operatorname{dist}(U, \partial\Omega)$ and the constant $C > 0$ does not depend on h.

P r o o f. Let us fix k and assume that $h \neq 0$ and $|h| \leq 1/2\,\operatorname{dist}(U, \partial\Omega)$. For $x \in U$ the difference operator D^h_k is well defined. For $x \notin U$ the products $\eta D^h_k(\cdot)$ are equal to zero. From the definition of the energy function we have

$$\begin{aligned}
\mathcal{E}(\eta D^h_k u, \eta D^h_k \varepsilon, \eta D^h_k \varepsilon_p, \eta D^h_k A)(t) = \int_\Omega \Big(&\eta^2\mu\|D^h_k\varepsilon - D^h_k\varepsilon_p\|^2 + \eta^2\frac{\lambda}{2}\operatorname{tr}[D^h_k\varepsilon]^2\\
&+ \mu_c\|\eta\,\operatorname{skew}(\nabla D^h_k u) - \eta D^h_k A + \operatorname{skew}(\nabla\eta \otimes D^h_k u)\|^2\\
&+ 2l_c\,\|\eta\nabla \operatorname{axl}(D^h_k A) + \nabla\eta \otimes \operatorname{axl}(D^h_k A)\|^2\Big)\,\mathrm{dx}\,.
\end{aligned}$$

Calculating the time derivative of the energy we obtain

$$\dot{\mathcal{E}}(\eta D^h_k u, \eta D^h_k \varepsilon, \eta D^h_k \varepsilon_p, \eta D^h_k A)(t) = \int_\Omega \eta^2 \Big(\langle 2\mu(D^h_k\varepsilon - D^h_k\varepsilon_p), D^h_k\dot{\varepsilon} - D^h_k\dot{\varepsilon_p}\rangle$$
$$+\lambda \mathrm{tr}\,[D^h_k\varepsilon]\mathrm{tr}\,[D^h_k\dot{\varepsilon}]\Big) + 2\mu_c\langle \eta\,\mathrm{skew}(\nabla D^h_k u) - \eta D^h_k A + \mathrm{skew}(\nabla\eta\otimes D^h_k u),$$
$$\eta\,\mathrm{skew}(\nabla D^h_k \dot{u}) - \eta D^h_k \dot{A} + \mathrm{skew}(\nabla\eta\otimes D^h_k \dot{u})\rangle$$
$$+4l_c\langle \eta\nabla \mathrm{axl}(D^h_k A) + \nabla\eta\otimes \mathrm{axl}(D^h_k A), \eta\nabla\mathrm{axl}(D^h_k\dot{A}) + \nabla\eta\otimes\mathrm{axl}(D^h_k\dot{A})\rangle\,\mathrm{dx}$$
$$= \int_\Omega \eta^2\Big(\langle 2\mu(D^h_k\varepsilon - D^h_k\varepsilon_p) + \lambda\mathrm{tr}\,[D^h_k\varepsilon] + 2\mu_c(\mathrm{skew}(\nabla D^h_k u) - D^h_k A), \nabla D^h_k\dot{u}\rangle\Big)\,\mathrm{dx}$$
$$-\int_\Omega \eta^2\langle 2\mu(D^h_k\varepsilon - D^h_k\varepsilon_p), D^h_k\dot{\varepsilon}_p\rangle dx - 2\int_\Omega \mu_c\eta^2\langle \mathrm{skew}(\nabla D^h_k u) - D^h_k A, D^h_k\dot{A}\rangle\,\mathrm{dx}$$
$$+4\int_\Omega \eta^2 l_c\langle\nabla\mathrm{axl}(D^h_k A), \nabla\mathrm{axl}(D^h_k\dot{A})\rangle\,\mathrm{dx} \tag{3.3}$$
$$+2\int_\Omega \mu_c\langle\eta\,\mathrm{skew}(\nabla D^h_k u) - \eta D^h_k A + \mathrm{skew}(\nabla\eta\otimes D^h_k u), \mathrm{skew}(\nabla\eta\otimes D^h_k\dot{u})\rangle\,\mathrm{dx}$$
$$+2\int_\Omega \mu_c\langle \mathrm{skew}(\nabla\eta\otimes D^h_k u), \eta\,\mathrm{skew}(\nabla D^h_k\dot{u}) - \eta D^h_k\dot{A}\rangle\,\mathrm{dx}$$
$$+4\int_\Omega l_c\langle\eta\nabla\mathrm{axl}(D^h_k A) + \nabla\eta\otimes\mathrm{axl}(D^h_k A), \nabla\eta\otimes\mathrm{axl}(D^h_k\dot{A})\rangle\,\mathrm{dx}$$
$$+4\int_\Omega l_c\langle\nabla\eta\otimes\mathrm{axl}(D^h_k A), \eta\nabla\mathrm{axl}(D^h_k\dot{A})\rangle\,\mathrm{dx}\,.$$

In the first integral on the right hand side of (3.3) we are using the balance of forces. Unfortunately, the term $2\mu(D^h_k\varepsilon - D^h_k\varepsilon_p) + \lambda\mathrm{tr}\,[D^h_k\varepsilon] + 2\mu_c(\mathrm{skew}(\nabla D^h_k u) - D^h_k A)$ is not equal to the difference quotient $D^h_k\sigma$ because the coefficients are not constant. By the property of the operator D^h_k similar to the product rule we have

$$D^h_k\sigma = 2\mu(D^h_k\varepsilon - D^h_k\varepsilon_p) + \lambda\mathrm{tr}\,[D^h_k\varepsilon] + 2\mu_c(\mathrm{skew}(\nabla D^h_k u) - D^h_k A)$$
$$+\,2D^h_k\mu(\varepsilon^h - \varepsilon^h_p) + D^h_k\lambda\mathrm{tr}\,[\varepsilon^h] + 2D^h_k\mu_c(\mathrm{skew}(\nabla u^h) - A^h)\,, \tag{3.4}$$

where the **superscript** $(\cdot)^h$ **denotes the shifted function** $(\cdot)(x + he_k)$. In a similar manner we transform the integrand from the second integral on the right hand side of (3.3)

$$D^h_k T_E = D^h_k(2\mu(\varepsilon - \varepsilon_p)) = 2\mu D^h_k(\varepsilon - \varepsilon_p) + 2D^h_k\mu(\varepsilon^h - \varepsilon^h_p)\,. \tag{3.5}$$

In the same manner calculating D^h_k of the both sides of the equation for the microrotation we obtain that

$$-\,\mathrm{Div}(l_c\nabla\mathrm{axl}(D^h_k A)) - \mathrm{Div}(D^h_k l_c\nabla\mathrm{axl}(A^h))$$
$$= \mu_c\mathrm{axl}(\mathrm{skew}(\nabla D^h_k u) - D^h_k A) + D^h_k\mu_c\mathrm{axl}(\mathrm{skew}(\nabla u^h) - A^h).$$

In the fourth integral on the right hand side of (3.3) we integrate by parts to obtain

$$\int_\Omega \eta^2 l_c\langle\nabla\mathrm{axl}(D^h_k A), \nabla\mathrm{axl}(D^h_k\dot{A})\rangle\,\mathrm{dx} = -\int_\Omega \eta^2\langle\mathrm{Div}(l_c\nabla\mathrm{axl}(D^h_k A)), \mathrm{axl}(D^h_k\dot{A}\rangle\,\mathrm{dx}$$
$$-\int_\Omega \eta\langle\nabla\mathrm{axl}(D^h_k A), \mathrm{axl}(D^h_k\dot{A}\otimes 2l_c\nabla\eta)\rangle\,\mathrm{dx}\,. \tag{3.6}$$

Inserting (3.4)–(3.6) into (3.3) and using the balance of forces we obtain

$$\begin{aligned}
&\dot{\mathcal{E}}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) = \int_\Omega \eta^2 \langle D_k^h f, D_k^h \dot{u}\rangle \,\mathrm{dx} - \int_\Omega 2\eta \langle D_k^h \sigma, D_k^h \dot{u} \otimes \nabla\eta\rangle \,\mathrm{dx} \\
&- \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \mathrm{tr}\,[\varepsilon^h] + 2D_k^h \mu_c(\mathrm{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u}\rangle \,\mathrm{dx} \\
&- \int_\Omega \eta^2 \underbrace{\langle D_k^h T_E, D_k^h \dot{\varepsilon}_p\rangle}_{\text{flow-rule}\geq 0} \,\mathrm{dx} + \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot{\varepsilon}_p\rangle \,\mathrm{dx} \\
&- 4\int_\Omega \mu_c \eta^2 \langle \mathrm{axl}(\mathrm{skew}(\nabla D_k^h u) - D_k^h A), \mathrm{axl}(D_k^h \dot{A})\rangle \,\mathrm{dx} \\
&- 4\int_\Omega \eta^2 \langle \mathrm{Div}(l_c \nabla \mathrm{axl}(D_k^h A)), \mathrm{axl}(D_k^h \dot{A}\rangle \,\mathrm{dx} \\
&- 4\int_\Omega \eta \langle \nabla \mathrm{axl}(D_k^h A), \mathrm{axl}(D_k^h \dot{A} \otimes 2l_c \nabla\eta)\rangle \,\mathrm{dx} \qquad (3.7) \\
&+ 2\int_\Omega \mu_c \langle \eta\, \mathrm{skew}(\nabla D_k^h u) - \eta D_k^h A + \mathrm{skew}(\nabla\eta \otimes D_k^h u), \mathrm{skew}(\nabla\eta \otimes D_k^h \dot{u})\rangle \,\mathrm{dx} \\
&+ 2\int_\Omega \mu_c \langle \mathrm{skew}(\nabla\eta \otimes D_k^h u), \eta\, \mathrm{skew}(\nabla D_k^h \dot{u}) - \eta D_k^h \dot{A}\rangle \,\mathrm{dx} \\
&+ 4\int_\Omega l_c \langle \eta \nabla \mathrm{axl}(D_k^h A) + \nabla\eta \otimes \mathrm{axl}(D_k^h A), \nabla\eta \otimes \mathrm{axl}(D_k^h \dot{A})\rangle \,\mathrm{dx} \\
&+ 4\int_\Omega l_c \langle \nabla\eta \otimes \mathrm{axl}(D_k^h A), \eta \nabla \mathrm{axl}(D_k^h \dot{A})\rangle \,\mathrm{dx}\,.
\end{aligned}$$

Next, using the balance of angular momentum equation for the microrotation and the monotonicity of the flow rule after integration over the time interval $(0, t)$ we arrive at the inequality

$$\begin{aligned}
&\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(t) \leq \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(0) \\
&+ \int_0^t \int_\Omega \eta^2 \langle D_k^h f, D_k^h \dot{u}\rangle \,\mathrm{dx}\, d\tau - \int_0^t \int_\Omega 2\eta \langle D_k^h \sigma, D_k^h \dot{u} \otimes \nabla\eta\rangle \,\mathrm{dx}\, d\tau \\
&- \int_0^t \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h) + D_k^h \lambda \mathrm{tr}\,[\varepsilon^h] + 2D_k^h \mu_c(\mathrm{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u}\rangle \,\mathrm{dx}\, d\tau \\
&+ \int_0^t \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot{\varepsilon}_p\rangle \,\mathrm{dx}\, d\tau \\
&- 4\int_0^t \int_\Omega \eta \langle \nabla \mathrm{axl}(D_k^h A), \mathrm{axl}(D_k^h \dot{A} \otimes 2l_c \nabla\eta\rangle \,\mathrm{dx}\, d\tau \qquad (3.8) \\
&+ 4\int_0^t \int_\Omega \eta^2 \langle \mathrm{Div}(D_k^h l_c \nabla \mathrm{axl}(A^h)) + D_k^h \mu_c \mathrm{axl}(\mathrm{skew}(\nabla u^h) - A^h), \mathrm{axl}(D_k^h \dot{A})\rangle \,\mathrm{dx}\, d\tau \\
&+ 2\int_0^t \int_\Omega \mu_c \langle \eta\, \mathrm{skew}(\nabla D_k^h u) - \eta D_k^h A + \mathrm{skew}(\nabla\eta \otimes D_k^h u), \mathrm{skew}(\nabla\eta \otimes D_k^h \dot{u})\rangle \,\mathrm{dx}\, d\tau \\
&+ 2\int_0^t \int_\Omega \mu_c \langle \mathrm{skew}(\nabla\eta \otimes D_k^h u), \eta\, \mathrm{skew}(\nabla D_k^h \dot{u}) - \eta D_k^h \dot{A}\rangle \,\mathrm{dx}\, d\tau \\
&+ 4\int_0^t \int_\Omega l_c \langle \eta \nabla \mathrm{axl}(D_k^h A) + \nabla\eta \otimes \mathrm{axl}(D_k^h A), \nabla\eta \otimes \mathrm{axl}(D_k^h \dot{A})\rangle \,\mathrm{dx}\, d\tau \\
&+ 4\int_0^t \int_\Omega l_c \langle \nabla\eta \otimes \mathrm{axl}(D_k^h A), \eta \nabla \mathrm{axl}(D_k^h \dot{A})\rangle \,\mathrm{dx}\, d\tau\,.
\end{aligned}$$

By the regularity of ε_p^0 and $f(0)$ we conclude that the initial value $\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)(0)$ is bounded by $C\,\mathcal{E}_{U^*}(u_{x_k}, \varepsilon_{x_k}, \varepsilon_{p,x_k}, A_{x_k})(0)$, where $C > 0$ does not depend on h. Next, we are going to estimate all integrals in the right hand side on (3.8). To estimate the first integral we use the regularity of f and $L^2(H^1)$ regularity of the velocity $\dot{u}$. Hence, we obtain

$$\int_0^t \int_\Omega \eta^2 \langle D_k^h f, D_k^h \dot{u}\rangle \,\mathrm{dx}\, d\tau \le C \|f_{x_k}\|_{L^2((0,t)\times\Omega)} \|\dot{u}_{x_k}\|_{L^2((0,t)\times\Omega)}\,. \tag{3.9}$$

The second integral can be estimated as follows

$$\begin{aligned}
&-\int_0^t \int_\Omega 2\eta \langle D_k^h \sigma, D_k^h \dot{u} \otimes \nabla\eta\rangle \,\mathrm{dx}\, d\tau \le C(\|\eta D_k^h \sigma\|^2_{L^2((0,t)\times U)} + \|\dot{u}_{x_k}\|^2_{L^2((0,t)\times\Omega)})\\
&\le C\left(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)\, d\tau + \int_0^t \mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A)\, d\tau + \|\dot{u}_{x_k}\|^2_{L^2((0,t)\times\Omega)}\right)\,,
\end{aligned} \tag{3.10}$$

where $C > 0$ does not depend on h. In the last estimate we have used (3.4) and the regularity of μ, λ, μ_c and l_c. To estimate the third integral we integrate by parts with respect to τ and get

$$\begin{aligned}
&-\int_0^t \int_\Omega \eta^2 \langle 2D_k^h\mu(\varepsilon^h - \varepsilon_p^h) + D_k^h\lambda\,\mathrm{tr}\,[\varepsilon^h] + 2D_k^h\mu_c(\mathrm{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u}\rangle \,\mathrm{dx}\, d\tau\\
&= \int_0^t \int_\Omega \eta^2 \langle 2D_k^h\mu(\dot{\varepsilon}^h - \dot{\varepsilon}_p^h) + D_k^h\lambda\,\mathrm{tr}\,[\dot{\varepsilon}^h] + 2D_k^h\mu_c(\mathrm{skew}(\nabla \dot{u}^h) - \dot{A}^h), \nabla D_k^h u\rangle \,\mathrm{dx}\, d\tau\\
&\quad - \int_\Omega \eta^2 \langle 2D_k^h\mu(\varepsilon^h - \varepsilon_p^h) + D_k^h\lambda\,\mathrm{tr}\,[\varepsilon^h] + 2D_k^h\mu_c(\mathrm{skew}(\nabla u^h) - A^h), \nabla D_k^h u\rangle \,\mathrm{dx}\\
&\quad + \int_\Omega \eta^2 \langle 2D_k^h\mu(\varepsilon^h - \varepsilon_p^h) + D_k^h\lambda\,\mathrm{tr}\,[\varepsilon^h] + 2D_k^h\mu_c(\mathrm{skew}(\nabla u^h) - A^h), \nabla D_k^h u\rangle_{|t=0} \,\mathrm{dx}\\
&\le C\left(\int_0^t \mathcal{E}_{U^*}(\dot{u},\dot{\varepsilon},\dot{\varepsilon_p},\dot{A})\, d\tau + \|\eta D_k^h \nabla u\|^2_{L^2((0,t)\times U)} + C(\alpha)\mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A)(t)\right.\\
&\quad \left. + \alpha\|\eta D_k^h \nabla u\|^2_{L^2(U)} + \mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A)(0) + \|\eta D_k^h \nabla u(0)\|^2_{L^2(U)}\right)\,,
\end{aligned} \tag{3.11}$$

where $\alpha > 0$ is an arbitrary constant and $C(\alpha)$ depends on α only. By the regularity of the initial data the last two terms on the right hand side of (3.11) are bounded by a constant which is independent of h. Using the coerciveness of the energy function we have

$$\begin{aligned}
&\|\eta D_k^h \nabla u\|^2_{L^2(U)} \le \|\eta D_k^h \varepsilon\|^2_{L^2(U)} + \|\eta D_k^h \mathrm{skew}\nabla u\|^2_{L^2(U)}\\
&\le C\Big(\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) + \mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A)\Big)\,.
\end{aligned} \tag{3.12}$$

Inserting (3.12) into (3.11) we arrive at the inequality

$$\begin{aligned}
&-\int_0^t \int_\Omega \eta^2 \langle 2D_k^h\mu(\varepsilon^h - \varepsilon_p^h) + D_k^h\lambda\mathrm{tr}\,[\varepsilon^h] + 2D_k^h\mu_c(\mathrm{skew}(\nabla u^h) - A^h), \nabla D_k^h \dot{u}\rangle \,\mathrm{dx}\, d\tau\\
&\le C\left(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)\, d\tau + \alpha\mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)\right.\\
&\quad \left. + \int_0^t \mathcal{E}_{U^*}(\dot{u},\dot{\varepsilon},\dot{\varepsilon_p},\dot{A})\, d\tau + C(\alpha)\mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A) + 1\right)\,.
\end{aligned} \tag{3.13}$$

In the same manner we estimate the fourth integral from the right hand side of (3.8)

$$\int_0^t \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \dot\varepsilon_p \rangle \,\mathrm{dx}\, d\tau = -\int_0^t \int_\Omega \eta^2 \langle 2D_k^h \mu(\dot\varepsilon^h - \dot\varepsilon_p^h), D_k^h \varepsilon_p \rangle \,\mathrm{dx}\, d\tau$$

$$+ \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \varepsilon_p \rangle \,\mathrm{dx} - \int_\Omega \eta^2 \langle 2D_k^h \mu(\varepsilon^h - \varepsilon_p^h), D_k^h \varepsilon_p \rangle_{|t=0} \,\mathrm{dx}$$

$$\le \hat C \left(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)\, d\tau + \beta \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) \right.$$

$$\left. + \int_0^t \mathcal{E}_{U^*}(\dot u, \dot\varepsilon, \dot{\varepsilon_p}, \dot A)\, d\tau + C(\beta)\mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) + 1 \right) , \tag{3.14}$$

where $\beta > 0$ is an arbitrary constant, $C(\beta) > 0$ depends on β only and $\hat C$ does not depend on h. The fifth integral from the right hand side of (3.8) can be estimated as follows

$$-\int_0^t \int_\Omega \eta \langle \nabla \mathrm{axl}(D_k^h A), \mathrm{axl}(D_k^h \dot A \otimes 2 l_c \nabla\eta) \rangle \,\mathrm{dx}\, d\tau$$

$$\le C \left(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)\, d\tau + \|\dot A_{x_k}\|^2_{L^2((0,t)\times\Omega)} \right) , \tag{3.15}$$

where again the constant $C > 0$ does not depends on h. For the sixth integral from the right hand side of (3.8) using that $l_c \in C^2(\Omega)$ and that A^h satisfies the equation for microrotation shifted by h we conclude that

$$\int_0^t \int_\Omega \eta^2 \langle \mathrm{Div}(D_k^h l_c \nabla \mathrm{axl}(A^h)) + D_k^h \mu_c(\mathrm{axl}\,\mathrm{skew}(\nabla u^h) - A^h), \mathrm{axl}(D_k^h \dot A) \rangle \,\mathrm{dx}\, d\tau$$

$$\le C \int_0^t (\mathcal{E}_{U^*}(\dot u, \dot\varepsilon, \dot{\varepsilon_p}, \dot A) + \mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A))\, d\tau \ , \tag{3.16}$$

where the positive constant C does not depend on h. Next, the seventh integral from the right hand side of (3.8) can be estimated immediately using the energy

$$\int_0^t \int_\Omega \mu_c \langle \eta\, \mathrm{skew}(\nabla D_k^h u) - \eta D_k^h A + \mathrm{skew}(\nabla\eta \otimes D_k^h u), \mathrm{skew}(\nabla\eta \otimes D_k^h \dot u) \rangle \,\mathrm{dx}$$

$$\le C \left(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A)\, d\tau + \|\dot u_{x_k}\|^2_{L^2((0,t)\times\Omega)} \right) , \tag{3.17}$$

where the constant C does not depend on h. Integrating by parts with respect to time in the eighth integral from the right hand side of (3.8) we have

$$\int_0^t \int_\Omega \mu_c \langle \mathrm{skew}(\nabla\eta \otimes D_k^h u), \eta\, \mathrm{skew}(\nabla D_k^h \dot u) - \eta D_k^h \dot A \rangle \,\mathrm{dx}\, d\tau$$

$$= -\int_0^t \int_\Omega \mu_c \langle \mathrm{skew}(\nabla\eta \otimes D_k^h \dot u), \eta\, \mathrm{skew}(\nabla D_k^h u) - \eta D_k^h A \rangle \,\mathrm{dx}\, d\tau$$

$$+ \int_\Omega \mu_c \langle \mathrm{skew}(\nabla\eta \otimes D_k^h u), \eta\, \mathrm{skew}(\nabla D_k^h u) - \eta D_k^h A \rangle \,\mathrm{dx}$$

$$- \int_\Omega \mu_c \langle \mathrm{skew}(\nabla\eta \otimes D_k^h u), \eta\, \mathrm{skew}(\nabla D_k^h u) - \eta D_k^h A \rangle_{|t=0} \,\mathrm{dx}$$

$$\le C \Big(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) + \mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A)\, d\tau + \|\dot u_{x_k}\|^2_{L^2((0,t)\times\Omega)} + 1 \Big)$$

$$+ \gamma \mathcal{E}(\eta D_k^h u, \eta D_k^h \varepsilon, \eta D_k^h \varepsilon_p, \eta D_k^h A) + C(\gamma) \mathcal{E}_{U^*}(u, \varepsilon, \varepsilon_p, A) , \tag{3.18}$$

where the constant $\gamma > 0$ is arbitrary, $C(\gamma)$ depends on γ only and C does not depend on h. The last two integrals from the right hand side of (3.8) can be estimated in the same manner as the seventh and the eighth integral respectively. Hence, we obtain

$$\begin{aligned}&\int_0^t\int_\Omega l_c\langle \eta\nabla \mathrm{axl}(D_k^h A) + \nabla\eta\otimes \mathrm{axl}(D_k^h A), \nabla\eta\otimes\mathrm{axl}(D_k^h\dot{A})\rangle\,\mathrm{dx}\,d\tau\\ &\quad\le C\left(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h\varepsilon, \eta D_k^h\varepsilon_p, \eta D_k^h A)\,d\tau + \|\dot{A}_{x_k}\|^2_{L^2((0,t)\times\Omega)}\right),\end{aligned}\tag{3.19}$$

where $C > 0$ does not depend on h and finally

$$\begin{aligned}&\int_0^t\int_\Omega l_c\langle \nabla\eta\otimes \mathrm{axl}(D_k^h A), \eta\nabla\mathrm{axl}(D_k^h\dot{A})\rangle\,\mathrm{dx}\,d\tau\\ &\quad\le C\Big(\int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h\varepsilon, \eta D_k^h\varepsilon_p, \eta D_k^h A) + \mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A)\,d\tau + \|\dot{A}_{x_k}\|^2_{L^2((0,t)\times\Omega)} + 1\Big)\\ &\qquad + \delta\mathcal{E}(\eta D_k^h u, \eta D_k^h\varepsilon, \eta D_k^h\varepsilon_p, \eta D_k^h A) + C(\delta)\mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A),\end{aligned}\tag{3.20}$$

where $\delta > 0$ is arbitrary, $C(\delta)$ depends on δ only and C does not depend on h. Let us choose now α, β, γ and δ such that $C\alpha + \hat{C}\beta + 2\gamma + 4\delta < 1$ where the constants C and $\hat{C}$ are from inequality (3.13) and (3.14) respectively. On inserting (3.9)-(3.10) and (3.13)-(3.20) into (3.8) we obtain

$$\begin{aligned}&\mathcal{E}(\eta D_k^h u, \eta D_k^h\varepsilon, \eta D_k^h\varepsilon_p, \eta D_k^h A)(t) \le C\Big(\mathcal{E}_{U^*}(u_{x_k}, \varepsilon_{x_k}, \varepsilon_{p,x_k}, A_{x_k})(0)\\ &\quad + \int_0^t \mathcal{E}(\eta D_k^h u, \eta D_k^h\varepsilon, \eta D_k^h\varepsilon_p, \eta D_k^h A)\,d\tau + \int_0^t (\mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A) + \mathcal{E}_{U^*}(\dot{u},\dot{\varepsilon},\dot{\varepsilon}_p,\dot{A}))\,d\tau\\ &\quad + \mathcal{E}_{U^*}(u,\varepsilon,\varepsilon_p,A) + \|f_{x_k}\|^2 + \|\dot{u}_{x_k}\|^2_{L^2((0,t)\times\Omega)} + \|\dot{A}_{x_k}\|^2_{L^2((0,t)\times\Omega)} + 1\Big).\end{aligned}\tag{3.21}$$

Finally, using the Gronwall Lemma and the inequality

$$\mathcal{E}_V(D_k^h u, D_k^h\varepsilon, D_k^h\varepsilon_p, D_k^h A) \le \mathcal{E}(\eta D_k^h u, \eta D_k^h\varepsilon, \eta D_k^h\varepsilon_p, \eta D_k^h A),$$

we easily complete the proof. □

Corollary 3.3. *Assuming that the initial plastic strain ε_p^0 and the external force f satisfy all requirements of Theorem 3.2 the solution to system (2.1) is more regular: $u \in L^\infty(0,T;H^2_{\rm loc}(\Omega;\mathbb{R}^3))$, $\sigma, \varepsilon_p \in L^\infty(0,T;H^1_{\rm loc}(\Omega;\mathrm{Sym}(3)))$, $A \in L^\infty(0,T;H^3_{\rm loc}(\Omega;\mathfrak{so}(3)))$.*

Proof. By Theorem 3.2 we immediately have that for all subsets $V \Subset \Omega$ all functions appearing in the energy function have the regularity $\varepsilon - \varepsilon_p \in L^\infty(0,T;H^1(V;\mathrm{Sym}(3)))$, $\mathrm{tr}\,\varepsilon \in L^\infty(0,T;H^1(V;\mathbb{R}))$ and $\mathrm{skew}(\nabla u - A) \in L^\infty(0,T;H^1(V;\mathfrak{so}(3)))$. From the definition of σ we obtain immediately that $\sigma \in L^\infty(0,T;H^1(V;\mathrm{Sym}(3)))$. Using the coerciveness of the energy function we conclude that $\varepsilon \in L^\infty(0,T;H^1(V;\mathrm{Sym}(3)))$, which implies that $u \in L^\infty(0,T;H^2_{\rm loc}(\Omega;\mathbb{R}^3))$. The $H^3_{\rm loc}$-regularity of A follows by $H^2_{\rm loc}$-regularity of u and the standard regularity theory of elliptic equations. □

4 $H^1_{\rm loc}$-regularity in the dynamic case

In the dynamical setting the system of equations is in the form

$$\begin{aligned}\ddot{u} - \mathrm{Div}\sigma &= f,\\ \sigma &= 2\mu\,(\varepsilon - \varepsilon_p) + 2\,\mu_c\,(\mathrm{skew}(\nabla u) - A) + \lambda\,\mathrm{tr}\,[\varepsilon]\cdot\mathbb{1},\\ \mathrm{axl}(\ddot{A}) - \mathrm{Div}(l_c\,\nabla\mathrm{axl}(A)) &= \mu_c\,\mathrm{axl}(\mathrm{skew}(\nabla u) - A),\\ \dot{\varepsilon}_p &\in \mathfrak{f}(T_E), \quad T_E = 2\mu\,(\varepsilon - \varepsilon_p),\\ u_{|\partial\Omega} &= u_d, \quad A_{|\partial\Omega} = A_d,\\ u(0) = u^0, \quad \dot{u}(0) = u^1, &\quad A(0) = A^0, \quad \dot{A}(0) = A^1, \quad \varepsilon_p(0) = \varepsilon_p^0,\end{aligned}\tag{4.1}$$

where f is a given volume force u_d, A_d are given boundary data and $u^0, u^1, A^0, A^1, \varepsilon_p^0$ are given initial data. This initial boundary-value problem was studied in [16] and the Main Theorem from [16] yields an existence and uniqueness result similar to Theorem 2.1. The energy function associated with system (4.1) is in the form

$$\mathcal{E}(u,\varepsilon,\varepsilon_p,A)(t) := \int_\Omega \Bigg(\frac{1}{2}\|\dot{u}\|^2 + 2\|\mathrm{axl}(\dot{A})\|^2 + \mu\,\|\varepsilon-\varepsilon_p\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[\varepsilon]^2 + \mu_c\|\mathrm{skew}(\nabla u) - A\|^2 + 2l_c\,\|\nabla \mathrm{axl}(A)\|^2 \Bigg)\,\mathrm{dx}\,.$$

This function is also coercive which means that $\mathcal{E}$ satisfies the statements of Theorem 3.1. Using the same methods as in Sect. 2 we can conclude the following regularity result for the initial boundary-value problem (4.1).

Theorem 4.1. *Let us assume that* $u^0 \in H^2_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$, $u^1 \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3)$ $\varepsilon_p^0 \in H^1_{\mathrm{loc}}(\Omega;\mathrm{Sym}(3))$, $A^0 \in H^2(\Omega;\mathfrak{so}(3))$, $A^1 \in H^1(\Omega;\mathfrak{so}(3))$ *and* $f \in L^2(0,T;H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3))$. *If* μ,λ,μ_c *are continuous, positive on* $\overline{\Omega}$, *locally Lipschitz and* $l_c \in C^2(\Omega)\cap C^1(\overline{\Omega})$ *then the solution to* (4.1) *is more regular:* $u \in L^\infty(0,T;H^2_{\mathrm{loc}}(\Omega;\mathbb{R}^3))$, $\sigma,\varepsilon_p \in L^\infty(0,T;H^1_{\mathrm{loc}}(\Omega;\mathrm{Sym}(3)))$.

5 Quasistatic case with a general flow rule

In [3] the quasistatic problem was studied with the following general flow rule of monotone type

$$z_t \in \mathfrak{f}(-\nabla_z\psi(\varepsilon,z,A))\,,$$

where $z = (\varepsilon_p,\tilde{z})$ is the vector of internal variables, $Bz = \varepsilon_p$ is the projector on the direction of the plastic strain, $\mathfrak{f}$ is a maximal monotone mapping satisfying $\{0\} \in \mathfrak{f}(0)$ and ψ is the free energy function. The free energy considered in this article is in the form

$$\psi(\varepsilon,z,A) = \mu\|\varepsilon-\varepsilon_p\|^2 + \frac{\lambda}{2}\mathrm{tr}\,[\varepsilon]^2 + \mu_c\|\mathrm{skew}(\nabla u) - A\|^2 + 2l_c\|\nabla\mathrm{axl}(A)\|^2 + \langle Lz,z\rangle\,,$$

where L is a symmetric and semi-positive matrix. Using the same procedure as in Sect. 3 the following regularity result can be obtained

Theorem 5.1. *If the initial value* $z^0 \in H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *and* $f \in L^2(0,T;H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^3))$ *then the solution to the quasistatic problem in the Cosserat plasticity with a general flow rule of monotone type is more regular:* $\sigma \in L^\infty(0,T;H^1_{\mathrm{loc}}(\Omega;\mathrm{Sym}(3)))$ *and* $Lz \in L^\infty(0,T;H^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N))$. *Moreover, the coerciveness of the energy function yields additionally that* $u \in L^\infty(0,T;H^2_{\mathrm{loc}}(\Omega;\mathbb{R}^3))$ *and* $\varepsilon_p \in L^\infty(0,T;H^1_{\mathrm{loc}}(\Omega;\mathrm{Sym}(3)))$.

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