Applied Mathematical Sciences, Vol. 5, 2011, no. 65, 3201 - 3210

Haar Wavelet Method to Solve Volterra Integral Equations with Weakly Singular Kernel by Collocation Method

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Abstract

In this work, we present a computational method for solving Volterra integral equations of the second kind with weakly singular kernel which is based on the use of Haar wavelets and properties of Block-Pulse-Functions(BPF). Error analysis is worked out that shows efficiency and the order of convergence of the method. Finally, we also give some numerical examples.

Mathematics Subject Classification: 45A05; 45D05; 45E99

Keywords: Volterra integral equations; Haar wavelet; Error analysis; Haar coefficient matrix; Block-Pulse-Function; Collocation points

1 Introduction

Volterra integral equations arise in many problems pertaining to mathematical physics like heat conduction problems. Several numerical methods for approximating the solution of Volterra integral equations are known [1-10]. This paper is focused on the solution of Volterra integral equations of the second kind with weakly singular kernel via Haar function by taking advantage of the nice properties of Haar wavelets.

In this paper we consider the following equation

$$u(x) = f(x) + \int_0^x \frac{K(x,t)}{(x-t)^\beta} u(t)dt, \quad 0 \leq t < x \leq 1, \quad 0 < \beta < 1, \tag{1}$$

where functions $f \in C[0,1]$ and $K \in C[0,1]^2$ are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $u \in C[0,1]$, where $C[0,1]$ denotes the space of all continuous functions defined on $[0,1]$. (see [1], [5]).

2 Haar wavelet

Beginning from 1991 the wavelet method has been applied for solving integral equations, a short survey on these applications can be found in [10]. The solutions are often quite complicated and the advantages of the wavelet method get lost, therefore any kind of simplifications are welcome. One possibility for it is to make use of the Haar wavelets. In fact, Haar wavelets have a number of advantages, including: simplicity, orthogonality and very compact support. The main advantages of the Haar wavelets method are sparse representation, fast transformation and possibility of implementation of fast algorithm in matrix representation. The Haar basis is simplest instance of spline wavelets, resulting when the polynomial degree is set to zero, so computational costs with Haar wavelets is very low. So we use them for solving equation (1).

Definition: The Haar wavelet is the function defined on the real line $\mathbb{R}$ as:

$$H(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0, & \text{elsewhere} \end{cases}$$

now for $n = 1, 2, \ldots,$ write $n = 2^j + k$ with $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^j - 1$ and define $h_n(t) = 2^{\frac{j}{2}} H(2^j t - k)|_{[0,1]}$. Also, define $h_0(t) = 1$ for all t. Here the integer 2^j, $j = 0, 1, \ldots,$ indicates the level of the wavelet and $k = 0, 1, \ldots, 2^j - 1$ is the translation parameter. It can be shown that the sequence $\{h_n\}_{n=0}^{\infty}$ is a complete orthonormal system in $L^2[0,1]$ and for $f \in C[0,1]$, the series $\sum_n < f, h_n > h_n$ converges uniformly to f [12], where $< f, h_n >= \int_0^1 f(x) h_n(x) dx$.

3 Function Approximation

A function $u(t)$ defined over the interval $[0,1)$ may be expanded as:

$$u(t) = \sum_{n=0}^{\infty} u_n h_n(t), \tag{2}$$

in practice, the first k-term of (1) are considered, where k is power of 2, so,

$$u(t) \simeq u_k(t) = \sum_{n=0}^{k-1} u_n h_n(t) = \mathbf{u}^t \mathbf{h}(t), \tag{3}$$

where, $\mathbf{u} = [u_0, u_1, \ldots, u_{k-1}]^t$ and $\mathbf{h}(t) = [h_0(t), h_1(t), \ldots, h_{k-1}(t)]^t$. Similarly, $K(x,t) \in L^2[0,1)^2$ may be approximated as:

$$K(x,t) \simeq \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} K_{ij} h_i(x) h_j(t) = \mathbf{h}^t(x)\mathbf{K}\mathbf{h}(t)$$

where $\mathbf{K} = [K_{ij}]_{0 \leq i,j \leq k-1}$ and $K_{ij} =< h_i(x), < K(x,t), h_j(t) >>$, approximation of the kernel $K(x,t)$ by wavelets is known as standard representation. It is a wavelet image of the kernel and is usually a sparse matrix.

4 Block-Pulse-Functions

We define a k-set of Block-Pulse-Functions (BPF) over the interval $[0,T)$ as:

$$B_i(t) = \begin{cases} 1, & \frac{(i-1)T}{k} \leq t < \frac{iT}{k}, \text{ for } i = 1, 2, \ldots, k \\ 0, & \text{elsewhere} \end{cases} \tag{4}$$

also, B_i is the i-th Block-Pulse-Function. In this paper, it is assumed that T = 1, so BPFs are defined over [0, 1). The most important properties of BPFs are disjointness, orthogonality and completeness. The disjointness property can be obtained from the definition of BPFs:

$$B_i(t)B_j(t) = \begin{cases} 0, & i \neq j \\ B_i(t), & i = j \end{cases} \tag{5}$$

where, $i, j = 1, 2, ..., k$.
The other property is orthogonality:

$$< B_i(t), B_j(t) >= \begin{cases} 0, & i \neq j \\ \frac{1}{k}, & i = j \end{cases} \tag{6}$$

the third property is completeness. For every $u \in L^2[0,1)$, when k approaches to the infinity, Parseval's identity holds:

$$\int_0^1 u^2(t)dt = \sum_{i=1}^{\infty} u_i^2 \|B_i(t)\|^2.$$

Now we present the Haar coefficient matrix $\mathbf{H}$; it is a $k \times k$ matrix with the elements

$$\mathbf{H} = [h_n(t_j)]_{0 \le n \le k-1, 1 \le j \le k},$$

where the points t_j are the collocation points

$$t_j = \frac{j - \frac{1}{2}}{k}, \quad j = 1, 2, \ldots, k.$$

It can be shown that $\mathbf{h}(t) = \mathbf{H}\mathbf{B}(t)$ [11], where, $\mathbf{B}(t) = [B_1(t), \ldots, B_k(t)]^t$, so $u(t)$ and $K(x,t)$ can be re-approximated as:

$$\begin{aligned} u(t) &\simeq \mathbf{u}^t \mathbf{h}(t) \\ &= \mathbf{B}^t(t)\mathbf{H}^t\mathbf{u}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} K(x,t) &\simeq \mathbf{h}^t(x)\mathbf{K}\mathbf{h}(t) \\ &= \mathbf{B}^t(x)\mathbf{H}^t\mathbf{K}\mathbf{H}\mathbf{B}(t). \end{aligned} \tag{8}$$

Substituting (7) and (8) into (1) we get

$$\begin{aligned} \mathbf{B}^t(x)\mathbf{H}^t\mathbf{u} &= f(x) + \mathbf{B}^t(x)\mathbf{H}^t\mathbf{K}\mathbf{H}(\int_0^x \frac{\mathbf{B}(t)\mathbf{B}^t(t)}{(x-t)^\beta}dt)\mathbf{H}^t\mathbf{u} \\ &= f(x) + \mathbf{B}^t(x)\mathbf{H}^t\mathbf{K}\mathbf{H}\mathbf{G}(x)\mathbf{H}^t\mathbf{u} \end{aligned} \tag{9}$$

where,

$$\mathbf{G}(x) = \int_0^x \frac{\mathbf{B}(t)\mathbf{B}^t(t)}{(x-t)^\beta}dt.$$

Simply, $t \in [\frac{j-1}{k}, \frac{j}{k})$, $j = 1, 2, \ldots, k$, implies that $\mathbf{B}(t) = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j-th column of the identity matrix of order k. Now by evaluating (9) at the collocation points $x_j = \frac{j-\frac{1}{2}}{k}, \quad j = 1, 2, \ldots, k$ we obtain

$$\mathbf{e}_j^t\mathbf{H}^t\mathbf{u} = f(x_j) + \mathbf{e}_j^t\mathbf{H}^t\mathbf{K}\mathbf{H}\mathbf{G}(x_j)\mathbf{H}^t\mathbf{u}, \tag{10}$$

now we have to evaluate $\mathbf{G}(x_j), \quad j = 1, 2, \ldots, k$. Simply, the disjointness property of BPFs implies that

$$\mathbf{B}(t)\mathbf{B}^t(t) = \begin{bmatrix} B_1(t) & & & O \\ & B_2(t) & & \\ & & \ddots & \\ O & & & B_k(t) \end{bmatrix}$$

therefore,

$$\begin{aligned}\mathbf{G}(x_j) &= \int_0^{\frac{j-1/2}{k}} \frac{\mathbf{B}(t)\mathbf{B}^t(t)}{(\frac{j-1/2}{k}-t)^\beta}dt \\ &= \int_0^{\frac{1}{k}} \frac{\mathbf{B}(t)\mathbf{B}^t(t)}{(\frac{j-1/2}{k}-t)^\beta}dt + \cdots + \int_{\frac{j-2}{k}}^{\frac{j-1}{k}} \frac{\mathbf{B}(t)\mathbf{B}^t(t)}{(\frac{j-1/2}{k}-t)^\beta}dt + \int_{\frac{j-1}{k}}^{\frac{j-1/2}{k}} \frac{\mathbf{B}(t)\mathbf{B}^t(t)}{(\frac{j-1/2}{k}-t)^\beta}dt\end{aligned}$$

$$= \begin{bmatrix} \int_0^{\frac{1}{k}}(\frac{j-1/2}{k}-t)^{-\beta}dt & & 0 \\ & 0 & \\ & & \ddots \\ & 0 & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ & & \int_{\frac{j-2}{k}}^{\frac{j-1}{k}}(\frac{j-1/2}{k}-t)^{-\beta}dt & \\ & & 0 & \\ & & & \ddots \\ 0 & & & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ & & 0 & \\ & & \int_{\frac{j-1}{k}}^{\frac{j-1/2}{k}}(\frac{j-1/2}{k}-t)^{-\beta}dt & \\ & & 0 & \\ & & & \ddots \\ 0 & & & 0 \end{bmatrix}$$

$$= \frac{1}{(1-\beta)(2k)^{1-\beta}}\mathbf{D}^j,$$

where,

$$\mathbf{D}^j = Diag[(2j-1)^{1-\beta}-(2j-3)^{1-\beta},(2j-3)^{1-\beta}-(2j-5)^{1-\beta},\dots,3^{1-\beta}-1,1,0,\dots,0]_{k\times k},$$

in fact, the diagonal matrix $\mathbf{D}^j$, $\quad j = 1, 2, ..., k$ is defined as follows:

$$\mathbf{D}^j_{mn} = \begin{cases} (2j-(2m-1))^{1-\beta}-(2j-(2m+1))^{1-\beta}, & m=n=1,2,\dots,j-1, \\ 1, & m=n=j, \\ 0, & m=n=j+1,\dots,k. \end{cases}$$

Substituting evaluated $\mathbf{G}(x_j)$ into (10) gives

$$\mathbf{e}_j^t\mathbf{H}^t\mathbf{u} = f(x_j) + \frac{1}{(1-\beta)(2k)^{1-\beta}}\mathbf{e}_j^t\mathbf{H}^t\mathbf{K}\mathbf{H}\mathbf{D}^j\mathbf{H}^t\mathbf{u},$$

or

$$\mathbf{A}^j\mathbf{u} = f(x_j), \quad j = 1, 2, ..., k, \tag{11}$$

where,

$$\mathbf{A}^j = \mathbf{e}_j^t\mathbf{H}^t\{\mathbf{I} - \frac{1}{(1-\beta)(2k)^{1-\beta}}\mathbf{KHD}^j\mathbf{H}^t\}.$$

Solving linear system of equations (11) gives column vector $\mathbf{u}$, therefore from (3) we can obtain desired approximation $u_k(t)$ for $u(t)$ at every point $t \in [0, 1)$.

5 Error Analysis

In this section we assume that $u(t)$ is a differentiable function with bounded first derivative on (0,1), that is,

$$\exists M > 0; \quad \forall t \in (0, 1): \quad |u'(t)| \leq M.$$

We may proceed as follows:

$$u_k(t) = \sum_{n=0}^{k-1} u_n h_n(t)$$

where, $k = 2^{\alpha+1}$, $\alpha = 0, 1, 2, \ldots$, then

$$\begin{aligned}\|u(t) - u_k(t)\|_E^2 &= \int_0^1 (u(t) - u_k(t))^2 dt \\ &= \sum_{n=2^{\alpha+1}}^{\infty} \sum_{n'=2^{\alpha+1}}^{\infty} u_n u_{n'} \int_0^1 h_n(t)h_{n'}(t)dt \\ &= \sum_{n=2^{\alpha+1}}^{\infty} u_n^2\end{aligned}$$

substituting $h_n(t) = 2^{\frac{j}{2}}\ H(2^j t - k)$, $k = 0, 1, \ldots, 2^j - 1$, $j = 0, 1, \ldots$, implies

$$u_n = \int_0^1 2^{\frac{j}{2}} u(t)H(2^j t - k)dt,$$

but

$$H(2^j t - k) = \begin{cases} 1, & k.2^{-j} \leq t < (k + \frac{1}{2})2^{-j} \\ -1, & (k + \frac{1}{2})2^{-j} \leq t < (k + 1)2^{-j} \\ 0, & \text{elsewhere} \end{cases} \tag{12}$$

which implies that

$$u_n = 2^{\frac{j}{2}} \left\{ \int_{k.2^{-j}}^{(k+\frac{1}{2})2^{-j}} u(t)dt - \int_{(k+\frac{1}{2})2^{-j}}^{(k+1)2^{-j}} u(t)dt \right\}$$

using mean value theorem we have:

$$\exists t_1, t_2 : k2^{-j} \leq t_1 < (k+\frac{1}{2})2^{-j}, (k+\frac{1}{2})2^{-j} \leq t_2 < (k+1)2^{-j}$$

such that

$$\begin{aligned} u_n &= 2^{\frac{j}{2}} \left\{ ((k+\frac{1}{2})2^{-j} - k.2^{-j})u(t_1) - ((k+1)2^{-j} - (k+\frac{1}{2})2^{-j})u(t_2) \right\} \\ &= 2^{-\frac{j}{2}-1}(u(t_1) - u(t_2)). \end{aligned}$$

Using the mean value theorem

$$\begin{aligned} u_n^2 &= 2^{-j-2}(t_2 - t_1)^2 u'^2(t_0) \quad (t_1 < t_0 < t_2) \\ &\leq 2^{-j-2}.2^{-2j}.M^2 \\ &= 2^{-3j-2}M^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u(t) - u_k(t)\|_E^2 &= \sum_{n=2^{\alpha+1}}^{\infty} u_n^2 \\ &= \sum_{j=\alpha+1}^{\infty} (\sum_{n=2^j}^{2^{j+1}-1} u_n^2) \\ &\leq M^2 \sum_{j=\alpha+1}^{\infty} 2^{-3j-2}(2^{j+1} - 1 - 2^j + 1) \\ &= \frac{M^2}{3} \frac{1}{k^2}, \end{aligned}$$

hence, $\|u(t) - u_k(t)\|_E = O(\frac{1}{k})$.

6 Numerical Examples

Example 1 [7]:

$$u(x) = x^7(1 - \frac{4096}{6435}\sqrt{x}) + \int_0^x \frac{u(t)}{\sqrt{x-t}}dt,$$

with exact solution $y(x) = x^7$.

Example 2 [9]:

$$u(x) = \frac{1}{\sqrt{x+1}} + \frac{\pi}{8} - \frac{1}{4}\arcsin(\frac{1-x}{1+x}) - \frac{1}{4}\int_0^x \frac{u(t)}{\sqrt{x-t}}dt,$$

with exact solution $u(x) = \frac{1}{\sqrt{x+1}}$.

Table 1 and Table 2 shows the analytic and approximated solution for the example 1 and example 2 at $t = 0.1i$ for $i = 0, 1, ..., 9$ with $k = 64$ respectively.

Table 1

t	Approximated solution	Analytic solution
0.0	0.00000	0.00000
0.1	0.00000	0.00000
0.2	0.00001	0.00001
0.3	0.00024	0.00021
0.4	0.00160	0.00163
0.5	0.00877	0.00781
0.6	0.02870	0.02799
0.7	0.07907	0.08235
0.8	0.21977	0.20971
0.9	0.47522	0.47829

Table 2

t	Approximated solution	Analytic solution
0.0	0.98088	1.00000
0.1	0.95282	0.95346
0.2	0.91469	0.91287
0.3	0.87548	0.87705
0.4	0.84562	0.84515
0.5	0.81438	0.81649
0.6	0.79019	0.79056
0.7	0.76803	0.76696
0.8	0.74439	0.74535
0.9	0.72758	0.72547

7 Conclusion

In [4] we used Haar wavelets for solving nonlinear Fredholm integral equations. In present paper, Haar wavelets were used to solve linear integral equations with weakly singular kernels. The benefits of the Haar wavelet method are sparse matrices of representation, fast transformation and possibility of implementation of fast algorithms. Therefore, we apply the fast, local and multiplicative properties of Haar wavelets for solving linear Volterra integral equations with weakly singular kernels. Example 1 is solved in [7] using Bernstein polynomials and example 2 is solved in [9] using the application of transformations of Korobov, Laurie and Sidi type in combination with the trapezoidal quadrature rule, evidently, in both cases, the methods are somewhat more accurate than our method. However, haar wavelet method is simplest and needs less computations. In this article detailed error analysis is carried out that shows high order convergence can be obtain easily by increasing the value of parameter k, for obtaining the desired approximation.

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Received: April, 2011