

# HADAMARD GAP SERIES AND NORMAL FUNCTIONS

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In this note we prove the following theorem.

**THEOREM 1.** *Let  $f$  be an analytic function defined in the unit disc  $\{z \mid |z| < 1\}$  by  $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$  where  $n_{k+1}/n_k \geq q > 1$ . Then  $f$  is normal if and only if  $f$  is Bloch if and only if  $\limsup |c_k| < \infty$ .*

J. S. Hwang [3] has proved the above theorem in the special case where  $c_k = n_k^m$  for some  $m > 1$  and  $(n_{k+1}/n_k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

The proof of Theorem 1 is based on ideas of W. Fuchs and techniques which go back to Hardy and Littlewood. Throughout the remainder of the paper  $p_0$  will denote the maximum of  $2 \log 20/(q-1-\log q)$  and  $\log 10/(\log q-1+q^{-1})$ .

Consider the associated real valued function  $F$  defined by

$$F(x) = \sum_{n=0}^{\infty} x^{n_k}, \quad n_{k+1}/n_k > 1.$$

By accentuating the dominance of the largest term of this series by successive differentiations Fuchs [2] proved for all integers  $p$  with  $p \geq p_0$  that  $F^{(p)}(x)$  behaves in certain intervals like a single term. To be precise, if  $p \geq p_0$  and  $v$  is sufficiently large, then for any  $s$  in the interval

$$\exp\left(\frac{-p}{n_v}\right) \leq s \leq \exp\left(\frac{-p}{2n_v}\left(1 + \frac{\log q}{q-1}\right)\right),$$

we have the inequality

$$\sum_{k \neq r} n_k(n_k-1) \dots (n_k-p+1) s^{n_k} \leq 1/4 n_r(n_r-1) \dots (n_r-p+1) s^{n_r}. \quad (1)$$

This observation about the associated real valued function  $F$  lets us prove the following important result.

**THEOREM 2.** *Let  $f$  be an analytic function defined in  $\{z \mid |z| < 1\}$  by  $f(z) = \sum c_k z^{n_k}$ ,  $n_{k+1}/n_k \geq q > 1$ , where  $\limsup |c_k| = \infty$ . Then for all integers  $p$  with  $p \geq p_0$  and all  $M > 0$ , there is an  $s$  such that*

$$(1 - |z|)^p |f^{(p)}(z)| \geq M$$

for every  $z$  on the circle  $|z| = s$ .

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*Proof.* Let  $\mu(r) = \sup_k |c_k| r^{nk}$ . The analyticity of  $f$  in  $|z| < 1$  and  $\limsup |c_k| = \infty$  together imply that  $\mu(r)$  is a monotone increasing function which tends to infinity as  $r \rightarrow 1$ .

Let  $p$  be an arbitrary integer greater than  $p_0$ . Choose  $r_0$  such that  $\mu(r_0) \geq 4M(2e)^p/p!$ . Let  $v$  be the largest integer such that

$$|c_v| r_0^{nv} \geq \mu(r_0)/2. \tag{2}$$

We note that  $r_0$  can be chosen near enough to 1 so that  $n_v \geq p$ , and we assume henceforth that  $r_0$  is so chosen. Now set  $s = s_0 r_0$  where  $s_0 = \exp(-p/n_v)$ . Then

$$|f^{(p)}(se^{i\theta})| \geq n_v(n_v-1) \dots (n_v-p+1) |c_v| s^{n_v-p} - \sum_{k \neq v} n_k(n_k-1) \dots (n_k-p+1) |c_k| s^{n_k-p} \tag{3}$$

Using (1) and (2) we obtain

$$\begin{aligned} \sum_{k \neq v} n_k(n_k-1) \dots (n_k-p+1) |c_k| (s_0 r_0)^{n_k-p} \\ \leq \sup_k (|c_k| r_0^{nk}) r_0^{-p} s_0^{-p} \sum_{k \neq v} n_k(n_k-1) \dots (n_k-p+1) s_0^{n_k} \\ \leq 1/2 n_v(n_v-1) \dots (n_v-p+1) |c_v| s_0^{n_v-p} r_0^{n_v-p}. \end{aligned} \tag{4}$$

Thus from (3) and (4) we find

$$\begin{aligned} |f^{(p)}(se^{i\theta})| &\geq 1/2 n_v(n_v-1) \dots (n_v-p+1) s_0^{n_v-p} r_0^{n_v-p} |c_v| \\ &\geq n_v(n_v-1) \dots (n_v-p+1) M 2^p/p! \end{aligned}$$

And therefore

$$\begin{aligned} (1-s)^p |f^{(p)}(se^{i\theta})| &\geq (1-s_0)^p n_v(n_v-1) \dots (n_v-p+1) M 2^p/p! \\ &\geq \left(\frac{p}{n_v}\right)^p n_v(n_v-1) \dots (n_v-p+1) M/p! \\ &= \frac{M p^p}{p!} \left(1 - \frac{1}{n_v}\right) \left(1 - \frac{2}{n_v}\right) \dots \left(1 - \frac{p-1}{n_v}\right) \\ &\geq \frac{M p^p}{p!} \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right) \dots \left(1 - \frac{p-1}{p}\right) \equiv M, \end{aligned}$$

which concludes the proof of Theorem 2.

*Proof of Theorem 1.* Pommerenke [8] proved  $\limsup |c_k| < \infty$  implies  $f$  is Bloch, and it is well known that Bloch functions are normal [9, p. 268]. It therefore suffices to prove that  $\limsup |c_k| = \infty$  implies  $f$  is not normal.

Fix an integer  $p$  for which  $p > p_0$ . Lappan [4] proved that if  $f$  is normal, then there is a finite constant  $K$  such that

$$\frac{|f^{(p)}(z)|(1-|z|)^p}{1+|f(z)|^{p+1}} \leq K \tag{5}$$

for all  $z$  in the unit disc. According to Theorem 2 there is a sequence of radii  $s_n$  such that for  $|z| = s_n$

$$(1 - |z|)^p |f^{(p)}(z)| \geq n.$$

If  $\min |f(z)|$  on  $|z| = s_n$  tends to  $\infty$ , then  $f$  has Koebe arcs and is therefore non-normal [9, p. 267]. If  $\min |f(z)|$  on  $|z| = s_n$  does not tend to  $\infty$ , then by passing to a subsequence we can find an integer  $M$  and a sequence of points  $z_n, |z_n| = s_n$ , such that  $|f(z_n)| \leq M < \infty$ . For this sequence of points

$$\frac{(1 - |z_n|)^p |f^{(p)}(z_n)|}{1 + |f(z_n)|^{p+1}} \geq \frac{n}{1 + M^{p+1}},$$

which proves that (6) cannot hold. Therefore  $f$  must be non-normal.

**COROLLARY.** *Let  $f$  be defined in the unit disc by  $f(z) = \sum_{k=1}^{\infty} k^k z^{2^k}$ . Then all of its derivatives and all of its integrals are non-normal functions which are analytic in  $\{z \mid |z| < 1\}$ .*

*Remark 1.* Motivated by a result of MacLane [5, p. 46], Bonar [1, p. 59] posed the following question. If  $f$  is a strongly annular function, can  $f$  be written as  $f(z) = g(z) + h(z)$  where  $g(z) = \sum a_k z^{\mu_k}$ ,  $\liminf \mu_{k+1}/\mu_k > 3$ , and  $h(z)$  is bounded in the unit disc? The answer is no. To see this, let  $F(z) = \sum 2^k z^{2^k}$ . Since  $n_{k+1}/n_k = 2$ , and  $|c_k| \rightarrow \infty$ , it is an easy consequence of Theorem 2 that for an integer  $p > p_0$ , there is an increasing sequence of positive numbers  $s_n$  such that  $(1 - |z|)^p |F^{(p)}(z)| \geq n$  for all  $z$  on the circle  $|z| = s_n$ . Thus  $F^{(p)}$  is strongly annular. Suppose  $F^{(p)}(z)$  could be written as  $g(z) + h(z)$  where  $g(z) = \sum a_j z^{\mu_j}$ ,  $\liminf \mu_{j+1}/\mu_j > 3$ , and  $h$  is bounded in the unit disc. The coefficients in the power series expansion of  $h$  about zero must go to zero since  $h$  is bounded. Therefore for  $k$  sufficiently large the index set  $\{\mu_j\}$  must contain all exponents of the form  $\{2^j - p\}_{j=k}^{\infty}$ . Consequently,  $3 < \liminf \mu_{j+1}/\mu_j \leq 2$ , which is absurd. An appropriate modification works for any  $q > 1$ .

*Remark 2.* Piranian [7] asked whether a bounded function of finite area must have a normal derivative. Theorem 1 lets us answer this in the negative. Let  $f$  be defined by

$$f(z) = \sum_{n=1}^{\infty} n(2^n + 1)^{-1} z^{2^n+1} = \sum_{j=1}^{\infty} a_j z^j.$$

Then  $f$  is bounded by  $\sum n 2^{-n}$ , and  $\sum j |a_j|^2 \leq \sum n^2 2^{-n}$  shows that  $f$  has finite area. Theorem 1 guarantees that  $f'(z) = \sum n z^{2^n}$  defines a non-normal function in the unit disc.

We close with two open questions.

*Question 1.* If  $f(z) = \sum c_k z^{n_k}$ ,  $n_{k+1}/n_k \geq q > 1$ ,  $\limsup |c_k| = \infty$ , must  $f$  be annular? If the maximum modulus of  $f$  grows rapidly enough, the answer is yes [c.f. 6, Thm. 4].

*Question 2.* What is the best value for  $p_0$  for which Theorem 2 is true?

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