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HADAMARD MATRICES AND δ-CODES OF LENGTH $3n$

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ABSTRACT. It is found that four-symbol δ-codes of length $t = 3n$ can be composed for odd $n \leq 59$ or $n = 2^a 10^b 26^c + 1$, where all a, b and $c \geq 0$. Consequently new families of Hadamard matrices of orders $4tw$ and $20tw$ can be constructed, where w is the order of Williamson matrices.

Introduction. An Hadamard matrix $H_n = (h_{ij})$ of order n is an $n \times n$ matrix with entries 1 or -1 such that $H_n H_n^T = nI_n$, where I_n is the $n \times n$ identity matrix and T indicates the transposed matrix. In H_n, row vectors $v_i = (h_{i1}, h_{i2}, \ldots, h_{in})$ are orthogonal, i.e. $v_i \cdot v_j \equiv \sum_{k=1}^n h_{ik} h_{jk} = 0$, $i \neq j$. H_n exists only if $n = 1, 2$, or $4k$.

A sequence of vectors $V = (v_k)_n \equiv (v_1, v_2, \ldots, v_n)$, where v_k is one of m orthonormal vectors $i_1, i_2, \ldots, i_m$ or their negatives, is said to be an m-symbol δ-code of length n, if

(I) $v(j) = 0$ for $j \neq 0$, where $v(j) \equiv \sum_{k=1}^{n-j} v_k \cdot v_{k+j}$ is the nonperiodic auto-correlation function of V. Another characterization of $V = (v_k)_n$ being an m-symbol δ-code is that its associated polynomial $V(z) \equiv \sum_{k=1}^n v_k z^{k-1} = \sum_{j=1}^m P_j(z) i_j$, where $P_j(z) = \sum_{k=1}^n p_{jk} z^{k-1}$, $1 \leq j \leq m$, satisfies

(II) $p_{jk} \in \{0, 1, -1\}$ and $\sum_{j=1}^m |p_{jk}| = 1$ $(1 \leq k \leq n)$; and

(III) $\sum_{j=1}^m |P_j(z)|^2 = n$, for any z on the unit circle $K = \{z \in \mathbf{C}\colon |z| = 1\} = \{z = \exp(ix)\colon 0 \leq x \leq 2\pi\}$, where $\mathbf{C}$ is the complex field and $i = \sqrt{-1}$.

Hadamard matrices of orders $4tw$ and $20tw$ can be composed if there exist a four-symbol δ-code of length t and Williamson matrices of order w (see [**1**]).

For four-symbol δ-codes, we can let $i_1 = (1,0,0,0)$, $i_2 = (0,1,0,0)$, $i_3 = (0,0,1,0)$, $i_4 = (0,0,0,1)$ and $v_k = (q_k, r_k, s_k, t_k)$. Then

$$q_k, r_k, s_k, t_k \in \{0, 1, -1\} \quad \text{and} \quad |q_k| + |r_k| + |s_k| + |t_k| = 1, \tag{1}$$

which corresponds to condition (II). Condition (I) becomes

$$q(j) + r(j) + s(j) + t(j) = 0 \quad \text{for } j \neq 0, \tag{2}$$

where $p(j)$ is the auto-correlation function of a sequence $P = (p_k)$. And (III) becomes

$$|Q|^2 + |R|^2 + |S|^2 + |T|^2 = n \quad \text{for any } z \in K,$$

where P stands for the associated polynomial $P(z)$ of a sequence (p_k). From now on we shall use the same P to represent both a sequence (p_k) and its associated polynomial $\sum p_k z^{k-1}$.

Four sequences Q, R, S and T of length n satisfying conditions (1) and (2) are called *Turyn sequences* (or T-sequences) of length n (abbreviated as $TS(n)$).

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Four $(1,-1)$ sequences $U=(u_k)_{m+p}$, $W=(w_k)_{m+p}$; $X=(x_k)_m$ and $Y=(y_k)_m$ (where $p\geq 0$) will be called *Turyn base sequences* for length $2m+p$ (abbreviated as $TBS(2m+p)$) if they satisfy

$$u(j)+w(j)+x(j)+y(j)=0 \quad \text{for } j\neq 0. \tag{3}$$

Condition (3) is also equivalent to

$$|U|^2+|W|^2+|X|^2+|Y|^2=2(2m+p) \quad \text{for any } z\in K.$$

If $TBS(2m+p)$: U,W; X and Y exist, then $TS(2m+p)$ can be formed (cf. [**1**]) as follows: $\frac{1}{2}(U+W,0)$, $\frac{1}{2}(U-W,0)$, $\frac{1}{2}(0',X+Y)$, and $\frac{1}{2}(0',X-Y)$, where $0=0_m$ (the sequence of zeros of length m) and $0'=0_{m+p}$.

THEOREM. *Let $U=(u_k)_{m+p}$, $W=(w_k)_{m+p}$; $X=(x_k)_m$ and $Y=(y_k)_m$ be $TBS(n)$ for $n=2m+p$. Then the following are $TS(3n)$:* [2]

$$\begin{aligned}
Q&=\frac{1}{2}(U+W,X+Y;0',0;(U-W)^*,0),\\
R&=\frac{1}{2}(U-W,X-Y;0',0;-(U+W)^*,0),\\
S&=\frac{1}{2}(0',0;U+W,-(X+Y);0',(X-Y)^*),\\
T&=\frac{1}{2}(0',0;U-W,-(X-Y);0',-(X+Y)^*),
\end{aligned}\tag{4}$$

or

$$\begin{aligned}
Q&=\frac{1}{2}((U-W)^*,0;U+W,X+Y;0',0),\\
R&=\frac{1}{2}(-(U+W)^*,0;U-W,X-Y;0',0),\\
S&=\frac{1}{2}(0',(X-Y)^*;0',0;U+W,-(X+Y)),\\
T&=\frac{1}{2}(0',-(X+Y)^*;0',0;U-W,-(X-Y)),
\end{aligned}\tag{5}$$

where $A^*=(a_N,a_{N-1},\ldots,a_1)$ is the reverse of $A=(a_1,a_2,\ldots,a_N)$.

LEMMA. *Let a, b, c and d be polynomials with real coefficients in $z\in K$. And let $e=a+b+c$, $f=a-b+d$, $g=a-c-d$, and $h=b-c+d$. Then*

$$|e|^2+|f|^2+|g|^2+|h|^2=3(|a|^2+|b|^2+|c|^2+|d|^2) \quad \textit{for any } z\in K.$$

The Lemma can be proved easily by straightforward computations and by observing that $|p|^2=pp'$, where $p'=p(z^{-1})$ for any $z\in K$.

PROOF OF THEOREM. Let $a=U$, $b=-z^{n-m}X$, $c=-z^{2n-m}Y^*$, and $d=-z^{2n}W^*$ in the Lemma. Then as sequences, $e=(U,-X;0',-Y^*)$, $f=(U,X;0',0;-W^*)$, $g=(U,0;0',Y^*;W^*)$ and $h=(0,-X;0',Y^*;-W^*)$. Consequently $g^*=(W,Y;0',0;U^*)$ and $h^*=(-W,Y;0',-X^*;0')$. In case (4), we have $Q=(f+g^*)/2$, $R=(f-g^*)/2$, $S=z^n(e-h^*)/2$, and $T=z^n(e+h^*)/2$. By noting that $|z|=1$ and $|p^*|^2=|p|^2$ since $|p^*(z)|=|p(z^{-1})|$, we obtain $|Q|^2+$

[2] This neat form of case (4), which contains less $*$'s than my original one, was suggested by R. J. Turyn.

$|R|^2 + |S|^2 + |T|^2 = (|e|^2 + |f|^2 + |g|^2 + |h|^2)/2 = 3(|a|^2 + |b|^2 + |c|^2 + |d|^2)/2 = 3(|U|^2 + |W|^2 + |X|^2 + |Y|^2)/2 = 3n$, for any $z \in K$. Similarly we can establish case (5) by letting $a = X^*$, $b = z^m U^*$, $c = -z^{n+m} W$ and $d = z^{2n} Y$ in the Lemma.

Since $TBS(n)$ are known to exist for odd $n \leq 59$ or $n = 2^a 10^b 26^c + 1$ (cf. [**1**, **2**, **3**]), $TS(3n)$ can be composed for these n. Consequently four-symbol δ-codes of length $3n$ can be found for these n.

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