Probab. Theory Relat. Fields (2013) 155:35–69
DOI 10.1007/s00440-011-0391-2

Hadamard's formula and couplings of SLEs with free field

Konstantin Izyurov · Kalle Kytölä

Received: 20 September 2010 / Revised: 29 September 2011 / Published online: 21 October 2011
© Springer-Verlag 2011

Abstract The relation between level lines of Gaussian free fields (GFF) and SLE_4-type curves was discovered by O. Schramm and S. Sheffield. A weak interpretation of this relation is the existence of a coupling of the GFF and a random curve, in which the curve behaves like a level line of the field. In the present paper we study these couplings for the free field with different boundary conditions. We provide a unified way to determine the law of the curve (i.e. to compute the driving process of the Loewner chain) given boundary conditions of the field and to prove existence of the coupling. The proof is reduced to the verification of two simple properties of the mean and covariance of the field, which always relies on Hadamard's formula and properties of harmonic functions. Examples include combinations of Dirichlet, Neumann and Riemann–Hilbert boundary conditions. In doubly connected domains, the standard annulus SLE_4 is coupled with a compactified GFF obeying Neumann boundary conditions on the inner boundary. We also consider variants of annulus SLE coupled with free fields having other natural boundary conditions. These include boundary conditions leading to curves connecting two points on different boundary components with prescribed winding as well as those recently proposed by C. Hagendorf, M. Bauer and D. Bernard.

Mathematics Subject Classification (2000) 60J67 · 60G60

K. Izyurov (✉)
Section de Mathématiques, Université de Genève, 1211 Geneva 4, Switzerland
e-mail: Konstantin.Izyurov@unige.ch

K. Kytölä
Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68 (Gustaf Hällströmin katu 2b), 00014 University of Helsinki, Finland
e-mail: kalle.kytola@helsinki.fi

1 Introduction

The topic of conformally invariant random processes in two dimensions has received a lot of attention during the past decade. Recent developments have enabled a probabilistic approach to problems traditionally studied in theoretical physics by means of conformal field theory.

Two fundamental examples of random conformally invariant objects are Schramm-Loewner evolutions (SLE) and Gaussian free fields (GFF). SLE are random fractal curves described by growth processes encoded in Loewner chains, discovered in [13]. Their most important characteristics are captured by one parameter, a positive real number κ, but still in different setups one needs different variants of SLE_κ as we will again see in this article. The GFF is a statistical model that fits naturally both in the setup of conformal field theory and in that of probability theory: it is essentially the simplest Euclidean quantum field theory, which describes the free massless boson, but it also admits an easy interpretation as a random generalized function.

Informally speaking, the Gaussian free field Φ in a planar domain Ω is a collection of Gaussian random variables indexed by the points of the domain, $\Phi = (\Phi(z))_{z\in\Omega}$, such that

- The mean $\mathsf{E}[\Phi(z)] = M(z)$ is a harmonic function.
- The covariance $\mathsf{E}[(\Phi(z_1) - M(z_1))(\Phi(z_2) - M(z_2))] = C(z_1, z_2)$ is a Green's function in Ω.

To obtain an unambiguous definition of the GFF, one has to specify which harmonic function to choose and what is meant by the Green's function. We will usually specify M by its boundary conditions. The Green's functions will be solutions to $-\triangle G(\cdot, z_2) = \delta_{z_2}(\cdot)$ with prescribed boundary conditions. From the definition one immediately sees that GFFs will posses conformal invariance properties—indeed, harmonic functions and Green's functions are simply transported by conformal maps. If $\phi : \Omega \to \Omega'$ is a conformal map and Φ is a GFF in Ω', then $\Phi \circ \phi$ is a GFF in Ω, boundary conditions in Ω being the pullback of those in Ω'. We will mostly deal with boundary conditions that transform nicely under conformal maps.

Note that Φ being Gaussian, the law is indeed determined by its mean and covariance. Due to the logarithmic blowup of the covariance as $|z_1 - z_2| \to 0$, however, the field Φ is not a random function but rather a random distribution (a generalized function). We postpone a formal definition of GFF to Sect. 2.3.

A typical example of how the mean M and covariance C are specified appears in the works of Schramm and Sheffield [14,16] which first established a relation between the Gaussian free field and SLE. In a simply connected domain Ω with boundary $\partial\Omega$ divided into two complementary arcs l_1 and l_2 one defines M and C by

$$\begin{cases} \triangle M(z) = 0 & \text{for } z \in \Omega \\ M(z) = +\lambda & \text{for } z \in l_1 \\ M(z) = -\lambda & \text{for } z \in l_2 \end{cases} \quad \text{and} \quad \begin{cases} \triangle_z C(z, z_1) = -\delta_{z_1}(z) & \text{for } z \in \Omega \\ C(z, z_1) = 0 & \text{for } z \in \partial\Omega. \end{cases} \tag{1.1}$$

For a suitable analogous discrete GFF these boundary conditions force the existence of a unique random curve, interpreted as a zero-level line of the field, which joins the endpoints of the boundary arcs l_1 and l_2, and has positive field values on all the neighbors on one side and negative values the neighbors on the other side. Schramm and Sheffield showed that chordal SLE_4 describes the scaling limit of these zero-level lines with the above boundary conditions, when the parameter λ has the particular value $\lambda = \sqrt{\pi/8}$. In particular, the free field is naturally coupled with a chordal SLE_4, and in the scaling limit the level lines of discrete GFF become discontinuity lines of the GFF of jump 2λ.

We will be interested in couplings of different variants of GFF with random growth processes of SLE type. A variant of GFF is a rule associating with any domain Ω with $n+1$ marked points $x, x_1, x_2, \ldots, x_n \in \partial\Omega$ a free field $\Phi_{\Omega;x,x_1,\ldots,x_n}$. For instance, in the above example (1.1) the marked points are the two endpoints of the boundary arcs and they correspond to jumps in piecewise constant boundary data, whereas in the examples further on in this article the marked points may have a change between more general boundary conditions of the corresponding GFF. Take a domain $(\Omega; x, x_1, x_2, \ldots)$ and suppose we have a random curve $\gamma \subset \Omega$ growing from x. The main property we require of the coupling is

> Conditionally on the random curve $\gamma \subset \Omega$ starting from $x \in \partial\Omega$, the law of the free field $\Phi_{(\Omega;x,x_1,\ldots,x_n)}$ is the same as that of the free field $\Phi_{(\widetilde{\Omega};\tilde{x},x_1,\ldots,x_n)}$ in the domain $\widetilde{\Omega} = \Omega \setminus \gamma$, where $\tilde{x} \in \partial\widetilde{\Omega}$ is the tip of the curve γ.

This property also immediately suggests a constructive way of producing the coupling given the random curve and the laws of the free fields in different domains:

> Sample the random curve γ, and then sample independently the free field $\Phi_{(\widetilde{\Omega};\tilde{x},x_1,\ldots,x_n)}$ in the slitted domain $\widetilde{\Omega} = \Omega \setminus \gamma$. The law of the resulting field is the same as the free field $\Phi_{(\Omega;x,x_1,\ldots,x_n)}$ in the original domain.

Note that this sampling produces randomness in two stages: the random field in $\widetilde{\Omega} = \Omega \setminus \gamma$ is integrated over the randomness of the curve γ. The motivation for imposing these properties of the coupling is the example of Schramm and Sheffield, in which the discontinuity line of the free field satisfies them. The present article exhibits numerous variations of that basic example.

The article is organized as follows: Sections 1.1 and 1.2 recall necessary background on Loewner chains and SLE in simply and multiply connected domains. Section 2 is devoted to the general setup for establishing couplings of SLEs and free fields. Section 2.1 writes the two basic conditions that we will verify in each case to prove the existence of couplings, and Sect. 2.2 concretely illustrates these conditions in the simplest example case (1.1) of Schramm and Sheffield. We define the GFFs in Sect. 2.3 and show that the two basic conditions imply a weak form of coupling. Next, in Sect. 2.4 we recall and prove in a setup appropriate for the present purpose the Hadamard's formula, whose variants are crucial to the verification of the basic conditions in all cases.

The concrete examples are divided into two sections, treating simply connected domains and doubly connected domains separately. Section 3 presents free fields with different boundary conditions in simply connected domains. The examples here include coupling of the dipolar SLE_4 with GFF having combined jump-Dirichlet and Neumann boundary conditions and the coupling of $SLE_4(\rho)$ with GFF having combined jump-Dirichlet and Riemann–Hilbert boundary conditions. We also show in the presence of more complicated combinations of boundary conditions how the coupling determines the law of the curve, i.e. how to compute the Loewner driving process. Section 4 treats examples in doubly connected domains. We warm up with a simple case in the punctured disc, giving a short proof that the radial SLE_κ is coupled with a compactified free field with jump-Dirichlet boundary conditions as stated in [6]. In an annulus with jump-Dirichlet boundary conditions on one boundary component and Neumann boundary conditions on the other, we show that the compactified free field is coupled with the standard annulus SLE_4 introduced in [1,19]. We also review the SLE_4 variants proposed in [7] on grounds of free field partition functions and show that they indeed admit couplings with the non-compactified free fields with corresponding boundary conditions. Another new example consists in imposing jump-Dirichlet boundary conditions on both boundary components for a compactified free field, leading to a curve with prescribed winding. In Sect. 4.5 we show that the cases with Dirichlet boundary conditions admit generalizations to $\kappa \neq 4$.

Appendix A explains why extensions at $\kappa \neq 4$ do not work with all boundary conditions, and Appendix B contains the proof of a property of Loewner chains we need in conjunction with general Hadamard's formulas.

Relation to other work

We note that the relation of the free field and SLE has already been explored beyond the basic example of Schramm and Sheffield. One research direction (see [6]) has been establishing the coupling in a stronger sense, i.e. extending the field to the whole domain (not merely subdomains almost surely untouched by the curve, as in the present paper) and proving that the curve is actually *determined* by the field.

The effect of the boundary conditions of the free field on the law of the curve is another important generalization of the basic example, and this is the direction we systematically pursue also in the present article. Earlier work in this direction concerns especially the appropriate SLE variants when one allows several jumps in the Dirichlet boundary conditions, discussed in some cases already in [5,16] and developed in more generality in [6]. Recently, Hagendorf, Bauer and Bernard [7] proposed natural SLE variants in annulus based on computations of free field partition functions with combined jump-Dirichlet and Neumann boundary conditions. Our examples cover also these cases explicitly.

It is also worth noting that Schramm and Sheffield [15] themselves indicated how their coupling can be extended to chordal SLE_κ with $\kappa \neq 4$ by modifying the conformal transformation property of the field in the manner dictated by the Coulomb gas formalism of conformal field theory. In our examples which involve piecewise Dirichlet boundary conditions we show how to treat $\kappa \neq 4$, and we give a non-commutation argument explaining why one is constrained to $\kappa = 4$ in the presence of other boundary conditions.

Generalizations to massive free fields have been treated in [3,11]. Many aspects of SLE_4 related conformal field theories are considered in the forthcoming articles [8,12].

1.1 Growth processes and Loewner evolutions

The Loewner evolution is a way of describing growth processes, curves in particular, in terms of conformal maps. In the case when Ω is a simply connected domain with analytic boundary, a setup convenient for our purposes is as follows. To each point of the boundary $x \in \partial\Omega$ we associate a Loewner vector field $V_x(z)\partial_z$, satisfying the following properties:

- $V_x(z)$ is analytic inside the domain Ω and up to the boundary apart from the point x;
- for $z \in \partial\Omega \setminus \{x\}$ the vector field $V_x(z)\partial_z$ is tangential to the boundary;
- $V_x(z)$ has a simple pole at x with $\mathrm{Res}_x\,(V_x(z)) = 2\,\tau_x^2$, where τ_x is a unit tangent to $\partial\Omega$ at x;
- $V_x(z)$ is bounded apart from the neighborhood of x.

Given a continuous function $t \mapsto X_t \in \partial\Omega$ called the driving process, the Loewner's differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = V_{X_t}(g_t(z)), \quad g_0(z) = z \tag{Loe}$$

where the initial condition is a point of the domain, $z \in \Omega$. For all $t \geq 0$ we let $K_t \subset \Omega$ be the set of points z for which the solution fails to exist up to time t. The hulls $(K_t)_{t\geq 0}$ form a growth process, $K_{t_1} \subset K_{t_2}$ for $t_1 < t_2$. The solution $(g_t)_{t\geq 0}$ is called a Loewner chain for the growth process.

Familiar examples in the half-plane, disc and strip are, respectively,

Domain	**Vector fields**	**Flow**	
$\mathbb{H} = \{\Im\mathrm{m}\, z > 0\}$	$V_x(z) = \frac{2}{z-x}$	$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \frac{2}{g_t(z) - X_t}$	(Loe-$\mathbb{H}$)
$\mathbb{D} = \{\lvert z\rvert < 1\}$	$V_x(z) = -z\frac{z+x}{z-x}$	$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = g_t(z)\frac{X_t + g_t(z)}{X_t - g_t(z)}$	(Loe-$\mathbb{D}$)
$\mathbb{S} = \{0 < \Im\mathrm{m}\, z < \pi\}$	$V_x(z) = \coth\left(\frac{z-x}{2}\right)$	$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = \coth\left(\frac{g_t(z) - X_t}{2}\right).$	(Loe-$\mathbb{S}$)

The first flow in $\mathbb{H}$ fixes the point ∞, the second in $\mathbb{D}$ fixes 0 and the third in $\mathbb{S}$ fixes both $\pm\infty$. These properties make the chosen flows convenient, but we remark that the choices are by no means unique. In particular, it is worth noting that a growth process can be described by several different Loewner chains. In what follows we assume that $V_x(z)$ depends sufficiently nicely on x, as is the case with the three examples.

The following proposition is standard, and in concrete examples we only use it with the three vector fields listed above.

Proposition 1 *For all $t > 0$, $\Omega \setminus K_t$ is simply-connected, and $z \mapsto g_t(z)$ is a conformal map from $\Omega_t := \Omega \setminus K_t$ to Ω. Moreover, $\partial_t \mathrm{lhcap}_{X_0}(K_t)|_{t=0} = 2$, where* lhcap *is the local half-plane capacity.*

The local half-plane capacity of K_t is informally defined as follows: if the boundary near the point X_0 is a straight line, translate and rotate the domain so that it would actually become a part of $\mathbb{R}$ and K_t would become a subset of $\mathbb{H}$, then take the half plane capacity. If the boundary is not a straight line, use a conformal map f to $\mathbb{H}$ such that $|f'(x_0)| = 1$. Roughly speaking, the last statement of the Proposition means that for small values of t, the size of the hull does not depend much on global structure of the domain and the evolution—to first order it is completely determined by the residue of $V_x(z)$ (which in turn is fixed by our conventions for Loewner vector fields). We postpone the formal definition of local half-plane capacity along with the proof of the Proposition to Appendix B.

Note that the map g_t maps the tip of the growing hull K_t to the point X_t. The notion of the tip is intuitive if the hulls are growing curves, $K_t = \gamma[0, t]$. The tip is, however, always well defined, since Loewner chains satisfy the *local growth property* (see, e.g., the discussion after Example 4.12 in [10]): $\lim_{\varepsilon\searrow 0}(\overline{K_{t+\varepsilon} \setminus K_t})$ is always a boundary point of $\Omega \setminus K_t$ (more precisely, a prime end). We call this point the tip of the hull and denote it by $\tilde{x}(t)$.

Loewner chains in doubly connected domains

For multiply connected domains the Loewner flow $V_x(z)$ cannot be tangential to the boundary on all boundary components—once we start growth, the conformal moduli of the domain change, and $z \mapsto g_t(z)$ cannot be a map from $\Omega \setminus K_t$ onto Ω anymore. Hence, instead of one domain, we fix a family of representatives of conformal equivalence classes, and g_t maps to one of these. In doubly connected case, a natural family is provided by the annuli $\mathbb{A}_r = \{z \in \mathbb{C} \ : \ e^{-r} < |z| < 1\}$, $r > 0$, with the unit circle as their common boundary component. For the Loewner flow to preserve this family, the radial component of the vector field should be constant on the inner boundary circle. This is equivalent to the condition $\Re\mathrm{e}\,(V_x^r(z)/z) = C$ on $|z| = e^{-r}$. On the outer component of the boundary, we want $V_x^r(z)\partial_z$ to be tangential to the boundary, meaning $\Re\mathrm{e}\,(V_x^r(z)/z) = 0$ for $|z| = 1$, $z \neq x$. For any value of the constant C, there exists a unique harmonic function with such boundary conditions and desired singularity at x, but only at one value of C the harmonic conjugate becomes a single-valued function. Namely, there exists a unique function $S_x^r(z)$ (Schwarz kernel) satisfying the following properties:

- $S_x^r(z)$ is analytic in the annulus $\mathbb{A}_r$;
- $\Re\mathrm{e}\, S_x^r(z) = \delta_x(z)$ on the outer boundary $\{|z| = 1\}$;
- $\Re\mathrm{e}\, S_x^r(z) = \frac{1}{2\pi}$ on the inner boundary $\{|z| = e^{-r}\}$.

There is a complicated explicit expression for $S_x^r(z)$ [1,19], but we will not need it.

We define Loewner vector fields as $V_x^r(z) = 2\pi z S_x^r(z)$. With this choice the modulus r decreases at unit speed under the flow analogous to (Loe): if $\Omega = \mathbb{A}_p$, then $g_t(\Omega \setminus K_t) = \mathbb{A}_{p-t}$. The modulus, therefore, directly serves as a time parametrization of the Loewner chain.

Domains	**Vector fields**	**Flow**	
$\mathbb{A}_r = \{e^{-r} < \lvert z\rvert < 1\}$	$V_x^r(z) = 2\pi\, z\, S_x^r(z)$	$\frac{\mathrm{d}}{\mathrm{d}t} g_t(z) = 2\pi\, z\, S_{X_t}^{p-t}(z).$	(Loe-$\mathbb{A}$)

The analog++ of Proposition 1 remains valid for this Loewner chain on the time interval $t \in [0, p)$.

1.2 Schramm-Loewner evolutions

Stochastic Loewner evolutions (or SLE) are random growth processes defined via a Loewner chain with random driving process. The random driving process is chosen so that the growth process satisfies two fundamental properties: *conformal invariance* and *domain Markov property*—the reader is referred to one of the many excellent introductions to SLE for details, e.g. [2,10,18]. In particular, the driving process will always be chosen to be a semimartingale (living on the boundary of the domain) whose quadratic variation grows at constant speed $\kappa > 0$, indicated by a subscript SLE$_\kappa$. In the following well-known examples the driving process is simply a Brownian motion on $\partial\Omega$ with the appropriate speed—here and in the sequel, $(B_t)_{t\geq 0}$ stands for a standard Brownian motion on $\mathbb{R}$:

- *Chordal SLE$_\kappa$ in $\mathbb{H}$ from* 0 *to* ∞*:*
  The Loewner chain is (Loe) with the driving process $X_t = \sqrt{\kappa}\, B_t$.
- *Radial SLE$_\kappa$ in $\mathbb{D}$ from* 1 *to* 0*:*
  The Loewner chain is (Loe) with the driving process $X_t = \exp(\mathrm{i}\sqrt{\kappa}\, B_t)$.
- *Dipolar SLE$_\kappa$ in $\mathbb{S}$ from* 0 *to* $\mathbb{R} + \mathrm{i}\pi$*:*
  The Loewner chain is (Loe) with the driving process $X_t = \sqrt{\kappa}\, B_t$. Note also that this is a special case of the example of SLE$_\kappa(\rho)$ in $\mathbb{S}$ below, with $\rho = \frac{\kappa-6}{2}$.

In other examples the driving process X may have a drift. For instance, if the domain has marked points $x, x_1, x_2, \ldots, x_n$ on the boundary, then the slitted domain $(\Omega_t, \tilde{x}(t), x_1, \ldots, x_n)$ is in general not conformally equivalent to $(\Omega, x_1, x_2, \ldots, x_n)$. The drift term of the Itô diffusion may therefore depend on conformal moduli of this configuration, as in the first of the following two examples:

- *SLE$_\kappa(\overline{\rho})$ in $\mathbb{H}$ started from* 0*:*
  Here $\overline{\rho} = (\rho_1, \rho_2, \ldots, \rho_n)$ is an n-tuple of real parameters. The marked points other than $x = 0$ are $x_1, x_2, \ldots, x_n \in \mathbb{R}$ on the boundary. The Loewner chain is (Loe) with driving process obeying the Itô diffusion $\mathrm{d}X_t = \sqrt{\kappa}\, \mathrm{d}B_t + \sum_j \frac{\rho_j}{X_t - g_t(x_j)}\, \mathrm{d}t$, with $X_0 = x = 0$.
- *SLE$_\kappa(\rho)$ in $\mathbb{S}$ started from* 0*:*
  In the above example if $n = 1$ it is convenient to perform a coordinate change from $\mathbb{H}$ to $\mathbb{S}$ sending $0 \mapsto 0$, $x_1 \mapsto \pm\infty$ and $\infty \mapsto \mp\infty$, see, e.g. [9]. The resulting growth process is described, up to a time reparametrization, by a Loewner chain (Loe) with driving process $X_t = \sqrt{\kappa}\, B_t \mp (\rho + \frac{6-\kappa}{2})\, t$.

The simplest example of SLE$_\kappa$ in doubly connected domains is the following, proposed independently in [1,19]:

- *Standard annulus SLE$_\kappa$ in $\mathbb{A}_p$ started from* 1*:*
  The Loewner chain is (Loe) with driving process $X_t = \exp(\mathrm{i}\sqrt{\kappa}\, B_t)$.

There are, of course, more variants of SLE$_\kappa$. We will find natural free fields admitting coupling with each of the above examples—and when boundary conditions of the free field are more complicated, we find other variants.

2 Couplings of SLEs and Gaussian free fields

2.1 Basic equations

Recall that we are interested in GFFs coupled with random curves or growth processes in the way described in the introduction. Suppose we have a rule associating with each domain Ω (with marked points) a free field Φ_Ω, determined by a harmonic function $M_\Omega : \Omega \to \mathbb{R}$ and a Green's function $C_\Omega : \Omega \times \Omega \setminus \{z_1 = z_2\} \to \mathbb{R}$. Consider a random growth process of hulls $(K_t)_{t\in[0,\sigma]}$ in a domain Ω_0 and let $\Omega_t = \Omega_0 \setminus K_t$. Now construct a field $\widetilde{\Phi}$ by first sampling the final random hull K_σ, and then on the remaining random domain Ω_σ sampling an independent free field with the law of Φ_{Ω_σ}. Does the law of $\widetilde{\Phi}$ coincide with the law of Φ_{Ω_0}, at least on a subset of Ω_0 that is almost surely untouched by K_σ?

A necessary condition for the field $\widetilde{\Phi}$ to have the same law as Φ_{Ω_0} is that the mean and covariance coincide, which can be written as

$$M_{\Omega_0}(z) \stackrel{?}{=} \mathsf{E}[M_{\Omega_\sigma}(z)] \tag{2.1}$$

$$C_{\Omega_0}(z_1, z_2) + M_{\Omega_0}(z_1)M_{\Omega_0}(z_2) \stackrel{?}{=} \mathsf{E}[C_{\Omega_\sigma}(z_1, z_2) + M_{\Omega_\sigma}(z_1)M_{\Omega_\sigma}(z_2)] \tag{2.2}$$

for all z in the domain where $\widetilde{\Phi}$ is defined. The expected values here refer to averages over the random hull K_σ.

If we knew a priori that $\widetilde{\Phi}$ is Gaussian, then the conditions (2.1) and (2.2) would imply the desired coincidence of laws of $\widetilde{\Phi}$ and Φ_{Ω_0}, since the two Gaussian variables would have equal means and covariances. We will actually impose the following stronger conditions, from which the coincidence of laws will follow, as will be proven in Sect. 2.3. We require that

$$M_t(z) := M_{\Omega_t}(z) \text{ are uniformly bounded continuous martingales} \tag{M-cond}$$

$$\text{such that } \langle M(z_1), M(z_2)\rangle_t = C_{\Omega_0}(z_1, z_2) - C_{\Omega_t}(z_1, z_2). \tag{C-cond}$$

Here $\langle \cdot, \cdot \rangle_t$ denotes the quadratic cross variation—the second condition is therefore equivalent to $t \mapsto C_{\Omega_t}(z_1, z_2)$ being a process of finite variation such that $M_t(z_1)M_t(z_2) + C_{\Omega_t}(z_1, z_2)$ is martingale. Note that by optional stopping theorem the above conditions guarantee (2.1) and (2.2). We remark here that this martingale condition itself is quite natural: if the coupling is valid for all t, then $M_t(z)$ and $M_t(z_1)M_t(z_2) + C_{\Omega_t}(z_1, z_2)$ may be viewed as conditional expectations given the initial segment of the curve, and as such they are martingales with respect to this curve.

In practice verifying the two basic conditions becomes rather explicit. We mostly deal with strictly conformally invariant boundary conditions in the following sense. Consider simply connected domains Ω with $n+1$ marked points $x, x_1, x_2, \ldots, x_n \in \partial\Omega$, and associate with them harmonic functions $M_{(\Omega;x,x_1,\ldots,x_n)}$ defined on Ω and Green's functions $C_{(\Omega;x_1,\ldots,x_n)}$ (we assume the Green's function not to depend on the marked point x). Suppose these are chosen so that for any conformal map $\phi : \Omega \to \Omega'$ sending $x, x_1, \ldots, x_n$ to $x', x_1', \ldots, x_n'$ we have

$$\begin{aligned} M_{\Omega;x,x_1,\ldots,x_n}(z) &= M_{\Omega';x',x_1',\ldots,x_n'}(\phi(z)) \text{ and} \\ C_{\Omega;x_1,\ldots,x_n}(z_1, z_2) &= C_{\Omega';x_1',\ldots,x_n'}(\phi(z_1), \phi(z_2)). \end{aligned} \tag{conf.inv.}$$

In particular, taking $\phi = g_t$, the conditions (M-cond) and (C-cond) require the processes

$$M_{\Omega_0;X_t,g_t(x_1),\ldots,g_t(x_n)}(g_t(z)) \tag{2.3}$$

$$M_{\Omega_0;X_t,\ldots,g_t(x_n)}(g_t(z_1))\, M_{\Omega_0;X_t,\ldots,g_t(x_n)}(g_t(z_2)) + C_{\Omega_0;g_t(x_1),\ldots,g_t(x_n)}(g_t(z_1), g_t(z_2)) \tag{2.4}$$

to be martingales. Since $(X_t)_{t\in[0,\sigma]}$ is a semimartingale and the flow $(g_t)_{t\in[0,\sigma]}$ is governed by Eq. (Loe), computing the Itô derivatives of the two processes is now easy.

Write first of all the Itô diffusion of the driving process as

$$\mathrm{d}X_t = \mathrm{d}W_{\kappa t} + \tau_{X_t} D_t \,\mathrm{d}t \tag{2.5}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $\partial\Omega_0$ and τ_x are positively oriented unit tangents to $\partial\Omega_0$ at x. Then write $M_{\Omega_0;x,x_1,\ldots,x_n}(z)$ as the imaginary part of an analytic function $F(z; x, x_1, \ldots, x_n)$ on Ω_0, which we assume to depend smoothly also on the marked points $x, x_1, \ldots, x_n \in \partial\Omega_0$. The Itô derivative of (2.3) can be read from the imaginary part of

$$\begin{aligned} &\mathrm{d}F(g_t(z); X_t, g_t(x_1), \ldots, g_t(x_n)) = (\sqrt{\kappa}\partial_x F)\,\mathrm{d}B_t \\ &\quad + \left\{ V_{X_t}(g_t(z))\partial_z F + \frac{\kappa}{2}\partial_{xx} F + D_t \partial_x F + \sum_{j=1}^{n} V_{X_t}(g_t(x_j)) \frac{1}{\tau_{g_t(x_j)}} \partial_{x_j} F \right\} \mathrm{d}t, \end{aligned} \tag{2.6}$$

the right-hand side being evaluated at $(g_t(z); X_t, g_t(x_1), \ldots, g_t(x_n))$.

Remark 1 Note that in the above formula and in what follows, x and x_i are points on the boundary and the derivatives ∂_x and ∂_{xx} should be understood as first and second derivatives with respect to length parameter on the boundary (in the direction of the unit tangent τ). We do not assume analyticity with respect to those points. Also, B_t is a standard Brownian motion on $\mathbb{R}$ such that $\sqrt{\kappa} B_t$ is the length parameter of $W_{\kappa t}$.

In view of Eq. (2.6), the condition of the mean (2.3) being a local martingale is

$$\Im\mathrm{m} \left\{ V_x(z)\partial_z F + \frac{\kappa}{2}\partial_{xx} F + D_t \partial_x F + \sum_{j=1}^{n} V_x(x_j) \frac{1}{\tau_{x_j}} \partial_{x_j} F \right\} = 0. \tag{M-cond'}$$

If this equation is satisfied, then the drift of $M_t(z)$ vanishes and we have

$$\mathrm{d}M_t(z) = \sqrt{\kappa}\,\Im\mathrm{m}\,(\partial_x F(g_t(z); X_t, \ldots, g_t(x_n)))\,\mathrm{d}B_t,$$

and the Itô derivative of (2.4) significantly simplifies due to the following:

$$\mathrm{d}(M_t(z_1)\,M_t(z_2)) = (\cdots)\,\mathrm{d}B_t + \kappa\;\Im\mathrm{m}\,(\partial_x F(g_t(z_1); \ldots))\;\Im\mathrm{m}\,(\partial_x F(g_t(z_2); \ldots))\;\mathrm{d}t.$$

The condition (C-cond) thus reduces to the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{\Omega_t; x_1,\ldots,x_n}(z_1, z_2) \qquad \text{(C-cond')}$$
$$= -\kappa\;\Im\mathrm{m}\,(\partial_x F(g_t(z_1); X_t, \ldots, g_t(x_n)))\;\Im\mathrm{m}\,(\partial_x F(g_t(z_2); X_t, \ldots, g_t(x_n)))\,.$$

In the strictly conformally covariant cases (conf.inv.), the verification of the two basic conditions (M-cond) and (C-cond) therefore boils down simply to Eqs. (M-cond') and (C-cond') as well as appropriate boundedness of M.

2.2 Example: chordal SLEs and Gaussian free fields

2.2.1 Basic conditions for chordal SLE$_4$ and GFF with jump-Dirichlet boundary conditions

We will now illustrate the general idea in the example (1.1) of Schramm and Sheffield, by checking the conditions (M-cond) and (C-cond) in this simplest case.

We take the upper half-plane as the starting domain $\Omega_0 = \mathbb{H}$, and we have two marked points 0 and ∞. Our Loewner chain (Loe) constructed by the vector fields $V_x(z) = \frac{2}{z-x}$ preserves one of the marked points (infinity), and the other corresponds to the tip of the curve. The driving process of chordal SLE$_4$ is $X_t = \sqrt{\kappa} B_t$ with $\kappa = 4$. Concretely, Eqs. (M-cond') and (C-cond') now have the following simple form:

$$\Im\mathrm{m}\left(\frac{\kappa}{2}\partial_{xx} F + \frac{2}{z-x}\,\partial_z F\right) = 0 \quad \text{and} \tag{2.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{\Omega_t}(z_1, z_2) = -\kappa\;\Im\mathrm{m}\,(\partial_x F(g_t(z_1); X_t))\;\Im\mathrm{m}\,(\partial_x F(g_t(z_2); X_t)). \tag{2.8}$$

The harmonic function M in $\mathbb{H}$ determined by boundary conditions (1.1) is $\frac{2\lambda}{\pi} \arg(z - x) - \lambda$; hence, $F(z; x) = \frac{2\lambda}{\pi} \log(z - x) - \lambda$ and an easy calculation confirms the validity of Eq. (2.7) when $\kappa = 4$. The Dirichlet Green's function in the half-plane is explicitly

$$C(z_1, z_2) = -\frac{1}{2\pi}\;\Re\mathrm{e}\;\log\left(\frac{z_1 - z_2}{z_1 - \bar{z}_2}\right). \tag{2.9}$$

Applying conformal invariance of C, (2.4), and computing the time derivative of $C(g_t(z_1), g_t(z_2))$, we find that (2.8) also holds true provided that the jump size is adjusted to the value found by Schramm and Sheffield, $\lambda = \pm\sqrt{\frac{\pi}{8}}$.

Remark 2 The above calculation is rather rigid in the following sense: Suppose we want to find some coupling of SLE with a free field whose covariance is given by the Dirichlet Green's function (2.9). Then the equation (2.8) in fact determines the function F up to a sign and an additive constant. One then only has to check that such F is a martingale for the SLE, i.e. that Eq. (2.7) is satisfied.

Remark 3 The right-hand side of (2.8) is $-\frac{16\lambda^2}{\pi^2}\,\Im\mathrm{m}\,(\frac{1}{z_1-x})\,\Im\mathrm{m}\,(\frac{1}{z_2-x})$. Terms in this product have invariant meaning. Namely, they are multiples of Poisson's kernel with zero Dirichlet boundary conditions. This is a general phenomenon and a consequence of a formula of Hadamard type which we discuss in Sect. 2.4.

2.2.2 Modification to chordal SLE$_\kappa$ for $\kappa \neq 4$

There is a way to save the validity of the basic conditions for chordal SLE$_\kappa$, $\kappa \neq 4$, if one relaxes the assumption (conf.inv.) of strict conformal invariance of M. By Remark 2 the choice of the Dirichlet Green's function together with Eq. (2.8) implies we should take the same F as before but with $\lambda = \lambda_\kappa = \pm\sqrt{\frac{\pi}{2\kappa}}$. However, Eq. (2.7) fails for general κ and we have instead

$$\Im\mathrm{m}\left(\frac{\kappa}{2}\partial_{xx}F + \frac{2}{z-x}\,\partial_z F\right) = \frac{(4-\kappa)\lambda_\kappa}{\pi}\,\Im\mathrm{m}\left(\frac{1}{(z-x)^2}\right) \neq 0.$$

We therefore adjust the definition of the mean $M_{\Omega_t}(z)$ for the field in the new domain Ω_t as follows:

$$M_{\Omega_t}(z) = \Im\mathrm{m}\,(F(g_t(z); X_t) + E_t(z)), \tag{2.10}$$

where the extra term $E_t(z)$ is taken to be the integral of the missing part

$$\frac{\mathrm{d}}{\mathrm{d}t}E_t(z) = \frac{(\kappa-4)\lambda_\kappa}{\pi}\,\frac{1}{(g_t(z)-X_t)^2}, \quad E_0(z) = 0. \tag{2.11}$$

This guarantees that $M_{\Omega_t}(z)$ are local martingales. Condition (M-cond) follows for appropriate stopping times σ, and since the added term $E_t(z)$ is of finite variation, the computation leading to Eq. (2.8) remains unchanged and implies (C-cond).

The definition (2.11) can be explicitly integrated to give

$$E_t(z) = \frac{(4-\kappa)\lambda_\kappa}{2\pi}\,\log g_t'(z), \tag{2.12}$$

simply using $\frac{\mathrm{d}}{\mathrm{d}t}g_t'(z) = \frac{-2g_t'(z)}{(g_t(z)-X_t)^2}$. In particular, $\Im\mathrm{m}\,(E_t(z))$ is determined by the domain Ω_t only and could be interpreted as a multiple of the harmonic interpolation of the argument of the tangent vector τ of $\partial\Omega_t$ ("winding of the boundary") if the boundary would be smooth. In Appendix A we show that for general boundary conditions the mean M_{Ω_t} defined as in (2.10) can depend on the full history of the Loewner chain $(g_s)_{0\leq s\leq t}$ and not be determined by the domain Ω_t only.

We remark also that the additional term (2.12) is what the Coulomb gas formalism of conformal field theory dictates in the presence of a background charge which modifies the central charge c to its correct value $c(\kappa) = 1 - 6\,(\frac{\kappa-4}{2\sqrt{\kappa}})^2$.

2.3 Basic equations imply coupling

2.3.1 Definition of the free fields

Let us now give a precise definition of our free fields Φ. It is common to define them as random tempered distributions, although they are almost surely somewhat more regular objects. We denote by $\mathcal{S}$ the Schwarz class of functions of rapid decrease on $\mathbb{C} = \mathbb{R}^2$ and by $\mathcal{S}'$ the tempered distributions. Define the function $W : \mathcal{S} \to \mathbb{C}$ which will be the characteristic function of Φ

$$W(f) = \exp\left(\mathrm{i}\int_{\Omega} M(z)f(z)\,\mathrm{d}z - \frac{1}{2}\iint_{\Omega\times\Omega} f(z)C(z,w)f(w)\,\mathrm{d}z\,\mathrm{d}w\right).$$

We clearly have $W(0) = 1$. All of our choices of functions M and C will satisfy the properties

- The function $M : \Omega \to \mathbb{R}$ is locally integrable and has at most polynomial growth at infinity
- The function C is locally integrable and has at most polynomial growth at infinity

which imply that W is continuous. Furthermore, all of our choices of C will have the property

- For all $f_1, \ldots, f_n \in \mathcal{S}$ the $n \times n$ real matrix with entries $C_{j,k} = \iint f_j(z)C(z,w)\,f_k(w)\,\mathrm{d}z\,\mathrm{d}w$ is positive semi-definite,

so a standard argument shows that for all $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and $f_1, \ldots, f_n \in \mathcal{S}$ we have $\sum_{j,k} \zeta_j \overline{\zeta}_k W(f_j - f_k) \geq 0$. These conditions guarantee, by Minlos' theorem, that W is indeed a characteristic function of a probability measure on $\mathcal{S}'$, which is our definition of the law of the massless free field with mean M and covariance C.

It is evident from the definition of W that the free field is almost surely supported on $\overline{\Omega}$.

2.3.2 Coupling

Theorem 2 *Let $A \subset \Omega$ be compact and $B \subset \Omega$ an open neighborhood of A. Let $(g_t)_{t\geq 0}$ be a random Loewner chain of hulls $(K_t)_{t\geq 0}$ and suppose that σ is a stopping time for which $\overline{K}_\sigma \cap (B \cup \{x_1, \ldots, x_n\}) = \emptyset$. Denote the tip of the hull at time t by $\tilde{x}(t)$*

and the complement by $\Omega_t = \Omega \setminus K_t$. *Assume conditions (M-mgale) and (C-mgale). Then the random Loewner chain* $(g_t)_{t\in[0,\sigma]}$ *can be coupled with a free field* $\widetilde{\Phi}$ *defined on A such that the following holds:*

- *Let* $\sigma' \leq \sigma$ *be a stopping time. Conditionally on* $(g_s)_{0\leq s\leq\sigma'}$*, the law of* $\widetilde{\Phi}$ *is the restriction to A of the free field corresponding to the domain* $\Omega_{\sigma'}$*, that is the free field with mean* $M_{(\Omega_{\sigma'},\tilde{x}(\sigma'),x_1,\ldots,x_n)}$ *and covariance* $C_{(\Omega_{\sigma'},\tilde{x}(\sigma'),x_1,\ldots,x_n)}$.

The proof is closely parallel to the proof of what is called the local invariance of the free field under SLE dynamics in [6].

Proof The theorem will be proved by showing that for any test function f the expectation $\mathsf{E}_{\text{GFF}}[\exp(\mathrm{i}\,\langle\widetilde{\Phi}_t, f\rangle)]$ is a martingale, where $\widetilde{\Phi}_t$ has the law of the free field in $\Omega_t = \Omega \setminus K_t$.

Denote by M_t the mean associated with the domain Ω_t with marked points $\tilde{x}(t), x_1, \ldots, x_n$, and by C_t the covariance associated to that domain. Define $\widetilde{\Phi}$ by sampling a free field in Ω_σ with mean and covariance M_σ and C_σ, and then restricting to A.

Given $f \in \mathcal{S}$, $\mathrm{supp}(f) \subset A$, we define first of all the process

$$L_t = \int_A M_t(z) f(z)\,\mathrm{d}z.$$

By the assumption (M-mgale) $(L_t)_{t\in[0,\sigma]}$ is a bounded continuous martingale. Its quadratic variation follows from assumption (C-mgale)

$$\langle L, L\rangle_t = \iint_{A\times A} f(z)(C_0(z,w) - C_t(z,w)) f(w)\,\mathrm{d}z\,\mathrm{d}w.$$

For any $f \in \mathcal{S}$ such that $\mathrm{supp}(f) \subset A$, define the random process $(\widetilde{W}_t(f))_{t\in[0,\sigma]}$ by

$$\widetilde{W}_t(f) = \exp\left(\mathrm{i}\int_A M_t(z) f(z)\mathrm{d}z - \frac{1}{2}\iint_{A\times A} f(z) C_t(z,w) f(w)\,\mathrm{d}z\,\mathrm{d}w\right).$$

Note that $\widetilde{W}_t(f)$ is up to a multiplicative constant the exponential martingale $\exp(\mathrm{i}\,L_t + \frac{1}{2}\langle L, L\rangle_t)$, so in particular it is a bounded martingale.

It is now easy to describe the law of the random distribution $\widetilde{\Phi}$ conditionally on $(g_s)_{0\leq s\leq t}$. The law is encoded in the characteristic function $\mathsf{E}[\exp(\mathrm{i}\,\langle\widetilde{\Phi}, f\rangle) \,|\, \mathcal{F}_t]$. At time $t = \sigma$ this is exactly the characteristic function of $\widetilde{\Phi}$, that is $\widetilde{W}_\sigma(f)$. By construction it is a bounded martingale and therefore coincides with

$$\mathsf{E}[\exp(\mathrm{i}\,\langle\widetilde{\Phi}, f\rangle) \,|\, \mathcal{F}_t] = \widetilde{W}_t(f).$$

Since $\widetilde{W}_t$ is by construction the characteristic function of the free field with mean and covariance M_t and C_t the assertion follows. □

2.4 Hadamard's variational formulas for Loewner chains

Hadamard's formula gives the variation of the Green's function in a smooth domain when the boundary evolves in a smooth way. In this section we prove a version of Hadamard's formula for Loewner chains. The need for this stems from the second basic condition for coupling—in Eq. (C-cond') we need the derivative of Green's functions in domains Ω_t with respect to the time t of the Loewner chain.

Theorem 3 *Let $(g_t)_{t\geq 0}$ be a Loewner chain in a simply or doubly connected domain as in Sect. 1.1, and let $\Omega_t = \Omega_0 \setminus K_t$. Let $G_{\Omega_t}(z_1, z_2)$ be the Green's function in Ω_t with zero-Dirichlet boundary values. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} G_{\Omega_t}(z_1, z_2)\big|_{t=0} = -2\pi \; P_{\Omega_0}(X_0, z_1) \; P_{\Omega_0}(X_0, z_2), \tag{2.13}$$

where P_Ω is the Poisson kernel in Ω.

Recall that the Poisson kernel $P_\Omega(\cdot, z)$ can be defined either as the density of the harmonic measure in Ω seen from $z \in \Omega$ with respect to the length measure of the boundary $\partial\Omega$, or as the inwards normal derivative of the Green's function.

Proof Fix the point z_2. The difference $\Gamma_{z_2}(z_1) := G_{\Omega_0}(z_1, z_2) - G_{\Omega_t}(z_1, z_2)$ is harmonic in Ω_t as function of z_1, since the singularities at z_2 cancel, and thus it can be represented as the integral of its boundary values against the harmonic measure:

$$\Gamma_{z_2}(z_1) = \int\limits_{\partial\Omega_t} G_{\Omega_0}(z, z_2)\, \mathrm{d}\omega_{z_1}^{\Omega_t}(z) = \int\limits_{\partial K_t} G_{\Omega_0}(z, z_2)\, \mathrm{d}\omega_{z_1}^{\Omega_t}(z), \tag{2.14}$$

where the second equality follows since the boundary values are zero everywhere but on ∂K_t.

By conformal invariance we may assume that $X_0 = 0$, $\Omega_0 \subset \mathbb{H}$ and Ω_0 coincides with the upper half-plane $\mathbb{H}$ in some neighborhood of $X_0 = 0$. If $x + iy = z \in \partial K_t$, then

$$G_{\Omega_0}(z, z_2) = G_{\Omega_0}(x, z_2) + y\; \partial_n G_{\Omega_0}(x, z_2) + o(y), \quad y \to 0.$$

The first term on the right-hand side is equal to 0, and the normal derivative of the Green's function is the Poisson kernel $P_{\Omega_0}(x; z_2)$, which is roughly the same as $P_{\Omega_0}(0; z_2)$. More precisely, one has

$$G_{\Omega_0}(z, z_2) = y\; P_{\Omega_0}(0; z_2) + y\; O(x) + o(y).$$

Hence, (2.14) reads

$$\Gamma_{z_2}(z_1) \approx P_{\Omega_0}(0; z_2) \int\limits_{\partial K_t} \Im\mathrm{m}\,(z)\, \mathrm{d}\omega_{z_1}^{\Omega_t}(z) =: P_{\Omega_0}(0, z_2)\; \Psi(z_1).$$

The notation "≈" means that the ratio of two expressions tends to 1 as the size of the hull tends to zero. Now, take a small $r > 0$ such that K_t is inside the semicircle $T_r(0)$ of radius r around 0. Denote $\Omega^{(r)} := \Omega_0 \setminus B_r(0)$. Write $\Psi(z_1)$ as

$$\Psi(z_1) = \int\limits_{T_r(0)} \Psi(z)\, \mathrm{d}\omega_{z_1}^{\Omega^{(r)}}(z). \tag{2.15}$$

We are going to factor out a term that captures the dependence of the latter integral on z_1. To this end, we apply the map $\psi_r(z) := z + \frac{r^2}{z}$ which maps $\mathbb{H} \setminus B_r(0)$ onto $\mathbb{H}$ (and Ω_0 onto some domain $\psi(\Omega_0)$). Now, conformal invariance of the harmonic measure yields

$$\mathrm{d}\omega_{z_1}^{\Omega^{(r)}}(z) = \mathrm{d}\omega_{\psi_r(z_1)}^{\psi_r(\Omega^{(r)})}(\psi_r(z)) = P_{\psi_r(\Omega)}(\psi_r(z); \psi_r(z_1))\, \mathrm{d}x.$$

Since $\psi_r(z) - z$ is small when r is small, we have

$$P_{\psi_r(\Omega)}(\psi_r(z); \psi_r(z_1)) \approx P_{\psi_r(\Omega)}(0; \psi_r(z_1)) \approx P_\Omega(0; z_1). \tag{2.16}$$

Hence Eq. (2.15) reads

$$\Psi(z_1) \approx P_\Omega(0, z_1) \int\limits_{\theta=0}^{\pi} \Psi(r\mathrm{e}^{\mathrm{i}\theta}) 2 \sin(\theta) r\, \mathrm{d}\theta.$$

The integral on the right-hand side is by definition equal to $\pi\, L^\Omega_{K_t, r}$, where $L^\Omega_{K_t, r}$ is the local half-plane capacity, see (B.1), and we apply Proposition 1 to finish the proof. □

It is easy to generalize this theorem to other boundary conditions. One possible generalization is as follows. Let the boundary of the domain $\Omega = \Omega_0$ consist of several connected components, that in turn are divided into several arcs each. Without loss of generality, assume $\partial\Omega$ to be piecewise smooth, and let $\tilde{G}_\Omega(z_1, z_2))$ be the Green's function with zero Dirichlet boundary conditions on some of those arcs and Neumann boundary conditions on others. Let (Ω_t) be a family of domains defined by a Loewner chain (the setup for the chain being analogous to that of Sect. 1.1, with residue of absolute value 2 at the marked point). We demand that the point of growth $X_0 \in \partial\Omega$ of the Loewner chain would belong to the "Dirichlet" part of the boundary, and by definition $\tilde{G}_{\Omega_t}(z_1, z_2))$ assumes zero Dirichlet boundary values on K_t. Then we have the following proposition:

Proposition 4

$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{G}_{\Omega_t}(z_1, z_2))\big|_{t=0} = -2\pi\ \tilde{P}_\Omega(X_0; z_1)\, \tilde{P}_\Omega(X_0; z_2). \tag{2.17}$$

where $\tilde{P}_\Omega$ is the Poisson kernel with the same boundary conditions as $\tilde{G}$.

Proof The proof literally repeats the one of Theorem 3; there are only two places where we have used a specific nature of the boundary conditions far away from the point X_0. One is the continuity or the Poisson kernel with respect to small variations of the domain [Eq. (2.16)]. This is also clear in the present case. Another one is the definition of $\mathrm{lhcap}(K_t)$, namely, the boundary conditions for Ψ in (B.1). It is clear, however, that they can be replaced by Neumann ones far from the point X_0, and the difference at the distance r from a is of order $o(r^2)$, which is negligible when computing $\partial_t \mathrm{lhcap}(K_t)|_{t=0}$. □

3 Various boundary conditions in simply connected domains

3.1 SLE$_4$ in the strip and Riemann–Hilbert boundary conditions

In this subsection, we develop a coupling of SLE$_4$ and GFF in the following situation. We take Ω to be a simply connected domain with three marked points x_0, x_1, x_2 on the boundary, dividing the boundary into three arcs l_{12}, l_{01}, l_{20}. The mean $M(z) = M_{\Omega;x_0,x_1,x_2}(z)$ of the field will be a harmonic function determined by the boundary conditions

$$\begin{cases} M(z) = -\lambda & \text{for } z \in l_{01} \\ M(z) = \lambda & \text{for } z \in l_{20} \\ \alpha\, \partial_n M(z) + \beta\, \partial_\tau M(z) = 0 & \text{for } z \in l_{12}. \end{cases} \tag{3.1}$$

Thanks to Cauchy–Riemann equations, the third condition can be reformulated in the following way: if $M(z) = \Im\mathrm{m}\, F(z)$, then on l_{12} the derivative of F in the direction of the boundary has a constant argument modulo π. If at some point of the arc l_{12} the function F vanishes, this implies that F itself has the same argument (modulo π) on l_{12}. As the covariance $C(z_1, z_2) = C_{\Omega;x_1,x_2}(z_1, z_2)$ we take the Green's function in Ω having zero Dirichlet boundary conditions on l_{20} and l_{01} and the above type of Riemann–Hilbert boundary conditions on l_{12}: for all $z_2 \in \Omega$ we require

$$\begin{cases} C(\cdot, z_2) = 0 & \text{on } l_{20} \cup l_{01} \\ \alpha\, \partial_n C(\cdot, z_2) + \beta\, \partial_\tau C(\cdot, z_2) \equiv 0 & \text{on } l_{12}. \end{cases} \tag{3.2}$$

These boundary conditions are conformally invariant in the sense of Eq. (conf.inv.), so the essential part of establishing a coupling consists of verifying Eqs. (M-cond') and (C-cond'). We already remark that Dirichlet and Neumann boundary conditions on l_{12} are two particular cases corresponding to vanishing α and vanishing β, respectively. After we have made the coupling explicit, we return to comment on an interpolation between the two.

The convenient choice of Loewner chain for the domains $(\Omega; x_0, x_1, x_2)$ with three marked boundary points is to keep x_1 and x_2 as fixed points. We therefore take the initial domain to be the strip, $\Omega_0 = \mathbb{S}$, with x_1 and x_2 at $+\infty$ and $-\infty$, respectively, and we use (Loe) to encode the growth process. We furthermore choose $X_0 = x_0 = 0$. Below $F(\cdot; x)$ denotes an analytic function in $\mathbb{S}$ whose imaginary part is the harmonic function $M_{\mathbb{S};x,+\infty,-\infty}$ determined by (3.1).

As the marked points x_1, x_2 are chosen to be fixed by the Loewner flow, the basic equations (M-cond') and (C-cond') have a simple form

$$\Im\mathrm{m}\left\{2\,\partial_{xx}F(z;x)+\coth\left(\frac{z-x}{2}\right)\,\partial_z F(z;x)+D_t\,\partial_x F(z;x)\right\}=0\quad\text{and}\tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{\Omega_t}(z_1,z_2)=-4\,\Im\mathrm{m}\,(\partial_x F(g_t(z_1);X_t))\,\Im\mathrm{m}\,(\partial_x F(g_t(z_2);X_t))\,.\tag{3.4}$$

Proposition 4 combined with conformal invariance readily gives the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{\Omega_t}(z_1,z_2)=-2\pi\;\tilde{P}(X_t;g_t(z_1))\;\tilde{P}(X_t;g_t(z_2)),$$

where $\tilde{P}$ is the Poisson kernel in $\mathbb{S}$ having the same boundary conditions as the Green's function (3.2). As before, Eq. (3.4) therefore determines $\partial_x F(z;x)$ up to a sign and a constant

$$\partial_x F(z,x)=\pm\mathrm{i}\sqrt{\frac{\pi}{2}}\;\tilde{S}_x(z)+\text{real constant},\tag{3.5}$$

where $\tilde{S}_x(z)$ is the Schwarz kernel corresponding to the present boundary conditions—an analytic function in $\mathbb{S}$ such that $\Re\mathrm{e}\,(\tilde{S}_x(z))=\tilde{P}_x(z)$.

We should then verify (3.3). Note first that our function F is invariant under shifts

$$\partial_x F+\partial_z F=0,$$

and hence (3.3) reads equivalently

$$\Im\mathrm{m}\left\{2\,\partial_{xx}F(z;x)-\coth\left(\frac{z-x}{2}\right)\,\partial_x F(z;x)+D_t\,\partial_x F(z;x)\right\}=0.\tag{3.6}$$

This identity could be checked for correctly chosen D_t by a direct calculation using an explicit expression for $\tilde{S}$, but we prefer an argument which identifies the drift D_t in a way that generalizes directly to other cases where explicit expressions may in practise be unavailable. A similar technique was used by Zhan in the context of loop-erased random walks in multiply connected domains [21].

The function on the left-hand side of (3.6) is harmonic in $\mathbb{S}$; it is zero on $\mathbb{R}$ and bounded apart from a possible singularity at x. On the upper part of the boundary, the first and third terms clearly satisfy the (α,β) Riemann–Hilbert boundary condition. In order to prove the same condition for the second term, recall that $\partial_x F$ was defined up to a real constant. If we now choose that constant so that $\Re\mathrm{e}\,(\partial_x F)=0$ at $-\infty$, then clearly $\partial_x F=0$ at $-\infty$, and the Riemann–Hilbert boundary condition for $\Im\mathrm{m}\,(\partial_x F)$ can be stated in the form that $\arg\partial_x F$ modulo π is fixed on $\mathbb{R}+\mathrm{i}\pi$. Since $\coth(\frac{z-x}{2})$ is purely real, multiplication by it does not harm this condition.

It remains to prove that singularities of the left-hand side of (3.6) at the point x actually cancel out. Expansions at x for the Schwarz kernel and the Loewner vector field give

$$\partial_x F(z;x) = \frac{C}{z-x} + C\,\mu + o(1),$$
$$\coth\left(\frac{z-x}{2}\right) = \frac{2}{z-x} + o(1),$$

where C and μ are real since the Schwarz kernel $\tilde{S}_x(z)$ is purely imaginary on the real line. Hence, the left-hand side of (3.6) is bounded if and only if

$$D_t \equiv 2\mu,$$

which determines the drift D_t of the driving process (2.5) and establishes the condition (M-cond') for the correctly chosen drift.

In order to find μ in terms of α and β, we need the explicit formula for the function $\partial_x F$. Note that for $-\frac{1}{2} < \theta < \frac{1}{2}$ the expression

$$\tilde{S}_x(z) = \frac{\mathrm{i}}{2\pi} \frac{\mathrm{e}^{\theta(z-x)}}{\sinh(\frac{z-x}{2})} \tag{3.7}$$

gives a Schwarz kernel in $\mathbb{S}$ satisfying

$$\arg \partial_\tau \tilde{S} = \pi\theta \quad \mod \pi \quad \text{on } \mathbb{R} + \mathrm{i}\pi.$$

so we find that for such boundary conditions $\mu = \theta$. We have proven the following proposition:

Proposition 5 *Choose* $\lambda = \sqrt{\pi/8}$ *and*

$$\alpha = \cos(\pi\theta), \quad \beta = -\sin(\pi\theta)$$

and let Φ *be the GFFs with means* $M_{\Omega;x_0,x_1,x_2}(z)$ *determined by boundary conditions* (3.1)*, and covariances* $C_{\Omega;x_1,x_2}$ *determined by* (3.2)*. Then* Φ *are coupled, in the sense of Theorem* 2*, with the* $SLE_4(\rho)$ *in* $\mathbb{S}$ *with* $\rho = 2\theta - 1$.

Remark 4 Free fields and SLEs are conformally invariant if we allow for (random) time reparametrizations of the Loewner chains, so the given coupling works in any other domain $(\Omega; x_0, x_1, x_2)$, too.

Remark 5 Both cases $\theta \to \pm\frac{1}{2}$ correspond to Dirichlet boundary conditions also on $l_{12} = \mathbb{R} + \mathrm{i}\pi$. Correspondingly, the curves become just chordal SLE_4 in the strip from 0 to $\pm\infty$, and these cases can be seen as mere coordinate changes of the case of Schramm and Sheffield discussed in Sect. 2.2.

Remark 6 The symmetric value $\theta = 0$ corresponds to Neumann boundary conditions on $\mathbb{R} + \mathrm{i}\pi$. The drift D_t then vanishes and the curve is a dipolar SLE_4. It appears that this case was first conjectured in [4].

Remark 7 As θ varies from $-\frac{1}{2}$ to $\frac{1}{2}$, the free fields and the curves interpolate between the above cases. This was suggested in [9], where $\tilde{S}_x(z)$ was also used to give a formula for left passage probability of the $\mathrm{SLE}_4(\rho)$ curve.

One would like, as in the chordal case, to extend the coupling to $\kappa \neq 4$. Again, Eq. (3.4) and Hadamard's formula for C leave us essentially no choice but $\partial_x F(z; x) = 2\mathrm{i}\,\lambda_\kappa\,\tilde{S}_x(z)$ with $\lambda_\kappa = \sqrt{\frac{\pi}{2\kappa}}$. Eq. (3.3) then fails, giving instead

$$\Im\mathrm{m}\left\{\frac{\kappa}{2}\,\partial_{xx}F(z;x) + \coth\left(\frac{z-x}{2}\right)\,\partial_z F(z;x) + D_t\,\partial_x F(z;x)\right\}$$
$$= (\kappa - 4)\lambda_\kappa\,\Im\mathrm{m}\,(\mathrm{i}\,\partial_x \tilde{S}_x(z)) \neq 0$$

As in (2.10), we could try to save the basic conditions by adding a non-conformally invariant term E_t to the mean of the field: $M_{\Omega_t}(z) = \Im\mathrm{m}\,\big(F(g_t(z); X_t) + E_t(z)\big)$, now taken to be

$$E_t(z) = (4-\kappa)\lambda_\kappa \int_0^t (\mathrm{i}\,\partial_x \tilde{S}_{X_s}(g_s(z)))\,\mathrm{d}s. \tag{3.8}$$

One observes that E_t, thus defined, satisfies the following properties:

- $\Im\mathrm{m}\,(E_t)$ has the same Riemann–Hilbert boundary conditions as $\Im\mathrm{m}\,(F)$ on $\mathbb{R} + \mathrm{i}\pi$
- $\Im\mathrm{m}\,(E_t) \equiv 0$ on $\partial\Omega_t \cap \mathbb{R}$
- If $z \in \partial K_t$ for some t, then $\Im\mathrm{m}\,(E_s(z)) = \Im\mathrm{m}\,(E_t(z))$ for all $s > t$ unless the point z is swallowed by time s. Thus, the boundary value of $\Im\mathrm{m}\,E$ on the curve is determined at the instant the point becomes a part of the boundary. Note that this property also held for the winding boundary conditions (2.12) which generalized the chordal coupling to $\kappa \neq 4$.

Despite the above properties, there is a crucial difference to the case of jump-Dirichlet boundary conditions: the mean (2.10) will be determined by the domain only if the commutation condition of Appendix A is satisfied—and for Eq. (3.8) it is not.

3.2 More marked points

In this section, we show how to compute the driving process of the SLE_4 variant coupled with free field whose boundary conditions change also at additional marked points $-\infty + \mathrm{i}\,\pi = x_0, x_1, x_2, \ldots, x_{n+1} = \infty + \mathrm{i}\,\pi$ on the upper boundary of $\mathbb{S}$. In our example, the mean of the field will satisfy the following boundary conditions:

$$\begin{cases} M(z, x, x_1, \ldots, x_n) = -\lambda & \text{for } z \in (x, +\infty) \\ M(z, x, x_1, \ldots, x_n) = +\lambda & \text{for } z \in (-\infty, x) \\ M(z, x, x_1, \ldots, x_n) \text{ obeys } \mathrm{BC}_i & \text{for } z \in l_i := (x_i, x_{i+1}) \subset \mathbb{R} + \mathrm{i}\pi \end{cases} \tag{3.9}$$

- Here BC_i may stand either for constant Dirichlet condition $M \equiv \lambda_i$, or zero Neumann boundary condition $\partial_n M \equiv 0$.

The covariance $C(z_1, z_2; x, x_1, \ldots, x_n)$ is taken to have zero Dirichlet boundary conditions on $\mathbb{R}$, and BC_i' on l_i, where BC_i' stands for the homogeneous condition corresponding to BC_i. We have only given the mean and covariance in $(\mathbb{S}; x, -\infty, x_1, \ldots, x_n, +\infty)$, but it is understood that the definitions are transported to other domains with marked points by Eq. (conf.inv.).

The initial position of the growth is $X_0 = 0$. Let $\tilde{M}$ be the harmonic conjugate to M normalized to be equal to 0 at $-\infty$, and let $\tilde{S}_x(z)$ be the Schwarz kernel with BC_i' boundary conditions on the corresponding segments of the upper boundary and with the same normalization at $-\infty$.

We have the following proposition:

Proposition 6 *For $\lambda = \sqrt{\frac{\pi}{8}}$, there exists a unique function $D(x, x_1, x_2, \ldots, x_n)$ such that the SLE_4 variant defined by* (Loe) *with the driving process*

$$\mathrm{d}X_t = 2\,\mathrm{d}B_t + D(X_t, g_t(x_1), \ldots, g_t(x_n))\,\mathrm{d}t$$

is coupled with the GFF described above. The function $D(x, x_1, x_2, \ldots, x_n)$ is given by

$$D(x, x_1, \ldots, x_n) = 2\,\mu(x, x_1, \ldots, x_n) - 2\sum_{i=0}^{n} \partial_{x_i} \tilde{M}(x, x, x_1, \ldots, x_n), \quad (3.10)$$

where μ is the second coefficient in the expansion at $z = x$ of the Schwarz kernel $\tilde{S}_x(z)$

$$\mu := \frac{\pi}{\mathrm{i}} \lim_{z \to x} \left(\tilde{S}_x(z) - \frac{\mathrm{i}}{\pi(z - x)} \right).$$

Proof Hadamard's formula implies that Eq. (C-cond') will hold provided that when we write $M = \Im\mathrm{m}\,(F)$ the function F satisfies $\partial_x F = 2\mathrm{i}\lambda\,\tilde{S}$, where $\tilde{S}$ is the Schwarz kernel with corresponding boundary conditions. The first equation (M-cond') now reads

$$\Im\mathrm{m}\left\{2\,\partial_{xx} F + \coth\left(\frac{z-x}{2}\right)\,\partial_z F + \sum_i \coth\left(\frac{x_i - x}{2}\right)\,\partial_{x_i} F + D_t\,\partial_x F\right\} = 0. \quad (3.11)$$

Obviously, the function in the parentheses in (3.11) is purely real when $z \in \mathbb{R} \setminus \{x\}$. We show that it satisfies the homogeneous BC_i' boundary conditions on the upper part of the boundary and that for the appropriate choice of D_t the singularities at x cancel out. It is clear that if the function $\Im\mathrm{m}\,(F)$ satisfies Dirichlet or zero Neumann boundary conditions on $l_i = (x_i, x_{i+1}) \subset \mathbb{R} + \mathrm{i}\pi$, then $\Im\mathrm{m}\,(\partial_{xx} F)$, $\Im\mathrm{m}\,(\partial_{x_i} F)$ and $\Im\mathrm{m}\,(\partial_x F)$

satisfy corresponding homogeneous conditions. The function $\partial_z F$ is purely real where BC_i is Dirichlet and purely imaginary where BC_i is Neumann. Multiplication by real constant does not affect homogeneous Dirichlet or Neumann boundary conditions, and multiplication by the real function $\coth(\frac{z-x}{2})$ does not change the argument of $\partial_z F$ modulo π. So all terms in (3.11) satisfy BC_i' on l_i.

Our function F is invariant under simultaneous translation of all arguments, i.e.

$$\partial_z F = -\partial_x F - \sum_i \partial_{x_i} F.$$

So, we rewrite the equation (3.11) as

$$\Im\mathrm{m} \left\{ 2\,\partial_{xx} F - \coth\left(\frac{z-x}{2}\right)\,\partial_x F - \coth\left(\frac{z-x}{2}\right)\sum_i \partial_{x_i} F \right. \\ \left. + \sum_i \coth\left(\frac{x_i - x}{2}\right)\,\partial_{x_i} F + D_t\,\partial_x F \right\} = 0 \tag{3.12}$$

Note that $\partial_{x_i} F$ might have a singularity at x_i—however, it can only be of order $O((z-x_i)^{-1})$, so in the above expression these singularities cancel out, and the function is bounded near x_i's. It remains to handle the singularity at x. To do so, note that the expansion of $\partial_x F$ at x is

$$\partial_x F = \frac{C}{z-x} + C\,\mu + o(1),$$

where C is a real constant and μ is as specified in the statement. Hence the second-order singularities, which only come from the first two terms, cancel out. The first-order singularities come from the second, the third and the last terms in (3.12). Clearly, there is a unique choice of D_t, specified in the statement of the proposition, for which they also cancel out. □

Remark 8 If one wishes, one may allow some of the BC_i's be Riemann–Hilbert boundary conditions. The proof is similar to the above one, and we leave it to the reader. We do not focus on this case to avoid discussing existence and positivity of the Green's function with these boundary conditions and uniqueness of solutions to the boundary value problem.

Remark 9 A comparison of two simple particular cases of the Proposition leads to a curious observation. Take the entire upper boundary with homogeneous Dirichlet or homogeneous Neumann boundary conditions. In both cases the drift D_t vanishes. These two GFFs with mutually singular laws are therefore both coupled with dipolar SLE_4.

4 Couplings in doubly connected domains

In this section we address the question of couplings of SLE and GFF in doubly connected domains. We first consider punctured disc and exhibit a coupling of GFF with radial SLE and then consider annuli $\mathbb{A}_p$. The non-simply connectedness requires in many cases non-trivial monodromies of the free field—to obtain couplings with single-valued fields we need to compactify the field, that is consider free field with values on a circle. In physics literature considerations of lattice model height functions in multiply connected domains or in the presence of vortices, and considerations of operator algebra and modular invariance of conformal field theories have both led to the study of such compactified free fields. We also remark that the martingale property for harmonic functions similar to those appearing in this section was pointed out by Zhan [20].

Throughout this chapter, $x, x_1, \ldots$ denote points on the boundary of an annulus $\mathbb{A}_r$ for some $r > 0$, and the derivatives $\partial_x, \partial_{x_1}, \ldots$ will be taken in counterclockwise direction, both for inner and outer boundary.

4.1 Compactified GFF and radial SLE$_4$

We first investigate the solutions to the basic equations (M-cond') and (C-cond') in the radial case. We will see that the solution to these equations will not be a harmonic function in the disc, but rather a harmonic function with monodromy. This situation has also been considered in [6].

We use the Loewner chain (Loe) to describe the growth process in $\Omega_0 = \mathbb{D}$, and for radial SLE$_4$ we have the driving process $X_t = \exp(\mathrm{i}2B_t)$. So in the absence of other marked points but the tip of the growth, the basic equations (M-cond') and (C-cond') read

$$\Im\mathrm{m}\left\{2\,\partial_{xx}F + z\frac{x+z}{x-z}\,\partial_z F\right\} = 0 \quad \text{and} \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_{\Omega_t}(z_1, z_2) = -4\,\Im\mathrm{m}\,(\partial_x F(z_1; x))\,\Im\mathrm{m}\,(\partial_x F(z_2; x)). \tag{4.2}$$

As usually, with C_{Ω_t} the Dirichlet Green's function, Hadamard's formula suggests the solution to (4.2), with the ambiguity of a sign and an additive real constant. Namely, we have

$$\partial_x F(z; x) = \pm 2\mathrm{i}\lambda\; S_x(z) + \text{const.} = \pm\mathrm{i}\;\frac{\lambda}{\pi}\;\frac{x+z}{x-z} + \text{const.},$$

expressed in terms of the Schwarz kernel $S_x(z)$ in the unit disc. The sign of $\partial_x F$ is unimportant, but the constant will have to vanish. We warn the reader that here, for the first time, it is important not to confuse the derivatives ∂_x w.r.t. the length parameter on the boundary with derivatives w.r.t. the position of the marked point x.

The function F can be taken invariant under rotations, and we get

$$\mathrm{i}\,z\,\partial_z F + \partial_x F = 0. \tag{4.3}$$

Using this in Eq. (4.1), and integrating explicitly gives

$$F(z; x) = \frac{\lambda}{\pi}(2\log(x - z) - \log(z))$$

and $M(z; x) = \frac{\lambda}{\pi}(-\arg(z) + 2\arg(x - z))$. The function M is not single-valued. However, all the formulas we have used make sense: as soon as we fix the branch of $M(z)$, the branch of $M(g_t(z))$ will also be fixed by continuity. We can thus define a multi-valued harmonic function $M(z)$ in the punctured disc $\mathbb{D} \setminus \{0\}$, such that it has monodromy of $2\lambda = \sqrt{\frac{\pi}{2}}$ around zero, and the boundary conditions have a jump of 2λ at the point x, being otherwise locally constant. Adding this function to a zero Dirichlet boundary valued GFF in $\mathbb{D}$, one obtains a multi-valued GFF Φ of the same monodromy, which could be interpreted as a single-valued free field with values in $\mathbb{R}/2\lambda\mathbb{Z}$.

Remark 10 Theorem 2 is not directly formulated for multivalued free fields, but this problem is superficial. It is easy to see that the corresponding single-valued free field on the universal cover of $\mathbb{D} \setminus \{0\}$ (with periodic covariance, in particular) is coupled with the growth process obtained by lifting the radial SLE_4 to the universal cover. Here and in the sequel we nevertheless prefer to talk about either multivalued free field or free field with values on a circle $\mathbb{R}/2\lambda\mathbb{Z}$.

4.2 Compactified GFF and standard annulus SLE_4

A natural generalization of the radial SLE to annuli of finite modulus is the standard annulus SLE. We will now show that at $\kappa = 4$ it is coupled with a multivalued free field having Neumann boundary conditions on the inner boundary component of the annulus and jump-Dirichlet boundary conditions on the outer boundary component.

The starting domain is taken to be $\Omega_0 = \mathbb{A}_p$, and we use the Loewner chain (Loe) to describe the growth process. The conformal maps $g_t : \mathbb{A}_p \setminus K_t \to \mathbb{A}_{p-t}$ uniformize the complements of the hull to thinner annuli, so even with strict conformal invariance we have to specify the mean and the covariance for all annuli $\mathbb{A}_{p-t}$. We take the covariance $C_t = C_{\mathbb{A}_{p-t}}$ to be the Green's function with Dirichlet boundary conditions on $\{|z| = 1\}$ and Neumann boundary conditions on $\{|z| = e^{-p+t}\}$. The mean $M_t = M_{\mathbb{A}_{p-t}}$ will be represented as the imaginary part of a multivalued analytic function F_t defined on $\mathbb{A}_{p-t}$. Correspondingly, Eq. (M-cond') should be generalized to the form

$$\Im\mathrm{m}\left\{\frac{\kappa}{2}\,\partial_{xx}F_t(z; x) + V_x^{p-t}(z)\,\partial_z F_t(z; x) + \partial_t F_t(z; x)\right\} = 0, \tag{4.4}$$

where both the function F and the vector field V now depend explicitly on t (this dependence should be kept in mind throughout the section). Recall that $V_x^{p-t}(z) = 2\pi\, z\, S_x^{p-t}(z)$ where $S_x^{p-t}(z)$ is the Schwarz kernel in the annulus $\mathbb{A}_{p-t}$, as specified in Sect. 1.1. Equation (C-cond') is exactly the same as before and Hadamard's formula applies, so we find $\Im\mathrm{m}\,(\partial_x F_t(z; x))$ to be equal to a multiple of the Poisson kernel in the annulus $\mathbb{A}_{p-t}$ with zero Dirichlet boundary conditions on the outer boundary and Neumann boundary conditions on the inner one. Let $\tilde{S}_x$ be the corresponding Schwarz kernel, the holomorphic function whose real part is the Poisson kernel with these boundary conditions, that is: $\Re\mathrm{e}\,(\tilde{S}_x(z)) = \delta_x(z)$ on $\{|z| = 1\}$ and $\Im\mathrm{m}\,(\tilde{S}_x) = 0$ on $|z| = \mathrm{e}^{t-p}$. Then we can write $\partial_x F_t(z; x) = 2\mathrm{i}\lambda\ \tilde{S}_x^{p-t}(z)$.

As in the radial case, we have rotational invariance (4.3) which allows us to rewrite (4.4) as

$$\Im\mathrm{m}\,(H) \equiv 0, \quad \text{where} \tag{4.5}$$
$$H := 2\mathrm{i}\ \partial_x \tilde{S}_x^{p-t}(z) - 2\pi\ S_x^{p-t}(z)\ \tilde{S}_x^{p-t}(z) + \frac{1}{2\lambda}\ \partial_t F_t(z; x)$$

We now prove that $\Im\mathrm{m}\,(H)$ is a harmonic function in the annulus satisfying

- $\Im\mathrm{m}\,(H) = 0$ on the outer part of the boundary
- $\partial_n \Im\mathrm{m}\,(H) = 0$ on the inner part of the boundary
- $\Im\mathrm{m}\,(H)$ is bounded.

This will imply Eq. (4.5), and consequently establish (M-cond) and (C-cond). The first two boundary conditions for $\Im\mathrm{m}\,(H)$ obviously hold on $\partial\mathbb{A}_{p-t} \setminus \{x\}$: if M_{Ω_t} satisfies those conditions for all t, then so does its drift $\Im\mathrm{m}\,(H)$. So, we only need to prove that $\Im\mathrm{m}\,(H)$ has no singularity at x. Without loss of generality, assume $t = 0$. The expansions of the two Schwarz kernels at $z = x$ coincide up to constant order

$$S_x(z) = \frac{-x}{\pi(z-x)} - \frac{1}{2\pi} + \mathcal{O}(z-x) \quad \text{and} \quad \tilde{S}_x(z) = \frac{-x}{\pi(z-x)} - \frac{1}{2\pi} + \mathcal{O}(z-x),$$

as follows from the condition that their real parts give the delta function on the outer boundary. Plugging these into (4.5) shows that the possible singularities at x cancel out.

We summarize the result of this section in the following proposition:

Proposition 7 *For any $p > 0$, let $M^p(z; x)$ be the unique multi-valued harmonic function in the annulus $\mathbb{A}_p$ satisfying the following properties:*

- *$M^p(z; x)$ obeys zero Neumann boundary conditions on the inner boundary circle $\{|z| = e^{-p}\}$;*
- *$M^p(z; x)$ has a jump-Dirichlet boundary conditions on the outer boundary circle, namely, for any branch of $M^p(z; x)$ there exist $n \in \mathbb{Z}$ such that $M^p(xe^{\pm i\theta}; x) \equiv \mp\lambda + 2\lambda n = \mp\sqrt{\frac{\pi}{8}} + 2\sqrt{\frac{\pi}{8}}n$ for small positive θ, and $M^p(x, z)$ is locally constant on $\{|z| = 1\} \setminus \{x\}$.*

As the free field Φ in $\mathbb{A}_p$, $p > 0$, take the sum $\Phi(z) = \Phi_0(z) + M^p(z; x)$, where Φ_0 is a GFF in $\mathbb{A}_p$ with zero Neumann boundary conditions on $\{|z| = e^{-p}\}$ and zero Dirichlet boundary conditions on $\{|z| = 1\}$. In other domains define the free field in the same manner, using conformal invariance. Then the standard annulus SLE_4 is coupled with these free fields in the sense of Theorem 2.

4.3 More marked points on the outer boundary

In this subsection, we extend the result above to the case of additional marked points $x_1, x_2, \dots$ on the outer boundary of the annulus $\mathbb{A}_p$. The free field will have locally constant Dirichlet boundary conditions with jumps $2\lambda = \sqrt{\frac{\pi}{2}}, 2\lambda_1, 2\lambda_2, \dots$ at $x, x_1, x_2, \dots$. If we impose zero Neumann boundary conditions on the inner boundary, then, for any choice of (λ_j) we find a variant of SLE_4 which is coupled with this field. If jumps add up to zero, one can also impose Dirichlet boundary condition on the inner boundary. In all cases, drifts of driving processes are computed explicitly.

We start with the case of one additional marked point and Neumann boundary conditions on the inner boundary. Let $\tilde{S}_x(z)$ be as in the previous section.

Proposition 8 *For any $p > 0$, let $M^p(z; x, x_1)$ be the multi-valued harmonic function in the annulus $\mathbb{A}_p$ satisfying the following properties:*

- *$M^p(z; x, x_1)$ obeys zero Neumann boundary conditions on the inner boundary $\{|z| = e^{-p}\}$;*
- *$M^p(z; x, x_1)$ has a jump-Dirichlet boundary conditions on the outer part of the boundary with jumps $-2\lambda = -\sqrt{\frac{\pi}{2}}$ at x and $-2\lambda_1$ at x_1.*

Let Φ_0 be a GFF in $\mathbb{A}_p$ with zero Neumann boundary conditions on $\{|z| = e^{-p}\}$ and zero Dirichlet boundary conditions on $\{|z| = 1\}$. In $\mathbb{A}_p$, $p > 0$, take the GFF as $\Phi(z) = M^p(x, z) + \Phi_0(z)$, and for other domains use conformal invariance. These free fields are coupled with an annulus SLE_4 variant defined using (Loe) with the driving process

$$\mathrm{d}X_t = \mathrm{d}W_{4t} - \mathrm{i}\pi\rho\, \tilde{S}^{p-t}_{g_t(x_1)}(X_t)\, \tau_{X_t}\, \mathrm{d}t,$$

where W stands for the Brownian motion on $\{|z| = 1\}$ and $\rho = 2\frac{\lambda_1}{\lambda}$.

Remark 11 The letter ρ is used here analogously to the case of ordinary $\text{SLE}_\kappa(\rho)$. Indeed, the drift of the driving process of $\text{SLE}_\kappa(\rho)$ in $\mathbb{H}$ is $\frac{\rho}{X_t - g_t(x_1)} = -\mathrm{i}\pi\rho\, S^{\mathbb{H}}_{g_t(x_1)}(X_t)$, where $S^{\mathbb{H}}_x(z)$ is the Schwarz kernel in $\mathbb{H}$ with Dirichlet boundary conditions. The value $\rho = 2\frac{\lambda_1}{\lambda}$ is also what one gets in the case of simply connected domains and piecewise constant Dirichlet boundary conditions with jump of size $2\lambda_1$ at a marked point.

Proof The proof essentially repeats the one of Proposition 7. Let us stress the differences. The first basic equation (M-cond') now reads

$$\Im\mathrm{m}\left\{\frac{\kappa}{2}\partial_{xx}F + V_x(z)\partial_z F + \partial_t F + \frac{D_t}{\mathrm{i}x}\,\partial_x F - 2\pi\mathrm{i}S^{p-t}_x(x_1)\,\partial_{x_1}F\right\} = 0, \quad (4.6)$$

whereas the second one (C-cond') is exactly the same as in Proposition 7. Hence we should choose $\partial_x F_t(z; x) = 2\mathrm{i}\lambda\, \tilde{S}_{x,p-t}(z)$ with the same $\tilde{S}$ (note that this identity holds true for the choice of M made in the assertion). The rotational invariance (4.3) now reads

$$\mathrm{i}z\partial_z F + \partial_x F + \partial_{x_1} F = 0$$

and we rewrite (4.6) as

$$\Im\mathrm{m}\,(H) = 0, \quad \text{where} \tag{4.7}$$
$$H := 2\mathrm{i}\;\partial_x \tilde{S}_x^{p-t}(z) - 2\pi\;\; S_x^{p-t}(z)\;\tilde{S}_x^{p-t}(z)$$
$$+\frac{1}{2\lambda}(2\pi\mathrm{i}\;S_x^{p-t}(z)\;\partial_{x_1}F + \partial_t F_t + D_t\;\partial_x F - 2\pi\mathrm{i}\;S_x^{p-t}(x_1)\;\partial_{x_1}F).$$

As before, it suffices to show that for an appropriate choice of D_t the holomorphic function H is bounded and has zero imaginary part on the outer part of the boundary and constant real part on the inner one. The boundary conditions on $\partial\mathbb{A}_{p-t} \setminus \{x, x_1\}$ follow immediately (see the proof of Proposition 7). At x and x_1, the terms in 4.7 have singularities, so we need to check that they disappear in the total sum. The only two terms that might produce a first-order singularity at x_1 are those containing $\partial_{x_1}F$, but the coefficients in front of $\partial_{x_1}F$ cancel each other as $z \to x_1$. As we have seen in the proof of Proposition 7, $2\mathrm{i}\,\partial_x \tilde{S}_x^{p-t}(z) - 2\pi\; S_x^{p-t}(z)\;\tilde{S}_x^{p-t}(z)$ is bounded near x; thus, H might only have a first-order singularity at x produced by $2\pi\mathrm{i}\,S_x^{p-t}(z)\partial_{x_1}F + D_t\partial_x F$. The choice of D_t made in the assertion is exactly to guarantee that it vanishes. □

We now consider the case of Dirichlet boundary conditions on the inner boundary circle.

Proposition 9 *For any $p > 0$, let $M^p(z; x, x_1)$ be the unique harmonic function in the annulus $\mathbb{A}_p$ satisfying the following boundary conditions:*

- $M^p(z; x, x_1) = \lambda = \sqrt{\frac{\pi}{8}}$ *on counterclockwise arc from x_1 to x and*
- $M^p(z; x, x_1) = -\lambda$ *on counterclockwise arc from x to x_1*
- $M^p(z; x, x_1) = \mu \in \mathbb{R}$ *on the inner boundary circle $\{|z| = e^{-p}\}$*

Let $\Phi_0(z)$ be a GFF in A_p with zero Dirichlet boundary conditions. Then the GFF $M^p(x, x_1, z) + \Phi_0(z)$, transported to other domains using conformal invariance, is coupled in the sense of Theorem 2 with the following annulus SLE$_4$ variant. The driving process X_t in (Loe) is given by

$$\mathrm{d}X_t = \mathrm{d}W_{4t} + D_t\,\tau_{X_t}\,\mathrm{d}t,$$

where W_t stands for the Brownian motion on $\{|z| = 1\}$, and the drift is explicitly

$$D_t = -\mathrm{i}\pi\rho\;S_{g_t(x_1)}^{p-t}(X_t) + \frac{2\pi}{p-t}\left(\frac{\mu}{2\lambda} + \frac{L_{[X_t, g_t(x_1)]} - \pi}{2\pi}\right)$$

with $\rho = -2$ and $L_{[x,x_1]}$ denoting the length of the counterclockwise boundary arc from x to x_1.

Remark 12 Since $\rho = -2 = \kappa - 6$, it is easy to show using coordinate changes of the kind described in [17] that in the limit $p \to \infty$ one recovers a chordal SLE$_4$ in $\mathbb{D}$ from x to x_1. This limit therefore degenerates to the basic example of Schramm and Sheffield discussed in Sect. 2.2.

Remark 13 The annulus SLE with the above driving process was proposed in [7], based on considerations of regularized free-field partition function with these boundary conditions. That article also computes the probabilities that the curve passes to the left or right of the inner boundary circle and finds that there is a non-zero probability for the curve to touch the inner circle only if $-\lambda < \mu < \lambda$—as anticipated for a discontinuity line of the free field between the levels $\pm\lambda$.

Proof For the above choice of M, if F is a holomorphic function such that $M = \Im\mathrm{m}\,(F)$, we have that $\Im\mathrm{m}\,(\partial_x F)$ is equal to the Dirichlet boundary valued Poisson kernel $P_x^{p-t}(z) = \Re\mathrm{e}\,(S_x^{p-t}(z) + \frac{1}{2\pi(p-t)}\log(z))$, exactly as required by (C-cond') and Hadamard's formula.

Observe that the harmonic conjugate of M is not a single-valued function and one should be careful defining $\Re\mathrm{e}\,(F)$. A rotationally invariant definition of F is given by the following formula:

$$F_t(z; x, x_1) := -\lambda \mathrm{i} \int_x^{x_1} S_w^{p-t}(z)\,|\mathrm{d}w| + \lambda \mathrm{i} \int_{x_1}^{x} S_w^{p-t}(z)\,|\mathrm{d}w| + \frac{\mathrm{i}\,\log(z/x)}{(t-p)} \times \left(\mu - 2\lambda \frac{\pi - L_{[x,x_1]}}{2\pi}\right),$$

the integrals being along the boundary in a counterclockwise direction. The first two terms above produce a function with correct boundary values on the outer boundary and constant (but different from μ) imaginary part on the inner boundary. The third term is introduced to fix the values on the inner boundary back to μ. We have the following expressions for the derivatives of F:

$$\partial_x F = 2\lambda \mathrm{i}\,(S_x^{p-t}(z) + R_1) \tag{4.8}$$

$$\partial_{x_1} F = 2\lambda \mathrm{i}\,(-S_{x_1}^{p-t}(z) + R_2), \quad \text{where} \tag{4.9}$$

$$R_1(x, x_1, z, t) = -\mathrm{i}\,\frac{\frac{\mu}{2\lambda}}{t-p} + \mathrm{i}\,\frac{\pi - L_{[x,x_1]}}{2\pi(t-p)} - \frac{\log(z/x)}{2\pi(t-p)}$$

$$R_2(x, x_1, z, t) = \frac{\log \frac{z}{x}}{2\pi(t-p)}$$

The remaining steps closely follow the proof of Proposition 8. We write the first basic equation (M-cond) using the above expressions and rotational invariance as

$$\Im\mathrm{m}\,(H) = 0, \quad \text{where} \tag{4.10}$$

$$\begin{aligned} H := {} & 2\mathrm{i}\,\partial_x(S_x^{p-t}(z) + R_1) - 2\pi S_x^{p-t}(z)(S_x^{p-t}(z) + R_1) \\ & + 2\pi(S_x^{p-t}(x_1) - S_x^{p-t}(z))(-S_{x_1}^{p-t}(z) + R_2) \\ & + \frac{1}{2\lambda}(\mathrm{i}D_t\ (S_x^{p-t} + R_1) + \partial_t F) \end{aligned}$$

Now the function H is possibly multi-valued because of the logarithm appearing in the definition of F. To prove (4.10) we check that the holomorphic function H satisfies the following properties:

- $\Im\mathrm{m}\,(H) = 0$ on $\partial\mathbb{A}_{p-t} \setminus \{x, x_1\}$;
- Any branch of $\Im\mathrm{m}\,(H)$ is bounded near x and x_1.

Since H clearly cannot grow faster than linearly at infinity on the universal cover, these conditions guarantee (4.10). Exactly as in the proof of Propositions 7 and 8, we show the first condition above and the boundedness near x_1. It remains to handle possible singularities at x. The fact that $2\mathrm{i}\,\partial_x(S_x^{p-t}(z) + R_1) - 2\pi\; S_x^{p-t}(z)\, S_x^{p-t}(z)$ is bounded near x follows from the explicit expansion of $S_x^p(z)$ as $z \to x$ given in the Proof of Proposition 7. So we look at the remaining three terms in H that have (first-order) singularities at x:

$$-2\pi\; S_x^{p-t}(z)\, R_1 + 2\pi\; S_x^{p-t}(z)\, (S_{x_1}^{p-t} - R_2) + \mathrm{i}\, D_t\, S_x^{p-t}(z).$$

We see that since $R_1 + R_2$ does not contain $\log z$, the choice $\mathrm{i}\, D_t = 2\pi(-S_{x_1}(x) + (R_1 + R_2)|_{z=x})$ guarantees vanishing of the singularity for all branches of H, and we are done. □

Remark 14 The extension of Propositions 8 and 9 to the case of several marked point $x_1, x_2, \ldots$ with jumps $2\lambda_1, 2\lambda_2, \ldots$ is straightforward; proofs are literally the same. The drift term D_t is just the sum $D_t = \sum_j \frac{\lambda_j}{\lambda} D_t^j$ where $D_t^j(x, x_j)$ is the drift we would have if we only had one jump of size 2λ at x_j. Indeed, one may observe that procedure of determining the drift term for F is in fact linear in F. One should remember, however, that in the Dirichlet case the construction only makes sense if all jumps add up to 0, including the jump of size $2\lambda = \sqrt{\frac{\pi}{2}}$ at x.

4.4 Compactified GFF with a marked point on the inner boundary

In this section we consider the case when an additional marked point x_1 is on the inner part of the boundary. The mean of the field M will be a multi-valued harmonic function obeying Dirichlet boundary conditions with jumps -2λ both at x and x_1. However, these conditions do not define M completely, we should also define the increment of a fixed branch of the function M along the radius, say, from e^{-p} to 1.

Let $\hat{M}^p(z)$ be the unique multi-valued harmonic function in $\mathbb{A}_p$ such that any continuous branch in any sector $\{r\mathrm{e}^{\mathrm{i}\theta} : \mathrm{e}^{-p} < r < 1,\; \theta_1 < \theta < \theta_2\}$ determined by angles $\theta_2 \in [0, 2\pi[$ and $\theta_1 \in]\theta_2 - 2\pi, \theta_2[$, has boundary values $\lambda\,(\mathrm{sign}(\theta) + 2n)$ at $z = \mathrm{e}^{\mathrm{i}\theta}$ and $z = \mathrm{e}^{-p+\mathrm{i}\theta}$. The function $M_t(z; x, x_1)$ in $\mathbb{A}_{p-t}$ is constructed by continuously moving the discontinuity points of boundary conditions of $\hat{M}^{p-t}$ from 1 to x and from e^{-p} to x_1. Hence, M is in fact a multi-valued harmonic function in z that depends on x, x_1, t and the choice of $\arg x_1 - \arg x$. More precisely, represent $\hat{M}^p$ as the imaginary part of a multivalued analytic function $\hat{F}^p$, and M_t as the imaginary part of

$$F_t(z; \arg x, \arg x_1) = \hat{F}^{p-t}(z) + 2\lambda\mathrm{i} \int_0^{\arg x} S^{p-t}_{\mathrm{e}^{\mathrm{i}\theta}}(z)\, \mathrm{d}\theta + 2\lambda\mathrm{i} \int_0^{\arg x_1} S^{\mathrm{inv.};p-t}_{\mathrm{e}^{\mathrm{i}\theta+t-p}}(z)\, \mathrm{d}\theta$$
$$- 2\lambda\mathrm{i} \frac{\log \frac{z}{x}}{2\pi(t-p)} \arg x - 2\lambda\mathrm{i} \frac{t-p-\log\frac{z}{x}}{2\pi(t-p)} \arg x_1.$$

Here we use $S^{\mathrm{inv.};p}_y(z) := S^p_{\mathrm{e}^{-p}/y}(\mathrm{e}^{-p}/z)$. With this definition, the function F is invariant under rotations. We will sometimes write it as function of x and x_1 where the branch of the argument will be clear from the context. We have the following proposition:

Proposition 10 *Let M be as above, and let $\Phi_0(z)$ be a GFF in $\mathbb{A}_p$ with zero Dirichlet boundary conditions. Consider the multi-valued GFF defined in $\mathbb{A}_p$ as $M^p(x, x_1, z) + \Phi_0(z)$ and in other domains by conformal invariance. It is coupled in the sense of Theorem 2 with the annulus SLE$_4$ variant whose driving process X_t in (Loe) satisfies*

$$\mathrm{d}X_t = \mathrm{d}W_{4t} + D_t\, \tau_{X_t}\, \mathrm{d}t \qquad \textit{with}$$
$$D_t = -2\pi\mathrm{i}\left(S^{\mathrm{inv.};p-t}_{g_t(x_1)}(X_t) - \frac{1}{2\pi}\right) + \frac{\arg g_t(x_1) - \arg X_t}{p-t}.$$

Proof The proof literally repeats the one of Proposition 9. We now have, as in Eqs. (4.8) and (4.9),

$$\partial_x F = 2\lambda\mathrm{i}\, (S^{p-t}_x(z) + R_1)$$
$$\mathrm{e}^{-p}\, \partial_{x_1} F = 2\lambda\mathrm{i}\, (S^{\mathrm{inv.};p-t}_{x_1}(z) + R_2), \quad \text{where}$$
$$R_1(x, x_1, z, t) = -\frac{\log\frac{z}{x}}{2\pi(t-p)} + \mathrm{i}\,\frac{(\arg x - \arg x_1)}{2\pi(t-p)}$$
$$R_2(x, z, t) = -\frac{1}{2\pi}\left(1 - \frac{\log\frac{z}{x}}{t-p}\right).$$

and we find that the basic equations are verified provided that

$$\mathrm{i}\, D_t = 2\pi(S^{\mathrm{inv.}}_{x_1}(x) + (R_1 + R_2)|_{z=x}).$$

□

Remark 15 One can write explicitly the stochastic differential equation satisfied by the process $\arg(g_t(x_1)) - \arg(X_t)$. It turns out to be a Brownian bridge which at time $t = p$ hits 0. Therefore, at $t = p$, the curve hits x_1 with a winding determined by the initial choice of $\arg(x_1) - \arg(x)$.

4.5 Generalizations to $\kappa \neq 4$ for Dirichlet boundary conditions

In the case of Dirichlet boundary conditions, one can generalize the previous couplings to $\kappa \neq 4$. As the example of the section (2.2) shows, the rule that associates a field to a domain is not conformally invariant: if we have a conformal map $\varphi : \Omega_1 \to \Omega_2$, then

$$M_{\Omega_1;x_1,x_2,\ldots}(z) = M_{\Omega_2;\varphi(x_1),\varphi(x_2),\ldots}(\varphi(z)) + \alpha_\kappa \ \arg \varphi'(z), \tag{4.11}$$

where $\alpha_\kappa = \frac{4-\kappa}{2\sqrt{2\pi\kappa}}$ as in Eq. (2.12). The covariance, however, is still the Dirichlet Green's function. Consider the annulus $\mathbb{A}_p$ with two marked points $x \in \{z : |z| = 1\}$, $x_1 \in \partial\mathbb{A}_p$, and let $M_4^p(z, x, x_1)$ be one of the functions M^p defined in Proposition 9 or 10. We define

$$M_\kappa^p(z, x, x_1) := \sqrt{\frac{4}{\kappa}} M_4^p(z, x, x_1) - \alpha_\kappa \ \arg z.$$

This is a multi-valued harmonic function (with a single-valued derivative); the monodromy is equal to $(\kappa - 6)\lambda_\kappa$. For an arbitrary doubly connected domain, we define the mean of the field by conformal map to an annulus and the rule (4.11); in particular, for $\mathbb{A}_p \backslash K_t$ we have

$$M_\kappa^{\mathbb{A}_p \backslash K_t}(z, x, x_1) = \sqrt{\frac{4}{\kappa}} M_4^{p-t}(g_t(z); g_t(x), g_t(x_1)) + \alpha_\kappa \ (\arg g_t'(z) - \arg g_t(z)). \tag{4.12}$$

We have the following proposition:

Proposition 11 *A GFF defined as above (with marked point on the outer or inner boundary) is coupled with annulus SLE defined using* (Loe) *with the driving process*

$$\mathrm{d}X_t = \mathrm{d}W_{\kappa t} + D_t \, \tau_{X_t} \, \mathrm{d}t,$$

D_t being the same as in Proposition 9 or 10 correspondingly.

Proof The additional term $\alpha_\kappa \ (\arg g_t'(z) - \arg g_t(z))$ has finite variation; hence the proof of (C-cond') will be the same as before (we have adjusted the coefficient in front of M to compensate the change of speed for $W_{\kappa t}$). Note, however, that without that term the proof of Proposition 9 (correspondingly Proposition 10) would fail for $\kappa \neq 4$ because the coefficient in front of the first term of the definition of $\tilde{F}$ [see Eq. (4.10)] changes from 2 to $\frac{\kappa}{2}$; hence the second-order singularities at x would not cancel out anymore. We now show that the additional term exactly compensates this effect, without destroying zero Dirichlet boundary conditions elsewhere.

Simple geometric considerations show that $\mathrm{d}(\arg g_t'(z) - \arg g_t(z)) = 0$ when $g_t(z) \in \partial\mathbb{A}_{p-t}\backslash\{X_t\}$. One has

$$\partial_t \log g_t'(z) = V_{X_t}'(g_t(z)) = 2\pi g_t(z) S_{X_t}'(g_t(z)) + 2\pi S_{X_t}(z) \quad \text{and}$$
$$\partial_t \log g_t(z) = 2\pi S_{X_t}(z).$$

Recall the rotational invariance of the Schwarz kernel: $\partial_x S_x(z) + \mathrm{i}z\, S_x'(z) = 0$, and the fact that the second-order singularity of H comes from its first term $\frac{\kappa}{2}\mathrm{i}\partial_x S_{x,p-t}(z)$. Comparing the coefficients finishes the proof. □

4.6 Some remarks about the multiply connected case

It is natural to ask whether the approach of the present paper generalizes to multiply connected cases or to the case of Riemannian surfaces. It turns out that indeed, given a free field with some reasonable boundary conditions, one can find a unique Loewner chain coupled with that field. Although in general we are unable to provide explicit expressions for the drift, the following simple observation readily generalizes to the multiply-connected case:

Proposition 12 *Let $M^p(z, x, \ldots)$ be any of the one-point function that appeared in Sect. 4.3 or 4.4. Let $\tilde{M}$ be its harmonic conjugate, and let $N(t)$ be the increment of $\tilde{M}^{p-t}(g_t(z), X_t, \ldots)$ around the hole. Then $N(t)$ is a local martingale.*

This proposition is almost trivial, since M is a martingale for any given point, and N can be expressed as a linear functional of M. We leave the details to the reader.

In the annulus case the monodromy of $\tilde{M}^p$ can be expressed in terms of p and the difference of mean values of M^p on the inner and outer circles. Hence the proposition provides another way to derive the Brownian bridge law mentioned in the end of Sect. 4.4, as well as to generalize this law to the case of arbitrary number of marked points both on the inner and outer boundary.

Acknowlegments Work supported by Swiss National Science Foundation and ERC AG CONFRA.

A Non-commutation at $\kappa \neq 4$ for general boundary conditions

This appendix discusses a difference between Dirichlet boundary conditions and other boundary conditions concerning the couplings with SLEs at $\kappa \neq 4$. In the case of Schramm and Sheffield treated in Sect. 2.2 as well as those of Sect. 4.5 we have remarked that for the coupling with SLE variants with $\kappa \neq 4$, it suffices to modify the boundary conditions of the one point function M by a harmonic interpolation of the winding of the boundary. In other cases no such claims were made, and we now explain why these cases indeed do not admit a generalization of this sort.

For the sake of concreteness we detail the argument only in the simplest case of combined jump-Dirichlet and Riemann–Hilbert boundary conditions as treated in Sect. 3.1. Recall that $\partial_x F$ is determined by (C-cond') and the Hadamard formula. One then defines, as in (2.10),

$$M_{\Omega_t}(z) = \Im\mathrm{m}\,(F(g_t(z); X_t) + E_t(z)),$$

which contains a process of finite variation $(E_t(z))_{t\geq 0}$ introduced to restore the martingale property of the mean (M-cond') at the cost of relaxing strict conformal invariance. Concretely,

$$E_t(z) = \int_0^t \Im\mathrm{m}\,(J_{X_s}(g_s(z)))\,\mathrm{d}s, \tag{A.1}$$

where $J_x(z)$ is a multiple of the derivative of the appropriate Schwarz kernel, see Eqs. (2.11) and (3.8). A question naturally arises: is the modified formula for M_{Ω_t} consistent with having a function $M_{\Omega;x,x_1,\ldots,x_n}$ associated with any domain with marked points? Does (A.1) depend on the full history $(g_s)_{s\in[0,t]}$ of the Loewner chain, or can it be expressed as a function of domain Ω_t only, as is the case in (2.12)?

Imagine two different Loewner chains that in the end uniformize the same hull. The prototype is a hull $K = K_- \cup K_+$ consisting of two small pieces K_+, K_- away from each other, located roughly at $\xi_+, \xi_- \in \partial\Omega_0$. We can uniformize K by first uniformizing one piece and then what remains of the other. Suppose that the local half plane capacities of K_+ and K_- are ε_+ and ε_-, respectively. In the calculations below we keep track of terms of order $\varepsilon_\pm$ as well as the second-order cross terms of type $\varepsilon_+\varepsilon_-$, but we omit other second-order and higher-order terms. Write the uniformizing maps of complements of $K_\pm$ constructed by a Loewner chain (Loe) as

$$\begin{aligned} &g_\pm : \Omega_0 \setminus K_\pm \to \Omega_0 \\ &g_\pm(z) \approx z + \varepsilon_\pm\, V_{\xi_\pm}(z) + \cdots . \end{aligned}$$

After having thus removed one piece $K_\pm$, we are left with the hull $\widetilde{K}_\mp = g_\pm(K_\mp)$ whose local half plane capacity is

$$\widetilde{\varepsilon}_\mp \approx \varepsilon_\mp\, |(g_\pm)'(\xi_\mp)|^2 + \cdots \approx \varepsilon_\mp + 2\varepsilon_\pm\varepsilon_\mp\, (V_{\xi_\pm})'(\xi_\mp) + \cdots$$

and the hull $\widetilde{K}_\mp$ can be uniformized by a map constructed by the same Loewner fields

$$\begin{aligned} &\widetilde{g}_\mp : \Omega_0 \setminus \widetilde{K}_\mp \to \Omega_0 \\ &\widetilde{g}_\mp(z) \approx z + \widetilde{\varepsilon}_\mp\, V_{\widetilde{\xi}_\mp}(z) + \cdots , \end{aligned}$$

where $\widetilde{\xi}_\mp$ is the location of the hull $\widetilde{K}_\mp$

$$\widetilde{\xi}_\mp = g_\pm(\xi_\mp) \approx \xi_\mp + \varepsilon_\pm\, V_{\xi_\pm}(\xi^\mp) + \cdots .$$

We then have two conformal maps

$$\widetilde{g}_+ \circ g_- \ \text{ and } \ \widetilde{g}_- \circ g_+ \quad : \quad \Omega_0 \setminus K \to \Omega_0.$$

In practise the Loewner vector fields are chosen to be the unique ones preserving some normalization condition, so the two maps must actually be equal. In any case, we can ask whether formula (A.1) gives the same answer for the hull K built in the two possible ways. The two expressions for E_t are approximately

$$\varepsilon_{\mp}\, J_{\xi_{\mp}}(z) + \widetilde{\varepsilon}_{\pm}\, J_{\widetilde{\xi}_{\pm}}(g_{\mp}(z)),$$

so their difference can be expressed expanding in all small parameters

$$\begin{aligned}\Delta E_t \approx \varepsilon_+\varepsilon_- \Big\{ & 2\,(V_{\xi_-})'(\xi_+)\, J_{\xi_+}(z) - 2\,(V_{\xi_+})'(\xi_-)\, J_{\xi_-}(z)\\ & + V_{\xi_-}(\xi_+)\,\partial_x J_{\xi_+}(z) - V_{\xi_+}(\xi_-)\,\partial_x J_{\xi_-}(z)\\ & + V_{\xi_-}(z)\,\partial_z J_{\xi_+}(z) - V_{\xi_+}(z)\,\partial_z J_{\xi_-}(z)\Big\} + \cdots \end{aligned} \tag{A.2}$$

For E_t to be a function of the hull K only, and not of the history of the Loewner chain, it is necessary that J satisfies the functional equation that makes the above expression vanish identically.

As is already clear from considerations of the chordal SLE_κ coupling, in particular Eq. (2.12), the function $J_x(z) = \text{const.}\ \frac{1}{(z-x)^2}$ satisfies the appropriate equation with $V_x(z) = \frac{2}{z-x}$ chosen according to the Loewner flow (Loe).

In the strip $\mathbb{S}$ we considered jump-Dirichlet boundary conditions on $\mathbb{R}$ and Riemann–Hilbert on $\mathbb{R} + i\pi$. We chose correspondingly $J_x(z) = \text{const.}\ \partial_x \tilde{S}_x(z)$, where $\tilde{S}_x(z)$ is the Schwarz kernel (3.7) with the same boundary conditions. A direct computation shows that with the appropriate Loewner vector field $V_x(z) = \coth(\frac{z-x}{2})$; this $J_x(z)$ produces a non vanishing difference in (A.2). It is therefore not possible to generalize the coupling of Sect. 3.1 to $\kappa \neq 4$ in the manner analogous to Dirichlet boundary conditions.

B Local half-plane capacity and Proposition 1

Most of the statements of Proposition 1 are standard Loewner chain techniques (and may be found in the literature for all particular cases we deal with in this paper), so we leave the proof to the reader. We will only discuss the slightly less standard statement about the local half-plane capacity.

Let Ω be a planar domain, $x \in \partial\Omega$, and let $\partial\Omega$ be analytic in a neighborhood of x. Let (K_t) be a family of growing compact hulls in $\overline{\Omega}$, $\lim\limits_{t\to 0} K_t = \{x\}$. Henceforth we assume that $x = 0$, the tangent to the boundary at x is parallel to the real line, and that the inner normal at 0 points to the upper half-plane.

Let Ψ be a harmonic function in $\Omega\backslash K_t$ with the following boundary conditions:

- $\Psi(z) = \text{dist}(z, \partial\Omega)$ on ∂K_t
- $\Psi(z) = 0$ on $\partial\Omega\backslash K_t$

Let $r > 0$ be small enough, so that $\Omega \cap \{|z| = r\}$ consists of one arc $\{re^{i\theta} : \theta_1 < \theta < \theta_2\}$. If the diameter of K_t does not exceed r, define

$$L^{\Omega}_{K_t,r} = \frac{1}{\pi} \int_{\theta_1}^{\theta_2} \Psi(re^{i\theta}) r \sin(\theta)\, d\theta. \tag{B.1}$$

If $\Omega = \mathbb{H}$, then $L^{\Omega}_{K_t,r}$ is well-known to be the half-plane capacity of K_t. We will thus call this quantity the *local half-plane capacity at distance* r.

It is easy to see that $L^{\Omega}_{K_t,r}$ satisfies the following two properties that express its stability under slight changes of the domain:

- Let $\phi : \Omega_1 \to \Omega_2$ be a conformal map such that $\phi(0) = 0$ and $\phi'(0) = 1$. Then $|\frac{L^{\Omega_1}_{K_t,r}}{L^{\Omega_2}_{K_t,r}} - 1| \le Cr$.
- Let $R > r$, and $\Omega_1 \cap B_R(0) = \Omega_2 \cap B_R(0)$. Then $|\frac{L^{\Omega_1}_{K_t,r}}{L^{\Omega_2}_{K_t,r}} - 1| \le C\frac{r}{R}$.

These properties allow us to define $\partial_t \mathrm{lhcap}(K_t)|_{t=0} := \lim_{r\to 0} \partial_t L^{\Omega}_{K_t,r}$.

It remains unchanged under conformal maps ϕ as in the first property above, and is equal to the derivative of the half-plane capacity of K_t if $\partial\Omega$ coincides with the real line in some neighborhood of zero. Henceforth we assume without loss of generality that this is the case.

Now, let K_t be generated by a Loewner chain as in Proposition 1. We first claim that, when computing $\partial_t \mathrm{lhcap}(K_t)|_{t=0}$, we can replace $\Psi(z)$ by $\Im\mathrm{m}\, z - \Im\mathrm{m}\, g_t(z)$ in the integral (B.1). Indeed, the difference $H(z) := \Psi(z) - \Im\mathrm{m}\, z + \Im\mathrm{m}\, g_t(z)$ is a harmonic function; $H(z) \equiv 0$ on $\partial\Omega \cap B_R(0)$ for some constant R, and $|H(z)| \le Ct$ elsewhere on $\partial\Omega$. Hence $|H(re^{i\theta})| \le C\frac{r}{R}t$, and this is negligible when we take r to zero.

However, we have

$$\begin{aligned}\partial_t \Im\mathrm{m}\, g_t(z)|_{t=0} &= \Im\mathrm{m}\,(V_0(z)) \\ &= \Im\mathrm{m}\left(\frac{2}{z}\right) + O(1) = \partial_t \Im\mathrm{m}\, h_t(z)|_{t=0} + O(1), \quad r \to 0,\end{aligned}$$

where $h_t(z)$ is the conformal map from $\mathbb{H}\setminus K_t$ to $\mathbb{H}$ (i.e. the solution to the half-plane Loewner equation). Since in the half-plane the formula (B.1) defines the half-plane capacity, we are done.

References

1. Bauer, M., Bernard, D.: SLE, CFT and zig-zag probabilities. In: Proceedings of the conference 'Conformal Invariance and Random Spatial Processes', Edinburgh, 2003
2. Bauer, M., Bernard, D.: 2D growth processes: SLE and Loewner chains. Phys. Rep. **432**(3–4), 115–222 (2006). arXiv:math-ph/0602049
3. Bauer, M., Bernard, D., Cantini, L.: Off-critical SLE(2) and SLE(4): a field theory approach. J. Stat. Mech. P07037 (2009). arXiv:0903.1023

4. Bauer, M., Bernard, D., Houdayer, J.: Dipolar stochastic Loewner evolutions. J. Stat. Mech. (3), P03001, 18 pp (2005, electronic)
5. Cardy, J.: SLE(kappa,rho) and Conformal Field Theory (2004). arXiv:math-ph/0412033
6. Dubédat, J.: SLE and the free field: Partition functions and couplings. J. Am. Math. Soc. **22**, 995–1054 (2009). arXiv:0712.3018
7. Hagendorf, C., Bauer, M., Bernard, D.: The Gaussian free field and SLE(4) on doubly connected domains. J. Stat. Phys. **140**, 1–26 (2010). arXiv:1001.4501
8. Kang, N.-G., Makarov, N.: Gaussian free field and conformal field theory. arXiv:1101.1024
9. Kytölä, K.: On conformal field theory of SLE(kappa, rho). J. Stat. Phys. **123**(6), 1169–1181 (2006). arXiv:math-ph/0504057
10. Lawler, G.F.: Conformally invariant processes in the plane. In: Mathematical Surveys and Monographs, vol. 114. American Mathematical Society, Providence, RI, 2005
11. Makarov, N., Smirnov, S.: Off-critical lattice models and massive SLEs. In: Proceedings of ICMP, 2009 (to appear)
12. Makarov, N., Zhan, D.: in preparation
13. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. Israel J. Math. **118**, 221–288 (2000). arXiv:math/9904022
14. Schramm, O., Sheffield, S.: Harmonic explorer and its convergence to SLE_4. Ann. Probab. **33**(6), 2127–2148 (2005). arXiv:math.PR/0310210
15. Sheffield, S.: Local sets of the Gaussian Free Field. Presentation at "Percolation, SLE, and related topics Workshop", Fields Institute, Toronto, 2005. http://www.fields.utoronto.ca/audio/05-06/#percolation_SLE
16. Schramm, O., Sheffield, S.: Contour lines of the two-dimensional discrete Gaussian free field. Acta Math. **202**, 21–137 (2009). arXiv:math.PR/0605337
17. Schramm, O., Wilson, D.B.: SLE coordinate changes. New York J. Math. **11**, 659–669 (2005, electronic). arXiv:math.PR/0505368
18. Werner, W.: Random planar curves and Schramm-Loewner evolutions. In: Lectures on probability theory and statistics. Lecture Notes in Math., vol. 1840, pp. 107–195. Springer, Berlin (2004)
19. Zhan, D.: Stochastic Loewner evolution in doubly connected domains. Probab. Theory Relat. Fields **129**(3), 340–380 (2004). arXiv:math/0310350
20. Zhan, D.: Some properties of annulus SLE. Electron. J. Probab. **11**, Paper no. 41, 10691093 (2006). arXiv:math/0610304
21. Zhan, D.: The scaling limits of planar LERW in finitely connected domains. Ann. Probab. **36**(2), 467–529 (2008). arXiv:math/0610304